Representation theory for polyadic algebras*

by

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The notion of a polyadic algebra was introduced by Halmos to reflect algebraically the essentials of the first order predicate logic. That the notion is adequate for this purpose was demonstrated by Halmos's representation theorem for *locally finite* polyadic algebras of infinite degree (1). Algebraically, it is still interesting to ask for stronger representation results. Also, current increased interest in logic with infinitely long expressions makes the possibility of a stronger result interesting to logicians.

In this paper we shall prove the following strong extension of Halmos's result: every polyadic algebra of infinite degree is representable (2). Our proof is purely algebraic and is, generally speaking, close in method to Halmos's work in this field. In the paper [12] immediately following this paper, H. J. Keisler gives a metamathematical proof of this result, using the correspondence between polyadic algebras and infinitary logics (3).

We shall preface our proof with an outline of the elementary theory of polyadic algebras, in which, besides results of Halmos, we shall give some lemmas needed in the remainder of the paper (§ 1 and § 2). The first part of our proof consists in an embedding process known as *dilation*. This process is applicable to certain more general relational systems, and we shall first give a proof of the more general result (§ 3). The process is extended to the polyadic case in § 4. An alternative method of arriving at the main result of § 4 is outlined in § 5. The second part of the representation proof follows a pattern which may be traced back to the

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* The work of the first author on this paper was done while he was a fellow of the Summer Research Institute of the Canadian Mathematical Congress.

(1) See [5] and [7].

(2) See the abstract [2].

(3) Keisler's results were previously announced in [11]. Our main result was discovered independently by Daigneault and Keisler; Daigneault's proof was mathematical, and Keisler's metamathematical. After Keisler's result was announced, but before Daigneault's mathematical proof was announced, Monk devised a mathematical proof for the result. The present paper combines methods of Daigneault and Monk.

well-known completeness proofs of Henkin, and Rasiowa and Sikorski (§ 6) [4].

Besides the proof of the main theorem we prove other interesting results about dilations and representations. The reader interested only in the main representation result (Theorem 6.4) needs only to follow the following path: read § 1 and § 2; omit Theorem 3.4 in § 3; read § 4; omit § 5; omit Theorems 6.1, 6.2 and 6.3 in § 6. Those interested only in the existence of representations and not in the cardinality conditions of Theorem 6.4 may simplify the proof even further by eliminating the words "minimal" and "faithful" in our proof, making the corresponding simplifications.

We shall use common set-theoretical notions and terminology, but it is perhaps worthwhile to explicitly state some of the less common notations here [5]. The identity function on a set A, i.e., the set $\{(x, x): x \in A\}$, is denoted by δ_A. A function f restricted to a set A, i.e., the set $\{(x, y): f(x) = y \text{ and } x \in A\}$, is denoted by $f|A$. The set of all functions with domain I and range included in X is denoted by X^I or sometimes by $F(I, X)$. Cardinal numbers are denoted by small German letters $\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}$, etc. The least cardinal greater than a cardinal $\mathfrak{m}$ is denoted by $\mathfrak{m}^+$. Cardinal numbers are identified with initial ordinal numbers. A symbol like $\mathfrak{n}^{\mathfrak{s}}$ also denotes the cardinal of the set of all functions from $\mathfrak{s}$ to $\mathfrak{n}$. The letter ω is used to denote the set of all natural numbers, i.e., $\omega = \{0, 1, 2, \ldots\}$.

§ 1. Polyadic algebras. In this section the basic definitions of the theory of polyadic algebras are given. More details may be found in the papers [6] and [7] of Halmos.

A quantifier on a Boolean algebra A is a mapping $\exists$ of A into A such that the following conditions hold for all $p, q \in A$:

(Q₁) $\exists 0 = 0$;

(Q₂) $p \leqslant \exists p$;

(Q₃) $\exists(p \wedge \exists q) = \exists p \wedge \exists q$.

THEOREM 1.1. *If $\exists$ is a quantifier on a Boolean algebra A and $p, q \in A$, then*:

(i) $\exists 1 = 1$;

(ii) $\exists\exists = \exists$;

(iii) *if* $p \leqslant q$, *then* $\exists p \leqslant \exists q$;

(iv) $\exists(\exists p)' = (\exists p)'$;

(v) $\exists(p \vee q) = \exists p \vee \exists q$.

[4] See [9], [4], and [17]. The authors arrived at this second proof independently; the second author arrived at the proof by generalizing an unpublished proof of Tarski that locally finite infinite dimensional cylindric algebras are representable.

[5] In general we follow the notation of [8].

For the proof, see [6], pp. 220-222. This theorem will be used constantly without citation.

We now define a concrete operation which generates quantifiers. Let B be a Boolean algebra, and X and I non-empty sets. For each $J \subset I$ and $p \in F(X^I, B)$ let $\exists^f(J)p$ be the element of $F(X^I, B)$ defined by

$$(\mathrm{F}_1) \qquad \exists^f(J)p(x) = \bigvee \{p(y): x|(I-J) = y|(I-J)\},$$

provided this supremum exists for every $x \in X^I$.

Another operation which is useful in this context is defined as follows. For each $\tau \in I^I$ and $p \in F(X^I, B)$ let

$$(\mathrm{F}_2) \qquad S^f(\tau)p(x) = p(x\tau),$$

for all $x \in X^I$.

A *functional polyadic I-algebra with domain X and value algebra B* is a quadruple $(A, I, S^f, \exists^f)$ such that A is a Boolean subalgebra of $F(X^I, B)$, A is closed under the operation $S^f(\tau)$ for each $\tau \in I^I$, and $\exists^f(J)p$ exists and is in A whenever $J \subset I$ and $p \in A$. Of special interest is the case where B is the two element Boolean algebra, which is henceforth denoted by O.

The general notion of a polyadic algebra is obtained from this concept by abstraction. A *polyadic algebra* is a quadruple $(A, I, S, \exists)$ such that A is a Boolean algebra, S is a mapping from I^I to Boolean endomorphisms of A, $\exists$ is a mapping from the set of all subsets of I to quantifiers of A, and the following conditions hold:

(P₁) $S(\delta_I) = \delta_A$;

(P₂) $S(\sigma\tau) = S(\sigma)S(\tau)$ *whenever* $\sigma, \tau \in I^I$;

(P₃) $\exists(0) = \delta_A$;

(P₄) $\exists(J \cup K) = \exists(J)\exists(K)$ *whenever* $J, K \subset I$;

(P₅) *if* $\sigma, \tau \in I^I$, *if* $J \subset I$, *and if* $\sigma|(I-J) = \tau|(I-J)$, *then* $S(\sigma)\exists(J) = S(\tau)\exists(J)$;

(P₆) *if* $\tau \in I^I$, *if* $J \subset I$, *and if* $\tau|\tau^{-1}J$ *is biunique, then* $\exists(J)S(\tau) = S(\tau)\exists(\tau^{-1}J)$.

We shall say loosely that A is a *polyadic I-algebra*. Members of I are called *variables*. The cardinal number of I is called the *degree* of A.

The following theorem shows that the abstraction made in the preceding definition is sound.

THEOREM 1.2. *Every functional polyadic algebra is a polyadic algebra.*

§ 2. Preliminary results. Throughout this section let $(A, I, S, \exists)$ be a fixed but arbitrary polyadic algebra. If $J \subset I$, an element p of A is said to be *independent* of J if $\exists(J)p = p$. The dual concept is that of a support: a subset J of I supports $p \in \mathbf{A}$ if p is independent of $I-J$. The following theorem summarizes the most elementary facts about supports and independence.

THEOREM 2.1. (i) *If* $p \in \mathbf{A}$, *then* $\{J\colon p$ *is independent of* $J\}$ *is an ideal, and* $\{J\colon J$ *supports* $p\}$ *is a filter, in the Boolean algebra of all subsets of* I;

(ii) *if* $J \subset I$, *then* $\{p\colon p$ *is independent of* $J\}$ *and* $\{p\colon J$ *supports* $p\}$ *are Boolean subalgebras of* $\mathbf{A}$;

(iii) *if* p *is independent of* J *and* $K \subset I$, *then* $\exists(K)p = \exists(K-J)p$;

(iv) *if* J *supports* p *and* $K \subset I$, *then* $\exists(K)p = \exists(K \cap J)p$;

(v) *if* p *is independent of* J *and* $K \subset I$, *then* $\exists(K)p$ *is independent of* J *and of* $J \cup K$;

(vi) *if* J *supports* p *and* $K \subset I$, *then both* J *and* $J-K$ *support* $\exists(K)p$;

(vii) *if* p *is independent of* J *and* $\sigma, \tau \in I^I$ *with* $\sigma|(I-J) = \tau|(I-J)$, *then* $\mathbf{S}(\sigma)p = \mathbf{S}(\tau)p$.

The proof may be found in [7], pp. 276-277.

The following result of the same character requires a proof here, since Halmos obtained a weaker theorem [6].

THEOREM 2.2. *If* J *supports* p *and* $\tau \in I^I$, *then* τJ *supports* $\mathbf{S}(\tau)p$.

Proof. If $J = 0$, then by Theorem 2.1 (vii) we have $\mathbf{S}(\tau)p = \mathbf{S}(\delta_I)p = p$, and so $\tau J = 0$ supports $\mathbf{S}(\tau)p$. Hence assume that $J \neq 0$. Choose σ such that $\sigma|J = \tau|J$ and $\sigma(I-J) \subset \tau J$. Thus $\sigma^{-1}(I-\tau J) = 0$. Hence

$$\begin{aligned}
\exists(I-\tau J)\mathbf{S}(\tau)p &= \exists(I-\tau J)\mathbf{S}(\sigma)p && 2.1\ (\mathrm{vii})\\
&= \mathbf{S}(\sigma)\exists(0)p && (\mathrm{P}_6)\\
&= \mathbf{S}(\tau)p && 2.1\ (\mathrm{vii})
\end{aligned}$$

Q.E.D.

The next three somewhat involved lemmas will be found useful later.

LEMMA 2.3. *Suppose* $p \in \mathbf{A}$, $\sigma, \tau \in I^I$, $J, K \subset I$, *and the following conditions are satisfied:*

(i) $\sigma|J$ *is biunique;*

(ii) $\tau(K-J) \cap \sigma J = 0$;

(iii) K *supports* p;

(iv) $\sigma|(I-J) = \tau|(I-J)$.

Then $\mathbf{S}(\tau)\exists(J)p = \exists(\sigma J)\mathbf{S}(\sigma)p$.

Proof. Suppose first that $\sigma J = I$. Then by (ii), $K-J = 0$, so by (iii), J supports p. Because of (i) there is a biunique θ such that $\theta\sigma|J = \delta_J$. Hence

[6] See [7], Lemma 6.14. In Halmos's proof local finiteness is assumed.

$$\begin{aligned}
\exists(\sigma J)\mathbf{S}(\sigma)p &= \exists(I)\mathbf{S}(\sigma)p\\
&= \mathbf{S}(\theta)\exists(I)\mathbf{S}(\sigma)p && 2.1\ (\mathrm{vii})\\
&= \mathbf{S}(\theta)\exists(\theta^{-1}J)\mathbf{S}(\sigma)p\\
&= \exists(J)\mathbf{S}(\theta\sigma)p && (\mathrm{P}_6)\\
&= \exists(J)p && 2.1\ (\mathrm{vii})\\
&= \mathbf{S}(\tau)\exists(J)p && 2.1\ (\mathrm{vii})
\end{aligned}$$

Thus we may and do assume henceforth that $\sigma J \neq I$.

Choose $\varphi \in I^I$ such that $\varphi|J = \sigma|J$, $\varphi|(K-J) = \tau|(K-J)$, and $\varphi[I-(K \cup J)] \subset I-\sigma J$. Then by (iv), $\varphi|K = \sigma|K$. Also, $\varphi^{-1}\sigma J = J$ by (ii), and $\varphi|J$ is biunique by (i). Note that $K-J$ supports $\exists(J)p$ in virtue of (iii) and 2.1 (vi). Hence

$$\begin{aligned}
\mathbf{S}(\tau)\exists(J)p &= \mathbf{S}(\varphi)\exists(J)p && 2.1\ (\mathrm{vii})\\
&= \exists(\sigma J)\mathbf{S}(\varphi)p && (\mathrm{P}_6)\\
&= \exists(\sigma J)\mathbf{S}(\sigma)p\,. && 2.1\ (\mathrm{vii})
\end{aligned}$$

Q.E.D.

LEMMA 2.4. *Suppose* $p, q \in \mathbf{A}$, $\tau \in I^J$, $J, K \subset I$, *and the following conditions hold:*

(i) K *supports* p;

(ii) q *is independent of* τJ;

(iii) $\tau J \cap K = 0$;

(iv) $\tau|J$ *is biunique;*

(v) $\tau|(K-J) = \delta_{K-J}$;

(vi) $q \wedge \{[\exists(J)p]' \vee \mathbf{S}(\tau)p\} = 0$.

Then $q = 0$.

Proof. By (i), (iii) and 2.1 (v), (ii), $[\exists(J)p]'$ is independent of τJ. Hence by (ii), (Q_3), and 1.1 (v), upon applying $\exists(\tau J)$ to the equation (vi) we get

$$(*)\qquad q \wedge \{[\exists(J)p]' \vee \exists(\tau J)\mathbf{S}(\tau)p\} = 0\,.$$

By Lemma 2.3 we have $\exists(\tau J)\mathbf{S}(\tau)p = \mathbf{S}(\tau)\exists(J)p$. By (i) and 2.1 (vi), $K-J$ supports $\exists(J)p$, and so by (v) and 2.1 (vii), $\mathbf{S}(\tau)\exists(J)p = \mathbf{S}(\delta_I)\exists(J)p = \exists(J)p$. Hence $(*)$ yields $q = 0$, as desired.

LEMMA 2.5. *If* $p \in \mathbf{A}$, $\varrho, \sigma, \tau \in I^I$, $J, K \subset I$, J *supports* p, $\varrho|\sigma J$ *is biunique, and* $\tau\varrho|\sigma J = \delta_{\sigma J}$, *then* $\exists(K)\mathbf{S}(\sigma)p = \mathbf{S}(\tau)\exists[\varrho(K \cap \sigma J)]\mathbf{S}(\varrho\sigma)p$.

Proof. From Theorem 2.2 and the hypothesis we infer that σJ supports $\exists(K)\mathbf{S}(\sigma)p$. Hence

$$\begin{aligned}
\exists(K)\mathbf{S}(\sigma)p &= \mathbf{S}(\tau)\mathbf{S}(\varrho)\exists(K \cap \sigma J)\mathbf{S}(\sigma)p && 2.2,\ 2.1\ (\mathrm{vii})\\
&= \mathbf{S}(\tau)\exists[\varrho(K \cap \sigma J)]\mathbf{S}(\varrho\sigma)p\,. && 2.3
\end{aligned}$$

Q.E.D.

An *ideal* in a polyadic algebra $\boldsymbol{A}$ is a subset M of $\boldsymbol{A}$ which is a Boolean ideal in $\boldsymbol{A}$ and satisfies the condition

(R) if $p \in M$ then $\exists(I)p \in M$.

The algebra $\boldsymbol{A}$ is simple if $\{0\}$ is the only proper polyadic ideal in it.

The notions of *subalgebra, direct product, subdirect product, homomorphism*, and *isomorphism* are carried over in a natural fashion from universal algebra to apply to polyadic algebras. A *semisimple* polyadic algebra is one which is isomorphic to a subdirect product of simple polyadic algebras.

THEOREM 2.6. *Every polyadic algebra is semisimple. In fact, a polyadic algebra $\boldsymbol{A}$ is isomorphic to a subdirect product of simple algebras the index set of which is $\boldsymbol{A}-\{0\}$.*

A proof may be found in [7], p. 289.

THEOREM 2.7. *A direct product of $\boldsymbol{B}$-valued functional I-algebras with a common base X, over a set T, is isomorphic to a $\boldsymbol{B}^T$-valued functional I-algebra with domain X.*

Proof. For each $t \in T$ let $(\boldsymbol{A}_t, I, \boldsymbol{S}_t, \exists_t)$ be a $\boldsymbol{B}$-valued functional I-algebra with base X, and let $(\boldsymbol{A}, I, \boldsymbol{S}, \exists)$ be the product of these algebras (with $\boldsymbol{A} = \prod_{t \in T} \boldsymbol{A}_t$). Define F mapping $\boldsymbol{A}$ into $F(X^I, \boldsymbol{B}^T)$ by setting

$$(fp)(x)_t = p_t(x),$$

for each $p \in \boldsymbol{A}$, $x \in X^I$, and $t \in T$. The fact that f is the desired isomorphism is easily checked.

§ 3. Dilations of transformation systems. This section is devoted to the study of two complementary concepts, that of *dilation* and that of *compression*. Because this discussion is more appropriately presented in a context more general than that of polyadic algebras, we shall first ignore the fact that our algebras are polyadic or even Boolean and retain only the fact that they are relational systems endowed with a transformation structure. More precisely, we shall call an *I-transformation system*, a triple $(\boldsymbol{A}, I, \boldsymbol{S})$ where $\boldsymbol{A}$ is a relational system, I is a set, and $\boldsymbol{S}$ is an homomorphism from the semigroup I^I into the semigroup of all endomorphisms of $\boldsymbol{A}$. Often the system $(\boldsymbol{A}, I, \boldsymbol{S})$ will be denoted simply by $\boldsymbol{A}$. There are no restrictions on the set of operations and relations of $\boldsymbol{A}$ but for definiteness and simplicity's sake we shall assume that it consists of one binary operation V and of one binary relation R. The definition of a $(V, R)-I$-transformation system can thus be expressed symbolically as follows. Letting α, β range over I^I and p, p_1, p_2 over $\boldsymbol{A}$ we have:

(T_1) $\boldsymbol{S}(\delta_I)p = p$;

(T_2) $\boldsymbol{S}(\alpha\beta) = \boldsymbol{S}(\alpha)\boldsymbol{S}(\beta)$;

(T_3) $V(\boldsymbol{S}(\alpha)p_1, \boldsymbol{S}(\alpha)p_2) = \boldsymbol{S}(\alpha)V(p_1, p_2)$;

(T_4) $R(p_1, p_2)$ *implies* $R(\boldsymbol{S}(\alpha)p_1, \boldsymbol{S}(\alpha)p_2)$.

If I^+ is a superset of I, an *I^+-dilation* of $\boldsymbol{A}$ is an I^+-transformation system $(\boldsymbol{A}^+, I^+, \boldsymbol{S}^+)$ such that $\boldsymbol{A}^+$ is a (V, R)-system of which $\boldsymbol{A}$ is a subsystem and such that the following conditions hold for all $\alpha, \alpha_1, \alpha_2 \in I^{+I^+}$ and $p \in \boldsymbol{A}$:

(D_1) $\boldsymbol{S}^+(\alpha)p = \boldsymbol{S}(\alpha|I)p$ *whenever* $\alpha(I) \subset I$;

(D_2) $\boldsymbol{S}^+(\alpha_1)p = \boldsymbol{S}^+(\alpha_2)p$ *whenever* $\alpha_1|I = \alpha_2|I$.

"We note that when I is infinite, (D_2) becomes superfluous. Indeed we can then let γ be a biunique transformation of I^+ such that $\gamma(\alpha_1(I)) \subset I$ and, noting that $\gamma\alpha_1|I = \gamma\alpha_2|I$, write $\boldsymbol{S}^+(\gamma)\boldsymbol{S}^+(\alpha_1)p = \boldsymbol{S}(\gamma\alpha_1|I)p = \boldsymbol{S}(\gamma\alpha_2|I)p = \boldsymbol{S}^+(\gamma)\boldsymbol{S}^+(\alpha_2)p$ according to (D_1) and (T_2) applied to $\boldsymbol{A}^+$. Finally, if $\sigma \in I^{+I^+}$ is such that $\sigma\gamma = \delta_{I^+}$, we obtain ($D_2$) by applying $\boldsymbol{S}^+(\sigma)$ to $\boldsymbol{S}^+(\gamma)\boldsymbol{S}^+(\alpha_1)p = \boldsymbol{S}^+(\gamma)\boldsymbol{S}^+(\alpha_2)p$ and using (T_2) and (T_1)."

An example of transformation system is that of a functional transformation system. Let $\boldsymbol{B}$ be a (V, R)-relational system and I and X be sets; then the direct power $F(X^I, \boldsymbol{B})$ of all functions from X^I to $\boldsymbol{B}$ is in a natural manner a transformation system. More precisely $F(X^I, \boldsymbol{B})$ becomes a $(V, R)-I$-system if we set for all $x \in X^I$; $f, f_1, f_2 \in F(X^I, \boldsymbol{B})$ and $\alpha \in I^I$:

(FT_1) $V(f_1, f_2)(x) = V(f_1(x), f_2(x))$;

(FT_2) $R(f_1, f_2)$ *iff* $R(f_1(y), f_2(y))$ *for all* $y \in X^I$;

(FT_3) $(\boldsymbol{S}(\alpha)f)(x) = f(x\alpha)$.

Any subsystem of $F(X^I, \boldsymbol{B})$ is said to be a *$\boldsymbol{B}$-valued functional I-system with domain X.*

The following known and easy but basic result shows that the above example is universal.

THEOREM 3.1. *Any I-transformation system $\boldsymbol{A}$ is isomorphic to a functional I-transformation system.*

Proof. Let $X = I$ and define a mapping

$$H: \boldsymbol{A} \to F(X^I, \boldsymbol{A})$$

by setting $H(p)(x) = \boldsymbol{S}(x)p$ for all $p \in \boldsymbol{A}$ and $x \in X^I$. The verification that H is an I-isomorphism is straightforward. Q.E.D.

An I^+-dilation is said to be *minimal* if $\boldsymbol{A}^+$ has no proper subsystem containing $\boldsymbol{A}$, i.e., if any subset of $\boldsymbol{A}^+$ containing $\boldsymbol{A}$, and closed under V

and $S^+(\alpha)$ for all α, is equal to A^+. Quite obviously any dilation of A contains a minimal one.

We are now ready for the main theorem of this section.

THEOREM 3.2. *For any set I and superset I^+, any I-transformation system admits a minimal I^+-dilation.*

Proof. It suffices to show that any functional transformation system $F(X^I, B)$ admits a dilation. For this we define a mapping

$$H\colon F(X^I, B) \to F(X^{I^+}, B)$$

by setting $(H(f))(x) = f(x|I)$; $f \in F(X^I, B)$, $x \in X^{I^+}$. It is easy to verify that H is a (V, R)-isomorphism. After the usual identification of the elements of $F(X^I, B)$ with their images in $F(X^{I^+}, B)$, the conditions (D_1) and (D_2) read respectively: $S^+(\alpha)H(f) = H(S(\alpha|I)f)$ whenever $\alpha(I) \subset I$; and $S^+(\alpha_1)H(f) = S^+(\alpha_2)H(f)$ whenever $\alpha_1|I = \alpha_2|I$. We omit the details of these verifications. Q.E.D.

When I is infinite, a useful description of minimal dilations can be given.

THEOREM 3.3. *If I is infinite and C is an I^+-dilation of A then the minimal dilation A^+ of A in C consists of the elements of the form $S^+(\sigma)p$ where $p \in A$ and $\sigma \in I^{+I^+}$ is biunique on I. Furthermore, if $\overline{\overline{I^+}} > \overline{\overline{I}}$, then we may assume that σ is a permutation of I^+.*

Proof. We first show that the set of elements $S^+(\sigma)p$ with $p \in A$ and $\sigma \in I^{+I^+}$ is closed under V and $S^+(\alpha)$ for all $\alpha \in I^{+I^+}$. As to V this means that, if α_1 and α_2 are transformations of I^+ and if p_1 and p_2 are elements of A then $V(S^+(\alpha_1)p_1, S^+(\alpha_2)p_2)$ can be written in the form $S^+(\alpha)p$. Let γ and σ be transformations of I^+ such that γ is biunique on $J = \alpha_1(I) \cup \alpha_2(I)$, $\gamma J \subset I$ and $\sigma\gamma|J = \delta_J$. Hence we have successively

$$\begin{aligned} &\sigma\gamma\alpha_1|I = \alpha_1|I; && \\ &S^+(\sigma\gamma\alpha_1)p_1 = S^+(\alpha_1)p_1; && (D_2) \\ &S^+(\sigma\gamma\alpha_1)p_1 = S^+(\sigma)S(\gamma\alpha_1|I)p_1; && (T_2),\ (D_1) \end{aligned}$$

And similarly for α_2 and p_2. Finally using (T_3) we get

$$(1)\qquad V(S^+(\alpha_1)p_1, S^+(\alpha_2)p_2) = S^+(\sigma)V(S(\gamma\alpha_1|I)p_1, S(\gamma\alpha_2|I)p_2)\,.$$

Thus this case is settled with $p = V(S(\gamma\alpha_1|I)p_1, S(\gamma\alpha_2|I)p_2)$.

As to $S^+(\alpha)$, let $S^+(\sigma_1)p_1$, and α be given; $\alpha, \sigma_1 \in I^{+I^+}$, $p_1 \in A$. Set $\alpha\sigma_1 = \beta$ and $J = \beta(I)$, and choose γ and σ in I^{+I^+} such that $\gamma J \subset I$ and $\sigma\gamma|J = \delta_J$. Finally we have $S^+(\alpha)S^+(\sigma_1)p_1 = S^+(\beta)p_1 = S^+(\sigma)S(\gamma\beta|I)p_1$ and this case is settled with $p = S(\gamma\beta|I)p_1$.

When $\overline{\overline{I^+}} > \overline{\overline{I}}$ it is obvious that, in both cases above, γ can be chosen a permutation of I^+ and γ^{-1} can be used as σ. When it is not assumed that $\overline{\overline{I^+}} > \overline{\overline{I}}$ we can only show that any element $S^+(\alpha)p$ with $\alpha \in I^{+I^+}$ and $p \in A$ is equal to an element $S^+(\sigma)q$ with $\sigma \in I^{+I^+}$ biunique on I with $q \in A$. Indeed let σ be a transformation of I^+ biunique on I and such that $\sigma(I) \supset \alpha(I)$. Let also β be a transformation of I^+ such that $\beta(I) \subset I$, and $\sigma\beta|I = \alpha|I$. Setting $q = S(\beta|I)p$ we have,

$$S^+(\sigma)q = S^+(\sigma)S(\beta|I)p = S^+(\sigma)S^+(\beta)p = S^+(\sigma\beta)p = S^+(\alpha)p\,.$$

Q.E.D.

As a corollary to this theorem we obtain, provided I is infinite, a unicity statement that shows that, in this case, our definition of dilation leaves nothing to be desired.

THEOREM 3.4. *If I is infinite then any two minimal I^+-dilations of A are equivalent.*

Proof. Let (A_1^+, S_1^+) and (A_2^+, S_2^+) be two minimal I^+-dilations of A. Using the preceding theorem we wish to show that the mapping $S_1^+(\alpha)p \to S_2^+(\alpha)p$, $\alpha \in I^{+I^+}$, $p \in A$, from A_1^+ to A_2^+ is a surjective isomorphism leaving A elementwise fixed. The only non-trivial parts of the assertion are that the mapping is well-defined and biunique and that it preserves V and R.

To show that the mapping is well-defined and biunique, suppose that

$$(2)\qquad S_1^+(\alpha_1)p_1 = S_1^+(\alpha_2)p_2;\quad p_1, p_2 \in A;\quad \text{and}\quad \alpha_1, \alpha_2 \in I^{+I^+}.$$

We must see that this is equivalent to

$$(3)\qquad S_2^+(\alpha_1)p_1 = S_2^+(\alpha_2)p_2\,.$$

But, setting $J = \alpha_1(I) \cup \alpha_2(I)$ and letting γ and σ be transformations of I such that $\gamma J \subset I$ and $\sigma\gamma|J = \delta_J$, both (2) and (3) can be shown to be equivalent to

$$(4)\qquad S(\gamma\alpha_1|I)p_1 = S(\gamma\alpha_2|I)p_2\,.$$

For instance we obtain (4) by applying $S_1^+(\gamma)$ to (2) and using (D_1) and we go back to (2) by applying $S_1^+(\sigma)$ to (4) and using (D_2) together with $\sigma\gamma\alpha_1|I = \alpha_1|I$.

The fact that the mapping $A_1^+ \to A_2^+$ preserves V is seen immediately from (1).

Finally, that the mapping preserves R follows from the equivalence

$$(5)\qquad R(S_i^+(\alpha_1)p_1, S_i^+(\alpha_2)p_2)\quad \text{iff}\quad R(S(\gamma\alpha_1|I)p_1, S(\gamma\alpha_2|I)p_2)\,,$$

$i = 1, 2$, in which γ has the same meaning as above. The proof is based on (T_2), (T_4), (D_1) and (D_2). Q.E.D.

The concept of compression, complementary to that of dilation, necessitates a generalization of our polyadic notion of support. In the context of transformation systems we shall say that a subset J of I *supports* an element p of A if, whenever α_1 and α_2 are transformations of I such that $\alpha_1|J = \alpha_2|J$, we have $S(\alpha_1)p = S(\alpha_2)p$. That this notion is indeed a generalization of the polyadic one will be seen in the next section (Lemma 4.1). For the time being we will prove some basic desirable properties, the first two of which we need immediately.

LEMMA 3.5. *If $p \in A$, $J \subset I$ and J supports p, then $\alpha(J)$ supports $S(\alpha)p$ for all $\alpha \in I^I$.*

Proof. We want to show that if $\beta_1|\alpha(J) = \beta_2|\alpha(J)$ then $S(\beta_1)S(\alpha)p = S(\beta_2)S(\alpha)p$, knowing that $\alpha_1|J = \alpha_2|J$ entails $S(\alpha_1)p = S(\alpha_2)p$. But this is obvious owing to the fact that $\beta_1|\alpha(J) = \beta_2|\alpha(J)$ iff $\beta_1\alpha|J = \beta_2\alpha|J$. For, setting $\alpha_1 = \beta_1\alpha$, $\alpha_2 = \beta_2\alpha$ and using (T_2) we obtain the desired conclusion.

LEMMA 3.6. *If I is infinite, A^+ is a dilation of A, $p \in A$, $J \subset I$ and J supports p in A, then J supports p in A^+.*

Proof. Given α_1 and α_2 in I^{+I^+} such that $\alpha_1|J = \alpha_2|J$, set $K = \alpha_1(I) \cup \alpha_2(I)$ and let γ and σ be transformations of I^+ such that $\gamma K \subset I$ and $\sigma\gamma|K = \delta_K$. Then $S^+(\sigma\gamma\alpha_1)p = S^+(\alpha_1)p$ since $\sigma\gamma\alpha_1|I = \alpha_1|I$. Also, $S^+(\sigma\gamma\alpha_1)p = S^+(\sigma)S(\gamma\alpha_1|I)p$. Similarly, $S^+(\sigma\gamma\alpha_2)p = S^+(\sigma)S(\gamma\alpha_2|I)p = S^+(\alpha_2)p$. But $S(\gamma\alpha_1|I)p = S(\gamma\alpha_2|I)p$ since $\gamma\alpha_1|J = \gamma\alpha_2|J$ and J supports p. Therefore $S^+(\alpha_1)p = S^+(\alpha_2)p$. Q.E.D.

LEMMA 3.7. *If $p \in A$, then $\{J\colon J$ supports $p\}$ is a filter in the Boolean algebra of all subsets of I.*

Proof. That, if J supports p and $J_1 \supset J$, then J_1 supports p is obvious. We proceed to show that the set of supports of p is closed under intersection. Let J_1 and J_2 be supports of p and let α_1 and α_2 be transformations of I such that $\alpha_1|J = \alpha_2|J$ with $J = J_1 \cap J_2$. We want to show that $S(\alpha_1)p = S(\alpha_2)p$. Let $\alpha \in I^I$ be such that $\alpha|J_1 = \alpha_1|J_1$ and $\alpha|J_2 = \alpha_2|J_2$. Then $S(\alpha)p = S(\alpha_1)p$ since J_1 supports p and similarly $S(\alpha)p = S(\alpha_2)p$. Therefore $S(\alpha_1)p = S(\alpha_2)p$. The proof is complete.

For a subset J of I the *J-compression* of A is a system (A_J, J, S^-) defined as follows. A_J consists of the elements of A that J supports; that A_J is closed under V follows from (T_3). The relation R of A_J is by definition the intersection with A_J of the relation R of A. Finally, if for $\alpha \in J^J$, $\bar{\alpha}$ is any transformation of I such that $\bar{\alpha}|J = \alpha$ we set $S^-(\alpha)p = S(\bar{\alpha})p$ for $p \in A_J$. The definition is obviously independent of $\bar{\alpha}$ since J supports p. We may for instance take for $\bar{\alpha}$, α^+ which is defined by the equations $\alpha^+|J = \alpha$ and $\alpha^+|I-J = \delta_{I-J}$. The conditions (T_1)-(T_4) for A_J are obvious from the same relations for A and the equation $(\alpha\beta)^+ = \alpha^+\beta^+$ which holds for all α and β in J^J. It is also easily verified that A is an I-dilation of A_J. We wish to answer the question: when is A a *minimal* dilation of A_J? For that we need a definition. The *effective degree* of a transformation system A is the smallest cardinal $\mathfrak{e}$ such that any element of A admits a support whose cardinality does not exceed $\mathfrak{e}$.

THEOREM 3.8. *If I is infinite then the dilation A of A_J is minimal iff $\overline{\overline{J}} \geqslant \mathfrak{e}$.*

Proof. If the dilation is minimal, then any element of A has the form $S(\sigma)p$ for some $\sigma \in I^I$ and $p \in A_J$ (Theorem 3.3) and has therefore a support $\sigma(J)$ of cardinality at most $\overline{\overline{J}}$ (Lemma 3.5). Hence $\mathfrak{e} \leqslant \overline{\overline{J}}$.

Conversely, if $\mathfrak{e} \leqslant \overline{\overline{J}}$, let q be an arbitrary element of A. Let K be a support of q such that $\overline{\overline{K}} \leqslant \overline{\overline{J}}$ and let σ and γ be transformations of I such that $\sigma(K) \subset J$ and $\gamma\sigma|K = \delta_K$. Setting $p = S(\sigma)q$, we have $p \in A_J$ by Lemma 3.5 and, since $q = S(\gamma)p$, q is in the minimal dilation of A_J contained in A. Q.E.D.

When $\overline{\overline{J}} \geqslant \mathfrak{e}$ and I is infinite, the compression A_J will be said to be *faithful*. In this case A can be recovered from A_J as its unique minimal I-dilation.

Theorem 3.8 admits the following "converse". In general, if A^+ is a dilation of A, A is not a compression of A^+; but this is so if A^+ is minimal. More precisely we have:

THEOREM 3.9. *If I is infinite and if A^+ is a minimal I^+-dilation of A then A is a faithful compression of A^+.*

Proof. Because of (D_2) A is contained in the I-compression of A^+. To show that A is identical with this compression let $q = S^+(\sigma)p$, $p \in A$, $\sigma \in I^{+I^+}$, be an element of A^+ supported by I. Let $\alpha \in I^{+I^+}$ be such that $\alpha|I = \delta_I$ and $\alpha\sigma(I) \subset I$. Then, since I supports q, $S^+(\alpha)q = q = S^+(\alpha\sigma)p = S(\alpha\sigma|I)p \in A$. The transformation structures S and S^- of A coincide for, if $\alpha \in I^I$ and $p \in A$, $S^-(\alpha)p = S^+(\alpha^+)p = S(\alpha^+|I)p = S(\alpha)p$. The proof is complete.

The last two theorems are summed up by the statement that A^+ is a minimal dilation of A iff A is a faithful compression of A^+.

A functional representation of an I-system A yields a natural functional representation of its compressions.

THEOREM 3.10. *If A is a B-valued I-transformation system with domain X then any J-compression of A is isomorphic to a B-valued J-transformation system with domain X.*

Proof. Define $H\colon A_J \to F(X^J, B)$ by setting, for $f \in A_J$ and $x \in X^J$, $H(f)(x) = f(y)$ where $y \in X^I$ and $y|J = x$. To show that this is well defined we let $z \in X^I$ be such that $z|J = x$ and we choose $\sigma \in I^I$ such that $\sigma|J = \delta_J$ and $\sigma I \subset J$. Then $f(y) = \big(S(\sigma)f\big)(y) = f(y\sigma)$ and similarly $f(z) = f(z\sigma)$. But $z\sigma = y\sigma$. The verification that H is an isomorphism of J-systems is straightforward. Q.E.D.

The following theorem will be needed in the next section.

THEOREM 3.11. *Let I be infinite and I_1 and I_2 be supersets of I such that $I_1 \subset I_2$.*

(i) *If (A_2, I_2, S_2) is an I_2-dilation of (S, I, S) then the I_1-compression (A_1, I_1, S_1) of A_2 is an I_1-dilation of A, and if A_2 is a minimal dilation of A so is A_1.*

(ii) *If (A_1, I_1, S_1) is a (minimal) I_1-dilation of (A, I, S) and A_2 is a (minimal) I_2-dilation of A_1, then A_2 is a (minimal) I_2-dilation of A.*

Proof. Only two of the many verifications to make are not straightforward. The first of these two things is that, in (i), if A_2 is a minimal dilation of A so is A_1. Let $q = S_2(\alpha)p$, $\alpha \in I_2^{I_2}$, $p \in A$, be an element of A_2 which I_1 supports. We shall construct $\sigma_1 \in I_1^{I_1}$ such that $q = S_1(\sigma_1)p$. Let $\gamma \in I_2^{I_2}$ be such that $\gamma | I_1 = \delta_{I_1}$ and $\gamma\alpha(I_1) \subset I_1$. Then, since I_1 supports q, $S_2(\gamma)q = S_2(\delta_{I_2})q = q$. Hence, setting $\sigma = \gamma\alpha$, we have $q = S_2(\sigma)p$ with $\sigma(I_1) \subset I_1$. If, finally, we let $\sigma_1 = \sigma | I_1$ we have, since I supports p in A and hence in A_2, $S_1(\sigma_1)p = S_2(\sigma)p$ by definition of S_1.

The second non-trivial thing is the proof of (D_1) for the dilation A_2 of A in the context of (ii). For $p \in A$ and $\alpha \in I_2^{I_2}$ such that $\alpha(I) \subset I$, we have to show that $S_2(\alpha)p = S(\alpha | I)p$. Let $\beta \in I_2^{I_2}$ be such that $\beta | I = \alpha | I$ and $\beta(I_1) \subset I_1$. Since I supports p in A, it still does in A_1 and hence also in A_2. Therefore, $S_2(\alpha)p = S_2(\beta)p$. Now, since A_2 is a dilation of A_1 we have by (D_1), $S_2(\beta)p = S_1(\beta | I_1)p$. Again since A_1 is a dilation of A we have by (D_1), $S_1(\beta | I_1)p = S(\alpha | I)p$. Therefore, $S_2(\alpha)p = S(\alpha | I)p$.

We omit the other verifications. Q.E.D.

The last two theorems enable us to strengthen Theorem 3.3 a little: the requirement that "$\overline{\overline{I^+}} > \overline{\overline{I}}$" there can be replaced by "$\overline{\overline{I^+}} > \mathfrak{e}$". Letting J be a subset of I of cardinality $\mathfrak{e}$ it suffices to apply (ii) of Theorem 3.11 with A_J in the role of A, A in that of A_1 and A^+ in that of A_2, noting that, by Theorem 3.8, A is a minimal dilation of A_J.

Two concepts related to that of effective degree will be used in § 6. First, we shall call the *effective cardinality* $\mathfrak{c}$ of A, the cardinality of A_J where J is a subset of I such that $\overline{\overline{J}} = \mathfrak{e}$. The effective cardinality is independent of the choice of J since if J_1 and J_2 are subsets of I such that $\overline{\overline{J_1}} = \overline{\overline{J_2}}$, A_{J_1} and A_{J_2} are isomorphic (V, R)-systems. A_{J_1} is indeed mapped onto A_{J_2} by any automorphism $S(\alpha)$ of A arising from a permutation α of I such $\alpha(J_1) = J_2$. This follows from Lemma 3.5.

The second consept is that of *local degree* of A. This is the smallest cardinal $\mathfrak{m}$ such that each element of A has a support of cardinality less than $\mathfrak{m}$. Of course, $\mathfrak{m} \geqslant \mathfrak{e}$. The two possibilities "$\mathfrak{m} = \mathfrak{e}$" and "$\mathfrak{m} = \mathfrak{e}^+$" can be realized.

An element p of A is said to be *closed* if the null set supports it. This means that $S(\alpha)p = p$ for all $\alpha \in I^I$. A is said to be *degenerate* if all its elements are closed. In this case, $\mathfrak{c} = \overline{\overline{A}}$, $\mathfrak{m} = 1$, and $\mathfrak{e} = 0$.

THEOREM 3.12. *If I is infinite and A is non-degenerate then $\overline{\overline{A}} \geqslant \overline{\overline{I}}$.*

Proof. Let p be an element of A which is not supported by 0. To construct a family $\{p_i | i \in I\}$ of distinct elements of A, let $\{\alpha_i | i \in I\}$ be a family of elements of I^I biunique on I and such that $\alpha_i(I) \cap \alpha_j(I) = 0$ whenever $i \neq j$; and set $p_i = S(\alpha_i)p$ (7). Then, if $i \neq j$ and $p_i = p_j = q$, we have by virtue of Lemma 3.7, that 0 supports q. Finally, if $\sigma \in I^I$ is such that $\sigma\alpha_i = \delta_I$, we have $p = S(\sigma)q$ and hence, $\sigma(0) = 0$ supports p. This contradicts the choice of p.

The foregoing theorem will help to clarify certain cardinality conditions in the representation theorem. In particular we have that for a non-degenerate A, $\mathfrak{c} \geqslant \mathfrak{e}$.

§ 4. Dilations of polyadic algebras.

It is obvious from our constructions (in the proof of 3.1 and 3.2) that a dilation of a transformation Boolean algebra is a Boolean algebra since the class of Boolean algebras is closed under the taking of powers and the taking of subalgebras, i.e., subsets closed under all Boolean operations. For the same reason a compression of a transformation Boolean algebra is also a Boolean algebra to which we may apply the processes of compression and of dilation. In this section we will generalize these processes in order to take care of the quantifier structure. First we need to show the identity of the transformation and polyadic concepts of support.

LEMMA 4.1. *Let $(A, I, S, \exists)$ be a polyadic algebra with $\overline{\overline{I}} > 1$ and let $p \in A$ and $K \subset I$. Then $S(\alpha_1)p = S(\alpha_2)p$ whenever α_1 and α_2 are transformations of I such that*

$$\alpha_1 | K = \alpha_2 | K \quad \textit{iff} \quad \exists(I-K)p = p\,.$$

Proof. The condition is necessary. For assuming first $K \neq 0$, and letting $\alpha_1 = \delta$ and α_2 be such that $\alpha_2 | K = \delta_K$ and $\alpha_2(I-K) \subset K$, we have, using (P_6) with $J = I-K$; $\exists(I-K)p = \exists(I-K)S(\alpha_1)p = \exists(I-K)S(\alpha_2)p = S(\alpha_2)\exists(0)p = S(\alpha_2)p = S(\alpha_1)p = p$.

If $K = 0$, we have $S(\alpha_1)p = S(\alpha_2)p = S(\delta_I)p = p$ for all α_1 and α_2 in I^I. Let J_1 and J_2 be non null disjoint sets such that $J_1 \cup J_2 = I$ and let α_1 and α_2 be such that $\alpha_1(I) \subset J_1$ and $\alpha_2(I) \subset J_2$. Then $\exists(I)p = \exists(J_1)\exists(J_2)p = \exists(J_1)\exists(J_2)S(\alpha_1)p = \exists(J_1)S(\alpha_1)\exists(0)p = \exists(J_1)S(\alpha_2)p = S(\alpha_2)\exists(0)p = p$.

The sufficiency is obtained from (P_5) with $J = I-K$ thus: $S(\alpha_1)p = S(\alpha_1)\exists(I-K)p = S(\alpha_2)\exists(I-K)p = S(\alpha_2)p$. Q.E.D.

If $J \subset I$ then the compression (A_J, J, S^-) can be turned into a *polyadic compression* $(A_J, J, S^-, \exists^-)$ by setting for $K \subset J$, $\exists^-(K) = \exists(K) | A_J$. The proof that A_J thus becomes a polyadic algebra is a short verification carried out in § 11 of [7].

(7) The necessary partition of I can be induced, for instance, by any biunique mapping $I \times I \to I$.

A polyadic algebra $(A^+, I^+, S^+, \exists^+)$ is a *polyadic dilation* of $(A, I, S, \exists)$ if (A^+, I^+, S^+) is a (transformation) dilation of (A, I, S) and $\exists^+(K)|A = \exists(K)$ whenever $K \subset I$. Our terminology departs slightly from that of [7] where it is further required that A be the whole of the I-compression of A^+. This discrepancy vanishes (by virtue of Theorem 3.9 and the following theorem) if we consider only minimal polyadic dilations, i.e. dilations without proper polyadic subalgebras which are dilations.

THEOREM 4.2. *If I is infinite, and if $(A^+, I^+, S^+, \exists^+)$ is a minimal polyadic dilation of $(A, I, S, \exists)$ then (A^+, I^+, S^+) is a minimal transformation dilation of (A, I, S).*

Proof. According to Theorem 3.3 it suffices to show that the set of elements of A^+ of the form $S^+(\sigma)p$ with $p \in A$ and σ a transformation of I^+ biunique on I is closed under the quantifiers $\exists^+(K)$, $K \subset I^+$. Choose $J \subset I$ such that $\sigma J = K \cap \sigma I$. Then by Lemma 2.3 we have: $\exists^+(K)S^+(\sigma)p = \exists(K \cap \sigma I)\,S^+(\sigma)p = \exists^+(\sigma J)\,S^+(\sigma)p = S^+(\sigma)\,\exists^+(J)p = S^+(\sigma)q$ with $q = \exists(J)p \in A$. Q.E.D.

THEOREM 4.3. *Any polyadic algebra of infinite degree $(A, I, S, \exists)$ admits for any superset I^+ of I one and, to within equivalence, only one minimal polyadic dilation $(A^+, I^+, S^+, \exists^+)$.*

Proof. First we prove the unicity. Assume $\exists^+$ is a quantifier structure on the (unique) minimal transformation dilation (A^+, I^+, S^+) of (A, I, S) such that $(A^+, I^+, S^+, \exists^+)$ is a polyadic algebra and:

$$\exists^+(K)|A = \exists(K) \quad \text{whenever} \quad K \subset I. \tag{1}$$

Let $q = S^+(\sigma)p$ be an element of A^+; $\sigma \in I^{+I^+}$, $p \in A$. By Lemma 2.5, we have that

$$\exists^+(K)q = S^+(\tau)\,\exists^+[\varrho(K \cap \sigma I)]\,S^+(\varrho\sigma)p\,,$$

whenever τ and ϱ are transformations of I^+ such that $\tau\varrho|\sigma I = \delta_{\sigma I}$ and $\varrho|\sigma I$ is biunique. If moreover, $\varrho\sigma I \subset I$, then, by virtue of (1) and (D_1), this equation becomes,

$$\exists^+(K)\,S^+(\sigma)p = S^+(\tau)\,\exists[\varrho(K \cap \sigma I)]\,S(\varrho\sigma|I)p\,. \tag{2}$$

From equation (2) it is apparent that $\exists^+$ can be described in terms of the data $\exists$, S and S^+, and is therefore unique.

The unicity proof just given suggests the course to follow for the existence proof. We could indeed, define $\exists^+$ by means of equations (2) with τ and ϱ as before. However, in order to take advantage of a slight simplification that becomes possible if $\overline{\overline{I^+}} > \overline{\overline{I}}$, we shall make this assumption noting that when $\overline{\overline{I^+}} = \overline{\overline{I}}$ the existence of $\exists^+$ can be obtained from the other case by polyadic compression by virtue of part (i) of Theorem 3.11. The general case could be treated directly at the expense of more complicated computations. When $\overline{\overline{I^+}} > \overline{\overline{I}}$, ϱ in (2) can be chosen a permutation and we can use ϱ^{-1} as τ. Hence we define $\exists^+$ by means of the equation

$$\exists^+(K)\,S^+(\sigma)p = S^+(\varrho^{-1})\,\exists[\varrho(K \cap \sigma I)]\,S(\varrho\sigma|I)p\,, \tag{3}$$

in which the notation is a before and in particular $\varrho\sigma I \subset I$.

Of course we need to show that this definition is independent of ϱ, σ and p, that is to say, if for some σ_1, p_1 and ϱ_1, $q = S^+(\sigma)p = S^+(\sigma_1)p_1$ and $\varrho_1\sigma_1(I) \subset I$, then

$$S^+(\varrho^{-1})\,\exists[\varrho(K \cap \sigma I)]\,S(\varrho\sigma|I)p = S^+(\varrho_1^{-1})\,\exists[\varrho_1(K \cap \sigma_1 I)]\,S(\varrho_1\sigma_1|I)p_1\,. \tag{4}$$

To do that let α be a permutation of I^+ such that

$$\alpha\big(\varrho^{-1}(I) \cup \varrho_1^{-1}(I)\big) \subset I\,.$$

Then applying $S^+(\alpha)$ to (4) and using (D_1) yields an equation equivalent to (4) the left member of which is

$$S(\alpha\varrho^{-1}|I)\,\exists[\varrho(K \cap \sigma I)]\,S(\varrho\sigma|I)p\,. \tag{5}$$

Applying (P_6), we see that this expression is equal to

$$\exists(\alpha K \cap \alpha\sigma I)\,S^+(\alpha)q\,. \tag{6}$$

Now we have that $\alpha(\sigma I \cup \sigma_1 I) \subset I$ and hence $S^+(\alpha)q = S(\alpha\sigma|I)p = S(\alpha\sigma_1|I)p_1 \in A$. Therefore both $\alpha\sigma I$ and $\alpha\sigma_1 I$, and hence also $\alpha\sigma I \cap \alpha\sigma_1 I$, support $S^+(\alpha)q$. By virtue of (iv) of Theorem 2.1 this implies that the expression (6) is equal to

$$\exists(\alpha K \cap \alpha\sigma I \cap \alpha\sigma_1 I)\,S^+(\alpha)q\,.$$

Similarly the right member of the equation of which (5) is the left member could be shown equal to the same expression. This completes the proof of the unambiguity of the definition (3).

Next we have to show that the unary operations $\exists^+(K)$ just defined on A^+ are quantifiers and that $(A^+, I^+, S^+, \exists^+)$ is a polyadic algebra. That is to say, we have to verify (Q_1)-(Q_3) and (P_3)-(P_6). Both (Q_1) and (P_3) are obvious. We proceed to a stepwise verification of the remaining conditions.

Proof of (Q_2). We have to verify that $q \leqslant \exists^+(K)q$, with $K \subset I^+$ and $q \in A^+$. We have, with an obvious notation

$$q = S^+(\sigma)p\,,$$

$$S^+(\varrho)\,S^+(\sigma)p = S(\varrho\sigma|I)p \leqslant \exists[\varrho(K \cap \sigma I)]\,S(\varrho\sigma|I)p\,,$$

$$S^+(\sigma)p \leqslant S^+(\varrho^{-1})\,\exists[\varrho(K \cap \sigma I)]\,S(\varrho\sigma|I)p\,,$$

$$q \leqslant \exists^+(K)q\,.$$

Proof of (Q_3). We have to verify that

$$\exists^+(K)\big(q_1\wedge\exists^+(K)q_2\big)=\exists^+(K)q_1\wedge\exists^+(K)q_2\,, \tag{7}$$

with $q_1=S^+(\sigma_1)p_1$ and $q_2=S^+(\sigma_2)p_2$ in A^+. Let ϱ be a permutation of I such that $\varrho(\sigma_1 I\cup\sigma_2 I)\subset I$ and set $p=S^+(\varrho)[q_1\wedge\exists^+(K)q_2]$. We have, by definition and familiar rules,

$$p=S(\varrho\sigma_1|I)p_1\exists[\varrho(K\cap\sigma_2 I)]S(\varrho\sigma_2|I)p_2\,, \tag{8}$$

and therefore p is in A. The left member of (7) is l.m. $=\exists^+(K)S^+(\varrho^{-1})p$. To evaluate it we apply (3) with ϱ^{-1} in the role of σ and γ in that of ϱ, γ being a permutation of I^+ such that $\gamma\varrho^{-1}I\subset I$. We have l.m. $=S^+(\gamma^{-1})\exists[\gamma(K\cap\varrho^{-1}I)]S(\gamma\varrho^{-1}|I)p$. Replacing p by its value taken from (8) and setting $r=S(\gamma\varrho^{-1}|I)\exists[\varrho(K\cap\sigma_2 I)]S(\varrho\sigma_2|I)p_2$, this becomes

$$\text{l.m.}=S^+(\gamma^{-1})\exists[\gamma(K\cap\varrho^{-1}I)]\{S(\gamma\sigma_1|I)p_1\wedge r\}\,. \tag{9}$$

Both arguments of $\wedge$ in (9) are in A. r is supported by $\gamma(\sigma_2 I-K)$ (which is disjoint from γK) and is therefore independent of $\gamma(K\cap\varrho^{-1}I)$ (which is contained in γK). Hence, applying (Q_3) in A, (9) becomes

$$\text{l.m.}=S^+(\gamma^{-1})\exists[\gamma(K\cap\varrho^{-1}I)]S(\gamma\sigma_1|I)p_1\wedge\exists^+(K)q_2\,. \tag{10}$$

Since $\gamma\sigma_1 I$ supports $S(\gamma\sigma_1|I)p_1$, we have, using (iv) of 2.1 and noting that $\varrho^{-1}I\supset\sigma_1 I$,

$$\exists[\gamma(K\cap\varrho^{-1}I)]S(\gamma\sigma_1|I)p_1=\exists[\gamma(K\cap\sigma_1 I)]S(\gamma\sigma_1|I)p_1\,. \tag{11}$$

From (11) it follows that the first argument of $\wedge$ in (10) is $\exists^+(K)q_1$. The proof is complete.

Proof of (P_4). We have to verify that

$$\exists^+(J\cup K)S^+(\sigma)p=\exists^+(J)\exists^+(K)S^+(\sigma)p\,, \tag{12}$$

with $J,\ K\subset I^+$; $p\in A$; and $\sigma\in I^{+I^+}$ as before. The left member is l.m. $=S^+(\varrho^{-1})\exists[\varrho(J\cup K)\cap\varrho\sigma I]S(\varrho\sigma|I)p$, where ϱ is a permutation of I^+ such that $\varrho\sigma I\subset I$.

The right member of (12) can be written successively as

$$\exists^+(J)S^+(\varrho^{-1})\exists[\varrho(K\cap\sigma I)]S(\varrho\sigma|I)p;$$
$$S^+(\varrho^{-1})\exists(\varrho J\cap I)\exists[\varrho(K\cap\sigma I)]S(\sigma\varrho|I)p;$$
$$S^+(\varrho^{-1})\exists[(\varrho J\cap I)\cup(\varrho K\cap\varrho\sigma I)]S(\varrho\sigma|I)p\,.$$

This last expression is the same as the one for the left member above according to (iv) of 2.1 and the fact that $\varrho\sigma I$ supports $S(\varrho\sigma|I)p$.

Proof of (P_5). We have to verify that

$$S^+(\alpha_1)\exists^+(J)S^+(\sigma)p=S^+(\alpha_2)\exists^+(J)S^+(\sigma)p \tag{13}$$

where $p\in A$; $J\subset I^+$; $\alpha_1,\alpha_2\in I^{+I^+}$; $\alpha_1|I^+-J=\alpha_2|J^+-J$; and σ is a permutation of I^+. Let ϱ be a permutation of I^+ such that $\varrho\sigma I\subset I$. Setting $\alpha_1\varrho^{-1}=\gamma_1$, $\alpha_2\varrho^{-1}=\gamma_2$, $K=\varrho(J\cap\sigma I)$, $q=S(\varrho\sigma|I)p$, (13) becomes $S^+(\gamma_1)\exists(K)q=S^+(\gamma_2)\exists(K)q$ where $q\in A$, $K\subset I$, and $\gamma_1|I^+-\varrho J=\gamma_2|I^+-\varrho J$. The last equation is equivalent to

$$S(\beta\gamma_1|I)\exists(K)q=S(\beta\gamma_2|I)\exists(K)q \tag{14}$$

where β is a permutation of I^+ such that $\beta(\gamma_1 I\cup\gamma_2 I)\subset I$. Equation (14) is true by virtue of (P_5) in A since $\beta\gamma_1|I-\varrho J=\beta\gamma_2|I-\varrho J$ and $\varrho\sigma I-K\subset I-\varrho J$ supports $\exists(K)q$.

Proof of (P_6). We have to verify that

$$\exists^+(K)\,S^+(\tau)\,S^+(\sigma)p=S^+(\tau)\,\exists^+(\tau^{-1}K)\,S^+(\sigma)p \tag{15}$$

where $K\subset I^+$, $p\in A$, σ is a permutation of I^+, and τ is a transformation of I^+ biunique on $\tau^{-1}K$.

By definition, equation (15) means that

$$S^+(\varrho^{-1})\exists[\varrho(K\cap\tau\sigma I)]S(\varrho\tau\sigma|I)p=S^+(\tau\varrho^{-1})\exists[\varrho(\tau^{-1}K\cap\sigma I)]S(\varrho\sigma|I)p \tag{16}$$

where ϱ is a permutation of I^+ such that $\varrho(\tau\sigma I\cup\sigma I)\subset I$. Applying ($P_6$) in A to part of the left member of (16) we obtain

$$\exists[\varrho(K\cap\tau\sigma I)]S(\varrho\tau\sigma|I)p=S(\varrho\tau\sigma|I)\exists(\sigma^{-1}\tau^{-1}K\cap I)p\,.$$

Treating the right member in the same way and substituting the results back in (16), we get that both members of (16) are equal to $S^+(\tau\sigma)\exists(\sigma^{-1}\tau^{-1}K\cap I)p$.

It remains only to verify (1) above. Suppose $K\subset I$ and $p\in A$. Then with $\sigma=\varrho=\delta_{I^+}$, the definition (3) yields

$$\exists^+(K)p=\exists^+(K)\,S^+(\delta_{I^+})p=S^+(\delta_{I^+})\,\exists(K)\,S(\delta_I)=\exists(K)p\,.$$

This completes the proof of Theorem 4.3.

We say that a compression $(A_J,J,S^-,\exists^-)$ of $(A,I,S,\exists)$ is *faithful* if, as a transformation system, A_J is a faithful compression of A. A^+ is a minimal dilation of A iff A is a faithful compression of A^+.

A functional representation of A yields natural representations of any compression or minimal dilation.

THEOREM 4.4. *If A is a B-valued functional polyadic I-algebra with domain X and A^+ is a minimal I^+-dilation of A then A^+ is isomorphic to a B-valued functional polyadic I^+-algebra with domain X.*

Proof. We have seen in the proof of Theorem 3.2 that if $F(X^{I^+},B)$ denotes the functional I^+-transformation algebra of all functions from X^{I^+} to B and if we set for $g\in A$ and $x\in X^{I^+}$, $H(g)(x)=g(x|I)$, then $F(X^{I^+},B)$ becomes an I^+-dilation of (A,I,S) after identification of g with $H(g)$ for all $g\in A$.

Suppose for a moment that $\boldsymbol{B}$ is a complete Boolean algebra. Then $F(X^{I^+}, \boldsymbol{B})$ becomes a functional polyadic algebra whose quantifier structure we denote by $\exists^+$. Now it is easy to verify that

$$\exists^+(K)H(g) = H\big(\exists(K)g\big)$$

whenever $K \subset I$ and $g \in \boldsymbol{A}$ that is to say that equation (1) is satisfied after identification of $\boldsymbol{A}$ with $H(\boldsymbol{A})$. Therefore $F(X^{I^+}, \boldsymbol{B})$ is a polyadic dilation of $\boldsymbol{A}$ containing a minimal dilation of $\boldsymbol{A}$ which is functional. But, by the unicity part of Theorem 4.3, any two minimal dilations are equivalent.

If $\boldsymbol{B}$ is not complete, it can be replaced by its McNeille completion $\overline{\boldsymbol{B}}$, for the imbedding $\boldsymbol{B} \to \overline{\boldsymbol{B}}$ is sum preserving. As before we obtain a minimal dilation of $\boldsymbol{A}$ in $F(X^{I^+}, \overline{\boldsymbol{B}})$. That this minimal dilation is actually $\boldsymbol{B}$-valued follows from Theorem 4.2 together with the remarks at the beginning of the present proof.

The use of $\overline{\boldsymbol{B}}$ could be avoided at the expense of showing directly by computation using (2) that the suprema $\big(\exists^+(K)\,\boldsymbol{S}^+(\sigma)\,H(g)\big)(x)$ exist in $\boldsymbol{B}$ whenever $K \subset I^+$, $\sigma \in I^{+I^+}$, $g \in \boldsymbol{A}$ and $x \in X^{I^+}$. Q.E.D.

THEOREM 4.5. *If $\boldsymbol{A}$ is a $\boldsymbol{B}$-valued functional polyadic I-algebra with domain X then any J-compression of $\boldsymbol{A}$ is isomorphic to a $\boldsymbol{B}$-valued functional polyadic J-algebra with domain X.*

Proof. The proof is the same as that of Theorem 3.10 supplemented with the easy verification that

$$\big[H\big(\exists(K)f\big)\big](x) = [\exists(K)H(f)](x)$$

for all $x \in X^J$, $f \in \boldsymbol{A}_J$ and $K \subset J$. Q.E.D.

§ 5. An alternative proof. In this section we shall outline a different proof for the main result of § 4, which is the following consequence of Theorem 4.3:

(L1) *Any polyadic algebra of infinite degree $(\boldsymbol{A}, I, \boldsymbol{S}, \exists)$ admits for any superset I^+ of I a polyadic dilation $(\boldsymbol{A}^+, I^+, \boldsymbol{S}^+, \exists^+)$.*

Our new proof of (L1), like Halmos's proof of his dilation theorem (11.9) in [7], proceeds inductively. We shall describe our proof in a stepwise fashion.

1. We can easily prove (L1) in the special case $\overline{\overline{I}} = \overline{\overline{I^+}}$. Let γ be a biunique function mapping I^+ onto I. For each $\tau \in I^{+I^+}$ let $\widetilde{\boldsymbol{S}}(\tau) = \boldsymbol{S}(\gamma\tau\gamma^{-1})$, and for each $J \subset I^+$ let $\widetilde{\exists}(J) = \exists(\gamma J)$. Then it is easy to verify that $(\boldsymbol{A}, I^+, \widetilde{\boldsymbol{S}}, \widetilde{\exists})$ is a polyadic algebra. Moreover, it is also straight-forward to show that $\boldsymbol{S}(\gamma|I)$ is a polyadic isomorphism of $(\boldsymbol{A}, I, \boldsymbol{S}, \exists)$ into the I-compression of $(\boldsymbol{A}, I^+, \widetilde{\boldsymbol{S}}, \widetilde{\exists})$. Hence (L1) easily follows in this special case.

2. In the locally finite case (where the local degree of $\boldsymbol{A}$ is $\aleph_0$) it is possible to proceed from this special case as follows (after Halmos): we build a chain of dilations $\boldsymbol{A} = \boldsymbol{A}_0 \subset \boldsymbol{A}_1 \subset \boldsymbol{A}_2 \subset \ldots \subset \boldsymbol{A}_\omega \subset \ldots$, adding one new variable at each non-limit step, and at limit steps going in two stages; first to a quasi-polyadic algebra (see § 7 of [7]) by taking the union of preceding algebras, and then to a full polyadic algebra (see Theorem 7.6 of [7]).

Unfortunately, in the general case it is impossible to proceed in this way, except in the case where there are no singular cardinals $\mathfrak{m}$ such that $\overline{\overline{I}} < \mathfrak{m} \leqslant \overline{\overline{I^+}}$. The difficulty can be surmounted by using a direct limit construction which we shall presently outline. It is essential to have at hand a general notion of $\mathfrak{m}$-*quasi-polyadic algebras*. By definition, such an object is a quadruple $(\boldsymbol{A}, I, \boldsymbol{S}, \exists)$ such that $\boldsymbol{A}$ is a Boolean algebra, I is a non-empty set, $\boldsymbol{A}$ is a mapping from the set $\exp_{\mathfrak{m}}(I) = \{\tau \in I^I$: *there is a* $J \subset I$ *such that* $\overline{\overline{J}} < \mathfrak{m}$ *and* $\tau|I-J = \delta_{I-J}\}$ to Boolean endomorphisms of $\boldsymbol{A}$, and $\exists$ is a mapping from $S_{\mathfrak{m}}(I) = \{J \subset I\colon \overline{\overline{J}} < \mathfrak{m}\}$ to quantifiers on $\boldsymbol{A}$, such that (P_1)-(P_6) hold with "I^I" replaced by "$\exp_{\mathfrak{m}}(I)$" and "subset of I" by "member of $S_{\mathfrak{m}}(I)$". Halmos's discussion of quasi-polyadic algebras in § 7 of [7] generalizes to the present case; the point is to read "of cardinality $<\mathfrak{m}$" for "finite" throughout Halmos's discussion. In particular, the essential part of Theorem 7.6 of [7] may be generalized as follows:

(L2) *If* $(\boldsymbol{A}, I, \boldsymbol{S}, \exists)$ *is an* $\mathfrak{m}$-*quasi-polyadic algebra, and if for each* $p \in \boldsymbol{A}$ *there is a subset* J *of* I *such that* $\overline{\overline{I-J}} < \mathfrak{m}$ *and* $\exists(K)p = p$ *whenever* $K \in S_{\mathfrak{m}}(J)$, *then there are* $\widetilde{\boldsymbol{S}}$ *and* $\widetilde{\exists}$ *such that* (i) $(\boldsymbol{A}, I, \widetilde{\boldsymbol{S}}, \widetilde{\exists})$ *is a polyadic algebra,* (ii) $\widetilde{\boldsymbol{S}}|\exp_{\mathfrak{m}}(I) = \boldsymbol{S}$ *and* $\widetilde{\exists}|S_{\mathfrak{m}}I) = \exists$, *and* (iii) *the local degree of* $(\boldsymbol{A}, I, \widetilde{\boldsymbol{S}}, \widetilde{\exists})$ *is* $\leqslant \mathfrak{m}$.

3. Now we shall outline a proof of the following induction statement:

(L3) *Suppose that* I, $\mathfrak{K}$, $\boldsymbol{A}$ *and* I^+ *satisfy the following conditions:*

(i) $I \subset I^+$ *and* $\overline{\overline{I}} < \overline{\overline{I^+}}$;

(ii) $\mathfrak{K} = \{K\colon I \subset K \subset I^+$ *and* $\overline{\overline{K}} < \overline{\overline{I^+}}\}$;

(iii) *for every* $K \in \mathfrak{K}$, $\boldsymbol{A}_K$ *is a polyadic algebra with variables* K *which is a dilation of* $\boldsymbol{A}_I$ $(= \boldsymbol{A})$.

Then there is a polyadic algebra with variables I^+ which is a dilation of $\boldsymbol{A}_I$.

Proof. For each $K \in \mathfrak{K}$ let $\boldsymbol{A}_K$ be the algebra $(\boldsymbol{A}_K, K, \boldsymbol{S}_K, \exists_K)$. Form the Boolean algebra $\boldsymbol{C} = \Pi_{K \in \mathfrak{K}} \boldsymbol{A}_K$. Let $M = \{f\colon f \in \boldsymbol{C}$ and for some $K \in \mathfrak{K}$ we have $f_{K'} = 0$ whenever $K \subset K' \in \mathfrak{K}\}$. Clearly M is a Boolean ideal in $\boldsymbol{C}$. Let $\boldsymbol{D} = \boldsymbol{C}/M$. We shall define the structure of an $\overline{\overline{I^+}}$-quasi-polyadic algebra with variables I^+ on $\boldsymbol{D}$. For $\tau \in \exp_{\overline{\overline{I^+}}}(I^+)$ define $\varphi_\tau \in \boldsymbol{C}^{\boldsymbol{C}}$ as follows:

$$(\varphi_\tau f)_K = \begin{cases} \boldsymbol{S}_K(\tau|K)f_K & \text{if} \quad \tau K \subset K, \\ f_K & \text{otherwise,} \end{cases}$$

for all $K \in \mathfrak{K}$ and $f \in \boldsymbol{C}$. Then for each such τ there is an $\boldsymbol{S}(\tau) \in D^D$ such that

$$\boldsymbol{S}(\tau)(f/M) = \varphi_\tau f/M$$

for all $f \in \boldsymbol{C}$.

If $J \in S_{\overline{\overline{I^+}}}(I^+)$, we define $\chi_J \in \boldsymbol{C}^C$ by putting

$$(\chi_J f)_K = \mathfrak{I}_K(K \cap J) f_K ,$$

for all $K \in \mathfrak{K}$ and $f \in \boldsymbol{C}$. For each such J there is an $\widetilde{\mathfrak{I}}(J) \in \boldsymbol{D}^{\boldsymbol{D}}$ such that

$$\widetilde{\mathfrak{I}}(J)(f/M) = \chi_J f/M$$

for all $f \in \boldsymbol{C}$. It is a straight-forward matter to verify that $(\boldsymbol{D}, I^+, \widetilde{\boldsymbol{S}}, \widetilde{\mathfrak{I}})$ is an $\overline{\overline{I^+}}$-quasi-polyadic algebra.

For each $a \in \boldsymbol{A}_I$ we define $g \in \boldsymbol{C}$ by setting $g_a(K) = a$ for all $K \in \mathfrak{K}$. We define h mapping $\boldsymbol{A}_I$ into $\boldsymbol{D}$ by letting $h(a) = g_a/M$ for all $a \in \boldsymbol{A}_I$. Let $(\boldsymbol{E}, I^+, \widetilde{\boldsymbol{S}}, \widetilde{\mathfrak{I}})$ be the subalgebra of $(\boldsymbol{D}, I^+, \widetilde{\boldsymbol{S}}, \widetilde{\mathfrak{I}})$ generated by $h(\boldsymbol{A}_I)$. Then, using essentially a generalization of Theorem 2.2, we find that the conditions of (L2) are satisfied, and so we obtain $\hat{\boldsymbol{S}}, \hat{\mathfrak{I}}$ such that the conclusions of (L2) hold. Thus $(\boldsymbol{D}, I^+, \hat{\boldsymbol{S}}, \hat{\mathfrak{I}})$ is a polyadic algebra. It is easily verified that h is a polyadic isomorphism of $\boldsymbol{A}_I$ into the compression of $(\boldsymbol{D}, I^+, \widetilde{\boldsymbol{S}}, \widetilde{\mathfrak{I}})$ and the proof of (L3) is complete.

4. Now the proof of (L1) can be given. Suppose that I^+ is a superset of I of minimum cardinality such that the conclusion of (L1) fails. Then by step 1, $\overline{\overline{I}} < \overline{\overline{I^+}}$. But then the hypotheses of (L3) may be satisfied, and we get a contradiction.

§ 6. Representation.

We begin our discussion of the representation theory proper by dealing with the simpler and less important case of functional representation. First we have a generalization of Theorem 10.1 of [7].

THEOREM 6.1 [8]. *If $\boldsymbol{A}$ is a polyadic algebra with local degree $\mathfrak{m}$, and if M is a subset of I such that $\mathfrak{m} \leqslant \overline{\overline{M}}$, then*

$$(*) \qquad \boldsymbol{S}(\tau)\mathfrak{I}(J)p = \bigvee \{\boldsymbol{S}(\sigma)p\colon\ \sigma \in M^I \text{ and } \sigma|I-J = \tau|I-J\}$$

whenever $p \in \boldsymbol{A}$, $J \subset I$, and $\tau \in M^I$.

Proof. On the one hand, if $\sigma \in M^I$ and $\sigma|I-J = \tau|I-J$, we have $\boldsymbol{S}(\sigma)p \leqslant \boldsymbol{S}(\sigma)\mathfrak{I}(J)p = \boldsymbol{S}(\tau)\mathfrak{I}(J)p$. This proves half of the equality $(*)$.

To prove the other half, suppose $\boldsymbol{S}(\sigma)p \leqslant q$ for all $\sigma \in M^I$ such that $\sigma|I-J = \tau|I-J$; we want to show that $\boldsymbol{S}(\tau)\mathfrak{I}(J)p \leqslant q$. Let K be a support of p such that $\overline{\overline{K}} < \mathfrak{m}$. Then by the hypothesis of the theorem there is a $\sigma \in M^I$ such that q is independent of $\sigma(K \cap J)$, $\sigma(K \cap J) \subset M - \tau(K-J)$, $\sigma|(K \cap J)$ is biunique, and $\sigma|I-(K \cap J) = \tau|I-(K \cap J)$. Thus

$$\begin{aligned} \boldsymbol{S}(\tau)\mathfrak{I}(J)p &= \boldsymbol{S}(\tau)\mathfrak{I}(K \cap J)p && 2.1\ \text{(iv)} \\ &= \mathfrak{I}\big(\sigma(K \cap J)\big)\boldsymbol{S}(\sigma)p && 2.3 \\ &\leqslant \mathfrak{I}\big(\sigma(K \cap J)\big)\, q \\ &= q\,. \end{aligned}$$

Q.E.D.

[8] See [17] in which the earliest form of this theorem may be found.

From this theorem we easily obtain.

THEOREM 6.2. *If $\boldsymbol{A}$ is a polyadic algebra with local degree $\mathfrak{m}$, and if M is a subset of I such that $\mathfrak{m} \leqslant \overline{\overline{M}}$, then $\boldsymbol{A}$ is isomorphic to an $\boldsymbol{A}$-valued functional algebra with domain M.*

Proof. The isomorphism, f, is defined by the equation

$$(fp)(\tau) = \boldsymbol{S}(\tau)p\,,$$

valid for all $p \in \boldsymbol{A}$ and $\tau \in M^I$. The only step in the verification of the properties of f which differs in any way from Halmos's proof of (10.9) in [7] (except in using our Theorem 6.1 instead of Theorem 10.1 of [7]) is that f is biunique. Suppose that $fp = 0$. Let J be a support of p of cardinality $< \mathfrak{m}$, and let σ and τ be elements of I^I and M^I respectively such that $\sigma\tau|J = \delta_J$. We have $(fp)(\tau) = 0$, i.e., $\boldsymbol{S}(\tau)p = 0$. Hence $p = \boldsymbol{S}(\delta_I)p = \boldsymbol{S}(\sigma\tau)p = \boldsymbol{S}(\sigma)\boldsymbol{S}(\tau)p = 0$. Q.E.D.

We note that the proof of Theorem 4.4 combined with Theorems 6.1 and 6.2 provide us with a third proof of the existence of dilations, in the special but important case in which the local degree, $\mathfrak{m}$, is at most $\overline{\overline{I}}$. This is exactly a generalization of the dilation proof Halmos applied in [7].

Using our general theorem on dilations (Theorem 4.3), we can now prove the following functional representation Theorem.

THEOREM 6.3. *If $\boldsymbol{A}$ is a polyadic algebra with an infinite set I of variables, then $\boldsymbol{A}$ is isomorphic to a functional polyadic algebra whose domain has any specified power greater than or equal to the local degree of $\boldsymbol{A}$.*

The proof is an easy application of Theorem 4.3, Theorem 6.2, and Theorem 4.5.

We warn the reader that Theorem 6.3 does not settle the question of the existence of $\boldsymbol{O}$-valued representations of $\boldsymbol{A}$. One might think that by composing an $\boldsymbol{A}$-valued representation f with a homomorphism $\boldsymbol{A} \to \boldsymbol{O}$ one would get an $\boldsymbol{O}$-valued representation. But a quick check shows that this homomorphism would have to preserve suprema

$$\mathfrak{I}(J)p = \bigvee \{\boldsymbol{S}(\sigma)p\colon\ \sigma|I-J = \delta_{I-J}\}$$

for $J \subset I$ and $p \in \boldsymbol{A}$. The existence of such a homomorphism is indeed the whole problem.

Turning, finally, to $\boldsymbol{O}$-valued representations, we are now in a position to prove the main theorem of this paper.

THEOREM 6.4. *For any I-polyadic algebra $\boldsymbol{A}$ of infinite degree $\mathfrak{d}$, there exists an homomorphism h of $\boldsymbol{A}$ into an $\boldsymbol{O}$-valued functional algebra with domain X. If $\mathfrak{m}$ is the local degree of $\boldsymbol{A}$, $\mathfrak{c}$ is its effective cardinality and $\mathfrak{n}$ is any cardinal such that*

$$\mathfrak{n} \geqslant \mathfrak{c} \quad \text{and} \quad \sum_{\mathfrak{s}<\mathfrak{m}} \mathfrak{n}^{\mathfrak{s}} = \mathfrak{n},$$

then X can be chosen to be a set of cardinality $\mathfrak{n}$.

Proof. If $\mathfrak{d} \leqslant \mathfrak{n}$, $\boldsymbol{A}$ admits a minimal dilation $\boldsymbol{C}$ of degree $\mathfrak{n}$ of which $\boldsymbol{A}$ is a compression. If, on the other hand, $\mathfrak{d} \geqslant \mathfrak{n}$, $\boldsymbol{A}$ admits a faithful compression $\boldsymbol{C}$, also of degree $\mathfrak{n}$, of which $\boldsymbol{A}$ is a minimal dilation since $\mathfrak{n} \geqslant \mathfrak{c}$ and $\mathfrak{c} \geqslant \mathfrak{e}$ (Theorem 3.8 and Theorem 3.12) assuming that $\boldsymbol{A}$ is non-degenerate. In both cases a representation of $\boldsymbol{C}$ yields a representation of $\boldsymbol{A}$ by virtue of Theorem 4.4 and Theorem 4.5. Therefore, the case where $\boldsymbol{A}$ is degenerate being trivial it suffices to prove the theorem in the case where $\mathfrak{d} = \mathfrak{n}$. Note that $\mathfrak{m} \leqslant \mathfrak{n}$.

The choice of X is simply $X = I$. Since every element of $\boldsymbol{A}$ has a support of cardinality smaller than $\mathfrak{m}$, the only quantifiers $\exists(J)$ we need to care for are those for which $\overline{\overline{J}} < \mathfrak{m}$. With this in mind, we let $Z = \{(J, p) | J \subset I, \overline{\overline{J}} < \mathfrak{m}, p \in \boldsymbol{A}\}$.

In order to show that the cardinality of Z is at most $\mathfrak{n}$, we let K be a subset of I of cardinality $\mathfrak{e}$, the effective degree of $\boldsymbol{A}$. Then every element p of $\boldsymbol{A}$ is of the form $\boldsymbol{S}(\sigma)q$ with $q \in \boldsymbol{A}_K$ and $\sigma \in I^I$. Now, it is obvious that the number of subsets J of I of cardinality not exceeding a fixed cardinal $\mathfrak{s}$ is at most $\mathfrak{n}^{\mathfrak{s}}$ and therefore the number of subsets J of I such that $\overline{\overline{J}} < \mathfrak{m}$ is at most $\sum_{\mathfrak{s}<\mathfrak{m}} \mathfrak{n}^{\mathfrak{s}} = \mathfrak{n}$. Let q in $\boldsymbol{A}_K$ have a support of cardinality $\mathfrak{s} < \mathfrak{m}$. Then the number of distinct elements $\boldsymbol{S}(\sigma)q$ with $\sigma \in I^I$ is at most $\mathfrak{n}^{\mathfrak{s}} \leqslant \mathfrak{n}$. Hence there are at most $\mathfrak{n} \cdot \mathfrak{c}$ elements $\boldsymbol{S}(\sigma)q$ with $\sigma \in I^I$ and $q \in \boldsymbol{A}_K$. Therefore $\overline{\overline{Z}} \leqslant \mathfrak{n} \cdot \mathfrak{n} \cdot \mathfrak{c} = \mathfrak{n}$.

Hence Z can be indexed (possibly with repetitions) by the cardinal $\mathfrak{n}$ so that $Z = \{(J_\xi, p_\xi) \colon \xi < \mathfrak{n}\}$. Now there is a function τ with domain $\mathfrak{n}$ such that for each $\xi < \mathfrak{n}$, τ_ξ is an element of I^I, and satisfying the following conditions:

(1) $\tau_\xi | I - J_\xi = \delta_{I-J_\xi}$;

(2) $\tau_\xi | J_\xi$ is biunique;

(3) p_η is independent of $\tau_\xi J_\xi$ for each $\eta \leqslant \xi$;

(4) $\boldsymbol{S}(\tau_\eta) p_\eta$ is independent of $\tau_\xi J_\xi$ for each $\eta < \xi$.

The existence of τ follows easily from the cardinality hypothesis by a transfinite argument, but in order not to detract from the present line of proof we shall postpone it till the end of the proof of the theorem.

By Lemma 2.4 we easily infer that the set

$$\{[\exists(J_\xi)p_\xi]' \vee \boldsymbol{S}(\tau_\xi)p_\xi \colon \xi < \mathfrak{n}\}$$

generates a proper Boolean filter in $\boldsymbol{A}$. Indeed it suffices to show by induction on the finite number r that an intersection of r elements of this set is non-zero. The induction is immediate, using (1)-(4), if we let the q of Lemma 2.4 be the intersection of the $r-1$ of the r given elements which come first in the indexing by $\mathfrak{n}$, and p be the p_ξ of the last of these elements.

Let P be an ultrafilter in $\boldsymbol{A}$ containing the above set. Henceforth, the only relevant special property of the ultrafilter P is that for each element p of $\boldsymbol{A}$ and each subset J of I such that $\overline{\overline{J}} < \mathfrak{m}$ there exists $\varrho \in I^I$ such that

(5) $$\varrho | I - J = \delta_{I-J};$$

and

(6) $$[\exists(J)p]' \vee \boldsymbol{S}(\varrho)p \in P.$$

Condition (5) implies that $\boldsymbol{S}(\varrho)p \leqslant \exists(J)p$ which together with (6) entails

(7) $$\exists(J)p \in P \quad \text{iff} \quad \boldsymbol{S}(\varrho)p \in P.$$

Now we can define the desired homomorphism h. For each $p \in \boldsymbol{A}$ and $x \in X^I$ $(=I^I)$ we set

$$h(p)(x) = 1 \quad \text{iff} \quad \boldsymbol{S}(x)p \in P.$$

We claim that h is an homomorphism of $\boldsymbol{A}$ into the $\boldsymbol{O}$-valued functional algebra of all functions from X^I to $\boldsymbol{O}$; and we go through a stepwise verification of this fact.

1° h preserves $\vee$. We have for $p, q \in \boldsymbol{A}$ and $x \in X^I$

$$\begin{aligned} h(p \vee q)(x) = 1 \quad &\text{iff} \quad \boldsymbol{S}(x)(p \vee q) \in P, \\ &\text{iff} \quad \boldsymbol{S}(x)p \vee \boldsymbol{S}(x)q \in P, \\ &\text{iff} \quad \boldsymbol{S}(x)p \in P \text{ or } \boldsymbol{S}(x)q \in P, \\ &\text{iff} \quad h(p)(x) = 1 \text{ or } h(q)(x) = 1, \\ &\text{iff} \quad h(p)(x) \vee h(q)(x) = 1, \\ &\text{iff} \quad \big(h(p) \vee h(q)\big)(x) = 1. \end{aligned}$$

2° h preserves $'$. For $p \in \boldsymbol{A}$ and $x \in X^I$ we have

$$\begin{aligned} h(p')(x) = 1 \quad &\text{iff} \quad \boldsymbol{S}(x)p' \in P, \\ &\text{iff} \quad \big(\boldsymbol{S}(x)p\big)' \in P, \\ &\text{iff} \quad \boldsymbol{S}(x)p \notin P, \\ &\text{iff} \quad h(p)(x) = 0, \\ &\text{iff} \quad \big((hp)(x)\big)' = 1, \\ &\text{iff} \quad (hp)'(x) = 1. \end{aligned}$$

3° h preserves $S(\sigma)$, $\sigma \in I^I$. For $p \in A$ and $x \in X^I$ we have

$$\begin{aligned} h\big(S(\sigma)p\big)(x) = 1 \quad & \text{iff} \quad S(x)S(\sigma)p \in P\,, \\ & \text{iff} \quad S(x\sigma)p \in P\,, \\ & \text{iff} \quad h(p)(x\sigma) = 1\,, \\ & \text{iff} \quad \big(S^f(\sigma)h(p)\big)(x) = 1\,. \end{aligned}$$

4° h preserves $\exists(M)$, $M \subset I$. Let $p \in A$ and $x \in X^I$ be such that $h\big(\exists(M)p\big)(x) = 1$, that is to say

(8) $$S(x)\exists(M)p \in P\,.$$

In order to show that $\big(\exists^f(M)h(p)\big)(x) = 1$, we let K be a support of p such that $\overline{\overline{K}} < \mathfrak{m}$ and we set $J = M \cap K$, so that

(9) $$\exists(J)p = \exists(M)p\,.$$

Let $\sigma \in I^I$ be such that $\sigma|I-J = x|I-J$, $\sigma J \cap x(K-J) = 0$, and $\sigma|J$ is biunique. Then by Lemma 2.3 in which we let $\tau = x$,

(10) $$S(x)\exists(J)p = \exists(\sigma J)S(\sigma)p\,.$$

Now let ϱ satisfy (5) and (7) with σJ instead of J and $S(\sigma)p$ instead of p, i.e. let ϱ be such that $\varrho|I-\sigma J = \delta_{I-\sigma J}$ and $\exists(\sigma J)S(\sigma)p \in P$ iff $S(\varrho)S(\sigma)p \in P$. By (8)-(10) it follows that

(11) $$S(\varrho\sigma)p \in P\,.$$

Define $y \in X^I$ by the equations $y|M = \varrho\sigma|M$ and $y|I-M = x|I-M$. If $k \in K-M$, then $\varrho\sigma k = \varrho x k$ and, since $xk \notin \sigma J$, $\varrho x k = xk = yk$. This proves that $y|K = \varrho\sigma|K$. Since K supports p, we get by (11), $S(y)p \in P$. Thus $h(p)(y) = 1$ and since $y|I-M = x|I-M$, we have $\big(\exists^f(M)h(p)\big)(x) = 1$.

Conversely, suppose that $\big(\exists^f(M)h(p)\big)(x) = 1$. To show that $S(x)\exists(M)p \in P$, let $y \in X^I$ be such that $y|I-M = x|I-M$ and $h(p)(y) = 1$. Thus $S(y)p = P$, hence $S(y)\exists(M)p \in P$. But $S(x)\exists(M)p = S(y)\exists(M)p$ by (P_5).

This concludes the proof of 4°.

In order to terminate the proof of Theorem 6.4 we still have to show the existence of the function $\tau\colon \mathfrak{n} \to I^I$ satisfying conditions (1)-(4). This is a set-theoretic affair for which we need the elementary

LEMMA 6.5. *If $\mathfrak{n}$ and $\mathfrak{m}$ are cardinal numbers such that $\sum_{\mathfrak{s}<\mathfrak{m}} \mathfrak{n}^{\mathfrak{s}} = \mathfrak{n}$, if $\mu < \mathfrak{n}$, and if for each $\eta < \mu$, $\mathfrak{a}_\eta$ is a cardinal less than $\mathfrak{m}$, then*

(12) $$\sum_{\eta<\mu} \mathfrak{a}_\eta < \mathfrak{n}\,.$$

Proof. We have $\mathfrak{m} \leqslant \mathfrak{n}$, for otherwise we would have by the hypothesis, setting $\mathfrak{s} = \mathfrak{n}$, $\mathfrak{n}^{\mathfrak{n}} \leqslant \mathfrak{n}$, which is absurd. If $\mathfrak{m} < \mathfrak{n}$ we have immediately

$$\sum_{\eta<\mu} \mathfrak{a}_\eta \leqslant \mathfrak{m}\cdot\overline{\overline{\mu}} < \mathfrak{n}\,.$$

Now suppose $\mathfrak{m} = \mathfrak{n}$ and $\sum_{\eta<\mu} \mathfrak{a}_\eta = \mathfrak{n}$. Then

$$\mathfrak{n} < 2^{\mathfrak{n}} = \mathfrak{n}^{\mathfrak{n}} = \mathfrak{n}^{\sum_{\eta<\mu}\mathfrak{a}_\eta} = \prod_{\eta<\mu} \mathfrak{n}^{\mathfrak{a}_\eta} \leqslant \mathfrak{n}^{\overline{\overline{\mu}}} \leqslant \mathfrak{n}\,,$$

a contradiction. (We have set successively $\mathfrak{s} = \mathfrak{a}_\eta$ and $\mathfrak{s} = \overline{\overline{\mu}}$ and used the inequality $\mathfrak{n}^{\mathfrak{s}} \leqslant \mathfrak{n}$ for all $\mathfrak{s} < \mathfrak{n}$.)

This completes the proof of the lemma.

Let now $\mu < \mathfrak{n}$ and $\tau'\colon \mu \to I^I$ satisfy (1)-(4) with μ instead of $\mathfrak{n}$ and τ' instead of τ. For each $\eta \leqslant \mu$ let K_η be a support of p_η of cardinality $< \mathfrak{m}$. Then, by virtue of Lemma 6.5, $\Sigma_{\eta<\mu+1}\overline{\overline{K}}_\eta < \mathfrak{n}$. Similarly, for each $\eta < \mu$ let L_η be a support of $S(\tau_\eta)p_\eta$ of cardinality $< \mathfrak{m}$. Then $\Sigma_{\eta<\mu}\overline{\overline{L}}_\eta < \mathfrak{n}$. Hence $I - \big(\bigcup_{\eta<\mu+1} K_\eta \cup \bigcup_{\eta<\mu} L_\eta\big)$ is of cardinality $\mathfrak{n}$, and τ' can be extended to a mapping $\mu+1 \to I^I$ also satisfying the conditions (1)-(4). The existence of τ now follows by a simple application of the transfinite recursion theorem, [8] p. 70, or of Zorn's lemma.

This completes the proof of Theorem 6.4.

THEOREM 6.6. *Let A, X and $\mathfrak{c}$ be as in the previous theorem. Then there exists an isomorphism of A into a $O^{\mathfrak{c}}$-valued functional algebra with domain X.*

Proof. This follows immediately from 2.6, 2.7, 4.4, and 6.4. Q.E.D.

Theorem 6.4 can in turn be easily deduced from Theorem 6.6. For as it has already been hinted (after Theorem 6.3) a B-valued representation of A yields an O-valued representation when composed with a sum-preserving homomorphism $B \to O$. In the case where $B = O^{\mathfrak{c}}$ it suffices to use the natural homomorphism $B \to B/M$, with M a principal maximal ideal.

We may ask about improvements of Theorems 6.4 and 6.6, in two natural ways. First, can the restriction to infinite degree be eliminated? A negative answer has been given by one of us in [15]. Second, with the assumption of infinite degree can these theorems be extended to equality algebras? That the answer is negative has been known for some years, and a proof may be found essentially in the paper [18] of Słomiński. Thus, in a sense, the present result is the best possible. It is of interest, however, to ask whether or not the cardinality conditions stated in

Theorem 6.4 can be improved. We note that in the case of a denumerable locally finite algebra A we can take $\mathfrak{n} = \omega$, and that $\mathfrak{n} = 2^{\mathfrak{m}}$ is always a solution of the equation $\sum_{s<\mathfrak{m}} \mathfrak{n}^s = \mathfrak{n}$.

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A complete first-order logic with infinitary predicates

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It is well known that the first order predicate logic (with or without an identity symbol) has the following two properties:

(*) *each proof involves only finitely many formulas;*

(**) *a set of formulas is consistent if and only if it is satisfiable* (Gödel Completeness Theorem) (1).

In this paper we shall (in § 1) introduce a formal system L which has predicates with infinitely many argument places and quantifiers over infinite sets of variables, but which has only finitary propositional connectives and no identity symbol, and which satisfies (*). The system L is patterned after the finitary first-order system F_1 of Church in [1], and our notion of satisfaction is the natural extension of Tarski's definition (e.g. in [26], p. 193). Our main result, the *Completeness Theorem* (Theorem 3.1), is that L also satisfies (**) (2). The methods of proof are based upon the proofs of Henkin, and of Rasiowa and Sikorski, of the Gödel Completeness Theorem.

Generalizations of the Löwenheim-Skolem Theorem and of the Compactness Theorem to L (in § 3) will follow easily from 3.1. It is to be expected that many of the other familiar applications of the Gödel Completeness Theorem to first-order theory of models will eventually be generalized to the theory of models of the system L (3).

In § 4 we shall give some examples which indicate the difficulties encountered when one attempts to make various improvements of our main result.

In § 5 we shall introduce a more general formal system $L^{\#}$ which has, in addition to the expressions of L, functions and terms with in-

(1) See [5], [9], [17], [20], and [21].

(2) The main results of this paper were announced in abstracts [14].

(3) For an expository discussion of several applications of the Gödel completeness theorem, and for a historical account and references, we refer to [15]. For related results concerning the infinitary logics of [27] — which do not have property (*) — see [8] and [28]. We shall not here be concerned with the systematic development of the theory of models for L in the spirit of [21] or of [25].