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On the structure of homogroups with applications to the theory of compact connected semigroups

by

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This work is, for the most part, devoted to the study and application of a certain type of semigroup called a *homogroup*. By a homogroup, we mean a semigroup having a two-sided minimal ideal which is a group.

In the first part, we obtain some conditions under which certain semigroups become homogroups, and introduce the notion of maximal sub-homogroup and other notions which will be useful in what follows.

In the second part, we apply these results to the study of topological semigroups. In particular, we shall study the structure of certain compact connected semigroups. The results in this area quite naturally depend upon the nature of the canonical endomorphism associated with a homogroup. Under suitable conditions, this endomorphism, in the topological case, is a monotone. As we shall see, this fact enables one to construct various sub-semigroups including arcs. In this connection we shall show that a compact connected abelian semigroup (which is not a group), having an identity 1 contains a non-degenerate compact connected sub-semigroup whose intersection with the maximal subgroup at 1 is precisely 1.

Another application of this canonical endomorphism is a natural description of certain semigroups as coordinate bundles with connected fibres.

§ 1. Homogroups. The term homogroup was introduced by G. Thierrin [29] who studied their regular equivalences and made a detailed study of a special homogroup called *resorbing* (*résorbant*).

Earlier, A. H. Clifford and D. D. Miller, [3], had studied homogroups under the title "*semigroups with zeroide elements*". Let us recall that an element x of a semigroup D is called *net* or *zeroid* if for any d there exist elements s and t such that $ds = x$ and $td = x$. Now Clifford and Miller show that K, the set of net elements, if non-vacuous, is a two-sided ideal

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and is a group. Furthermore, if e is the identity of this group, then the application

$$x \to xe \quad (= ex),$$

is an endomorphism of D upon K.

On the other hand, suppose that D is a semigroup having a two-sided ideal K, which is a group. It is then easy to see that K is, in fact, the set of net elements. For let $k \in K$ and $d \in D$. Then $kd \in K$ and since K is a group, there is an element $s \in K$ such that

$$s(kd) = (sk)d = k.$$

Likewise there is a t such that $dt = k$.

It seems somewhat preferable for our purposes to view the notion of homogroup in terms of the existence of an ideal which is a group.

It is a virtually immediate fact that if an ideal L meets a subgroup G then L contains G. Hence, if I is an ideal and a group it is a minimal ideal. Thus we see that D is a homogroup if and only if D contains a minimal (two-sided) ideal which is a group. Thus, in the sense of Clifford [1], K is the kernel = minimal two-sided ideal of D. It is clear then, what we mean by the kernel of a homogroup. Following Thierrin [29], we shall call e, the idempotent in K, the *unitif élément* of the homogroup. (We reserve e for this rôle.) The canonical endomorphism,

$$x \to xe = ex,$$

shall be denoted, following [3], by ζ.

Certainly, if D is homogroup then D admits a homomorphism onto some group, namely K. And if D does not have a zero element this homomorphism is non-trivial.

On the other hand, suppose that A is a semigroup admitting a homomorphism f into a group G. Then $A \cup G$ can be given the structure of a homogroup by defining for $a \in A$ and $x \in G$

$$ax = f(a)\cdot x.$$

(See [3].) Of course, not every homogroup is obtainable in this way for G, the kernel in this construction, is prime which is not the case in general.

Let us recall the following quotient sets of Dubreil [5]. Let C be a subset of a semigroup D and $a \in D$.

$$C \centerdot\!\cdot a = \{y \mid ya \in C\},$$
$$C \cdot\!\centerdot a = \{x \mid ax \in C\}.$$

We denote the set

$$C \centerdot\!\cdot a \cap C \cdot\!\centerdot a \quad \text{by} \quad C^a.$$

Let D be a semigroup. By a *sub-group* or *sub-homogroup* we shall mean a sub-semigroup which is a group or a homogroup.

Let f be an idempotent element of the semigroup D. Then it is well known that there is a unique maximal sub-group containing f. It is somewhat customary to denote this group by H_f. It is easy to see that H_f may be given explicitly;

$$H_f = \{x \mid x \in fDf \text{ and } f \in xD \cap Dx\}.$$

It is also somewhat customary to denote the set of idempotent elements by E.

As we see in the first theorem each group H_f determines a particular sub-homogroup.

If the sets A and B have a non-vacuous intersection we shall write $A \between B$.

THEOREM 1.1. *Let D be a semigroup and M a subsemigroup of fDf where $f^2 = f$. Then*

$$M \centerdot\!\cdot f \cap M \cdot\!\centerdot f = M^f$$

is a sub-semigroup. If M is a sub-homogroup then so is M^f. In particular,

$$H_f \centerdot\!\cdot f \cap H_f \cdot\!\centerdot f$$

is a homogroup with minimal ideal H_f. Furthermore,

$$H_f^f$$

is the maximal sub-homogroup which has f as unitif element.

Let x be an element of $M \centerdot\!\cdot f \cap M \cdot\!\centerdot f$. Then

$$xf \in M \quad \text{and} \quad fx \in M.$$

Consequently,

$$fxf = f(xf) = xf$$

and

$$fxf = (fx)f = fx.$$

That is to say,

$$xf = fx.$$

Let x and y be points of $M \centerdot\!\cdot f \cap M \cdot\!\centerdot f$. Then

$$(xy)f = x(yf) = x(fyf) = (xf)(yf)$$

and since

$$(xf)(yf) \in MM \subseteq M,$$

we have

$$xy \in M \centerdot\!\cdot f \cap M \cdot\!\centerdot f.$$

Thus M^f is a sub-semigroup.

Now let M be a homogroup with K as kernel. For any $x \in M^f$ we have

$$xK = x(MK) = x(fM)K = (xf)MK \subseteq MMK = K.$$

Thus K is a left ideal of M^f and in the same manner is a right ideal. Since K is already a group, K is the kernel of M^f.

Finally, let F be a sub-homogroup whose kernel is G. If f is the unitif element, that is to say, if f is the identity of the group G then immediately $G \subseteq H_f$ since H_f is maximal. Since

$$fF \cup Ff \subseteq G\,,$$

we have

$$F \subseteq G \mathbin{.\cdot} f \cap G \mathbin{\cdot .} f \subseteq H_f \mathbin{.\cdot} f \cap H_f \mathbin{\cdot .} f\,.$$

Following the above theorem, if G is any sub-group whose identity is g then we shall designate the sub-homogroup

$$G \mathbin{.\cdot} g \cap G \mathbin{\cdot .} g$$

by

$$G^g\,.$$

For simplicity, we shall write H^g instead of H^g_g.

THEOREM 1.2. *Let h and f be idempotent elements of the semigroup D. If h commutes with each element of H_f then fh is an idempotent and*

$$H_f \subseteq H_{fh} \mathbin{.\cdot} h \cap H_{fh} \mathbin{\cdot .} h\,.$$

Consequently,

$$H_f \subseteq H^{fh}\,.$$

It is immediate that fh is idempotent. Now the application

$$x \to xh\,,$$

defined upon H_f is a homomorphism since h commutes with the elements of H_f. It follows that $(H_f)h$ is a sub-group of D and

$$fh \in (H_f)h\,.$$

Since H_{fh} is maximal,

$$(H_f)h \subseteq H_{fh}\,.$$

Similarly,

$$h(H_f) \subseteq H_{fh}\,.$$

For the second assertion one need only note,

$$(H_f)fh = (H_f)h \subseteq H_{fh}\,.$$

Thus

$$H_f \subseteq H^{fh}\,.$$

Let us note that the assumption that h commute with the elements of H_f is essential.

Consider the free semigroup S on four symbols x, f, y, h. Now impose the relations

$$x^2 = f^2 = f\,, \qquad xf = fx = x\,,$$
$$y^2 = h^2 = h\,, \qquad yh = hy = y\,.$$

In the semigroup S modulo these relations,

$$H_f = \{x\} \cup \{f\}\,, \qquad H_h = \{y\} \cup \{h\}\,.$$

However, $hf \neq fh$ and neither is idempotent.

THEOREM 1.3. *Let D be a semigroup containing at least one minimal left ideal and at least one minimal right ideal. If every two idempotents commute with one another then D is a homogroup.*

Clifford, [1], has shown that under our hypothesis a minimal ideal K exists and that K is the union of the minimal left ideals and is the union of the minimal right ideals. Furthermore, Clifford has shown that each minimal left ideal L is of the form Dg where $g^2 = g$. Likewise, each minimal right ideal has the form fD where $f^2 = f$. Now two distinct minimal left ideals must be disjoint. But if L and M are minimal left ideals then $L = Dg$ and $M = Df$ where $g^2 = g$ and $f^2 = f$ and hence,

$$fg = gf \in Dg \cap Df = L \cap M\,.$$

Hence K is itself a minimal left ideal. In the same way, we see that K is a minimal right ideal. It is then well known that K is a group, for in fact

$$Kx = K = xK$$

for each $x \in K$.

A subset N of a semigroup D is said to be *normal* if

$$dN = Nd \quad \text{for each} \quad d \in D\,.$$

We note the easily established fact that if D is normal itself then the idempotent elements of D commute with one another. For let f and g be idempotent. Now D_f contains gf and fD contains fg and $Df = fD$. Thus

$$fgf = f(gf) = gf$$

since $gf \in fD = Df$ and

$$fgf = (fg)f = fg$$

since $fg \in Df = fD$.

Hence we have the following.

COROLLARY. *Let D be a semigroup having at least one minimal left ideal. If D is normal then D is a homogroup.*

For with normality and minimal left ideal is also minimal right. Since the idempotents commute, D is a homogroup.

COROLLARY (Thierrin). *A finite abelian semigroup is a homogroup.*

According to Clifford [2], a semigroup D is said to *admit relative inverses* if for any $a \in D$ there is an idempotent e_a such that

$$ae_a = e_a a = a\,,$$

and an element a' such that

$$aa' = a'a = e_a\,.$$

It follows that if D admits relative inverses then D is the union of maximal subgroups H_g.

It should be noted that if X is any set we may define

$$ab = a \quad \text{for all} \quad a, b \in X .$$

X is then a semigroup which admits relative inverses. Hence, off hand, this hypothesis is not very restrictive. Certainly X is not a homogroup.

THEOREM 1.4. *Let D be a semigroup which admits relative inverses. Suppose that D contains an idempotent z such that $zg = gz = z$ for any idempotent g. Then D is a homogroup. More generally, if every one-sided ideal contains an idempotent then D is a homogroup.*

One sees without pain that if L is a left ideal and G is a subgroup of D then

$$L \mathrel{\supset\!\subset} G \quad \text{implies} \quad L \subseteq G .$$

Consequently, if D admits relative inverses then each left (right) ideal contains an idempotent. Since z annihilates each idempotent, each left (right) ideal contains z. It is now immediate that D contains both minimal left and minimal right ideals. According to [1], K, the minimal ideal, exists. As before, since K is the union of the minimal left ideals, and since any left ideal contains z, it follows that K is a minimal left ideal. Likewise K is a minimal right ideal. As before, K is a group, since

$$xK = K = Kx \quad \text{for} \quad x \in K .$$

Of course if D is a homogroup and admits relative inverses then the unitif element e has the property hypothesized for z in the previous result.

Again, let D be a homogroup, ζ the cannonical endomorphism, and e the unitif element. Clearly, $\zeta^{-1}(e)$ the inverse image of e is a sub-semigroup. Alternatively, we may describe this as

$$\{e\}\cdot\!.\, e, \quad \{e\}.\!\cdot\, e, \quad \text{or} \quad \{e\}^e .$$

However, we shall follow [3], and denote this set by J. Thus,

$$J = \{x|\ ex = xe = e\} .$$

It is immediate that J is a unitary sub-semigroup. That is to say,

$$xy \in J \text{ and } y \in J \quad \text{imply} \quad x \in J .$$

According to Clifford and Miller, J is the core and $K \cup J$ is the frame of D.

Many of our applications will center around the properties of J. We have already considered some of these in [10] and [11]. In particular, applications are made concerning the existence of arcs in topological

semigroups. The problem of arcs in semigroups is considered in [9], [10], [11], [12], [13], [14], [15], [16], [17], [19], [20], [24] and in various other works.

We recall that an element a is *left cancellative* if the application

$$x \to ax$$

is one-to-one. Equivalently,

$$ax = ay \quad \text{implies} \quad x = y .$$

Thus, each element of H_1 the maximal subgroup at 1 is left cancellative.

THEOREM 1.5. *Let D be a homogroup and P a sub-semigroup such that each element of P is left cancellative. Suppose further that*

$$\zeta|_P$$

is a monomorphism. If the application

$$\tau\colon PaJ \to D$$

is defined by

$$\tau(p, x) = px ,$$

then is one-to-one. If the elements of P and J commute with each other, and if multiplication in $P \times J$ is defined by

$$(p, x)(\bar{p}, \bar{x}) = (p\bar{p}, x\bar{x}) ,$$

then τ is a monomorphism.

Suppose $px = \overline{px}$. We then have

$$(px)e = (\overline{px})e = p(xe) = \bar{p}(\bar{x}e) = pe = \bar{p}e .$$

Since ζ is one-to-one on P, one has

$$p = \bar{p}$$

so that

$$px = p\bar{x}$$

and since p is left cancellative,

$$x = \bar{x} .$$

Thus, ζ is one-to-one.

Finally, if $q \in P$ and $y \in J$, implies $qy = yq$, one has

$$\tau(p, x)\tau(q, y) = (px)(qy) = p(xq)y = p(qx)y = (pq)(xy) = \tau(pq, xy) .$$

Thus, τ is a homomorphism.

Suppose that D is a homogroup with identity element 1 and, as usual, unitif element e. Consider the maximal subgroup H_1. Each element of this semigroup is left and right cancellative. Hence the previous result applies with $P = H_1$.

In any case if $\zeta|_{H_1}$ is not one-to-one, the application $\tau(p, x) = px$ where $p \in H_1$, $x \in J$, is still a homomorphism if H_1 and J commute elementwise.

Now let D be any semigroup. Let f and g be idempotents. We shall say that H_g is *under* H_f if $H_f \subset H^g$. If this is the case then g is under f in the usual sense. That is to say

$$gf = fg = g .$$

Moreover, since H_f is a homogroup, the application

$$x \to xg$$

defined upon H_f is a homomorphism. We shall denote this homomorphism from H_f to H_g by ζ_g^f. Thus with this notation, ζ_g^1 is the canonical endomorphism "cut down" to the maximal subgroup at 1.

The subset of D consisting of all x such that $x \in fDf$ and $gx = xg = g$ will, in accordance with our previous notation, be denoted by J_g^f.

A good deal of our later discussion will involve crucially the homomorphism ζ_e^1. Because of this, the following definitions will be useful.

Let D be a homogroup with identity 1 and unitif element e. Then we shall call D an *epi-group*, *mono-group*, *iso-group* respectively as ζ_e^1 is an epimorphism, monomorphism or an isomorphism.

We say that a homogroup D is *left* (*right*) *cylindrical* if $D = JH_1$ ($D = H_1J$). By *cylindrical* we mean both left and right cylindrical.

The following result is an immediate corollary to Theorem 1.5.

Let D be a cylindrical iso-group. If the elements of H_1 commute with the elements of J then

$$D \cong H_1 \times J .$$

THEOREM 1.6. *Let D be a homogroup and B a left ideal of J. That is to say, $B \subseteq J$ and $JB \subseteq B$. Then*

$$DB \cap J = JB .$$

It follows readily that

$$JB \subseteq DB \cap J .$$

Now let $x \in DB \cap J$. Then

$$x = db \in J .$$

Since J is unitary, as noted previously, $db \in J$ and $b \in J$ implies $d \in J$. Thus $db \in JB$ so $x \in JB$.

We recall that a semigroup D is left simple (with zero) if the only non-degenerate left ideal of D is D itself.

We recall the Rees quotient. Let I be an ideal of the semigroup D. We say that $a \equiv b$ modulo I if either $a = b$ or $a \in I$ and $b \in I$. It is easily checked that this is a congruence. The quotient semigroup is denoted by D/I or $D \bmod I$. We note that D/I has a zero, namely $\{I\}$.

THEOREM 1.7. *Let D be a homogroup. If D/K is left simple with zero then J is left simple with zero.*

Suppose, on the contrary, that $B \neq J$, $B \neq \{e\}$, $B \subset J$ and $JB \subseteq B$. Then $DB \cap J = JB$, from above. Now DB being a left ideal is either equal to D or DB is contained in K since D/K has no proper non-zero ideal. First, if $DB = D$ then

$$DB \cap J = D \cap J = J = JB \subseteq B$$

which is a contradiction. On the other hand, if $DB \subseteq K$ we see that $B \cup K$ is a left ideal so that $B \cup K = D$. It is then clear that this implies $B = J$.

Let D admit relative inverses. Then each element x is contained in some subgroup. We denote the maximal such subgroup by H_x.

THEOREM 1.8. *Let D be a semigroup which admits relative inverses. If the idempotent elements commute with one another then for each idempotent f we have*

$$H_f \cdot{}^\bullet f = H_f{}^\bullet \cdot f .$$

Furthermore, H^f consists of all elements x such that either

$$(H_x)f \supset H_f$$

or

$$f(H_x) \supset H_f .$$

According to lemma 3.1 or [2], each idempotent belongs to the center. Thus,

$$H_f \cdot{}^\bullet f = H_f{}^\bullet \cdot f .$$

The second part follows readily for H_x is a group and since f commutes with each element of this group, $(H_x)f$ is the homomorphic image of H_x under the application

$$\bar{x} \to \bar{x}f .$$

Clearly then, since H_f is maximal,

$$(H_x)f \supset H_f \quad \text{implies} \quad (H_x)f \subseteq H_f .$$

Thierrin, [29], calls a homogroup D *resorbing* if

(i) D admits relative inverses.

(ii) The product of two distinct idempotents is the unitif element.

As we shall see, a topological resorbing homogroup is of a rather limited sort. We shall consider these conditions, (i) and (ii), separately.

The following shows that certain naturally occurring sub-semigroups are resorbing.

THEOREM 1.9. *Let D be a semigroup and f and g idempotents such that fg commutes with each element of $H_f \cup H_g$. Then fg is idempotent and $H_f \cup H_g \cup H_{fg}$ is a resorbing homogroup with kernel H_{fg}.*

First of all,

$$(fg)^2 = (fg)(fg) = \big((fg)f\big)g = f(fg)g = fg\,.$$

Since $f(fg) = (fg)f = fg$, the rest follows from Theorem 1.2.

Rees, [26], defines a semigroup D to be *completely simple* if

1. D is simple.
2. $D = EDE$.
3. Under each idempotent there is a non-zero primitive idempotent.

We recall, [25], that a semigroup D is *simple* if D has no proper ideals except possibly zero and if, moreover, D is not the zero semigroup of order two. (By a *zero semigroup*, we mean one in which all products are zero.)

THEOREM 1.10. *Let D be a homogroup satisfying condition* (ii). *If K is a maximal ideal and D contains at least two idempotents then D/K is completely simple. (If $D^2 = D$ and each element in D is of finite order then there is an idempotent outside of K.)*

First of all we assert that D/K is a simple semigroup (with zero). To see this we note first that D/K cannot be the zero-semigroup of order two. For if D/K is composed of only two elements then for some point x we have $K \cup \{x\} = D$. Since D contains two idempotents, by hypothesis, and K being a group has only one idempotent, we conclude that $x^2 = x$. Thus, D/K is not a zero-semigroup. Certainly, since K is maximal, D/K can have no proper ideal except zero. Now let g be a non-zero idempotent in D/K. Then $g(D/K)g$ has only two idempotents because of condition (ii). Hence each idempotent in K/K which is non-zero must be primitive. According to Rees, [27], a simple semigroup with zero in which each non-zero idempotent is primitive and which contains a non-zero idempotent is completely simple.

Let us now consider the second assertion. Suppose first, that $D-K$ is degenerate say $\{x\} = D-K$. Since $D^2 = D$, we then must have $x^2 = x$ since $D = D^2 = (K \cup \{x\})^2 = K^2 \cup xK \cup Kx \cup x^2 = K \cup x^2$.

Suppose then that $D-K$ is non-degenerate. Let $A = D-K$. Now we cannot have $DAD \subset K$ since $D^3 = D^2 = D$. So we must have $DAD \between A$. Hence there is an $a \in A$ such that $DaD \between A$. Since K is maximal, we have $DaD \cup K = D$. Hence $a \in DaD$ so that $a = sat$ for some $s, t \in D$. Since each element has finite order, we see that since

$$a = sat = s^2at^2 = \ldots = s^nat^n = \ldots,$$

we may write

$$a = fag\,,$$

where f and g are idempotents. Now since $a \in K$, we see that say, $f \in K$.

Again let D be a homogroup with H_1 and J as usual. The products JH_1, H_1J play a particularly important rôle in the theory of topological semigroups. If D is compact and H_1 is finite dimensional then JH_1 can be given, in a natural way, the structure of a coordinate fibre in the sense of Steenrod as we shall later see. (The projection is taken as ζ.) The following definition will be useful in what follows:

We say that a homogroup D is *left* (*right*) *cylindrical* if $D = JH_1$ ($D = H_1J$). By *cylindrical* we mean both left and right cylindrical.

THEOREM 1.11. *Let D be a left* (*right*) *cylindrical homogroup. Then D is an epigroup.*

Let s and t be points of D. Clearly, if $tH_1 \between sH_1$ then $tH_1 = sH_1$. Thus if $dH_1 \between K$ then $dH_1 \between kH_1$, where $k \in dH_1 \cap K$. But then $dH_1 \subseteq K$. Now this is certainly impossible unless $d \in K$. Thus if $D = JH_1$ we must have $K = dH_1$ where $d \in J \cap K$. That is to say $d = e$. Thus $K = eH_1$.

THEOREM 1.12. *Let D be a semigroup such that each element of D generates a finite semigroup. Then D is the union of subsemigroups which are homogroups.*

We have only to recall that a finite abelian semigroup is a homogroup.

§ 2. We now turn our attention to the topological aspects of homogroups. Of course, by a topological semigroup we mean a Hausdorff space endowed with a continuous (associative) multiplication. In a topological homogroup the cannonical endomorphism ζ is continuous.

For a resumé of certain parts of the theory of topological semigroups in general, see [32].

We omit the adjective "topological" when there is no danger of confusion.

THEOREM 2.1. *Let D be a topological homogroup. If F is a compact sub-semigroup of D then*

$$F \cup \zeta(F)$$

is a sub-homogroup with kernel $\zeta(F)$.

The semigroup $\zeta(F)$ is compact since ζ is continuous. Now it is well known that a compact semigroup which is cancellative is a group. Now $\zeta(F)$ is cancellative, being a sub-semigroup of a group, and hence $\zeta(F)$ is a group. Clearly, $\zeta(F)$ is an ideal of the semigroup $F \cup \zeta(F)$ for if $x \in \zeta(F)$ and $y \in F$

$$xy = (xe)y = x(ey) \in \zeta(F)\,.$$

We shall denote the closure of a set X by X^*.

In the same manner as Theorem 2.1 we have

THEOREM 2.2. *Let D be a topological homogroup having a compact kernel. If F is a sub-semigroup of D then*

$$F \cup \zeta(F)^*$$

is a homogroup with kernel $\zeta(F)^*$.

THEOREM 2.3. *Let D be a compact topological semigroup. If the idempotent elements commute with one another then D is a homogroup.*

Since D is compact, there exists at least one minimal left (right) ideal. Theorem 1.3 then applies.

It is easy to see that Theorem 2.3 fails if the idempotents do not commute. One has but to take any set X and define $ab = a$ for all $a, b \in X$.

One may use Theorem 2.3 to obtain an analogue of Theorem 1.12 for topological semigroups.

Suppose that D is a (topological) semigroup having the property that each element x of D generates a semigroup whose closure is compact. Then D is the union of homogroups. Indeed, the closure of the set $x \cup x^2 \cup \cup x^3 \cup \ldots$ is an abelian compact semigroup and hence is a homogroup.

THEOREM 2.4. *Let D be a compact connected resorbing semigroup. Then D is a group.*

Suppose, on the contrary, that D is not a group. We note first that $D \neq K$. Since D admits relative inverses, there is an idempotent element f not in K. Now fDf is a compact connected semigroup meeting K and containing H_f. Since H_f is compact and H_f does not meet K, there is some point $x \in fDf$ such that $x \in H_f \cup K$. Now $x \in H_g$ for some idempotent g. Now if $fD \not\supset H_g$ then $fD \supseteq H_g$ since fD is a right ideal. Likewise $Df \not\supset H_g$ implies $Df \supseteq H_g$. Since $fDf = fD \cap Df$, it follows that $fDf \not\supset H_g$ implies $fDf \supseteq H_g$. Hence there is an idempotent g such that $g \neq e$, $g \neq f$ and $g \in fDf$. This is impossible since D is resorbant. Thus D is a group.

Theorem 2.4 fails without connectedness. For instance, let X be the semigroup consisting only of a zero and an identity.

We note also the following: *There exists a locally compact connected resorbing homogroup.* Indeed, let S be the set of points of the form $(e^{2\pi i S}, e^{-S})$ or $(e^{2\pi i S}, 0)$ where S is real. Then S is as desired the kernel being the points of the form $(e^{2\pi i S}, 0)$ and H_1 being the points of the form $(e^{2\pi i S}, e^{-S})$. We may describe S as a circle with a copy of the real line winding or spiraling down upon this circle.

At this point we recall the equivalences $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, of Green [7]. For any semigroup D, we define

$$a \equiv b(\mathcal{L}) \Leftrightarrow Da \cup a = Db \cup b\,,$$
$$a \equiv b(\mathcal{R}) \Leftrightarrow aD \cup a = bD \cup b\,,$$
$$a \equiv b(\mathcal{H}) \Leftrightarrow a \equiv b(\mathcal{L})\,, \quad \text{and} \quad a \equiv b(\mathcal{R}).$$

The $\mathcal{L}$-class, $\mathcal{R}$-class, or $\mathcal{H}$-class of an element x will be denoted respectively by L_x, R_x, and H_x.

It is well known, and not difficult to see, that the sets L_x, (R_x, H_x) form an upper semi-continuous decomposition of the compact semigroup D.

If g is an idempotent then $H_g = L_g \cap R_g$ is the maximal subgroup containing g. Hence our notation is consistent.

In the theory of semigroups the terms I-semigroup and standard thread are used to denote a semigroup which is topologically an arc and is such that one endpoint is a zero and the other is an identity. It is well known that such a semigroup must be abelian [6]. In the characterization of I-semigroups the following semigroups are canonical [24];

$I_1 =$ The usual unit interval with the ordinary multiplication,

$I_2 =$ The real interval $[\frac{1}{2}, 1]$ with the multiplication $x \cdot y = \max(\frac{1}{2}, xy)$,

$I_3 =$ The unit interval with the multiplication $x \cdot y = \min(x, y)$.

It is well known, [24], that if D is an I-semigroup (standard thread) containing no idempotents except zero and identity then D is isomorphic to either I_1 or I_2.

One important result about semigroups we will use is the following [6].

Let S be a compact connected semigroup. Let p be a point of S such that

$$S - p = A \cup B \quad \textit{mutually separate,}$$

with $K \subset A$. Then

$$Sp \cup pS \cup SpS \subseteq A^*.$$

It is immediate from this that if D is a standard thread from θ to 1 where θ is the zero of D. Then for any $x \in D$ the subarc from θ to x coincides with Dx. Actually if $[\theta, x]$ is this subarc we have

$$[\theta, x] = Dx = xD = DxD\,.$$

THEOREM 2.5. *Let S be a compact connected semigroup which admits relative inverses. Then E — the set of idempotent elements — is a continuum. Thus, if S is also a homogroup then ζ is monotone. Indeed, J is, in fact, arcwise connected containing a standard thread isomorphic to I_3 between e — the unitif idempotent — and any other idempotent in J. Hence, if K is arcwise connected so also is S.*

Let Δ be the cannonical application,

$$\Delta\colon\ S \to S/\mathcal{H}$$

defined, of course, by $\Delta(x) = \{H_x\}$. Since each set H_x has one and only one idempotent, the application,

$$r\colon E \to S/\mathcal{H}\,,$$

where r is defined by

$$r(f) = \Delta(f) = \{H_f\},$$

is one to one and onto.

Since E is compact, r is a homeomorphism. Since $S/\mathcal{H}$ is a continuum, E is a continuum.

Now, let S be a homogroup in addition. As always $E \subseteq J$. Now recall the partial order defined on the set E; f is under g, written $f \leqslant g$, if and only if $fg = gf = f$. This partial order is continuous. That is to say it has a closed graph. Otherwise said, if $f, g \in E$ with $f \not\leqslant g$ and $g \not\leqslant f$ then there are open sets U, V, such that $f \in U$, $g \in V$ and $a \in U$. $b \in V$ imply $a \not\leqslant b$ and $b \not\leqslant a$. All this follows quickly from the continuity of multiplication. Now e being a zero in J is, consequently, a minimal element in this partially ordered set E. Now, from a theorem of Koch [19] E contains a chain, T that is a linear ordered (under $\leqslant$) compact connected set from e — the minimal element — to say other $g \in E$. Thus if $t, s \in T$ either $st = ts = t$ or $ts = st = s$ as $t \leqslant s$ or $s \leqslant t$. It is clear now that ζ is monotone for consider $\zeta^{-1}(k) = \{x|\ ex = xe = k\}$. Let $t \in \zeta^{-1}(k)$. Let h be the idempotent in $H_t = H_h$. Now, from the above, there is an arc $[e, h]$ contained in the core. Now $t[e, h]$ contains an arc $[te, th] = [k, t]$. Hence, the inverse image of each point under ζ is, in fact, arcwise connected.

Let us weaken the rather strong condition that a homogroup be resorbing to the following.

A homogroup D is called *weakly resorbing* if

(1) $D = EDE$.

(2) *If f and g are any two distinct idempotents then $fg = gf = e$, where e is the unitif element.*

THEOREM 2.6. *Let S be a weakly resorbing homogroup. Then for each idempotent f, not in K, the set $fSf - K$ is open. Furthermore f is the only idempotent in $fSf - K$.*

Let x be a point of $fSf - K$. Suppose, on the contrary, that for any open set U about x such that $U \cap K = \square$ we have

$$U \not\subset (S - fSf).$$

Let y be a point of U not in fSf. Since S is weakly resorbant, there is an idempotent g such that

$$gyg = y.$$

Now

$$fy = f(gy) = (fg)y = ey \in K.$$

Since multiplication is continuous, and K is closed, we must therefore have

$$fx \in K.$$

But

$$fx = x \in K.$$

COROLLARY. *If S is weakly resorbing and separable metric and K is closed then E is at most countable.*

For each idempotent $f \notin K$ there is an open set $fSf - K$. If $h \notin K$ is another idempotent $hSh - K$ is open. Any two such open sets are mutually exclusive. Clearly if S is metric there are only countably many such open sets.

COROLLARY. *Let D be a locally compact resorbing homogroup with compact kernel. Then for each idempotent f the subsemigroup fDf is again a locally compact resorbing homogroup with compact kernel.*

Note that fDf is closed and hence locally compact. Now $fDf \cap K$ is a compact sub-semigroup of the group K and hence is itself a group. Hence fDf has a kernel $fDf \cap K$ which is a group. Since D is resorbing, so also is fDf.

It is to be noted that if fDf is connected then $H_f = fDf - K$ and is dense in fDf. This type of semigroup is considered in detail by Hoffmann [8].

At this point it is convenient to consider an example which is somewhat representative.

In the ordinary euclidean plane let (x_i, x_i^2) be a sequence of points converging to the origin, $x_i > 0$. Let A be the segment joining the origin with the point (x_i, x_i^2). Now let A_i be given the multiplication of I_2 or I_1 with the origin as zero. If $a_i \in A_i$ and $a_j \in A_j$ where $i \neq j$ define the product to be the origin. Let T be the union of the arcs A_i. Then T is a compact connected weakly resorbing semigroup.

THEOREM 2.7. *Let S be a compact connected weakly resorbing metric homogroup. Then S/K contains a sub-semigroup M such that*

(i) *M contains the set of idempotents of S/K.*

(ii) *M is a dendrite and is the sum of a countable number of arcs A_i such that each A_i is a sub-semigroup with the structure of I_1 or I_2 and two distinct arcs A_i, A_j meet only at the zero of S/K.*

If f is an idempotent not in K, then fSf has only two idempotents. Hence, fSf/K has only two idempotents one a zero the other an identity (for fSf/K). Hence, [23], there is an arc A_f, which is a sub-semigroup of fSf/K, which contains the zero and identity of fSf/K. Since A_f has but two idempotents, it is either I_2 or I_1. Now for each idempotent f choose precisely one A_f and let M be the union of the arcs A_f. Since S is compact, M is compact. Certainly M is a compact semigroup and $A_f \cap A_g$ contains only the zero element of S/K.

Let D be a homogroup and G a normal (invariant) subgroup of K. We define the congruence $\mathcal{K}_G$ by

$$x \equiv y\,(\mathcal{K}_G) \Leftrightarrow x = y$$

or

$$x \in K\,, \quad y \in K\,, \quad \text{and} \quad xG = yG\,.$$

It is straightforward to check that $\mathcal{K}_G$ is, in fact, a congruence. Furthermore, if D is a compact homogroup and G is a closed subgroup of K then D modulo $\mathcal{K}_G$, which we write $D/\mathcal{K}_G$, is a compact (topological) homogroup whose kernel is the quotient group $K/G = K$ modulo G. The natural homomorphism from D onto $D/\mathcal{K}_G$ is continuous.

In terms of this notation we have obviously

$$D/K \cong D/\mathcal{K}_K\,.$$

If f is a continuous mapping from the compact space X onto the space Y then it is well known that there is a space X'' and mappings g and h such that g is monotone, h is light and the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle h} & \\ & X'' & \end{array}$$

Moreover, [36], the space X'' is formed as the hyperspace of the upper semi-continuous decomposition of X formed by the components of the inverse images $f^{-1}(y)$. Now if C_1 and C_2 are components of the inverse images $f^{-1}(y_1)$ and $f^{-1}(y_2)$ then $C_1 \cdot C_2$ is contained in some component of the inverse image $f^{-1}(y_1 y_2)$. Thus, as first observed by Wallace, the analogue of the monotone-light factorization holds for compact semigroups. We shall make frequent use of this fact without further reference.

THEOREM 2.8. *Let K be a compact group. Suppose that*

$$\gamma\colon\ K \to C$$

is a continuous homomorphism onto C, the usual circle group. That is to say, suppose that γ is a character of K. Let $\gamma = \beta a$ be the monotone-light factorization of γ.

$$\begin{array}{ccc} K & \xrightarrow{\gamma} & C \\ & {\scriptstyle a}\searrow \quad \nearrow{\scriptstyle \beta} & \\ & P & \end{array}$$

Let G be a compact connected subgroup of K such that $\gamma(G) = C$. Then $a(G) = P$. Moreover P is of dimension one.

We assert that P does not contain a 2-cell. To see this note that the mapping β is light. Thus, if B were a 2-cell contained in P, $\beta(B)$ would be only one dimensional. But a light mapping cannot lower the dimension of B (of [18], pp. 91-92). Now the dimension of a compact topological group can be given in terms of the largest dimensional n-cell which it contains [25]. Thus P is at most one dimensional. Now, $a(G)$ is a non-degenerate compact connected subgroup of P so that

$$a(G) = P\,.$$

At this point, for the convenience of the reader we state the following:

PROPOSITION 1. *Let S be a compact connected normal semigroup with identity which is not a group. Let $\varphi = \beta a$ be the monotone-light factorization of φ where φ is the natural mapping onto $S/\mathcal{L}$.*

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & S/\mathcal{L} \\ & {\scriptstyle a}\searrow \quad \nearrow{\scriptstyle \beta} & \\ & S'' & \end{array}$$

Then both $S/\mathcal{L}$ and S'' contain standard threads from zero to identity. Thus there is a standard thread A in S'' such that $M = a^{-1}(A)$ meets K and contains the identity of S.

The above Proposition is merely a reformulation of some of the results in [10].

THEOREM 2.9. *Let S be compact connected normal homogroup with identity. If S is an epigroup, that is to say if ζ_e^1 is an epimorphism, then ζ is monotone.*

Let M be a continuum such as that whose existence is assured by Proposition 1. We assert that under the hypothesis of the theorem, the set $M \cap J$ is a continuum. We suppose then, on the contrary, that

$$M \cap J = C \cup B$$

with C and B mutually separate. Let e be a point of, say, C. Since $\varphi(M)$ is an arc from $\varphi(e)$ to $\varphi(1)$, there is a point $\varphi(b)$ which is the first point of $\varphi(B)$ in the order from $\varphi(e)$ to $\varphi(1)$. Since the set $\varphi(E)$ is compact, there is an idempotent h in $[\varphi(b), \varphi(1)]$ — the sub-arc of $\varphi(M)$ which is the first idempotent in the order from $\varphi(b)$ to $\varphi(1)$. Since

$$\varphi^{-1}(h) = L_d\,,$$

where $\varphi(d) = h$, is a compact sub-semigroup, it contains an idempotent g. And thus

$$\varphi^{-1}(h) = L_d = L_g$$

is a sub-group with identity g. It is immediate since $b \in gSg$ that

$$bg = gb = b\,.$$

Since multiplication is continuous, there are open sets U and V such that $b \in U$, $g \in V$, $bV \subset U$ and $U \cap C = \square$. Now, we assert there must exist a point p such that

$$p \in V \cap J$$

and

$$\varphi(p) < \varphi(g)$$

in the order from $\varphi(e)$ to $\varphi(1)$.

To see this, let us note that for any $x \in S$,

$$\zeta(L_1 x) = \zeta(L_1)\cdot\zeta(x) = \zeta_e^1(L_1)\cdot\zeta(x).$$

Since ζ_e^1 is an epimorphism, we have

$$\zeta_e^1(L_1)\zeta(x) = K\cdot\zeta(x) = K.$$

That is to say,

$$\zeta(L_1 x) = K$$

and thus for any x there is a point $q \in L_1 x$ such that

$$\zeta(q) = e$$

or what is the same

$$J \cap\!\!\!\!\subset L_x \quad \text{for all} \quad x.$$

In particular then, $J \cap\!\!\!\!\subset L_x$ for each x such that

$$\varphi(x) < \varphi(g) = h$$

in the order from $\varphi(e)$ to $\varphi(1)$. This fact and the compactness of J imply that there is a point t such that

$$t \in L_g \cap J$$

and such that for any open set W containing t, we have

$$\varphi^{-1}[\varphi(e), \varphi(g)] \cap J = gMg \cap J \cap\!\!\!\!\subset W - L_g.$$

Since $L_g \cap J$ is a compact sub-semigroup of a group, it is itself a group. Hence, there is a point s in $L_g \cap J$ such that

$$ts = g.$$

Multiplication being continuous, it follows that

$$(gMg \cap J)s \cap\!\!\!\!\subset V - L_g$$

Now if p is any point of $(gMg \cap J)s$ then

$$p \in V \cap J$$

and

$$\varphi(p) < \varphi(g)$$

in the order from $\varphi(e)$ to $\varphi(1)$.

Since $M \cap J$ is a sub-semigroup, $bV \subset U$, and $C \cap U = \square$, one has

$$bp \in B.$$

Now, since A is a standard thread from $\varphi(e)$ to $\varphi(1)$, we know that

$$\varphi(bp) = \varphi(b)\cdot\varphi(p) \leqslant \varphi(b)$$

in the order from $\varphi(e)$ to $\varphi(1)$.

Let us take first the case

$$\varphi(b)\cdot\varphi(p) = \varphi(b).$$

It then follows (p. 398, cor. 1, [21]) that

$$\varphi(b)r = \varphi(b),$$

where

$$r^2 = r \in \Gamma(\varphi(p)).$$

Since r is an idempotent, $\varphi^{-1}(r)$ is a group, as we have seen, so that for some $f^2 = f$

$$\varphi^{-1}(r) = L_f.$$

Now then the idempotent f is such that

$$\varphi(f) = r, \quad bf = fb = b,$$
$$\varphi(f) = r \in [\varphi(b), \varphi(1)],$$
$$\varphi(f) < \varphi(g) \quad \text{in the order from} \quad \varphi(b) \text{ to } \varphi(1).$$

This is in contradiction with the choice of g. Hence we assume that

$$\varphi(bp) = \varphi(b)\varphi(p) < \varphi(b).$$

But we already have

$$bp \in B$$

so that this is in contradiction to the choice of b. Thus, in either case we have a contradiction to the supposition that $M \cap J$ is not connected. Thus $M \cap J$ is a continuum contained in J and containing e and 1. Now since J contains a continuum between e and 1, it readily follows that each set $\zeta^{-1}(k)$ is a continuum by considering translates of $M \cap J$.

As it turns out, the previous theorem is crucial to our investigations. Now in a given homogroup S the natural homomorphism ζ_e^1 may, of course, not be onto, that is to say, S need not be an epigroup. However, in the homogroups which we shall consider, namely certain sub-homogroups near the identity, the natural endomorphism will at least be non-degenerate. This will make possible the transfer of the problem to a quotient homogroup which is an epigroup where the quotient is obtained by a congruence of the form $\mathcal{K}_G$ previously discussed. To do this much we rely rather heavily on the existence of enough characters to separate the points of an abelian K.

THEOREM 2.10. *Let S be a compact connected homogroup with identity. Let γ be a character of K such that*

$$\gamma(\zeta(H_1)) = C$$

where C is the circle group. If N is a normal subgroup of K such that

$$\operatorname{Ker}\alpha \subseteq N \subseteq \operatorname{Ker}\gamma\,,$$

where $\gamma = \beta\alpha$ is the monotone-light factorization of γ, then the semigroup $S/\mathcal{K}_N$ is an epigroup.

We have the following commutative diagram:

$$\begin{tikzcd} K \arrow[rr, "\gamma"] \arrow[dr, "\alpha"'] & & C \\ & P \arrow[ur, "\beta"] \arrow[r, "\sigma"'] & K/N \arrow[u, "\Delta"'] \end{tikzcd}$$

The maps σ and Δ are the natural homomorphisms. From Theorem 2.8 we know that $\alpha\zeta(H_1) = P$. Since $\beta = \Delta\sigma$, we see that

$$\beta\alpha\zeta(H_1) = \Delta\sigma\alpha\zeta(H_1) = C\,,$$

and hence

$$\sigma\alpha\zeta(H_1) = K/N\,.$$

Hence we see that every coset of N in K meets the group $\zeta(H_1)$. Clearly then, $S/\mathcal{K}_N$ is an epigroup.

THEOREM 2.11. *Let S be a compact homogroup with identity and abelian minimal ideal K. Let G be the component of e in $\zeta(H_1)$ and suppose that G is non-degenerate. Then there exists a character of K, say γ, such that $\gamma(G) = C$.*

If $g \in G$ and $g \neq e$ there is a character γ such that $\gamma(g) \neq \gamma(e)$. Hence $\gamma(G)$ is a non-degenerate compact connected subgroup of C and so $\gamma(G) = C$.

THEOREM 2.12. *Let S be a compact connected normal homogroup with identity and N a compact normal subgroup of K such that $S/\mathcal{K}_N$ is an epi-group. Then $K \cup \zeta^{-1}(N)$ is a compact connected homogroup.*

That $K \cup \zeta^{-1}(N)$ is a compact homogroup is clear. To see that is a continuum, note that since $S/\mathcal{K}_N$ is an epigroup, the associated endomorphism ζ_N (of $S/\mathcal{K}_N$) is monotone. Now the core of $S/\mathcal{K}_N$ and the space $K \cup \zeta^{-1}(N)$ modulo K are homeomorphic. Since K is a continuum, it is immediate that $K \cup \zeta^{-1}(N)$ is a continuum.

THEOREM 2.13. *Let S be a compact connected normal homogroup with identity such that $\zeta(H_1)$ is not totally disconnected and K is abelian. Then there is a compact invariant subgroup N of K such that*

$$K \cup \zeta^{-1}(N)$$

is a continuum and

$$\zeta^{-1}(N) \cap H_1 \neq H_1\,.$$

Let x be a point of the component of e in $\zeta(H_1)$ such that $x \neq e$. Since K is abelian, there is a character γ such that

$$\gamma(x) \neq \gamma(e)\,.$$

Let N be the component of e in $\operatorname{Ker}\gamma$. That is to say let $N = \operatorname{Ker}\alpha$ where

$$\begin{tikzcd} K \arrow[rr, "\gamma"] \arrow[dr, "\alpha"'] & & C \\ & P \arrow[ur, "\beta"'] & \end{tikzcd}$$

is the monotone-light factoring of γ. We now have the following diagram:

$$\begin{tikzcd} S \arrow[rr] \arrow[dr] & & S/\mathcal{K}_{\operatorname{Ker}\varphi} \\ & S/\mathcal{K}_N \arrow[ur] & \end{tikzcd}$$

where the mappings involved are all canonical homomorphisms. We know from Theorem 2.10 that $S/\mathcal{K}_N$ is an epigroup. From Theorem 2.12 we know that $K \cup \zeta^{-1}(N)$ is a continuum. Finally, $\zeta^{-1}(N) \cap H_1 \neq H_1$ since $x \in \operatorname{Ker}\gamma$ and so $x \notin N$ so $\zeta^{-1}(x) \cap H_1$ is not contained in $\zeta^{-1}(N)$.

Let S be a compact connected semigroup with identity. If S, as a continuum, is irreducible between some two points then S/K is an arc ([9]). Moreover, although irreducible subcontinua are available as usual, these may well not be sub-semigroups. Hence, if the notion of *irreducibility* is to be used in the theory of compact connected semigroups it might be revised somewhat to a more appropriate one. With this in mind, we make the following

DEFINITION. A compact connected semigroup D is said to be *algebraically irreducible* about (the subset) A if no proper compact connected subsemigroup of D contains A. Moreover D is said to be algebraically irreducible between the (mutually exclusive) subsets A and B if no proper compact connected subsemigroup of D meets both A and B.

Using the usual standard methods (Hausdorff maximality Principles or Zorn's Lemma) we have the following fact.

Let D be a compact connected semigroup and A and B two mutually exclusive compact subsets of D. Then there exists a compact connected subsemigroup of D which is algebraically irreducible between A and B.

These semigroups have been examined by N. J. Rothman and the author from the standpoint of character theory.

In connection with the notion of irreducibility it is of interest to point out the following result which follows from [14].

Let S be a compact connected semigroup such that $S = ESE$. If S, as a continuum, is irreducible about a finite set then either $S = K$ or S/K is a dendrite.

One important aspect of algebraic irreducibility is seen in the following result which is fundamental for our later investigations.

For various examples of semigroups which are algebraically irreducible, see the examples at the end of the paper and also those in [10] and [13].

THEOREM 2.14. *Let S be a compact connected abelian semigroup algebraically irreducible from K to 1. Then $H_1 = \{1\}$.*

We recall the monotone-light factorization of φ:

$$\begin{tikzcd} S \arrow[rr, "\varphi"] \arrow[dr, "\alpha"'] & & S/\mathcal{L} \\ & S'' \arrow[ur, "\beta"'] & \end{tikzcd}$$

We know from [10] that the semigroup S'' contains a thread from zero to identity, say M, and that $\alpha^{-1}(M)$ is thus a compact connected semigroup. Hence $S = \alpha^{-1}(M)$ and $S'' = M$.

Now suppose first that there is an open set V about 1 such that $V-\{1\}$ does not meet E. In this, it follows that there is a point $\alpha(x)$ of the arc M such that the sub-arc $[\alpha(x), \alpha(1)]$ contains no idempotents except $\alpha(1)$. Thus the semigroup S/Sx contains only two idempotents, one a zero — the element $\{Sx\}$ — the other the identity 1. Now from [23] we know that S/Sx contains a standard thread from zero to identity, say T. Let λ denote the canonical mapping from S to S/Sx. Then λ is one-to-one on $S-Sx$. It follows, since λ is monotone, that $\lambda^{-1}(T)$ is a compact connected semigroup and so $\lambda^{-1}(T) = S$. Since S is irreducible, we see that $H_1 = \{1\}$.

Hence, if there do not exist points of E arbitrarily close to 1, we are finished.

We suppose now that there do exist points of $E-\{1\}$ in any open set about 1. Or, what is the same thing, that in any sub-arc $[\alpha(x), \alpha(1)]$ of $M(S'')$ there are idempotents other than $\alpha(1)$. We now suppose that H_1 is non-degenerate. Since multiplication is continuous, it follows readily that for some idempotent f, where $\alpha(1) \neq \alpha(f) \in [\alpha(x), \alpha(1)]$, the homomorphism ζ_f^1 is non-trivial, that is that $fH_1 \neq \{f\}$. Now $\alpha^{-1}[\alpha(f), \alpha(1)] = H^f$ is a compact connected homogroup with minimal ideal $H_f = L_f$. Indeed, H^f satisfies the hypothesis of Theorem 2.13 so that there is a compact invariant subgroup of H_f, say N, such that $H_f \cup N^f$ is a compact connected semigroup and such that $N^f \cap H_1 \neq H_1$. Hence the semigroup $Sf \cup N^f$ is a continuum which is a proper subset of S. This contradiction completes the proof.

THEOREM 2.15. *Let S be a compact connected abelian semigroup with identity which is not a group. Then S contains a non-degenerate compact connected sub-semigroup X such that $X \cap H_1 = \{1\}$ and $X \supset\!\subset S-H_1$.*

One has only to choose as X any compact connected semigroup algebraically irreducible from K to $\{1\}$. From the previous results, the maximal subgroup of 1 in X is precisely $\{1\}$. Since $X \cap H_1$ is a compact sub-semigroup of a group, it also is a group. Hence $H \cap H_1 = \{1\}$.

COROLLARY. *Let S satisfy the conditions of Theorem 2.15. Then if G is any compact subgroup of H_1 there is a compact connected sub-semigroup X such that $X \cap H_1 = G$ and $X \supset\!\subset S-H_1$. If B is any compact subset of H_1 there is a subcontinuum X of S such that $X \supset\!\subset S-H_1$ (in fact $X \supset\!\subset K$) and $X \cap H_1 = B$.*

Without commutativity the best information on the existence of a continuum such as X is to be found in [12].

As we see in the next theorem, the class of compact connected semigroups which are algebraically irreducible from K to $\{1\}$ exhibits a number of properties similar to the class of standard threads.

THEOREM 2.16. *Let S be compact connected abelian semigroup algebraically irreducible from K to $\{1\}$ where $1 \notin K$. If $f \in E$ then*

(1) *H_f is a continuum.*

(2) *If $f \neq 1$ and $f \notin K$ then H_f separates S. Indeed, we have $S-H_f = (Sf-H_f) \cup (H^f-H_f)$ mutually separate.*

(3) *H^f is a compact connected semigroup algebraically irreducible from H_f to 1.*

As in the proof of Theorem 2.14, we note that $S = \alpha^{-1}(M)$ where α is monotone and M is an arc. Now $H_f = \alpha^{-1}(\alpha(f))$ and so is a continuum. It is clear that H_f separates S if $f \neq 1$ and $f \notin K$.

Now $\alpha(f)$ separates the arc M into $[\alpha(e), \alpha(f))$ and $(\alpha(f), \alpha(1)]$ mutually separate. We recall that the subarc $[\alpha(f), \alpha(1)]$ is a sub-semigroup with zero $\alpha(f)$ and $[\alpha(e), \alpha(f)]$ is a semigroup with $\alpha(f)$ as identity. It is immediate then that $H^f = \alpha^{-1}[\alpha(f), \alpha(1)]$. We have

$$S-H_f = \alpha^{-1}[\alpha(e), \alpha(f)) \cup \alpha^{-1}(\alpha(f), \alpha(1)]$$

mutually separate. Now if Sf meets the inverse image of an element in M, say $\alpha^{-1}(m)$, we must have $\alpha^{-1}(m) \subseteq Sf$. It follows that $\alpha^{-1}[\alpha(e), \alpha(f)] = Sf-H_f$. Now $H^f = \alpha^{-1}[\alpha(f), \alpha(1)]$ is a compact connected semigroup and $H^f \cap Sf = \alpha^{-1}(f)$. If H^f were to contain a proper subcontinuum, say Y, which is a sub-semigroup meeting H_f and containing 1 we see that $S = Sf \cup Y$ from the irreducibility. To see this we need only show that $Sf \cup Y$ is a semigroup. (It is certainly a continuum.) Let $a, b \in Sf \cup Y$. If either a or b is in Sf then so is ab since $Sf = fS$ is an ideal. If both are in Y so is ab since Y is by hypothesis a semigroup. If say $a \in Sf$ and $b \in Y$ then $ab = (af)b = a(fb) \in aH_f \subseteq Sf$. Likewise $ba \in Sf$. Now, by assumption Y is a proper subset of H^f. Since $Sf \cap H^f = H_f$, we see that $H^f - Y \subseteq H_f$. That is to say if Y omits a point h of H^f then $h \in H_f$. Now $Y \cap H_f$ is a compact sub-semigroup of H_f. In fact, $Y \cap H_f$ is an ideal of Y and so is a continuum. Since a compact sub-semigroup of a group is a semigroup with cancellation, it follows that it must already be a group. Hence, we see that $Y \cap H_f$ is a compact connected subgroup of H_f. From the corollary to Theorem 2.15, we see that $Sf = fSf$ contains a compact connected sub-semigroup Q meeting K and such that $Q \cap H_f = Y \cap H_f$. But then $h \notin Q \cup Y$ and yet $Q \cup Y = S$ using the irreducibility of S. With this contradiction our proof is complete.

Either as a consequence of the previous theorem or of [10], we have

COROLLARY. *Let S be a compact connected abelian semigroup algebraically irreducible from K to H_1. If each subgroup H_f, $f^2 = f$, is totally disconnected then S is an arc (i.e. a standard thread).*

THEOREM 2.17. *Let S and H_f be as in Theorem 2.16. Suppose that H_f is finite dimensional. If f is a limit point of $E \cap H^f$ then there is an element g in $E \cap H^f$ such that ζ_f^g is an epimorphism.*

Let us note first that $\alpha(E)$ is the set of idempotents of M where $\alpha\colon S \to M$ is the monotone homomorphism onto the standard thread M.

Let us note first that since α is monotone, $\operatorname{cl}\alpha^{-1}(\alpha(f), \alpha(1)]$ is a compact connected semigroup which meets H_f and contains 1. Hence from Theorem 2.16 we know that $H^f = \operatorname{cl}\alpha^{-1}(\alpha(f), \alpha(1)]$. In other words,

$$\operatorname{cl}\alpha^{-1}(\alpha(f), \alpha(1)] = \alpha^{-1}[\alpha(f), \alpha(1)]\,.$$

Now since H_f is finite dimensional, compact and connected, the union of the groups fH_h, $f < h$, is a compact group. We denote this by G. We assert that $H_f = G$. Suppose, on the contrary, that there is an element x in H_f but not in G. From above, we know that there are points of H^f arbitrarily close to x. Since $fx = x$, there is an open set V such that $fV \cap G = \square$, $x \in V$. Let t be an element of $V \cap H^f$. Now, by hypothesis there is an idempotent q such that $\alpha(q) \in [\alpha(f), \alpha(t)]$. Now, $qt \in Hq$ and so

$$ft = (fq)t = f(qt) \in fH_q \subseteq G\,.$$

Hence $ft \in G$ which is a contradiction. Hence $G = H_f$. Since the collection fH_h is ascending, we must have $fH_g = H^f$ for some g.

THEOREM 2.18. *Let S be a compact homogroup, such that $\operatorname{Ker}\zeta_e^1 = H_1 \cap J$ admits a local cross-section in H_1. Let $q\colon V \to H_1$ be this cross-section. Then the mapping*

$$\tau\colon V \times J \to S$$

defined by

$$\tau(v, j) = q(v) \cdot j$$

is a homeomorphism into.

It is clear that τ is a continuous mapping. To see that τ is 1 to 1 suppose that

$$q(v) \cdot j = q(\bar{v})\bar{j}\,.$$

Multiplying, we see that $q(v)(je) = q(\bar{v})(\bar{j}e) = q(v)e = q(\bar{v})e$. Since q is a cross section,

$$v = \bar{v}\,.$$

Then, $q(v) \cdot j = q(v)\bar{j}$ implies

$$j = \bar{j}$$

since $q(v) \in H_1$. Thus τ is a homeomorphism (into).

It is known that $\operatorname{Ker}\zeta_e^1$ admits a cross-section in H_1 whenever H_1 is finite dimensional.

Let us note also that $\operatorname{Ker}\zeta_e^1$ is an effective transformation group of J.

LEMMA. *Let all things be as in Theorem 2.18. For $h \in H_1$ let $hV = V_h$. Define $q_h\colon V_h \to H_1$ by*

$$q_h(v) = hq(h^{-1}v)\,.$$

Then q_h is 1 to 1. Define the coordinate function τ_h by

$$\tau_h(x, y) = q_h(x) \cdot y$$

where $x \in V_h$ and $y \in J$. Define $\tau_{h,x}$ by

$$\tau_{h,x}(y) = \tau_h(x, y)\,.$$

Then the homeomorphism

$$\tau_{h,x}^{-1}\tau_{g,x}\colon J \to J\,,$$

where $x \in V_h \cap V_g$, coincides with (the action of) an element of $\operatorname{Ker}\zeta_e^1$.

That q_h is 1 to 1 follows from Theorem 2.18. It is then easy to see that $\tau_{h,x}^{-1}\tau_{g,x}$ is a homeomorphism. Now we note that

$$(\tau_{h,x}^{-1}\tau_{g,x})(y) = \tau_{h,x}^{-1}(\tau_g(x, y))\,,$$
$$\tau_{h,x}^{-1}(g \cdot q(g^{-1}x)y) = q_h(x)^{-1} \cdot q_g(x) \cdot y\,.$$

Furthermore,

$$\begin{aligned} e(q_h(x) \cdot q_g(x)) &= e\{hq(h^{-1}x)\}^{-1} \cdot gq(g^{-1}x) \\ &= \{heq(h^{-1}x)\}^{-1} \cdot g \cdot eq(g^{-1}x) \\ &= \{h(h^{-1}x)\}^{-1} \cdot g(g^{-1}x) = x^{-1}x = e\,. \end{aligned}$$

Hence
$$q_g(x)[q_h(x)]^{-1}$$
coincides with an element of $\mathrm{Ker}\,\zeta_e^1$.

LEMMA. *Let all things be as in Theorem 2.18. Then the mapping*
$$\gamma_{h,g}\colon V_h \cap V_g \to \mathrm{Ker}\,\zeta_e^1$$
given by
$$\gamma_{h,g}(x) = \tau_{h,x}^{-1}\tau_{g,x}$$
is continuous.

This follows from the fact that $q_h q_g$ and inversion are continuous.

Hence we see that if $\mathrm{Ker}\,\zeta_e^1$ has a local cross-section in H_1 then $H_1\cdot J$ has in a natural way the structure of a coordinate fibre (bundle). The base being $\zeta_e^1(H_1) = eH_1 = H_1 e$, the fibre being J and the group $\mathrm{Ker}\,\zeta_e^1$ being the group of the bundle.

Now let f and g be idempotents with f under g that is
$$fg = gf = f.$$
If S is abelian then H_f is under H_g and then
$$\zeta_f^g \text{ is defined.}$$
Considering the semigroup $gSg \cap H^f$ it follows, from the above, that $H_g\cdot J_f^g$ has in a natural way the structure of a coordinate fibre (bundle). The base being $f\cdot H_g$, the projection ζ_f^g the fibre J_f^g, and the group $\mathrm{Ker}\,\zeta_f^g$.

THEOREM 2.19. *Let S be as in Theorem 2.16. Suppose that $f, g \in E$ are such that ζ_f^g is an epimorphism. Then*
$$a^{-1}[a(f), a(g)] = H_g\cdot J_f^g.$$
If $\mathrm{Ker}\,\zeta_f^g$ has a cross-section in H then $a^{-1}[a(f), a(g)]$ is given as a coordinate fibre in the sense of [28] where the base is H_f, the fibre is the continuum J_f^g, the group of the fibre is
$$\mathrm{Ker}\,\zeta_f^g = J_f^g \cap H_h.$$
Furthermore, J_f^g is itself algebraically irreducible about f and $\mathrm{Ker}\,\zeta_f^g$.

First of all, $J_f^g = gJ_f'$ is a continuum because of Theorem 2.9. Clearly
$$H_g\cdot J_f^g$$
is a compact connected semigroup. We know from Theorem 2.16 that
$$H^f = a^{-1}[a(f), a(1)]$$
is algebraically irreducible from H_f to 1. Now
$$H^f = a^{-1}[a(f), a(g)] \cup a^{-1}[a(g), a(1)].$$

We assert that
$$T = H_g\cdot J_f^g \cup a^{-1}[a(g), a(1)]$$
is a compact connected semigroup. To see this let $t_1, t_2 \in T$. If both points are in either $H_g\cdot J_f^g$ or $a^{-1}[a(g), a(1)]$ so is the product. Suppose $t_1 \in H_g\cdot J_f^g$ and $t_2 \notin H_g\cdot J_f^g$. Then $t_1\cdot t_2 = (t_1 g)t_2 = t_1(gt_2) \in t_1(H_g t_2) \in t_1 H_g \subseteq H_g J_f^g H_g = H_g J_f^g$. By the irreducibility of H^f, then, we see that
$$H_g J_f^g = a^{-1}[a(f), a(g)].$$
The description of $H_g\cdot J_f^g$ as a coordinate fibre has already been done.

Finally, suppose that Y is a proper subcontinuum of J_f^g which is a semigroup containing f and $\mathrm{Ker}\,\zeta_f^g$. Since J_f^g is a unitary sub-semigroup of
$$a^{-1}[a(f), a(g)],$$
it follows that if $x \in a^{-1}[a(f), a(g)]$ and $x \notin J_f^g$ then $xY \cap J_f^g = \square$. Now $H_g\cdot Y$ is a compact connected subsemigroup of $H_g J_f^g$ containing $H_f \cup H_g$. Hence $H_g Y = H_g J_f^g$. But since $H_g Y$ does not contain $J_f^g - Y$, we have a contradiction. Hence
$$J_f^g = Y.$$
As we have already noted if D is an iso-group then the mapping
$$\lambda\colon H_1 \times J \to D$$
defined by
$$\lambda(h, j) = h\cdot j$$
is one-to-one since
$$\bar{h}\bar{j} = hj$$
implies
$$\bar{h}\bar{j}e = hje = \bar{h}e = he.$$
Hence
$$h = \bar{h} \quad\text{and}\quad j = \bar{j}.$$
It follows that if D is a compact connected iso-group algebraically irreducible about K and H_1 then D is topologically a cartesian product. And, if D is abelian we have
$$D \cong H_1 \times J.$$
We consider at this point some applications to finite dimensional semigroups.

THEOREM 2.20. *Let S be a compact connected, normal, n-dimensional, homogroup with identity. If S is an epi-group and K has dimension $n-1$ then J contains a standard thread $[e, 1]$.*

We know that $H_1 J$ is a fibre bundle over the base $eH_1 = K$. Now J cannot have dimension $\geqslant 2$. For then the dimension of $J\cdot H_1$ would be at least $n-1+2=n+1$. To see this we recall, [25], that any open set in K contains an $n-1$ cell N. Hence if $\dim J=2$ then $\dim(J\times N)=\dim J+\dim N$. Hence we conclude that $\dim J<2$. Now since ζ_e^1 is an epimorphism J is a continuum, hence $\dim J=1$. Thus J is a compact connected one dimensional semigroup with zero e and identity 1. From [9] we know that J contains a standard thread.

THEOREM 2.21. *Let S be a compact connected normal, n-dimensional semigroup with identity. If H_1 is $n-1$ dimensional then there is a local thread at the identity.*

We know, as before, from Proposition 1, that S contains a continuum M such that $\alpha(M)$ is a standard thread where $\varphi\colon S\to S''$ is the cannonical homomorphism. If in M there are no other idempotents near 1, that is to say, if there exists an open set V such that $V\cap M\cap E=\{1\}$, then it is well known that M contains such a local standard thread. In fact, in this case M contains a local one-parameter semigroup [23]. Suppose then that there are idempotents $e_\alpha\neq 1$ in every neighborhood about of M about 1. For the remainder of the argument we restrict ourselves to points of M. As we know if $f\leqslant g$, that is if f is under g, $fg=gf=f$, or equivalently if $\varphi(f)\leqslant\varphi(g)$ in the order from $\varphi(e)$ to $\varphi(1)$ then

$$\operatorname{Ker}\zeta_g^1\subset\operatorname{Ker}\zeta_f^1 .$$

Now by continuity, the common part of the groups $\operatorname{Ker}\zeta_h^1$ must be precisely $\{1\}$. Hence there is an idempotent $p\neq 1$ such that $\operatorname{Ker}\zeta_p^1$ is zero dimensional. It then follows that $\zeta_p^1(H_1\cap M)$ is $n-1$-dimensional since

$$\dim H_1\cap M=\dim\operatorname{Ker}\zeta_p^1+\dim\zeta_p^1(H_1\cap M) .$$

Since $\zeta_p^1(H_1\cap M)$ is an $n-1$ dimensional subgroup of $H_p\cap M=L_p\cap M$ and since $H_p\cap M$ is at most $n-1$ dimensional for $p\notin K$, we see that H_p modulo $\zeta_p^1(H_1)$ is zero dimensional so that $\zeta_p^1(H_1\cap M)$ coincides with the identity component of $H_1\cap M$.

Now we know that in the monotone-light factorization of φ,

$$\begin{tikzcd} S \arrow[rr, "\varphi"] \arrow[dr, "\alpha"'] & & S/\mathcal{L} \\ & S'' \arrow[ur, "\beta"'] & \end{tikzcd}$$

that $\alpha(M)$ is a standard thread as is $\alpha^{-1}\big([\alpha(p),\alpha(1)]\big)$. Now since ζ_p^1 takes H_1 onto the identity component of H_p, it is well known that ζ_p^1 must map the identity component of H_1 onto the identity component of H_p. Thus, in the semigroup $\alpha^{-1}\big([\alpha(p),\alpha(1)]\big)$ the maximal subgroup at 1 is C_1 the identity component of H_1, and the minimal ideal is C_p — the identity component of H_p. And since

$$\zeta_p^1(C_1)=C_p=pC_1 ,$$

we see that

$$\alpha^{-1}\big([\alpha(p),\alpha(1)]\big)$$

satisfies the conditions of Theorem 2.20. For its kernel H_p is $n-1$ dimensional and its maximal subgroup at the identity is $n-1$ dimensional. Now it is a well-known result of Wallace [33] that if D is a compact connected n-dimensional semigroup with identity 1 then its maximal subgroup at 1 can be of dimension at most $n-1$. It follows from this that

$$\alpha^{-1}\big([\alpha(p),\alpha(1)]\big)$$

is n-dimensional.

In the next theorem we turn our attention to the special situation in which we deal with a plane semigroup. Here, quite naturally, the algebraic irreducibility is very restrictive topologically.

THEOREM 2.22. *Let S be a compact connected abelian semigroup with identity which is a subset of the plane and which is algebraically irreducible between K and 1. Then K is either the usual circle group or is degenerate. If K is a circle group then S is the union of a half open arc and a set which is either a simple closed curve (namely K), or is an annulus one of whose boundary curves is K the other a maximal subgroup H_f for some $f^2=f$.*

If K is degenerate, then S is either the union of a disc and a half open arc or is itself an arc. In case S contains a disc, the boundary curve of the disc is of the form H_f, $f^2=f$.

From Theorem 1.3 we see that K is a group. Now K is always a continuum if S is a continuum. Since K is a compact connected group embeddable in the plane, it must be the usual circle group.

Now we know as before, Proposition 1, that $S/\mathcal{L}$ is an arc from $\varphi(K)$ to $\varphi(1)$ and is monotone. In particular, then, since K does not separate S, we know that if K is a simple closed curve, $S-K$ is contained entirely in one of the complementary domains of K. In either case we assert that S/K is embeddable in the plane. For suppose first that $S-K$ is contained in D the bounded complementary domain of K. Now S/K is contained in the space formed by taking $D\cup K$ and shrinking K to a point. Thus S/K is topologically contained in the 2-sphere. Now S/K being a compact connected semigroup with identity cannot be the 2-sphere as is well known [32]. (For example, it is well known that $H^2(S/K)$ is trivial.) On the other hand, if $S-K$ is contained in the unbounded complementary domain of K then we note that S/K is naturally contained in the space formed by taking the plane and shrinking to a point the set $D\cup K$. This space is again the plane and so S/K in either case is embeddable in the plane.

Now if every maximal subgroup of S/K is totally disconnected then S/K contains a standard thread from zero to identity. By the irreducibility of S we see that S/K must then itself be a standard thread. Thus, in this case S is the sum of K and $S-K$ the latter being homeomorphic to $\varphi(S)-\varphi(K)$ which is a half open arc. On the other hand, if some maximal subgroup of S/K is not totally disconnected then from [10] there is a maximal subgroup H_g such that H_g is a usual circle group, the zero of S/K is contained in the bounded complementary domain of H_g and either S/Sg is a standard thread or $gS = gSg = S$ in which case S is topologically a disc.

$S =$

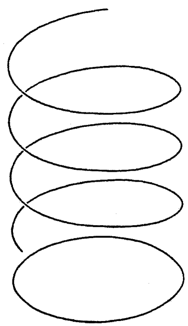

Fig. 1

Hence we see that if S/K is not an arc then there is a subgroup H_g such that K and H_g are the boundary curves of an annulus. Now this annulus must be a subset of S since S/K cannot separate the plane since $H_1(S/K) = 0$, [33]. Thus, in this case S is the sum of this annulus and a half-open arc.

Now if K is degenerate then the results of [10] applies immediately. There is a $g^2 = g$ such that either $gSg = S = a$ disc or S/Sg is a standard thread and H_g is a usual circle group and the disc it bounds is contained in S.

Now we know from Theorem 2.14 that $H_1 = \{1\}$ and hence in no case can S be a disc or an annulus for this situation would imply that H_1 is a simple closed curve.

The following result will prove useful in the verification that certain subsets of a semigroup are sub-semigroups.

PROPOSITION. *Let D be a semigroup. Let A and B be two sub-semigroups of D such that $A \cap B = C$ is such that*

(1) *it is an ideal of B,*

(2) *for each $a \in A$ there are elements g and h in C such that $ag = a$ and $ha = a$.*

Then $A \cup B$ is a sub-semigroup of D.

$$\begin{aligned}(A \cup B)^2 &= A^2 \cup AB \cup BA \cup B^2\\ &= A \cup (AC)B \cup B(CA) \cup B\\ &= A \cup B \cup A(CB) \cup (BC)A \cup B\\ &= A \cup B \cup AC \cup CA \cup B\\ &= A \cup B \cup A \cup A \cup B.\end{aligned}$$

EXAMPLE 1. Let T be composed of the points $(e^{2\pi iS}, e^{-S})$ where S is real and greater than or equal to zero, together with the usual circle group C-consisting of the points of the form $(e^{2\pi iS}, 0)$. We may describe T

as a half open arc spiraling down upon a circle C. Here, the circle is the usual circle group and the kernel of T. The above example is due to Wallace.

For another example we could include the whole complex disc, i.e. all complex z with $|z| \leqslant 1$. This semigroup is again algebraically irreducible and has a zero.

$T_1 =$

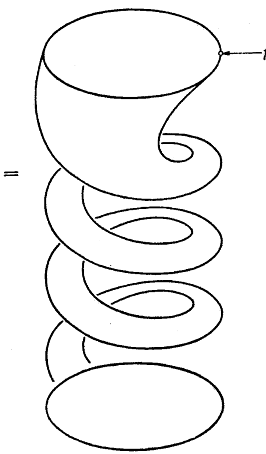

Fig. 2

$T_1' =$

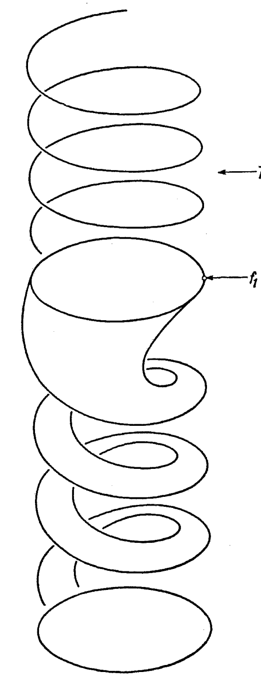

Fig. 3

EXAMPLE 2. We construct now a compact connected abelian semigroup algebraically irreducible from kernel to identity with the following property: There are idempotents f_i arbitrarily close to 1 (in fact, $\{f_i\}$ converges to 1), such that each H_{f_i} is the usual circle group. In particular then, none of the $\zeta^1_{f_i}$ are epimorphisms.

We begin by recalling the semigroup T of Example 1, an arc winding upon the circle group. Let C_1 be a copy of the circle group and form the semigroup $T \times C_1$. Now form the upper semicontinuous decomposition which shrinks to a point each set of the form $\{k\} \times C_1$ where $k \in K =$ the minimal ideal of T. (K is a circle group.) The resulting semigroup T_1

is a tube winding upon a circle. Let the identity of T_1 be denoted by f_1. Now using H_{f_1} in place of C in example construct, using H_{f_1} as kernel, a semigroup isomorphic to T.

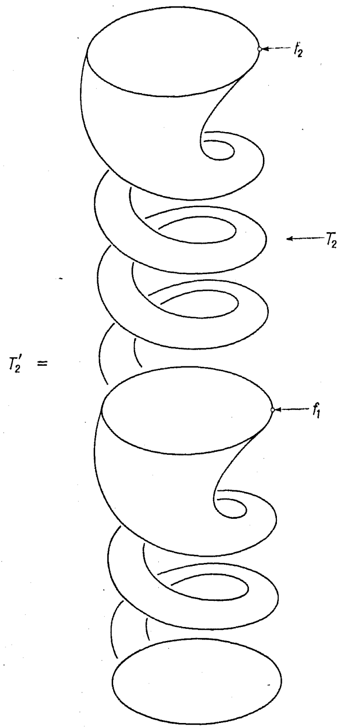

Fig. 4

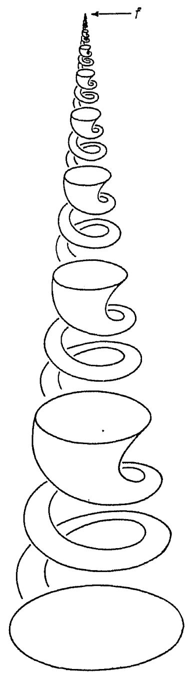

Fig. 5

Specifically, one can form $T_1 \times I$, where I is the unit interval, and select out the sub-semigroup

$$T_1 \times \{0\} \cup H_{f_1} \times I .$$

In the semigroup $H_{f_1} \times I$, construct a semigroup such as T. It follows that

$$T_1 \times \{0\} \cup T$$

is a sub-semigroup. Let us identify $T_1 \times \{0\}$ with T_1. The semigroup $T_1' = T_1 \cup T$ is pictured in figure 3. As before we form $T_1' \times C$ and shrink to a point each set $\{t_1\} \times C$. The resulting semigroup, we denote by T_2'.

We denote the identity of T_2 by f_2. The closure of the arc component of f_2 we denote by T_2. We continue this process as before. Let the T_i be taken so as to converge to a point f not in any T_i. Define $xf = fx = x$ for all $x \in T_i$. The resulting semigroup T is the desired one.

If we form $T \times C$ and then allow an arc to wind upon $f \times C$ so as to form a semigroup we see that we can construct an example in which H_f is non-trivial $f_i \to f$ and no $\zeta_{f_i}^{f}$ is an epimorphism.

In a compact connected semigroup with identity certain compact connected algebraically irreducible sub-semigroups may be standard threads while other such sub-semigroups may have a more complicated structure.

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Pointwise periodic groups

by

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1. Introduction. Let A be any Boolean ring, and let G be a group of automorphisms of A; for an element $a \in A$ and an element $g \in G$, we denote the image of a under g by a^g. The set $a^G = \{a : g \in G\}$ is called the *orbit* of a (under G). The group G is said to be *pointwise periodic* (on A) if every orbit is a finite set, and a pointwise periodic group is said to be *periodic* (on A) if there is an upper bound to the size of the orbits. (A group, all of whose elements have finite order, is sometimes called periodic. We shall call such groups *torsion groups*.) By the duality theory of M. H. Stone [6], any Boolean ring is isomorphic to the Boolean field of open compact sets in some locally compact, totally disconnected, Hausdorff space X. Following Stone, we shall call such spaces *Boolean spaces*, and, following Halmos, we shall the space X associated with a Boolean ring the *dual space* of the ring. Stone has shown that there is a natural isomorphism between the group of all automorphisms of a Boolean ring and the group of all homeomorphisms of its dual space. Consequently, for any group G of automorphisms of A there is an isomorphic group Γ of homeomorphisms of X. The notions of pointwise periodicity and periodicity for Γ have an obvious meaning.

The special case of a cyclic group has received some attention. More precisely, if g is an automorphism of A, and if γ is the dual homeomorphism of X, then the following is known:

(a) *The automorphism g is periodic if and only if γ is periodic*; this is an immediate consequence of the isomorphism given by Stone.

(b) *If γ is periodic, then, trivially, it is pointwise periodic.*

(c) *If γ is pointwise periodic, then g is pointwise periodic*; this result is due to A. D. Wallace [7], extending a theorem of Hall and Schweigert [2].

The converses of (b) and (c) are not, in general, valid. Counter-examples will be found below; the failure of the converse of (c) seems not to have been noticed.

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