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Generalizations of Bohr's theorem on Fourier series with independent characters (1)

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1. Introduction. A classical theorem of Bohr (1924) states that if $\sum a_n e^{i\lambda_n t}$ is the Fourier series of a uniformly almost periodic function f and if $\lambda_1, \lambda_2, \ldots$ are real numbers linearly independent over the field of rational numbers, then the series is absolutely convergent and $\sum |a_n| = \sup\{|f(t)| : -\infty < t < \infty\}$. The theorem may be formulated for an Abelian group G as follows:

If $\sum a_n \chi_n$ is the Fourier series of a uniformly almost periodic function f on G and if the characters (2) *$\chi_1, \chi_2, \ldots$ are independent, then $\sum |a_n| < \infty$.*

An exact formulation of this theorem requires a precise definition of independence of characters. The standard definition for characters $e^{i\lambda t}$ may be extended to the case of an arbitrary Abelian group in various ways; the equality $\sum |a_n| = \sup\{|f(t)| : t \in G\}$ occurs only when the strongest definition of independence is assumed. This fails to include certain important cases. The best constant K in the inequality $\sum |a_n| \leqslant K \sup |f(t)|$ with various notions of independence of $\chi_1, \chi_2, \ldots$ is discussed in Section 3.

Generalizations of the theorem of Bohr can be obtained by replacing characters by more general functions. E. g. S. Mazur and W. Orlicz [17] established similar theorems for bounded periodic functions on the line with linearly independent inverses of periods. S. Bochner ([2], p. 134) has proved the following generalization of Bohr's theorem:

Let $f \sim \sum_{\lambda \in \Lambda} a(\lambda) e^{i\lambda t}$ be a Bohr function such that Λ can be decomposed into a countable union $\Lambda = \bigcup \Lambda_n$ and $k_1\lambda_1 + \ldots + k_n\lambda_n = 0$, where $\lambda_1 \in \Lambda_1, \ldots, \lambda_n \in \Lambda_n$ and $k_1, \ldots, k_n$ are integers, implies $k_1 = \ldots = k_n = 0$. Then $\sum_{\lambda \in \Lambda_n} a(\lambda) e^{i\lambda t}$ are Fourier series of Bohr functions f_n $(n = 1, 2, \ldots)$, $\sum f_n = f$ and $\sum \sup |f_n| < \infty$.

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(2) By *characters* we shall always mean continuous characters; ι will denote the unit character. Group operations in G will be written additively and those in the dual $\hat{G}$ of G—multiplicatively.

If we consider this question for an Abelian group G, Bochner's assumptions have a natural group formulation as follows: Let $\{\chi_{11}, \chi_{12}, \ldots\}$, $\{\chi_{21}, \chi_{22}, \ldots\}, \ldots$ be a sequence of finite or infinite systems of characters on G. These systems $\Phi_n = \{\chi_{nm}\}_{m=1,2,\ldots}$ are called *independent* if

$$\prod_{n=1}^{N}\prod_{k=1}^{k_n}\chi_{nk}^{m_{nk}} = \iota \quad \text{implies} \quad \prod_{k=1}^{k_n}\chi_{nk}^{m_{nk}} = \iota \text{ for } n = 1, \ldots, N \tag{1}$$

for any integers $N \geqslant 1$, $k_1 \geqslant 1, \ldots, k_N \geqslant 1$, $m_{nk} = 0, \pm 1, \pm 2, \ldots$ $(n = 1, \ldots, N;\ k = 1, \ldots, k_n)$. In other words, if $[H]$ denotes the group generated by H, then the independence of $\Phi_1, \Phi_2, \ldots$ means that, for any finite disjoint sets $\{n_1, \ldots, n_p\}$ and $\{m_1, \ldots, m_q\}$ of indices, the intersection $[\bigcup \Phi_{n_i}] \cap [\bigcup \Phi_{m_j}]$ contains only the unit of the group. Suppose that f is a Bohr function on G and

$$f \sim \sum_{n,m} a_{nm}\chi_{nm}. \tag{2}$$

The problem is to find Bohr functions f_n such that $f = \sum f_n$ and

$$f_n \sim \sum_{m} a_{nm}\chi_{nm}. \tag{3}$$

M. Jerison and G. Rabson [10] have worked out an interesting approach to theorems of this kind for arbitrary locally compact groups. Their method (which is the development of ideas of B. Jessen [11]) may be reformulated as follows. Let us assume for simplicity that G is compact and Abelian, and let

$$G_n = \bigcap_{k=1}^{n}\bigcap_{m}\{x \in G:\ \chi_{km}(x) = 1\} \quad \text{and} \quad \Gamma_n = G/G_n.$$

Then (cf. [15], § 33A, p. 131) $\mu_{\Gamma_n}(E) = \mu_G(\pi_n^{-1}E)$ for any Borel set $E \subset \Gamma_n$ and

$$\int_{\Gamma_n}\Big[\int_{G_n} f(x+y)\mu_{G_n}(dy)\Big]\mu_{\Gamma_n}(dx) = \int_G f(x)\mu_G(dx)$$

for $f \in L(G)$, where π_n is the natural projection of G onto Γ_n and μ_H denotes the normalized Haar measure on H. If $\chi \in \bigcup_{k=1}^{n}\Phi_k$, then $\chi(x) = 1$ for all $x \in G_n$ whence $\int \chi\, d\mu_{G_n} = 1$. If $\chi \in \bigcup_{k=n+1}^{\infty}\Phi_k$, then $\chi(x_0) \neq 1$ for some $x_0 \in G_n$ (by double annihilator property) whence $\int \chi\, d\mu_{G_n} = 0$ by standard argument. Thus, if $f \sim \sum a_{nm}\chi_{nm}$ and $f \in C(G)$, then the coset averages

$$\bar{f}_n(x) = \int_{G_n} f(x+y)\mu_{G_n}(dy)$$

(defined for $x \in G$) are also continuous and they are partial sums of the Fourier series of f, namely

$$\bar{f}_n(x) = \int_{G_n}\sum_{k=1}^{\infty}\sum_{m} a_{km}\chi_{km}(x)\chi_{km}(y)\mu_{G_n}(dy) = \sum_{k=1}^{n}\sum_{m} a_{km}\chi_{km}(x),$$

equalities being meant in $L_2(G)$. Since $\bar{f}_n$ are averages of f on decreasing sequence of subgroups G_n and since f is constant on $\bigcap G_n$, $\bar{f}_n$ converge to f uniformly on G. Finally, $f_n = \bar{f}_n - \bar{f}_{n-1}$ are also continuous, $f_n = \sum a_{nm}\chi_{nm}$ and $\sum f_n$ is uniformly convergent to f. This convergence is actually unconditional because the order of $\Phi_1, \Phi_2, \ldots$ plays no role. Thus, if each Φ_n contains exactly one character χ_n, we get Bohr's theorem. This method is also valid if Φ_n's are not independent mutually but if each Φ_{n+1} is disjoint with the group generated by $\Phi_1 \cup \ldots \cup \Phi_n$, though we do not conclude that the convergence is unconditional in this case.

If G is considered as a probability space with respect to Haar measure μ_G and if $\mathscr{F}_n$ denotes the field of all sets of the form $\pi_n^{-1}(E)$, where E is a Borel set in Γ_n, then $\{\bar{f}_n, \mathscr{F}_n\}$ is a martingale (cf. Doob [3], Ch. VII). Applying theorems on convergence of martingales, Jerison and Rabson have proved some theorems on convergence of $\bar{f}_n$ for $f \in L_1(G)$. However, this belongs to another group of theorems which include the Rademacher-Kolmogorov theorem on convergence of Rademacher series and its generalizations for independent random variables, first group of Banach's equivalences for lacunary orthogonal series (cf. [1] and [12], p. 250-255) and so on. In this paper we consider only theorems concerning absolute convergence and bounded functions, i. e. theorems involving the spaces l_1 and l_∞, like Bohr's theorem, second group of Banach's equivalences and their generalizations.

S. Hartman and C. Ryll-Nardzewski [6] have proved that if Φ_1, $\Phi_2, \ldots$ are independent and $f, f_1, f_2, \ldots$ have the same meaning as above, then

$$\sum_{n=1}^{\infty}\|f_n\| < \infty \tag{4}$$

where $\|f_n\| = \sup\{|f_n(x)| : x \in G\}$. They embed G into a toroidal group and show that f_n depend on separate variables.

In this paper we give two new proofs of this theorem. Their starting point is the following remark. The theorem of Bohr means that if χ_1, $\chi_2, \ldots$ are independent characters on a compact Abelian group G, then the class of all uniformly almost periodic functions with Fourier series of the form $\sum a_n\chi_n$ is a closed subspace of $C(G)$ and is isomorphic to the space l_1.

A similar interpretation is valid for Hartman's and Ryll-Nardzewski's generalization of Bochner's theorem. Let us assume, for simplicity, that series (2) contains no constant term (i. e., that $\chi_{nm} \neq \iota$ for all n and m). (4) means that if $\Phi_1, \Phi_2, \ldots$ are independent, then the class E of all functions of $C(G)$ with Fourier series of the form (2) is a closed subspace of $C(G)$ and is the l_1-direct sum ([3]) of subspaces $E_1, E_2, \ldots$ of functions with Fourier series of the form (3), $n = 1, 2, \ldots$, respectively. Consequently, by Banach's theorem on the inversion of linear operations, there exists a constant $K > 0$ such that

$$(5) \qquad \sum_{n=1}^{\infty} \|f_n\| \leqslant K\|f\| \quad \text{for all} \quad f = \sum_{n=1}^{\infty} f_n \in E.$$

On the other hand, if inequality (5) is true for all finite systems $f_1, f_2, \ldots$ (i. e., such that $f_n = 0$ for $n > n_0$), it is true for infinite systems as well (whenever the series $\sum \|f_n\|$ converges), and E must be the l_1-direct sum of $E_1, E_2, \ldots$ Thus, we have only to prove inequality (5) for finite systems and for some $K > 0$.

First proof enables us to find the best constant K in (5); it turns out to be equal to π. The proof is elementary and uses only the abstract theorem of Kronecker on solvability of the moment problem $|\chi_n(t) - c_n| < \varepsilon$ $(n = 1, \ldots, N)$.

The same method can be used for sequences of bounded independent random variables with expected values 0. It is a natural generalization of Bohr's theorem, as independent characters on a compact group are stochastically independent with respect to Haar measure (D. A. Edwards [4]).

Second proof does not yield the best constant, but it does not use the fact that χ_{nk} are characters; it uses only some orthogonality relations (with respect to Haar measure on the Bohr compactification of G) which follow from (1). Thus, the second proof admits generalizations.

E. Hewitt and H. S. Zuckerman [9] have established a theorem that is a simultaneous extension of the theorem of Bohr and of the classical theorem of Sidon concerning absolute convergence of lacunary Fourier series. This theorem (proved for arbitrary compact groups) may be formulated for compact Abelian groups as follows. A set V of characters on G will be called HZ-*lacunary* if it can be decomposed into a finite number of subsets $V_1, \ldots, V_p$ so that, for each q $(1 \leqslant q \leqslant p)$, for each finite sequence $m_1, \ldots, m_N$ of numbers $0, 1, -1$ and for each finite set $\chi_1, \ldots, \chi_N$ of characters of V_q, the equality

$$\prod_{k=1}^{N} \chi_k^{m_k} = \chi \in V_q$$

implies $\chi = \chi_s$ for some $s \leqslant N$, $m_k = 0$ for $k \neq s$ and $\chi_s^{m_s} = \chi_s$ ([4]). Then, *if V is* HZ-*lacunary, there exists an $A > 0$ such that $\|\sum a_n \chi_n\| \geqslant A \sum \|a_n\|$ for any $\chi_1, \ldots, \chi_N \in V$ and for any complex coefficients $a_1, \ldots, a_N$*; moreover, $A \geqslant (3p)^{-1}$. This includes Bohr's theorem, for any set of independent characters is HZ-lacunary. If χ is of infinite order (i. e. if $\chi^n \neq \iota$ for all $n \neq 0$) and if $n_{k+1}/n_k \geqslant a > 1$, then the sequence $\{\chi^{n_k}\}_{k=1,2,\ldots}$ is HZ-lacunary, the converse implication being invalid (cf. [9], p. 8).

The last part of the present paper contains a generalization of this theorem to arbitrary bounded measurable functions, algebraic hypotheses about characters being changed to certain orthogonality relations.

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2. Elementary lemmas. Throughout this paper, ε_n^k will denote $\exp(2\pi ki/n)$; i. e. $\varepsilon_n^0 = 1, \varepsilon_n^1, \ldots, \varepsilon_n^{n-1}$ are the n-th roots of unity.

LEMMA 1. *The inequality*

$$\sup_{1 \leqslant k_1 < \ldots < k_s \leqslant n} \left| \sum_{r=1}^{s} a_{k_r} \right| \geqslant \frac{1}{\pi} \sum_{k=1}^{n} |a_k|$$

holds for every finite system $a_1, \ldots, a_n$ of complex numbers. The constant $1/\pi$ is the best possible; one can approach it considering sequences $a_k = \varepsilon_n^k$, $k = 1, \ldots, n$, with $n \to \infty$.

This lemma may be formulated for plane vectors as well; it has been proved as an extremal diagonal property of the convex polygon obtained by reordering of the vectors ([5]).

([3]) X being a normed linear space and $Y, X_1, X_2, \ldots$ being linear subsets of X, Y is called the l_1-*direct sum* of $X_1, X_2, \ldots$ if every element $x \in Y$ can be uniquely written in the form $x = \sum x_n$ with $x_n \in X_n$ and $\sum \|x_n\| < \infty$. If this is the case, then besides the norm induced by X, there is another one in Y defined by $\|x\|_0 = \sum \|x_n\|$. If $\langle X, \|\ \|\rangle$ is complete and $X_1, X_2, \ldots$ are closed, then $\langle Y, \|\ \|_0\rangle$ is complete.

([4]) If $\{\chi_n\}$ is HZ-lacunary, then all products $\prod \chi_k^{\delta_k}$ $(\delta_k = 0, +1, -1)$ are orthogonal to ι unless all δ_k's are 0.

([5]) See A. Rosenthal and O. Szász [19], K. Reinhardt [18]. For p-dimensional vectors $(p \geqslant 2)$ we have (A. E. Mayer [16]):

$$\inf_{v_1, \ldots, v_n} \sup_{\eta_k = 0,1} \frac{\|\sum \eta_k v_k\|}{\sum \|v_k\|} = \frac{1}{2\sigma(S_+)} \int_{S_+} u_1 \, d\sigma(u) = \frac{\Gamma(p/2)}{2\sqrt{\pi}\,\Gamma[(p+1)/2]} \sim \frac{1}{\sqrt{2\pi p}},$$

i. e. the best constant is equal to one half of the u_1-coordinate of the center of gravity of the hemisphere

$$S_+ = \{u : u = (u_1, \ldots, u_p),\ \|u\| = (\sum u_i^2)^{1/2} = 1,\ u_i \geqslant 0\},$$

σ being the ordinary surface measure on S_+.

The author is indebted to Professor Jean-Pierre Kahane for a proof of this theorem. An analogous method is used in the proof of Lemma 2.

LEMMA 2. *Let $q_1 \geqslant 2, \ldots, q_n \geqslant 2$ be any sequence of integers, and let $a_1, \ldots, a_n$ be any sequence of complex numbers. Then*

$$\sup_{m_1=1,\ldots,q_1} \ldots \sup_{m_n=1,\ldots,q_n} \Big|\sum_{k=1}^{n} a_k \varepsilon_{q_k}^{m_k}\Big| \geqslant \frac{1}{\pi} \sum_{k=1}^{n} |a_k| q_k \sin\frac{\pi}{q_k}.$$

Proof. We may assume $a_k \neq 0$ for $k = 1, \ldots, n$. Then

$$\sup_{m_k=1,\ldots,q_k} \Big|\sum_{k=1}^{n} a_k \varepsilon_{q_k}^{m_k}\Big| = \sup_{0\leqslant\varphi\leqslant 2\pi} \sup_{m_1=1,\ldots,q_1} \ldots \sup_{m_n=1,\ldots,q_n} \sum_{k=1}^{n} \mathrm{Re}(a_k \varepsilon_{q_k}^{m_k} e^{i\varphi})$$

$$= \sup_{\varphi} \sum_{k} \sup_{\{m_k\}} \mathrm{Re}(a_k \varepsilon_{q_k}^{m_k} e^{i\varphi}) \geqslant \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=1}^{n} |a_k| \sup_{\{m_k\}} \mathrm{Re}\Big(\frac{a_k}{|a_k|} \varepsilon_{q_k}^{m_k} e^{i\varphi}\Big) d\varphi$$

$$= \frac{1}{2\pi} \sum_{k} |a_k| \int_0^{2\pi} \sup_{\{m_k\}} \mathrm{Re}(\varepsilon_{q_k}^{m_k} e^{i\varphi}) d\varphi = \frac{1}{2\pi} \sum_{k} |a_k| \int_0^{2\pi} \cos\Big(\min_{\{m_k\}} \Big|\frac{2\pi m_k}{q_k} - \varphi\Big|\Big) d\varphi$$

$$= \frac{1}{2\pi} \sum_{k} |a_k| \cdot 2q_k \sin\frac{\pi}{q_k}.$$

Now, given any integer $q \geqslant 2$, let

$$C_q = \inf_{n} \inf_{\Sigma|a_k|=1} \sup_{m_k=1,\ldots,q} \Big|\sum_{k=1}^{n} a_k \varepsilon_q^{m_k}\Big|,$$

$$C'_q = \inf_{n} \inf_{\Sigma|a_k|=1} \inf_{q_k\geqslant q} \sup_{m_k=1,\ldots,q_k} \Big|\sum_{k=1}^{n} a_k \varepsilon_{q_k}^{m_k}\Big|.$$

LEMMA 3. *The following relations hold:*

(i) $C_q = C'_q = \dfrac{q}{\pi} \sin\dfrac{\pi}{q}$ *for* $q = 2, 3, \ldots$

(ii) $\dfrac{2}{\pi} = C_2 < C_3 < \ldots$

(iii) $\lim\limits_{q\to\infty} C_q = 1$.

Proof. The inequalities $C_q \geqslant C'_q \geqslant q\pi^{-1}\sin(\pi/q)$ follow from Lemma 2. To prove that $C_q \leqslant q\pi^{-1}\sin(\pi/q)$, let us consider

$$a_k = \frac{1}{n}\varepsilon_n^k, \quad k = 1, \ldots, n, \quad n = ql,$$

l being an integer; then

$$\sup_{m_k=1,\ldots,q} \frac{1}{n} \Big|\sum_{k=1}^{n} \varepsilon_n^k \varepsilon_q^{m_k}\Big| \geqslant \frac{q}{n} \Big|\sum_{k=1}^{l} \varepsilon_n^k\Big|$$

$$= \frac{q}{n}\Big[1-\cos\frac{2\pi}{q}\Big]^{1/2}\Big[1-\cos\frac{2\pi}{n}\Big]^{-1/2} \to \frac{q}{\pi}\sin\frac{\pi}{q} \quad \text{as} \quad l\to\infty.$$

Thus, we have proved (i). Conditions (ii) and (iii) are obvious.

3. Discussion of the theorem of Bohr. For an Abelian group G, several definitions of independence of a sequence $\chi_1, \ldots, \chi_n$ of characters arise naturally. The following ones appear in considerations ($m_1, \ldots, m_n$ denote any integers):

(I) *If* $\prod_{k=1}^{n} \chi_k^{m_k} = \iota$, *then* $m_1 = \ldots = m_n = 0$ (*full independence*).

(I_q) *All characters are of the same order q ($2 \leqslant q < \infty$) and the equality* $\prod_{k=1}^{n} \chi_k^{m_k} = \iota$ *implies* $m_1 \equiv \ldots \equiv m_n \equiv 0 \pmod q$ (*q-independence*).

(II) *No χ_k is equal to ι, and if* $\prod_{k=1}^{n} \chi_k^{m_k} = \iota$, *then* $\chi_1^{m_1} = \ldots = \chi_n^{m_n} = \iota$ (*rectangular independence or stochastic independence*).

(III) *No χ_s is equal to any product* $\prod_{k\neq s} \chi_k^{m_k}$ (*separation independence*).

The terminology in (II) and (III) will be justified in the sequel. Obviously (I) $\Rightarrow$ (II) $\Rightarrow$ (III) and (I_q) $\Rightarrow$ (II). Rectangularly independent characters are fully independent if and only if they are of infinite order, and are q-independent if and only if they are of the same finite order q. Thus, definitions (I) and (II) are equivalent for a compact group G if and only if G is connected (cf. [7], Theorem (24.25)).

A main tool in our considerations is an abstract version of the Kronecker approximation theorem [6].

THEOREM OF KRONECKER. *Given a sequence $\chi_1, \ldots, \chi_n$ of characters on a topological Abelian group G and complex numbers $c_1, \ldots, c_n$ of modulus 1, the following conditions are equivalent:*

(i) *For every $\varepsilon > 0$ there exists a point $t_\varepsilon \in G$ such that*

(6) $$|\chi_k(t_\varepsilon) - c_k| < \varepsilon \quad \textit{for} \quad k = 1, \ldots, n.$$

[6] If G is the real line and $\chi_k(t) = e^{i\mu_k t}$, we get a classical theorem of Kronecker. The abstract form is very close to a theorem of Hewitt and Zuckerman [8], cf. also [7], theorem (26.15), and has been formulated explicitly by Hartman and Ryll-Nardzewski [5].

(ii) *For every system $m_1, \ldots, m_n$ of integers*

$$\prod_{k=1}^{n} \chi_k^{m_k} = \iota \quad \textit{implies} \quad \prod_{k=1}^{n} c_k^{m_k} = 1. \tag{7}$$

If (ii) holds, we shall say that the numbers $c_1, \ldots, c_n$ are *compatible* with the characters $\chi_1, \ldots, \chi_n$. This condition means that if we consider the subgroup H of $\hat{G}$ generated by $\chi_1, \ldots, \chi_n$ (it consists of the characters of the form $\prod \chi_k^{m_k}$), then $\omega_0(\prod \chi_k^{m_k}) = \prod c_k^{m_k}$ is a well-defined character of H. The implication (i) ⇒ (ii) is trivial. For a compact group (6) can be replaced by the equalities $\chi_k(t) = c_k$ where t is a limit point of the t_s's and the implication (ii) ⇒ (i) is an immediate consequence of Pontriagin's duality theorem (cf. [7], § 24). Indeed, we extend ω_0 to a character ω on $\hat{G}$ ($\hat{G}$ being discrete, ω is continuous) and find $t \in G$ such that $\omega(\chi) = \chi(t)$ for all $\chi \in \hat{G}$ whence $\chi_k(t) = \omega(\chi_k) = \omega_0(\chi_k) = c_k$ for $k = 1, \ldots, n$.

If G is not compact, we may assume it to be discrete, as the topology of G does not matter in the theorem. Let $\bar{G}$ be the Bohr compactification (7) of G; G and $\bar{G}$ have the same dual $\hat{G}$ and $\chi_1, \ldots, \chi_n$ have unique extension on $\bar{G}$. This reduces the question to the preceding case. There exists $t_0 \in \bar{G}$ such that $\chi_k(t_0) = c_k$ for $k = 1, \ldots, n$. Since G is dense in $\bar{G}$ in the weak topology induced by $\hat{G}$, the neighborhood

$$\bigcap_{k=1}^{n} \{t \in \bar{G} : |\chi_k(t_0) - \chi_k(t)| < \varepsilon\}$$

contains a point t_1 of G. Obviously, $|\chi_k(t_1) - c_k| < \varepsilon$ for $k = 1, \ldots, n$.

LEMMA 4. *Let $\chi_1, \chi_2, \ldots$ be a countable set of characters on G. Let q_k be either equal to 0 or be the least positive integer q such that χ_k^q is equal to a finite product $\prod_{p \neq k} \chi_p^{m_p}$; if no such q exists, let q_k be any positive integer. Then, for any integers $s_1, \ldots, s_n$, the numbers $\varepsilon_{q_1}^{s_1}, \ldots, \varepsilon_{q_n}^{s_n}$ are compatible with $\chi_1, \ldots, \chi_n$.*

Proof. Suppose that $s_1, \ldots, s_n$ are given and that

$$\prod_{k=1}^{n} \chi_k^{m_k} = \iota.$$

For each p ($1 \leqslant p \leqslant n$), the equality $\chi_p^{m_p} = \prod_{k \neq p} \chi_k^{-m_k}$ implies $m_p = l_p q_p$, where l_p is an integer. Hence

$$\prod_{k=1}^{n} (\varepsilon_{q_k}^{s_k})^{m_k} = \prod_{k=1}^{n} \varepsilon_{q_k}^{s_k l_k q_k} = 1.$$

(7) For the basic properties of the Bohr compactification, see e. g. [7], (26.11).

In the sequel we shall assume G to be a compact Abelian group. This assumption simplifies the proofs and does not decrease the generality, as we may always consider all characters as extended over the Bohr compactification.

Now, let us consider the following conditions:

(I′) *Any numbers $c_1, \ldots, c_n$ of modulus 1 are compatible with $\chi_1, \ldots, \chi_n$.*

(I′$_q$) *$\chi_k^q = \iota$ for $k = 1, \ldots, n$ and any numbers $\varepsilon_q^{m_1}, \ldots, \varepsilon_q^{m_n}$ are compatible with $\chi_1, \ldots, \chi_n$.*

(II′) *$\chi_k \neq \iota$ and any numbers $c_1, \ldots, c_n$ belonging to the ranges of $\chi_1, \ldots, \chi_n$, respectively, are compatible with $\chi_1, \ldots, \chi_n$.*

(II″) *$\chi_k \neq \iota$ and for any points $t_1, \ldots, t_n$ of G there exists a point $t \in G$ such that $\chi_k(t_k) = \chi_k(t)$ for $k = 1, \ldots, n$.*

(II‴) *$\chi_k \neq \iota$ and $\chi_1, \ldots, \chi_n$ are mutually independent random variables with respect to Haar measure on G.*

(III′) *For each s ($1 \leqslant s \leqslant n$) there exist $t, u \in G$ such that $\chi_k(t) = \chi_k(u)$ for $k \neq s$ and $\chi_s(t) \neq \chi_s(u)$. In other words, no proper subset of $\{\chi_1, \ldots, \chi_n\}$ separates all points separated by the whole set.*

(III″) *There exist integers $q_1 \geqslant 2, \ldots, q_n \geqslant 2$ such that for every sequence $s_1, \ldots, s_n$ of integers the numbers $\varepsilon_{q_1}^{s_1}, \ldots, \varepsilon_{q_n}^{s_n}$ are compatible with $\chi_1, \ldots, \chi_n$.*

PROPOSITION 1. *The following equivalences hold:* (I) ⇔ (I′), (I$_q$) ⇔ (I′$_q$), (II) ⇔ (II′) ⇔ (II″) ⇔ (II‴), (III) ⇔ (III′) ⇔ (III″).

Proof. The equivalences (I) ⇔ (I′), (I$_q$) ⇔ (I′$_q$), (II) ⇔ (II′) ⇔ (II″) are well known and follow from the theorem of Kronecker. Implications (III″) ⇒ (III′) ⇒ (III) are obvious. (III) ⇒ (III″) is a consequence of Lemma 4.

Equivalence (II) ⇔ (II‴) has been proved by D. A. Edwards [4], but it seems worth while to give an elementary proof of the implication (II) ⇒ (II‴). Let $\chi_1, \ldots, \chi_n$ be independent in the sense (II). Suppose that $\chi_1, \ldots, \chi_p$ are of finite orders $q_1, \ldots, q_p$, respectively, and that $\chi_{p+1}, \ldots, \chi_n$ are of infinite order. Let

$$Q_{ml}^k = \left\{ t \in G:\ \chi_k(t) = e^{i\varphi},\ \frac{(2l-1)\pi}{m} \leqslant \varphi < \frac{(2l+1)\pi}{m} \right\}$$

and

$$P_{l_1, \ldots, l_n} = \bigcap_{k=1}^{n} Q_{q_k l_k}^k.$$

It is enough to prove that

$$\mu(P_{l_1, \ldots, l_n}) = \prod_{k=1}^{n} \mu(Q_{q_k l_k}^k) \quad (k = 1, \ldots, n;\ l_k = 1, \ldots, q_k), \tag{8}$$

where $q_{p+1}, \ldots, q_n$ are arbitrary positive integers. By (II'), there exists $u = u_{l_1,\ldots,l_n} \in G$ such that $\chi_k(u) = \varepsilon_{q_k}^{l_k}$ for $k = 1, \ldots, n$. Consequently, $Q^k_{q_k l_k} = Q^k_{q_k 0} + u$ and $P_{l_1,\ldots,l_n} = P_{0,\ldots,0} + u$. Since

$$\bigcup_{l=1}^{q_k} Q^k_{q_k l} = G = \bigcup_{l_1=1}^{q_1} \ldots \bigcup_{l_n=1}^{q_n} P_{l_1,\ldots,l_n}$$

and these are decompositions of G to disjoint sets and since μ is invariant, $\mu(Q^k_{q_k l_k}) = q_k^{-1}$ and $\mu(P_{l_1,\ldots,l_n}) = (q_1 \ldots q_n)^{-1}$ for all $l_k = 1, \ldots, q_k$, which gives (8).

We shall say that an infinite set of characters is *independent in the sense* (I), (I_q), (II) *or* (III) if any of its finite subsets is independent in the sense (I), (I_q), (II) or (III), respectively.

THEOREM 1. *Let $\sum a_n \chi_n$ be the Fourier series of a function $f \in L^\infty(G)$. Then* [8]

(i) *If there exists an index n_0 such that $\{\chi_n\}_{n \neq n_0}$ is fully independent, then*

$$\|f\| = \sum |a_n|. \tag{9}$$

Conversely, if (9) *is satisfied for all functions $f = \sum a_n \chi_n$ with $(a_n) \in l_1$, then there exists n_0 such that $\{\chi_n\}_{n \neq n_0}$ is fully independent.*

(ii) *If $\{\chi_n\}$ is q-independent, then*

$$\|f\| \leqslant \sum |a_n| \leqslant \frac{1}{C_q} \|f\|, \tag{10}$$

where the constant C_q^{-1} is as in Lemma 3 *and is the best possible.*

(iii) *If $\{\chi_n\}$ is independent in the sense* (II) *or* (III), *then*

$$\|f\| \leqslant \sum |a_n| \leqslant \frac{\pi}{2} \|f\| \tag{11}$$

and the constant $\pi/2$ is the best possible for either class.

(iv) *If $\{\chi_n\}$ is independent in the sense* (II) *and the order of each χ_n is greater than or equal to q, q being an integer and $q \geqslant 2$, then* (10) *holds.*

Proof. In each case the reasoning is as follows. First we prove that the desired inequality holds if f is a linear combination of a finite number of characters of $\{\chi_n\}$. Next we conclude that $\sum |a_n| < \infty$ for each $f \in L^\infty(G)$ with $f \sim \sum a_n \chi_n$ (cf. [9], p. 14) whence, by the completeness of the orthogonal system of all characters, $f = \sum a_n \chi_n$ almost everywhere and we may assume that $f \in C(G)$. Thus, by Fejér's theorem, f can be uniformly approximated by finite sums $\sum b_k \chi_k$ and the corresponding inequality must hold for infinite sums as well (the same argument is used in the proof of Theorem 3; if one of $\|f\|$, $\sum |a_n|$ is finite, so is the other).

Proof of (i). We may restrict ourselves to the case where $\{\chi_1, \ldots, \chi_n\}$ is not fully independent but $\{\chi_2, \ldots, \chi_n\}$ is; given complex numbers $a_1 = r_1 e^{i\varphi_1}, \ldots, a_n = r_n e^{i\varphi_n}$ and $f = \sum_{k=1}^n a_k \chi_k$, we have only to prove that $\|f\| \geqslant \sum |a_k|$, the converse inequality being obvious. There are integers $m_1, \ldots, m_n$ such that $m_1 > 0$, $\prod \chi_k^{m_k} = \iota$ and m_1 is the least integer with this property. Hence, by Lemma 4, the numbers $\varepsilon_{m_1}^s, 1, 1, \ldots, 1$ are compatible with $\chi_1, \ldots, \chi_n$ and, by Kronecker's theorem, there exist $t_1, \ldots, t_{m_1}$ in G such that $\chi_1(t_s) = \varepsilon_{m_1}^s$ and $\chi_k(t_s) = 1$ for $k = 2, \ldots, n$, $s = 1, \ldots, m_1$. Let $\psi = \sum m_k \varphi_k / \sum m_k$. By (I'), there exists $u \in G$ such that $\chi_k(u) = e^{i(\psi - \varphi_k)}$ for $k = 2, \ldots, n$, whence

$$f(u) = a_1 \varepsilon_{m_1}^s \exp\left[\frac{m_2\varphi_2 + \ldots + m_n\varphi_n - \psi(m_2 + \ldots + m_n)}{m_1} i\right] + \sum_{k=2}^n a_k e^{i(\psi - \varphi_k)},$$

where $\varepsilon_{m_1}^s$ appears as a result of taking m_1-th root while calculating $\chi_1(u)$. Thus, $f(u - t_s) = e^{i\psi} \cdot (|a_1| + \ldots + |a_n|)$ whence $\|f\| \geqslant \sum |a_k|$.

We have now to prove that if no $\{\chi_n\}_{n \neq n_0}$ is fully independent, then there exists a sequence $(a_n) \in l_1$ such that $\|\sum a_n \chi_n\| \neq \sum |a_n|$. By reordering we may assume that there exist integers $m_1, \ldots, m_n, l_2, \ldots, l_n$ such that

$$m_1 > 0, \quad l_2 > 0, \quad \prod_{k=1}^n \chi_k^{m_k} = \iota \quad \text{and} \quad \prod_{k=2}^n \chi_k^{l_k} = \iota.$$

Consider $f = a_1 \chi_1 + \chi_2 + \ldots + \chi_n$ where $a_1 = e^{i\alpha\pi}$, α being irrational, and $a_2 = \ldots = a_n = 1$. We claim that $\|f\| < \sum |a_k| = n$. Indeed, if there existed t_0 such that $|f(t_0)| = n$, then there would exist ψ such that $a_1 \chi_1(t_0) = \chi_2(t_0) = \ldots = \chi_n(t_0) = e^{i\psi}$ whence $\psi = 2\pi s (l_2 + \ldots + l_n)^{-1}$, s being an integer, and α would be rational. Thus, we have proved (i).

In order to prove (ii), (iii) and (iv), it is enough to show the following facts: 1° If χ_n are separation independent, then (11) holds. 2° If they are rectangularly independent and their orders are greater than or equal to q, then (10) holds. 3° For each $q \geqslant 2$ and $\varepsilon > 0$, there exist a compact group G, q-independent characters $\chi_1, \ldots, \chi_n$ on G and complex numbers $a_1, \ldots, a_n$ such that $\|\sum a_k \chi_k\| < (C_q + \varepsilon) \sum |a_k|$ (notice that $\sup\{C_q^{-1} : q = 2, 3, \ldots\} = \pi/2$).

Proof of 1°. Let $q_1, \ldots, q_n$ satisfy (III''). Then, for each system $s_1, \ldots, s_n$ of integers, there exists a point $t_{s_1,\ldots,s_n} \in G$ such that

[8] Throughout this paper, the symbol $\| \ \|$ will always mean the sup norm, i. e. $\|f\| = \sup |f(t)|$ for continuous functions and $\|f\| = \operatorname{es\,sup} |f(t)|$ for bounded measurable functions.

$\chi_k(t_{s_1,\ldots,s_n}) = \varepsilon_{a_k}^{s_k}$ for $k = 1, \ldots, n$. Consequently, by Lemma 2 and by $C_p \geqslant 2/\pi$,

$$\sum |a_k| \leqslant \frac{\pi}{2} \sup_{\{s_k\}} \left| \sum_{k=1}^{n} a_k \chi_k(t_{s_1,\ldots,s_n}) \right| \leqslant \frac{\pi}{2} \left\| \sum_{k=1}^{n} a_k \chi_k \right\|,$$

whence we have (11) for finite sums.

Proof of 2° is similar: we apply Lemmas 2, 3, 4.

Proof of 3°. Let Z_q be the cyclic group of order q and let G be the Cartesian product of $\aleph_0$ copies of Z_q (with product topology), i. e. G consists of the sequences $u = (u_1, u_2, \ldots)$ where $u_n = \varepsilon_q^{s_n}$, $1 \leqslant s_n \leqslant q$. Every character of G is of the form

$$\chi(u) = \chi_{n_1}^{m_1}(u) \cdot \ldots \cdot \chi_{n_k}^{m_k}(u) \quad (m_k = 0, \ldots, q-1),$$

where $\chi_n(u) = u_n$ are elementary characters of G, generators of $\hat{G}$.

Obviously, $\chi_1, \chi_2, \ldots$ are q-independent and (for $a_k = \varepsilon_{nq}^{k}$)

$$\frac{1}{nq} \left\| \sum_{k=1}^{nq} \varepsilon_{nq}^{k} \chi_k \right\| = \frac{1}{nq} \sup_{l_k = 0,\ldots,q-1} \left| \sum_{k=1}^{nq} \varepsilon_{nq}^{k} \varepsilon_q^{l_k} \right| = \frac{1}{nq} \left| \sum_{k=1}^{n} \varepsilon_{nq}^{k} \right| = C_q + O\left(\frac{1}{n}\right).$$

4. Main theorem. Complex-valued functions $f_1, \ldots, f_n$ on a set S will be called *rectangularly independent* if for any points $t_1, \ldots, t_n$ of S there exists $t_0 \in S$ such that $f_k(t_0) = f_k(t_k)$ for $k = 1, \ldots, n$; this means that

$$\{(f_1(t), \ldots, f_n(t)) : t \in S\} = \{f_1(t) : t \in S\} \times \ldots \times \{f_n(t) : t \in S\}.$$

We shall say that the values of a complex-valued function f on S *surround zero* if 0 belongs to the convex hull of the image of the function f, i. e. $0 \in \operatorname{conv}\{f(t) : t \in S\}$.

Proposition 2. *Let S be a compact Hausdorff space, $f \in C(S)$ and let $\mathfrak{M}$ be the set of all non-negative Radon measures on S such that $\mu(S) = 1$. Then the values of f surround zero if and only if there exists $\mu \in \mathfrak{M}$ such that $\int_S f\,d\mu = 0$.*

Proof. Sufficiency is trivial, to prove necessity let us consider $\Phi(\mu) = \int f\,d\mu$ as a map $\Phi: \mathfrak{M} \to \operatorname{conv} f(S)$. Obviously, $\Phi(\mathfrak{M})$ is convex, compact and contains $f(S)$ whence Φ is onto $\operatorname{conv} f(S)$.

Theorem 2. *Let S be compact, $f_1, \ldots, f_n \in C(S)$ and let $\|f_1\| = \ldots = \|f_n\| = 1$. If $f_1, \ldots, f_n$ are rectangularly independent and their values surround zero, then the inequality*

$$\left\| \sum_{k=1}^{n} a_k f_k \right\| \geqslant \frac{1}{\pi} \sum_{k=1}^{n} |a_k|$$

holds for every sequence $a_1, \ldots, a_n$ of complex numbers.

Proof. We may assume $a_k \neq 0$; let $g_k = a_k f_k$ and $g = \sum g_k$. There exist points $t_1, \ldots, t_n \in S$ such that $\|f_k\| = |f_k(t_k)|$ for $k = 1, \ldots, n$. Let $\eta_1, \ldots, \eta_n$ be a sequence of numbers 0 or 1 such that $\sup_{\{k_i\}} |\sum g_{k_i}(t_{k_i})|$ is attained, i. e.

$$\sup_{\delta_k = 0,1} \left| \sum_{k=1}^{n} \delta_k g_k(t_k) \right| = |b|, \quad \text{where} \quad b = \sum_{k=1}^{n} \eta_k g_k(t_k).$$

Suppose $\eta_l = 0$. Then $\sphericalangle\langle g_l(t_l), b\rangle > \pi/2$ where $\sphericalangle\langle a, b\rangle$ denotes the non-oriented angle between a and b treated as vectors. Since g_l has values surrounding zero, there exists $u_l \in S$ such that either $g_l(u_l) = 0$ or $\sphericalangle\langle g_l(u_l), b\rangle < \pi/2$. Let $A = \{l : \eta_l = 0\}$. Hence,

$$\text{either} \quad \sum_{l \in A} g_l(u_l) = 0 \quad \text{or} \quad \sphericalangle\left\langle \sum_{l \in A} g_l(u_l), b \right\rangle < \pi/2.$$

Let $v_k = t_k$ if $\eta_k = 1$ and $v_k = u_k$ if $\eta_k = 0$. By rectangular independence, there exists $t_0 \in S$ such that $f_k(t_0) = f_k(v_k)$ whence $g_k(t_0) = g_k(v_k)$ for $k = 1, \ldots, n$. Consequently, by Lemma 1,

$$\sum_{k=1}^{n} |a_k| = \sum_{k=1}^{n} |g_k(t_k)| \leqslant \pi |b| \leqslant \pi \left| b + \sum_{l \in A} g_l(u_l) \right|$$
$$= \pi \left| \sum_{k=1}^{n} g_k(v_k) \right| = \pi |g(t_0)| \leqslant \pi \left\| \sum_{k=1}^{n} a_k f_k \right\|.$$

Theorem 3. *Let $\Phi_n = \{\chi_{nm}\}$ be independent systems of characters and $\chi_{nk} \neq \iota$ for all n, k. Then, if a function $f \in C(G)$ has the Fourier series of the form (2), then there exist functions $f_1, f_2, \ldots$ in $C(G)$ such that (3) and $\sum \|f_n\| \leqslant \pi \|f\|$ hold. The constant π is the best possible, if one considers all Φ_n's and all compact Abelian groups.*

Proof. Let $P_m = \sum_{k=1}^{k_m} a_{mk} \chi_{mk}$. Then $P_1, \ldots, P_n$ are rectangularly independent (by the theorem of Kronecker) and, by Proposition 2, their values surround zero because $\int P_m\,d\mu = 0$, μ being the Haar measure on G. So we have $\sum \|P_m\| \leqslant \pi \|\sum P_m\|$.

Now, let E_m be the uniform closure of the class of all linear combinations of characters of Φ_m, and let E_∞ be the l_1-direct product of E_1, $E_2, \ldots$ Then $\|\sum f_n\| \leqslant \sum \|f_n\| \leqslant \pi \|\sum f_n\|$ holds on E_∞ and this means that the norm $\|\ \|_0$ defined by $\|f\|_0 = \sum \|f_n\|$ for $f = \sum f_n \in E_\infty$ is equivalent to $\|\ \|$. E_∞ being complete with respect to $\|\ \|_0$, it is complete with respect to $\|\ \|$ and closed in $C(G)$ as well. On the other hand, by Fejér's theorem, the class E_∞ is dense in the class E of all functions of $C(G)$ with Fourier series of the form (2); hence both classes coincide.

We have also to show that π is the best constant. Let G_0 be the real line and, for any $0<\delta<\frac{1}{2}$, let

$$\sigma_\delta(t)=\begin{cases}1 & \text{for } k\leqslant t\leqslant k+\delta,\ k=0,\pm1,\pm2,\ldots,\\ -\delta(1-\delta)^{-1} & \text{elsewhere on the line,}\end{cases}$$

and let $h_m(t)=\varepsilon_n^m\sigma_\delta(\lambda_m t)$ where $\lambda_1,\ldots,\lambda_n$ are rationally independent real numbers. Then

$$\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}h_m(t)\,dt=0,$$

$\|h_m\|=1$ for $m=1,\ldots,n$, and

$$\Big\|\sum_{m=1}^{n}h_m\Big\|=\sup_{\eta_m=0,1}\Big|\sum_{m=1}^{n}\eta_m\varepsilon_n^m\Big|+O(n\delta)=\frac{n}{\pi}+o(n)+O(n\delta),$$

so the ratio $\|\sum h_m\|(\sum\|h_m\|)^{-1}$ can approach $1/\pi$ arbitrarily as $\delta\to0$, $n\to\infty$. Approximating the functions h_m by suitable trigonometric polynomials

$$f_m=\sum_{k\neq0}a_{mk}e^{ik\lambda_m t}$$

we can approach $1/\pi$ with polynomials over independent sets of characters $\Phi_m=\{e^{ik\lambda_m t}\}_{k=\pm1,\pm2,\ldots}$ (without constant terms); finally we pass to the Bohr compactification.

5. Independent random variables. Actually, the method of Hartman and Ryll-Nardzewski [6] is founded on the following fact which is worth while stating explicitly (9):

PROPOSITION 3. *Let $\Phi_n=\{\chi_{nk}\}_{k=1,2,\ldots}$ be independent systems of characters on a compact Abelian group G (i. e. let (1) be satisfied) and let the Fourier series of functions $f_1,f_2,\ldots$ of $L_2(G)$ have the form (3). Then $f_1,f_2,\ldots$ are mutually independent random variables with respect to Haar measure μ on G.*

Proof. Let

$$G_0=\bigcap_{k,n}\{t\in G:\ \chi_{nk}(t)=1\}.$$

The homomorphisms

$$\psi(x)=\{\chi_{nk}(x)\}_{n,k=1,2,\ldots},\qquad \psi_n(x)=\{\chi_{nk}(x)\}_{k=1,2,\ldots}$$

(9) The following problem arises: Suppose $f_n=\sum_{k=1}^{k_n}a_{nk}\chi_{nk}$ are independent random variables on G, $n=1,\ldots,n_0$. Must $\Phi_1,\ldots,\Phi_{n_0}$ be independent in the sense (1)? Evidently, the answer is *yes* if we require the independence for all possible a_{nk}.

map G into some toroidal groups and G_0 is the kernel of ψ. G being compact, ψ is open (cf. [7], 5.29) and generates an isomorphism from G/G_0 onto $H=\psi(G)$. Since $H=\psi_1(G)\times\psi_2(G)\times\ldots$ (cf. [6], p. 291), the polynomials

$$P_n=\sum_{m=1}^{k_n}a_{nm}\chi_{nm}$$

correspond to functions on H which depend on different variables in the product $\psi_1(G)\times\psi_2(G)\times\ldots$ The Haar measure λ on H is the product of Haar measures λ_n on $\psi_n(G)$, whence the functions $P_1\psi^{-1},\ldots,P_n\psi^{-1},\ldots$ are independent with respect to λ. Since $\lambda(A)=\mu(\psi^{-1}A)$ for any Borel set A in H, $P_1,P_2,\ldots$ are independent with respect to μ. Finally, since $P_n\to f_n$ as $k_n\to\infty$ (convergence in measure), $f_1,f_2,\ldots$ are also independent.

PROPOSITION 4. *Let $f_1,\ldots,f_n$ be independent complex continuous random variables on a compact probability space (S,F,μ) such that $W\in F$ and $\mu(W)\neq0$ for any open non-void set W in S. Then $f_1,\ldots,f_n$ are rectangularly independent.*

Proof. The sets $A=\{(f_1(t),\ldots,f_n(t)):\ t\in S\}$ and $A_k=f_k(S)$ are compact. If A differed from $A_1\times\ldots\times A_n$, there would exist relatively open sets $V_1,\ldots,V_n$ of complex numbers such that $\emptyset\neq V_i\subset A_i$ and $(V_1\times\ldots\times V_n)\cap A=\emptyset$, whence

$$0\neq\prod_{i=1}^{n}\mu(f_i^{-1}(V_i))=\mu\Big[\bigcap_{i=1}^{n}f_i^{-1}(V_i)\Big]=\mu(\emptyset)=0.$$

It is known (cf. [6], p. 291) that if $h_1,h_2,\ldots$ are independent bounded random variables with expected values zero and if $\sum h_n(x)$ converges a. e., then $\sum\|h_n\|<\infty$, where $\|f\|=\operatorname{ess\,sup}|f(t)|$. This may be also treated as a generalization of the theorem of Bohr. On the other hand, Theorem 2 can be applied to this case as well.

THEOREM 4. *Let $f_1,f_2,\ldots$ be bounded mutually independent random variables on a probability space $(\Omega,\mathscr{F},\mu)$ and let $E(f_n)=0$ for $n=1,2,\ldots$ Then*

$$\sum_{n=1}^{\infty}\|f_n\|\geqslant\Big\|\sum_{h=1}^{\infty}f_n\Big\|\geqslant\frac{1}{\pi}\sum_{n=1}^{\infty}\|f_n\|$$

whenever $\sum f_n$ converges a. e. The constant $1/\pi$ is the best possible, if one considers all sequences $f_1,f_2,\ldots$ and all probability spaces.

Proof. First let us consider a finite sequence $f_1,\ldots,f_n$. Let S be the Stone space of the Boolean algebra $\mathscr{F}/\mathscr{I}$ where $\mathscr{I}=\{A:\ \mu(A)=0\}$. The functions $f_1,\ldots,f_n$ correspond to some continuous functions $g_1,\ldots,g_n$ on S and μ corresponds to a measure λ on S which does not vanish on open non-void sets. Thus, by Proposition 4, Proposition 2 and Theorem 2, we get $\|\sum f_m\| \leqslant \sum \|f_m\| = \sum \|g_m\| \leqslant \pi \|\sum g_m\| = \pi \|\sum f_m\|$ for finite sums.

We have to show that if $f = \sum_{n=1}^{\infty} f_n$ is essentially bounded, then $\sum \|f_n\| < \infty$. Let $s_n = f_1 + \ldots + f_n$, $r_n = f - s_n$ and $c = E(f)$. Then $f_1, \ldots, f_n, r_n - c$ are essentially bounded and independent, and $E(r_n - c) = 0$, whence, by the first part of the proof,

$$\sum_{m=1}^{n} \|f_m\| \leqslant \sum_{m=1}^{n} \|f_m\| + \|r_n - c\| \leqslant \pi \|s_n + r_n - c\| = \pi \|f - c\| \leqslant 2\pi \|f\| < \infty.$$

The last part of the theorem follows from previous considerations.

6. Multiplicative-orthogonal systems. In this section we shall give another proof of the theorem of Hartman and Ryll-Nardzewski. It is valid for bounded measurable functions; considering S as the Bohr compactification of G, μ as the Haar measure on S and φ_{nk} as characters, we get that theorem as a special case.

In the sequel S will be any set with a measure μ on a σ-field of subsets and $\mu(S) = 1$. L_∞ will mean the complex space $L_\infty(S, \mu)$ with norm $\|f\| = \operatorname{es\,sup}\{|f(t)| : t \in S\}$. Let $a^{\langle m \rangle} = a^m$ and $a^{\langle -m \rangle} = \overline{a}^m$ for $m = 1, 2, \ldots$ and let $a^{\langle 0 \rangle} = 1$, a being a complex number.

THEOREM 5. *Let $\{\varphi_{nk}\}_{n,k=1,2,\ldots}$ be a family of uniformly bounded measurable functions on S such that, for any integers m_{nk} $(n = 1, \ldots, N;\ k = 1, \ldots, k_n)$,*

$$\int_S \prod_{n=1}^{N} \prod_{k=1}^{k_n} \varphi_{nk}^{\langle m_{nk} \rangle} d\mu \neq 0 \quad \text{implies} \quad \prod_{k=1}^{k_n} \varphi_{nk}^{\langle m_{nk} \rangle} = 1 \quad \text{for} \quad n = 1, \ldots, N. \tag{12}$$

Next, let f be any function of L_∞ such that there exists a sequence h_m of linear combinations of functions φ_{nk} $(n, k = 1, 2, \ldots)$ convergent to f boundedly almost everywhere, i. e.

$$h_m \to f \ \text{ a. e. on } S \text{ and } \|h_m\| \leqslant C \ \text{ for } m = 1, 2, \ldots \tag{13}$$

Then there exist functions $f_n \in L_\infty$ such that $f = \sum f_n + c$, c being a constant, $\int_S f_n d\mu = 0$ for $n = 1, 2, \ldots$, each f_n can be approximated in $\| \ \|$ by linear combinations of $\varphi_{n1}, \varphi_{n2}, \ldots$, and

$$|c| + \sum_{n=1}^{\infty} \|f_n\| \leqslant 9\|f\|.$$

Proof. Let Ψ denote the set of all polynomials of the form

$$P_n(t) = a_0^{(n)} + \sum a_{m_1,\ldots,m_r}^{(n)} \prod_{k=1}^{r} [\varphi_{nk}(t)]^{\langle m_k \rangle},$$

where the summation is extended over a finite number of systems $m_1, \ldots, m_r$ such that $\prod_k \varphi_{nk}^{\langle m_k \rangle} \neq 1$. Then

$$\int_S \prod_{n=1}^{N} P_n(t)\, d\mu(t) = \prod_{n=1}^{N} a_0^{(n)} = \prod_{n=1}^{N} \int_S P_n(t)\, d\mu(t)$$

for any $P_n \in \Psi_n$ $(n = 1, \ldots, N)$. (In particular, if no P_n have constant terms, then they form a multiplicative-orthogonal system.) Consequently,

$$\Big\| \prod_{n=1}^{N} P_n \Big\| = \prod_{n=1}^{N} \|P_n\|.$$

Indeed, $Q_{np} = |P_n|^{2p} = P_n^{\langle p \rangle}, P_n^{\langle -p \rangle}$ belong to Ψ_n for $p = 1, 2, \ldots$, and

$$\Big\| \prod_{n=1}^{N} P_n \Big\| = \lim_{p \to \infty} \Big[\int_S \prod_{n=1}^{N} Q_{np}\, d\mu \Big]^{1/2p} = \prod_{n=1}^{N} \lim_{p \to \infty} \Big[\int_S Q_{np}\, d\mu \Big]^{1/2p} = \prod_{n=1}^{N} \|P_n\|.$$

The closures Y_n of Ψ_n in L_∞ are subrings of L_∞ and $\|\prod f_n\| = \prod \|f_n\|$ is still valid for $f_n \in Y_n$. Thus, arguing as Mazur and Orlicz do in [17], we get (for real $f_n \in Y_n$)

$$\operatorname*{es\,sup}_S \sum_{n=1}^{N} f_n = \sum_{n=1}^{N} \operatorname*{es\,sup}_S f_n, \qquad \operatorname*{es\,inf}_S \sum_{n=1}^{N} f_n = \sum_{n=1}^{N} \operatorname*{es\,inf}_S f_n.$$

Indeed, $\exp f_n \in Y_n$, whence $\exp(\operatorname{es\,sup} \sum f_n) = \operatorname{es\,sup}(\exp \sum f_n) = \|\prod \exp f_n\| = \prod \operatorname{es\,sup}(\exp f_n) = \prod \exp(\operatorname{es\,sup} f_n) = \exp(\sum \operatorname{es\,sup} f_n)$. If we consider complex-valued $f_n \in Y_n$ such that $\int f_n d\mu = 0$, then, arguing as Hartman and Ryll-Nardzewski do in [6] (p. 292), we get

$$\operatorname{es\,sup}(\operatorname{Re} f_n) \geqslant 0 \geqslant \operatorname{es\,inf}(\operatorname{Re} f_n),$$

whence

$$\begin{aligned}
\sum \|\operatorname{Re} f_n\| &\leqslant \sum \operatorname{es\,sup}(\operatorname{Re} f_n) - \sum \operatorname{es\,inf}(\operatorname{Re} f_n) \\
&\leqslant 2 \max \Big[\sum \operatorname{es\,sup}(\operatorname{Re} f_n), -\sum \operatorname{es\,inf}(\operatorname{Re} f_n) \Big] \\
&= 2 \operatorname{es\,sup} \Big[\max\Big(\sum \operatorname{Re} f_n, -\sum \operatorname{Re} f_n \Big) \Big] = 2 \Big\| \sum \operatorname{Re} f_n \Big\|.
\end{aligned}$$

A similar relation holds for imaginary parts, whence $\sum \|f_n\| \leqslant 4\|\sum f_n\|$ for $f_n \in Y_n$ with $\int f_n d\mu = 0$. If we consider any functions $g_n \in Y_n$, we may write $f = \sum g_n$ uniquely in the form $f = \sum f_n + c$ where $f_n \in E_n = \{g \in Y_n : \int g\, d\mu = 0\}$, c is a constant, $|c| = |\int f\, d\mu| \leqslant \|f\|$, and $\|f\| \leqslant \sum \|f_n\| + |c| \leqslant 4\|f - c\| + |c| \leqslant 9\|f\|$.

The next reasoning is as in Theorem 3. Nevertheless, we have to prove that the l_1-direct product E of $E_1, E_2, \ldots$ is closed not only with respect to $\| \;\|$ but also with respect to the convergence (13); this can however by proved by two-norm methods, cf. [21].

We shall now prove a generalization of the quoted theorem on HZ-lacunary sequences. It is also a generalization of Bohr's theorem, as any sequence of separation-independent characters is HZ-lacunary. We find Theorem 6 to be close to Theorem 5, but neither is a special case of the other. Namely, condition (12) will be assumed only for products of functions and their adjoints, but not for higher powers of them. Of course, this assumption is not enough to prove that the span of the functions in question is the l_1-direct product of corresponding subspaces (example: characteristic functions of disjoint non-zero sets in S), but an additional assumption $\inf \int_S |f_a|^2 d\mu > 0$ will be.

THEOREM 6. *Let $\{f_a\}_{a\in A}$ be a family of functions of $L_\infty(S, \mu)$ satisfying the following conditions:*

(14) $\|f_a\| \leqslant 1$, $\int_S f_a d\mu = 0$ *for* $a \in A$;

(15) $\int_S [f_a(t)]^2 d\mu = 0$ *unless* f_a *is real-valued;*

(16) *If $F = \{a_1, \ldots, a_n\}$ is any finite subset of A, if $\delta_1, \ldots, \delta_n$ are equal to $0, 1$ or -1 and if $\beta \in A$, then the condition*

$$\int_S \prod f_{a_k}^{\langle\delta_k\rangle} f_\beta d\mu \neq 0$$

implies $\delta_k = 0$ for $a_k \neq \beta$.

Then the inequality

$$\Big\| \sum_{k=1}^{n} a_k f_{a_k} \Big\| \geqslant \frac{1}{3} \inf_a \Big[\int_S |f_a|^2 d\mu \Big] \sum_{k=1}^{n} |a_k| \tag{17}$$

holds for any finite system $a_1, \ldots, a_n \in A$ and for any complex numbers $a_1, \ldots, a_n$.

Proof. Let

$$\varkappa_a = \int_S |f_a|^2 d\mu \quad \text{and} \quad \varkappa = \inf_a \varkappa_a > 0.$$

Let Ω be the Stone space of the Boolean algebra of μ-measurable subsets of S considered up to sets of μ-measure 0. The spaces $L_\infty(S, \mu)$ and $C(\Omega)$ are isometrically isomorphic. Let ψ_a be the continuous functions on Ω corresponding to f_a $(a \in A)$ and let λ be the Radon measure on Ω

which corresponds to μ. Conditions (14)-(17) may be written for ψ_a and λ in just the same form as for f_a and μ; so we can apply the method used by Hewitt and Zuckerman in [9].

Let $\{c_a\}$ be any complex-valued function on A such that $\sup |c_a| = \varkappa$. Let F be any finite set of indices of A. We define generalized Riesz products:

$$h_F(u) = 2 \prod_{a \in F_c} [1 + \varkappa_a^{-1} \operatorname{Re}(c_a \overline{\psi_a(u)})] \prod_{a \in F_r} [1 + \tfrac{1}{2} \varkappa_a^{-1} \operatorname{Re}(c_a) \psi_a(u)],$$

$$g_F(u) = \prod_{a \in F_r} [1 + \varkappa_a^{-1} \operatorname{Im}(c_a) \psi_a(u)],$$

where F_r is the set of all $a \in F$ such that ψ_a is real-valued and $F_c = F \setminus F_r$ (if F_r or F_c is void, let the corresponding product be equal to 1). Then

$$h_F = 2 + \sum_{a \in F_c} \varkappa_a^{-1} c_a \overline{\psi_a} + \sum_{a \in F_c} \varkappa_a^{-1} \overline{c_a} \psi_a + \sum_{a \in F_r} \varkappa_a^{-1} \operatorname{Re}(c_a) \psi_a + R,$$

where R is a linear combination of products $\prod_{a \in F} \psi_a^{\langle\delta_a\rangle}$ such that $\sum |\delta_a| \geqslant 2$. Hence, by (14), (15) and (16),

$$\int h_F \psi_\beta d\lambda = \begin{cases} c_\beta & \text{if} \quad \beta \in F_c, \\ \operatorname{Re} c_\beta & \text{if} \quad \beta \in F_r, \\ 0 & \text{if} \quad \beta \notin F. \end{cases}$$

Similarly,

$$\int g_F \psi_\beta d\lambda = \begin{cases} \operatorname{Im} c_\beta & \text{if} \quad \beta \in F_r, \\ 0 & \text{if} \quad \beta \in F_c \text{ or if } \beta \notin F. \end{cases}$$

Since $|c_a \psi_a(h)| \leqslant \varkappa_a$, h_F and g_F are non-negative on Ω. Consequently, the functionals

$$\xi_F(x) = \int_\Omega x(u) [h_F(u) + i g_F(u)] d\lambda(u)$$

are linear on $C(\Omega)$ and their norms $\int |h_F + i g_F| d\lambda$ do not exceed 3. Hence, by compactness argument, there exists a functional $\xi(x) = \int x d\lambda$ of norm $\leqslant 3$ and such that $\xi(\psi_a) = \int \psi_a d\lambda = c_a$ for every $a \in A$. Thus, (17) is a consequence of the following lemma [10]:

LEMMA 5. *Let T be a compact Hausdorff space and let $\{g_a\}_{a\in A}$ be a family of continuous functions on T such that $\|g_a\| \leqslant 1$ for $a \in A$ and there exist biorthogonal Radon measures $\{\mu_\beta\}_{\beta \in A}$ on T (i. e. $\int g_a d\mu_\beta = \delta_{a\beta}$ for $a, \beta \in A$). Then, given $M > 0$, the following conditions are equivalent:*

[10] This lemma is a generalization of an equivalence known for orthogonal systems (Kahane [13], p. 310, Hewitt and Zuckerman [9], p. 14, Rudin [20], p. 207).

(a) *The inequality*

$$\Big\|\sum_{k=1}^{n} a_k g_{a_k}\Big\| \geqslant M \sum_{k=1}^{n} |a_k|$$

holds for any finite system of indices $\alpha_1, \ldots, \alpha_n \epsilon A$ *and for any complex numbers* $a_1, \ldots, a_n$.

(b) *For every bounded complex function* $\{c_a\}_{a\epsilon A}$ *on* A *there exists a bounded complex Radon measure* λ *on* T *such that* $\int g_a d\lambda = c_a$ *for* $a \epsilon A$ *and* $|\lambda|(T) \leqslant M^{-1} \sup |c_a|$.

Proof. Let Z be the class of all functions of the form $x = \sum a_a g_a$ with $\{a_a\} \epsilon l(A)$ (summation is actually countable), and let $\|x\|_0 = \sum |a_a|$ (the existence of $\{\mu_\beta\}$ yields uniqueness of such representation). Then $\|x\| \leqslant \|x\|_0$ for all $x \epsilon Z$ and (a) means that $\|x\|_0 \leqslant M^{-1}\|x\|$, i. e. that the norms are equivalent. If this is the case, then any bounded function $\{c_a\}$ determines a linear functional $\xi_0(x) = \sum a_a c_a$ on $\langle Z, \|\ \|\rangle$ which can be extended to a linear functional $\xi(x) = \int x d\lambda$ on $C(T)$ so that $\int g_a d\lambda = \xi(g_a) = \xi_0(g_a) = c_a$ and $|\lambda|(T)$ (being the norm of ξ on $C(T)$) is equal to the norm of ξ_0 on $\langle Z, \|\ \|\rangle$ and does not exceed $M^{-1} \cdot \sup |c_a|$. Conversely, if for every bounded $\{c_a\}$ such a λ exists, then the conjugate spaces to $\langle Z, \|\ \|\rangle$ and $\langle Z, \|\ \|_0\rangle$ coincide and $\|x\| \geqslant M\|x\|_0$.

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