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On the divisibility of $\sigma_\nu(n)$*

by

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I. Introduction

The divisor function $\sigma_\nu(n)$ is defined by

$$\sigma_\nu(n) = \sum_{d|n} d^\nu$$

where the sum is over all positive integral divisors of n; in the following pages it will be assumed that ν is a positive integer. The aim of this paper is to investigate a certain divisibility property of $\sigma_\nu(n)$.

Let q be a prime and m a positive integer, and assume that both are fixed and independent of x. Denote by $D_m(\nu, q; x)$ the number of positive integers $n \leqslant x$ for which $q^m \| \sigma_\nu(n)$, where the notation $\|$ means that q^m divides $\sigma_\nu(n)$ but q^{m+1} does not. In this paper an asymptotic equation for $D_m(\nu, q; x)$ will be established. Define γ by $q^\gamma \| \nu$, and let $m' = [m/(\gamma+1)]$ and $h = (q-1)/(\nu, q-1)$. Then the precise result to be obtained is as follows:

THEOREM 1. (i) *If q and h are both odd, then, as $x \to \infty$,*

$$D_m(\nu, q; x) \sim A_1^{(m)} x.$$

(ii) *If q is odd and h is even, then, as $x \to \infty$,*

$$D_m(\nu, q; x) \sim A_2^{(m)} x (\log\log x)^{m'} (\log x)^{-1/h}.$$

(iii) *As $x \to \infty$,*

$$D_m(\nu, 2; x) \sim A_3^{(m)} x (\log\log x)^{m-1} (\log x)^{-1}.$$

$A_1^{(m)}$, $A_2^{(m)}$, $A_3^{(m)}$ are positive constants depending only on ν, q and m.

The corresponding results for the case $m = 0$ have been obtained by R. A. Rankin in a paper [1] published in 1961. The function $D_0(\nu, q; x)$

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represents the number of positive integers $n \leqslant x$ for which q does not divide $\sigma_\nu(n)$. If, more generally, $N(\nu, k; x)$, for any positive integer k, is defined to be the number of positive integers $n \leqslant$ for which k does not divide $\sigma_\nu(n)$, then

$$D_0(\nu, q; x) = N(\nu, q; x). \tag{1}$$

This latter notation is that used by Rankin in [1] when he proved that, as $x \to \infty$,

$$N(\nu, q; x) \sim \begin{cases} A_1^{(0)} x & \text{if } q \text{ and } h \text{ are odd,} \quad \text{(i)} \\ A_2^{(0)} x (\log x)^{-1/h} & \text{if } q \text{ is odd and } h \text{ is even,} \quad \text{(ii)} \\ A_3^{(0)} x^{1/2} & \text{if } q = 2, \quad \text{(iii)} \end{cases} \tag{2}$$

where $A_1^{(0)}$, $A_2^{(0)}$, $A_3^{(0)}$ are positive constants depending on ν and q; in fact $A_3^{(0)} = 1+2^{-1/2}$.

If $m \geqslant 2$, it follows from the definition that $N(\nu, q^m; x)$ is the number of positive integers $n \leqslant x$ for which one of $q^r \,\|\, \sigma_\nu(n)$, $r = 0, 1, 2, \ldots, m-1$, holds, and hence

$$N(\nu, q^m; x) = \sum_{r=0}^{m-1} D_r(\nu, q; x). \tag{3}$$

An asymptotic equation for $N(\nu, q^m; x)$ can be (and, in part V of this paper, will be) deduced from Theorem 1, (2) and (3), and the result obtained is an improvement, for the case $k = q^m$, in an estimate established for $N(\nu, k; x)$, when ν is odd, by G. N. Watson in a paper [2] published in 1935; corresponding improvements were deduced from (2) by Rankin. Watson proved that, when ν is odd and k is any positive integer,

$$N(\nu, k; x) = O\big(x(\log x)^{-1/\varphi(k)}\big) \tag{4}$$

as $x \to \infty$, where $\varphi(k)$ is Euler's function. When ν and q are odd, h is even and hence part (ii) of Theorem 1 and (2) are used to obtain the estimate, already mentioned as being more precise than (4), for $N(\nu, q^m; x)$. It is also possible to obtain from Theorem 1 and (2) an improvement of (4), and in some cases an asymptotic equation for $N(\nu, k; x)$, when k is not a power of a prime, and this is done in part V of this paper.

The proof of Theorem 1 falls into two parts. Define

$$a_m(n) = \begin{cases} 1 & \text{if } \quad q^m \,\|\, \sigma_\nu(n), \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly

$$D_m(\nu, q; x) = \sum_{n=1}^{x} a_m(n), \tag{5}$$

where without loss of generality x can be taken to be an integer, and to prove Theorem 1 it is necessary to find an estimate for the sum on the right of (5). The first stage in obtaining this estimate is to express the generating function

$$f_m(s) = \sum_{n=1}^{\infty} a_m(n) n^{-s} \qquad (s = \sigma + it)$$

in terms of the Riemann zeta-function and Dirichlet L-functions, and the following result is proved.

THEOREM 2. (i) *If q and h are both odd,*

$$f_m(s) = \zeta(s) g(s),$$

where $\zeta(s)$ is the Riemann zeta-function and $g(s)$ is holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \geqslant \frac{1}{2}+\delta$ $(\delta > 0)$.

(ii) *If q is odd and h is even,*

$$f_m(s) = \{\zeta(s)\}^{1-1/h} \sum_{u=0}^{m'} \{\log\zeta(s)\}^u H_u(s),$$

where each $H_u(s)$ $(0 \leqslant u \leqslant m')$ is a function involving Dirichlet L-functions associated with non-principal characters and functions satisfying the conditions on $g(s)$ in (i).

(iii) *If $q = 2$,*

$$f_m(s) = \sum_{u=0}^{m} \{\log\zeta(s)\}^u H_u(s),$$

where each $H_u(s)$ $(0 \leqslant u \leqslant m)$ satisfies the conditions given in (ii).

The second stage in estimating $D_m(\nu, q; x)$ entails deriving Theorem 1 from Theorem 2. Theorem 1 (i) follows immediately from Theorem 2 (i) and the Wiener-Ikehara Theorem (which is stated in Lemma 10). However another result has to be proved in order that the rest of Theorem 1 can be deduced. Let

$$h(s) = \{\zeta(s)\}^{1-\beta} \{\log\zeta(s)\}^u H(s),$$

where $0 < \beta \leqslant 1$, u is a non-negative integer and $H(s)$ is a product of powers of Dirichlet L-functions associated with non-principal characters, non-negative powers of the logarithm of such functions, and a function holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \geqslant \frac{1}{2}+\delta$ $(\delta > 0)$. Furthermore suppose that $h(s)$ can be expressed in the form

$$h(s) = \sum_{n=1}^{\infty} b(n) n^{-s},$$

where $b(n) \geqslant 0$. Then:

THEOREM 3. (i) *If* $0<\beta<1$ *and* $u\geqslant 1$, *then*

$$\sum_{n=1}^{x} b(n) = \frac{H(1)}{\Gamma(1-\beta)} x(\log\log x)^{u}(\log x)^{-\beta}+O\big(x(\log\log x)^{u-1/2}(\log x)^{-\beta}\big).$$

(ii) *If* $0<\beta<1$ *and* $u=0$, *then*

$$\sum_{n=1}^{x} b(n) = \frac{H(1)}{\Gamma(1-\beta)} x(\log x)^{-\beta}+O\big(x(\log x)^{-(1+\beta)/2}\big).$$

(iii) *If* $\beta=1$ *and* $u\geqslant 2$, *then*

$$\sum_{n=1}^{x} b(n) = uH(1)x(\log\log x)^{u-1}(\log x)^{-1}+O\big(x(\log\log x)^{u-3/2}(\log x)^{-1}\big).$$

(iv) *If* $\beta=1$ *and* $u=1$, *then*

$$\sum_{n=1}^{x} b(n) = H(1)x(\log x)^{-1}+O\big(x(\log\log x)^{1/2}(\log x)^{-3/2}\big).$$

(v) *If* $\beta=1$ *and* $u=0$, *then*

$$\sum_{n=1}^{x} b(n) = O\big(x(\log x)^{-3/2}\big).$$

A proof of part (iv) of this theorem with $H(s)=1$ forms part of one of the proofs of the Prime Number Theorem; Rankin [1] applied part (ii) of this result with $\beta=1/h$, and Watson's paper [2] includes the proof of a similar result with h replaced by $\varphi(k)$. However, although some cases of this theorem are already known, to the author's knowledge the statement and proof of the general result have not previously appeared in print.

Theorem 1 (ii) and (iii) will be deduced from Theorems 2 and 3 in part V of this paper. By the method of the following pages one can prove results analogous to Theorem 1 for the functions $d(n)$ and $\varphi(n)$, where $d(n)$ is the number of divisors of n and $\varphi(n)$ is Euler's function. The results which can be obtained in this way will be stated in part V. The proofs of Theorem 2 (i) and (ii) and Theorem 1 (i) are contained in part II. In order to simplify the details of the proof of Theorem 2, the case $q=2$, stated in part (iii), is proved separately in part III although the method used is essentially the same as that contained in part II. Part IV contains the proof of Theorem 3.

I wish to thank Professor R. A. Rankin for suggesting to me the problem from which this paper arose and for all his valuable advice, and Professor H. Halberstam for his helpful criticism of an earlier draft of this paper.

II. Proof of Theorem 2 (i) and (ii) and Theorem 1 (i)

The main object of part II is to prove Theorem 2 when q is an odd prime; hence we shall assume throughout part II that $q\neq 2$. Our aim is to find an expression for

$$f_m(s) = \sum_{n=1}^{\infty} a_m(n)n^{-s};$$

to do this we first find (in Lemma 5) the positive integers a for which $a_m(p^a)=1$, where p is a prime, and, since we shall see that $a_m(n)$ is not multiplicative, the next step, given in Lemma 8, is to express $a_m(n)$ in terms of $a_r(p^a)$ $(0\leqslant r\leqslant m)$. It will then be shown that the required result follows.

In § 4 we shall deduce Theorem 1 (i) from Theorem 2 (i) and the Wiener-Ikehara Theorem.

1. A preliminary result. Let p denote a prime. The purpose of this section is to obtain an expression for the order of p^ν modulo powers of q, and hence to prove Lemma 4, which will be needed later. Let g be a primitive root $(\bmod\ q^r)$ for all positive integers r. Then every prime p, $p\neq q$, satisfies a congruence relation of the form

$$p\equiv g^{c_p(r)} (\bmod\ q^r) \quad \text{where} \quad 1\leqslant c_p(r)\leqslant \varphi(q^r)=q^{r-1}(q-1). \tag{6}$$

If $r_2>r_1$, $p\equiv g^{c_p(r_2)}\equiv g^{c_p(r_1)} (\bmod\ q^{r_1})$ and hence

$$c_p(r_2)\equiv c_p(r_1) \big(\bmod\ \varphi(q^{r_1})\big). \tag{7}$$

Define $\varepsilon_p(r)$ to be the highest power of q dividing $c_p(r)$, so that $q^{\varepsilon_p(r)}\parallel c_p(r)$ where, clearly, $0\leqslant \varepsilon_p(r)\leqslant r-1$. From (7) it follows that

$$\varepsilon_p(r_1) = \min\big(\varepsilon_p(r_2), r_1-1\big), \tag{8}$$

and hence

$$\varepsilon_p(r+1)=\varepsilon_p(r) \quad \text{or} \quad \varepsilon_p(r)+1. \tag{9}$$

LEMMA 1. *If* $r\geqslant 2$ *and* $q\mid c_p(r)$, *then* $q\mid c_p(2)$ *and*

$$c_p(2)=qc_p, \tag{10}$$

where $c_p=c_p(1)$. *If* $r>2$ *and* $q^{r-1}\parallel c_p(r)$, *then*

$$c_p(r)=q^{r-1}c_p. \tag{11}$$

Proof. If we put $r_1=2$ and $r_2=r$ in (8), we see that if $q\mid c_p(r)$, then $q\mid c_p(2)$. On putting $r_1=1$ and $r_2=2$ in (7) and using the inequality in (6), we obtain

$$c_p(2)=c_p+u(q-1) \quad \text{where} \quad 0\leqslant u\leqslant q-1.$$

If $q \mid c_p(2)$, it follows that $q \mid (c_p - u)$; since $|c_p - u| < q$, we have $u = c_p$. This gives (10).

Suppose now that $r > 2$ and $q^{r-1} \parallel c_p(r)$. Then $\varepsilon_p(r) = r-1$, and it follows from (8) that $\varepsilon_p(r-1) = r-2$, $\varepsilon_p(r-2) = r-3, \ldots, \varepsilon_p(2) = 1$. If $3 \leqslant i \leqslant r$, we have from (7) that

$$c_p(i) = c_p(i-1) + u_i q^{i-2}(q-1) \quad \text{where} \quad 0 \leqslant u_i \leqslant q-1.$$

If $c_p(i-1) = q^{i-2}c_p$ and $q^{i-1} \parallel c_p(i)$, it follows that $u_i = c_p$ and $c_p(i) = q^{i-1}c_p$; this is true for $i = 3, 4, \ldots, r$. Since $q \mid c_p(r)$, (10) holds so that, when $i = 3$, $c_p(i-1) = q^{i-2}c_p$; hence if $r > 2$ and $q^{r-1} \parallel c_p(r)$, then

$$c_p(r) = q^{r-1}c_p.$$

We recall that $q^\gamma \parallel \nu$ and that $h = (q-1)/(\nu, q-1)$. Define $t = t(p)$ by

$$q^t \parallel (p^\nu - 1);$$

we shall assume now that $r \geqslant t$. Our next lemma gives an expression for the order of $p^\nu \pmod{q^r}$ when $t \geqslant 1$, and Lemma 3 gives a corresponding expression valid for $t \geqslant 0$. We adopt the convention that the order of $p^\nu \pmod{q^t}$ is 1; if $r < t$, then the order of $p^\nu \pmod{q^r}$ is not defined. If $r > t$, then clearly the order of $p^\nu \pmod{q^r}$ must exceed 1.

LEMMA 2. *If $r > t$ and $t \geqslant 1$, then the order of $p^\nu \pmod{q^r}$ is q^{r-t}.*

This result is proved by LeVeque [3] in Theorem 4-6, and will be deduced from Lemma 3.

LEMMA 3. *The order of $p^\nu \pmod{q^r}$ is*

$$\lambda_p(r)h/(h, c_p),$$

where

$$\lambda_p(r) = \begin{cases} q^{r-1-\gamma-\varepsilon_p(r)} & \text{if} \quad r-1-\gamma-\varepsilon_p(r) \geqslant 0, \\ 1 & \text{if} \quad r-1-\gamma-\varepsilon_p(r) \leqslant 0. \end{cases}$$

Proof. We shall use (6). The order of $g \pmod{q^r}$ is $\varphi(q^r)$ by definition of a primitive root. Hence the order of $g^\nu \pmod{q^r}$, $h(r)$ say, is given by

$$h(r) = \frac{\varphi(q^r)}{(\nu, \varphi(q^r))} = \frac{q^{r-1}}{(\nu, q^{r-1})} \cdot \frac{q-1}{(\nu, q-1)} = \begin{cases} q^{r-1-\gamma}h & \text{if} \quad r-1-\gamma \geqslant 0, \\ h & \text{if} \quad r-1-\gamma \leqslant 0. \end{cases}$$

It follows that, if $r-1-\gamma \geqslant 0$, the order of $g^{\nu c_p(r)} \pmod{q^r}$, that is the order of $p^\nu \pmod{q^r}$, is equal to

$$\frac{h(r)}{(h(r), c_p(r))} = \frac{q^{r-1-\gamma}}{(q^{r-1-\gamma}, c_p(r))} \cdot \frac{h}{(h, c_p(r))} = q^{r-1-\gamma-\varepsilon_p(r)} \frac{h}{(h, c_p)}$$

provided that $r-1-\gamma-\varepsilon_p(r) \geqslant 0$; the fact that $(h, c_p(r)) = (h, c_p)$, which is used in the last step, follows from (7) on putting $r_1 = 1$ and $r_2 = r$, since $h \mid (q-1)$. If $r-1-\gamma \leqslant 0$, replace $q^{r-1-\gamma}$ by 1, and if $r-1-\gamma-\varepsilon_p(r) \leqslant 0$, replace $q^{r-1-\gamma-\varepsilon_p(r)}$ by 1; thus in either of these cases the order of $p^\nu \pmod{q^r}$ is $h/(h, c_p)$. This completes the proof of the lemma.

We observe that, by (9), $\lambda_p(r+1) = q\lambda_p(r)$ or $\lambda_p(r)$ according as $\varepsilon_p(r+1) = \varepsilon_p(r)$ or $\varepsilon_p(r)+1$. It is not immediately evident that Lemmas 2 and 3 are equivalent if $t \geqslant 1$, so we shall now deduce Lemma 2 from Lemma 3. The order of $p^\nu \pmod{q^t}$ is 1, and so $h/(h, c_p) = 1$ and $\lambda_p(t) = 1$; if $r > t$, the order of $p^\nu \pmod{q^r}$ is $\lambda_p(r) = q^{r-1-\gamma-\varepsilon_p(r)}$ by Lemma 2, and $\lambda_p(r) > 1$. Hence, since $\lambda_p(t+1) > 1$ and $\lambda_p(t) = 1$, $\lambda_p(t+1) = q$ and $\varepsilon_p(t+1) = \varepsilon_p(t)$ by the remark at the beginning of this paragraph, so that $\varepsilon_p(t+1) \leqslant t-1 < t$. On putting $r_1 = t+1$ and $r_2 = r$ in (8), we obtain $\varepsilon_p(r) = \varepsilon_p(t)$. Hence, since $\lambda_p(t) = 1$ so that $t-1-\gamma-\varepsilon_p(t) = 0$, $\gamma+\varepsilon_p(r) = \gamma+\varepsilon_p(t) = t-1$, and so $\lambda_p(r) = q^{r-1-\gamma-\varepsilon_p(r)} = q^{r-t}$.

We define $\mu_p(r)$ $(r \geqslant 1)$ to be the order of $p^\nu \pmod{q^{r+t}}$; by Lemmas 2 and 3,

$$(12) \qquad \mu_p(r) = \begin{cases} \lambda_p(r)h/(h, c_p) = \lambda_p(r)\mu_p & \text{if} \quad p^\nu \not\equiv 1 \pmod{q}, \\ q^r & \text{if} \quad p^\nu \equiv 1 \pmod{q}, \end{cases}$$

where $\mu_p = \mu_p(1)$. We observe that $\mu_p(r) \geqslant \mu_p \geqslant 2$ always, and that $\mu_p(r) \geqslant \mu_p \geqslant 3$ if $p^\nu \equiv 1 \pmod{q}$.

LEMMA 4. *If h is even, then $\mu_p(r) = 2$ and $\mu_p(r+1) = q\mu_p(r)$ hold simultaneously if and only if $r \geqslant \gamma+1$ and p is congruent to one of $\varphi(q^{\gamma+1})(\nu, q-1)$ elements of a reduced residue system* $\pmod{q^{r+1}}$.

Proof. $\mu_p(r) = 2$ cannot hold unless $t = 0$; hence we may assume that $t = 0$. Clearly if $\mu_p(r) = 2$, then $\mu_p = 2$ and $\lambda_p(r) = 1$. Since $\mu_p = h/(h, c_p)$, $\mu_p = 2$ if and only if c_p is an odd multiple of $\frac{1}{2}h$; this is so when

$$c_p = \tfrac{1}{2}h(2u-1) \quad \text{where} \quad 1 \leqslant u \leqslant (\nu, q-1),$$

the bounds for the integer u following since $1 \leqslant c_p \leqslant q-1$, so that

$$\tfrac{1}{2}\left(\frac{2}{h}+1\right) \leqslant u \leqslant \tfrac{1}{2}\left(2(q-1)h^{-1}+1\right) = (\nu, q-1)+\tfrac{1}{2}.$$

Thus there are exactly $(\nu, q-1)$ values of c_p which are such that $\mu_p = 2$, and hence $\mu_p = 2$ if and only if p is congruent to one of $(\nu, q-1)$ elements of a reduced residue system $\pmod{q}$.

We now find the number of values of $c_p(r+1)$, corresponding to a given value of c_p, for which $\mu_p(r+1) = q\mu_p(r) = q\mu_p$. Clearly $\lambda_p(r+1) = q$ but $\lambda_p(r) = \lambda_p(r-1) = \ldots = \lambda_p(2) = 1$; thus

$$(r+1)-1-\gamma-\varepsilon_p(r+1) = 1 \quad \text{and} \quad r-1-\gamma-\varepsilon_p(r) = 0,$$

giving $\varepsilon_p(r+1) = \varepsilon_p(r) = r-1-\gamma$, provided $r \geqslant \gamma+1$. By (8)

$$\varepsilon_p(r-\gamma) = \min\big(\varepsilon_p(r),\, r-1-\gamma\big) = r-1-\gamma,$$

and hence $\varepsilon_p(r+1) = \varepsilon_p(r) = \ldots = \varepsilon_p(r-\gamma) = r-1-\gamma$. Therefore $q^{r-1-\gamma} \parallel c_p(r-\gamma)$, and, by (11), $c_p(r-\gamma) = q^{r-1-\gamma}c_p$; thus to each c_p there corresponds exactly one $c_p(r-\gamma)$. Now, by (7)

$$c_p(r+1) = c_p(r-\gamma)+uq^{r-1-\gamma}(q-1) \quad \text{where} \quad 0 \leqslant u < q^{\gamma+1},$$

so that

$$c_p(r+1) = q^{r-1-\gamma}\big(c_p+u(q-1)\big).$$

Hence, if $q^{r-1-\gamma} \parallel c_p(r+1)$, $q \nmid (c_p-u)$. This means that u can take any value between 0 and $q^{\gamma+1}-1$ except

$$c_p,\ c_p+q,\ \ldots,\ c_p+(q^{\gamma}-1)q,$$

and so u, and hence $c_p(r+1)$, can take $q^{\gamma+1}-q^{\gamma} = \varphi(q^{\gamma+1})$ values for each given value of c_p.

It follows that $\mu_p(r+1) = q\mu_p(r) = 2q$ if and only if p is congruent to one of $\varphi(q^{\gamma+1})(\nu, q-1)$ elements of a reduced residue system (mod q^{r+1}) provided $r \geqslant \gamma+1$. If $r < \gamma+1$, we observe that $\mu_p(r+1) = \mu_p(r) = \mu_p$ for all p, so that no p satisfies the required conditions. This completes the proof of the lemma.

2. The evaluation of $\sum\limits_{a=1}^{\infty} a_r(p^a)$.

We have already defined

$$a_r(n) = \begin{cases} 1 & \text{if} \quad q^r \parallel \sigma_\nu(n), \\ 0 & \text{otherwise} \end{cases}$$

for $r \geqslant 1$; we define also $a_0(n)$ by $a_0(n) = 1$ or 0 according as q does not divide or divides $\sigma_\nu(n)$. Clearly the definitions imply that $a_0(1) = 1$ and $a_r(1) = 0$ for $r \geqslant 1$. For convenience we shall frequently write $a(n)$ for $a_0(n)$. The results of this section and the next which involve $a(n)$, but not $a_r(n)$ for $r \geqslant 1$, are all proved by Rankin [1]; Lemmas 5 and 6, parts (i) and (ii), and Lemma 7 are proved in the first part of § 2 of his paper.

The next Lemma enables us to determine the form of a when $a_r(p^a) = 1$, $r \geqslant 0$.

LEMMA 5. (i) *If* $p \neq q$, $a(p^a) = 1$ *if and only if* $a \neq u\mu_p-1$ *for any integer* u.

(ii) $a(q^a) = 1$ *for all* a.

(iii) *If* $r \geqslant 1$, $p \neq q$ *and* $\mu_p(r+1) = q\mu_p(r)$, *then* $a_r(p^a) = 1$ *if and only if* $a = u\mu_p(r)-1$ *where* $(u, q) = 1$.

(iv) *If* $r \geqslant 1$ *and either* $p = q$ *or* $\mu_p(r+1) = \mu_p(r)$, *then* $a_r(p^a) = 0$ *for all* a.

Proof. We have

$$\sigma_\nu(p^a) = 1+p^\nu+p^{2\nu}+\ldots+p^{a\nu} = (p^{\nu(a+1)}-1)/(p^\nu-1),$$

and $q^t \parallel (p^\nu-1)$ where $t \geqslant 0$. For any $r \geqslant 0$, $q^r \parallel \sigma_\nu(p^a)$ implies that $q^{r+t} \parallel (p^{\nu(a+1)}-1)$, and this occurs if and only if the order of $p^\nu \pmod{q^{r+t}}$, which is $\mu_p(r)$ by definition, divides $a+1$ but the order of $p^\nu \pmod{q^{r+t+1}}$, which is $\mu_p(r+1)$, does not. (We recall that the order of $p^\nu \pmod{q^t}$ is 1, and we use this convention also when $t = 0$.)

(i) If $p \neq q$, $a(p^a) = 1$ if and only if $\mu_p \nmid (a+1)$, which gives the result.

(ii) $\sigma_\nu(q^a) \equiv 1 \pmod q$, and hence the result follows.

(iii) If the given conditions are satisfied, then from above $a_r(p^a) = 1$ if and only if

$$\mu_p(r) \mid (a+1) \quad \text{but} \quad \mu_p(r+1) \nmid (a+1).$$

Since $\mu_p(r+1) = q\mu_p(r)$, the result follows.

(iv) This part is an immediate consequence of the proof of (ii) if $p = q$ and of (iii) if $\mu_p(r+1) = \mu_p(r)$.

LEMMA 6. (i) *If* $p \neq q$,

$$\sum_{a=0}^{\infty} a(p^a)p^{-as} = (1-p^{-(\mu_p-1)s})/(1-p^{-s})(1-p^{-\mu_p s}).$$

(ii) $\sum\limits_{a=0}^{\infty} a(q^a)q^{-as} = (1-q^{-s})^{-1}$.

(iii) *If* $r \geqslant 1$, $p \neq q$ *and* $\mu_p(r+1) = q\mu_p(r)$, *then*

$$\sum_{a=1}^{\infty} a_r(p^a)p^{-as} = (1-p^{-(q-1)\mu_p(r)s})p^{-(\mu_p(r)-1)s}/(1-p^{-\mu_p(r)s})(1-p^{-q\mu_p(r)s}).$$

(iv) *If* $r \geqslant 1$ *and either* $p = q$ *or* $\mu_p(r+1) = \mu_p(r)$, *then*

$$\sum_{a=1}^{\infty} a_r(p^a)p^{-as} = 0.$$

Proof. This Lemma follows from the previous one. For example, to prove (iii) we have, if the given conditions hold, that

$$\sum_{a=1}^{\infty} a_r(p^a)p^{-as} = \sum_{\substack{u=1\\(u,q)=1}}^{\infty} p^{-(u\mu_p(r)-1)s} = p^s\Big\{\sum_{u=1}^{\infty} p^{-u\mu_p(r)s} - \sum_{u=1}^{\infty} p^{-uq\mu_p(r)s}\Big\},$$

and we obtain the result on summing these two geometric series.

3. The generating functions. Since $\sigma_\nu(n)$ is multiplicative, we can write

$$\sigma_\nu(n) = \prod_{p^a \| n} \sigma_\nu(p^a),$$

where the product is over all distinct primes dividing n. From this it follows that $a(n)$ is multiplicative; for $q \nmid \sigma_\nu(n)$ if and only if $q \nmid \sigma_\nu(p^a)$ for every $p^a \| n$. Hence

$$a(n) = \prod_{p^a \| n} a(p^a).$$

Let

$$f(s) = \sum_{n=1}^{\infty} a(n) n^{-s};$$

then we have

LEMMA 7. $f(s) = \zeta(s) \prod_{p \neq q} (1-p^{-(\mu_p - 1)s})/(1-p^{-\mu_p s})$.

Proof. Since $a(n)$ is multiplicative, we have by Lemma 6 (i) and (ii) that

$$f(s) = \sum_{n=1}^{\infty} a(n) n^{-s} = \prod_p \Big\{ \sum_{a=0}^{\infty} a(p^a) p^{-as} \Big\}$$
$$= (1-q^{-s})^{-1} \prod_{p \neq q} \frac{1-p^{-(\mu_p-1)s}}{(1-p^{-s})(1-p^{-\mu_p s})} = \zeta(s) \prod_{p \neq q} \frac{1-p^{-(\mu_p-1)s}}{1-p^{-\mu_p s}}.$$

However, although $a(n)$ is multiplicative, $a_m(n)$, $m \geqslant 1$, is not; for $q^m \| \sigma_\nu(n)$ certainly does not hold if $q^m \| \sigma_\nu(p^a)$ for every $p^a \| n$ (unless $n = p^a$). Nevertheless we can obtain an expression for $a_m(n)$ in terms of $a(n_1)$ and $a_r(p^a)$, where $n_1 \mid n$, $p^a \| n$ and $r \leqslant m$. In the following lemma we assume that $p_i^{a_i} \| n$ for all i (with or without a suffix), and that two primes p with different suffixes are distinct. Let R_k denote a set $r_1, r_2, \ldots, r_k$ of positive integers, with $1 \leqslant r_1 \leqslant r_2 \leqslant \ldots \leqslant r_k$, and let $\mathfrak{R}(m)$ represent the collection of sets R_k whose members satisfy $r_1 + r_2 + \ldots + r_k = m$, where k takes all possible values; clearly $1 \leqslant k \leqslant m$. Let $\mathfrak{P}_k$ denote an ordered set of distinct primes $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$, and $\mathfrak{P}_{k_2} - \mathfrak{P}_{k_1}$ (where $k_2 > k_1$ and where the first k_1 primes of $\mathfrak{P}_{k_2}$ are those of $\mathfrak{P}_{k_1}$) denote the ordered set of primes $p_{k_1+1}, \ldots, p_{k_2}$.

LEMMA 8. *If* $m \geqslant 1$,

$$a_m(n) = \sum_{R_k \in \mathfrak{R}(m)} \{\varrho^*(R_k)\}^{-1} \sum_{\mathfrak{P}_k} a_{r_1}(p_{i_1}^{a_{i_1}}) a_{r_2}(p_{i_2}^{a_{i_2}}) \ldots a_{r_k}(p_{i_k}^{a_{i_k}}) a(n p_{i_1}^{-a_{i_1}} \ldots p_{i_k}^{-a_{i_k}}),$$

where (i) *the set* R_k *ranges over all sets belonging to* $\mathfrak{R}(m)$, *which are such that* k *does not exceed the number of primes dividing* n, (ii) *the sum over* $\mathfrak{P}_k$ *represents the sum over all sets* $\mathfrak{P}_k$ *consisting of* k *of the distinct primes dividing* n, *and* (iii) $\varrho^*(R_k)$ *is defined below.*

Proof. Since $\sigma_\nu(n)$ is multiplicative, we may write

$$\sigma_\nu(n) = \prod_{j=1}^{k} \sigma_\nu(p_{i_j}^{a_{i_j}}) \sigma_\nu(n p_{i_1}^{-a_{i_1}} \ldots p_{i_k}^{-a_{i_k}}),$$

where $q^{r_j} \| \sigma_\nu(p_{i_j}^{a_{i_j}})$, $j = 1, 2, \ldots, k$, and $q \nmid \sigma_\nu(n p_{i_1}^{-a_{i_1}} \ldots p_{i_k}^{-a_{i_k}})$; then we have $a_{r_j}(p_{i_j}^{a_{i_j}}) = 1$, $j = 1, 2, \ldots, k$, and $a(n p_{i_1}^{-a_{i_1}} \ldots p_{i_k}^{-a_{i_k}}) = 1$. It follows that $q^m \| \sigma_\nu(n)$, so that $a_m(n) = 1$, if and only if $r_1 + \ldots + r_k = m$, and that there will be only one set $R_k \in \mathfrak{R}(m)$, which we denote by $\hat{R}_k$, and certain sets $\mathfrak{P}_k$, which we denote by $\hat{\mathfrak{P}}_k$, for which this holds.

Consider now the expression

$$M(R_k, \mathfrak{P}_k) = a_{r_1}(p_{i_1}^{a_{i_1}}) \ldots a_{r_k}(p_{i_k}^{a_{i_k}}) a(n p_{i_1}^{-a_{i_1}} \ldots p_{i_k}^{-a_{i_k}}).$$

From above $a_m(n) = 1$ if and only if $M(R_k, \mathfrak{P}_k) = M(\hat{R}_k, \hat{\mathfrak{P}}_k) = 1$. We shall now calculate the number $\varrho^*(R_k)$ of sets $\mathfrak{P}_k$ which, for fixed R_k, leave $M(R_k, \mathfrak{P}_k)$ essentially unaltered, that is unaltered except for a rearrangement in the order of its factors. Suppose that R_k has τ distinct elements $r_1^*, \ldots, r_\tau^*$, where $r_1^* < r_2^* < \ldots < r_\tau^*$, occurring $l_1, l_2, \ldots, l_\tau$ times respectively in the set; clearly $l_1 + \ldots + l_\tau = k$. If we rearrange the primes of the set $\mathfrak{P}_{l_1}$ amongst themselves (and there are $l_1!$ different arrangements of $\mathfrak{P}_{l_1}$), $M(R_k, \mathfrak{P}_k)$ will remain essentially unaltered (since every member of $\mathfrak{P}_{l_1}$ is associated with r_1^* in $M(R_k, \mathfrak{P}_k)$); similarly if we rearrange the members of the set $\mathfrak{P}_{l_{j+1}} - \mathfrak{P}_{l_j}$ $(1 \leqslant j \leqslant \tau - 1)$ amongst themselves, $M(R_k, \mathfrak{P}_k)$ will remain essentially unaltered. Hence we can arrange the members of the set $\mathfrak{P}_k$ in $\varrho^*(R_k)$ ways, where

$$\varrho^*(R_k) = l_1! \, l_2! \ldots l_\tau!,$$

without essentially altering $M(R_k, \mathfrak{P}_k)$. However, if we alter the order of the members of $\mathfrak{P}_k$ in any way other than those mentioned above, or if we replace R_k by another set R_k' (possibly having a different number of members) and/or replace $\mathfrak{P}_k$ by another set $\mathfrak{P}_k'$, we shall essentially alter $M(R_k, \mathfrak{P}_k)$.

If $a_m(n) = 1$ and $R_k \in \mathfrak{R}(m)$, it follows from above that $M(R_k, \mathfrak{P}_k) = 0$ unless R_k is the set $\hat{R}_k$ and $\mathfrak{P}_k$ is one of the $\varrho^*(\hat{R}_k)$ sets $\hat{\mathfrak{P}}_k$; if $a_m(n) = 0$, $M(R_k, \mathfrak{P}_k) = 0$ for all $R_k \in \mathfrak{R}(m)$ and all $\mathfrak{P}_k$. Hence

$$\sum_{R_k \in \mathfrak{R}(m)} \sum_{\mathfrak{P}_k} M(R_k, \mathfrak{P}_k) = \begin{cases} \varrho^*(\hat{R}_k) a_m(n) & \text{if} \quad a_m(n) = 1, \\ 0 & \text{if} \quad a_m(n) = 0. \end{cases}$$

Unless $a_m(n) = 1$ and $R_k = \hat{R}_k$, the inner sum on the left is zero (for at least one factor of each term $M(R_k, \mathfrak{P}_k)$ is zero), hence the result follows.

We are now in a position to find the generating function

$$f_m(s) = \sum_{n=1}^{\infty} a_m(n) n^{-s}$$

for $m \geqslant 1$. Let $S(r)$ be the set of all primes p for which $\mu_p(r+1) = q\mu_p(r)$. Then we have

LEMMA 9.

$$f_m(s) = f(s) \sum_{R_k \in \mathfrak{R}(m)} \{\varrho^*(R_k)\}^{-1} \sum_{p_1} P\big(p_1, \mu_{p_1}(r_1); s\big) \sum_{p_2} P\big(p_2, \mu_{p_2}(r_2); s\big) \ldots \\ \ldots \sum_{p_k} P\big(p_k, \mu_{p_k}(r_k); s\big),$$

where

$$P\big(p, \mu_p(r); s\big) = \frac{(1-p^{-s})(1-p^{-\mu_p s})(1-p^{-(q-1)\mu_p(r)s})\, p^{-(\mu_p(r)-1)s}}{(1-p^{-(\mu_p-1)s})(1-p^{-\mu_p(r)s})(1-p^{-q\mu_p(r)s})},$$

and where the sum over p_i $(i = 1, 2, \ldots, k)$ *is over all primes* $p \in S(r_i)$ *except* $q, p_1, p_2, \ldots, p_{i-1}$.

Proof. By Lemma 8,

$$(13)\quad f_m(s) = \sum_{n=1}^{\infty} \Big\{ \sum_{R_k \in \mathfrak{R}(m)} \{\varrho^*(R_k)\}^{-1} \sum_{\mathfrak{P}_k} M(R_k, \mathfrak{P}_k) \Big\} n^{-s}$$

$$= \sum_{n=1}^{\infty} \sum_{R_k \in \mathfrak{R}(m)} \{\varrho^*(R_k)\}^{-1} \sum_{\mathfrak{P}_k} a_{r_1}(p_{i_1}^{a_{i_1}}) p_{i_1}^{-a_{i_1}s} a_{r_2}(p_{i_2}^{a_{i_2}}) p_{i_2}^{-a_{i_2}s} \ldots \\ \ldots a_{r_k}(p_{i_k}^{a_{i_k}}) p_{i_k}^{-a_{i_k}s}\, a(np_{i_1}^{-a_{i_1}} \ldots p_{i_k}^{-a_{i_k}})\, (np_{i_1}^{-a_{i_1}} \ldots p_{i_k}^{-a_{i_k}})^{-s}$$

$$= \sum_{R_k \in \mathfrak{R}(m)} \{\varrho^*(R_k)\}^{-1} \sum_{p_1} \sum_{a_1=1}^{\infty} a_{r_1}(p_1^{a_1}) p_1^{-a_1 s} \sum_{p_2} \sum_{a_2=1}^{\infty} a_{r_2}(p_2^{a_2}) p_2^{-a_2 s} \ldots \\ \ldots \sum_{p_k} \sum_{a_k=1}^{\infty} a_{r_k}(p_k^{a_k}) p_k^{-a_k s} \sum_{\substack{n=1 \\ (n, p_1 p_2 \ldots p_k)=1}}^{\infty} a(n) n^{-s}$$

where the sum over p_i $(i = 1, 2, \ldots, k)$ is over all primes except $p_1, \ldots, p_{i-1}$.

By Lemma 6 (i) and 7, we have if $p_i \neq q$, $i = 1, 2, \ldots, k$, that

$$(14)\quad \sum_{\substack{n=1 \\ (n, p_1 p_2 \ldots p_k)=1}}^{\infty} a(n) n^{-s} = \prod_{\substack{p \\ p \neq p_1, \ldots, p_k}} \Big\{ \sum_{a=0}^{\infty} a(p^a) p^{-as} \Big\}$$

$$= f(s) \prod_{i=1}^{k} \Big\{ \sum_{a_i=0}^{\infty} a(p_i^{a_i}) p_i^{-a_i s} \Big\}^{-1}$$

$$= f(s) \prod_{i=1}^{k} (1-p_i^{-s})(1-p_i^{-\mu_{p_i} s})(1-p_i^{-(\mu_{p_i}-1)s})^{-1}.$$

We do not need to consider the above sum with any p_i equal to q; for if $p_j = q$, $a_{r_j}(p_j^{a_j}) = a_{r_j}(q^{a_j}) = 0$ for all a_j by Lemma 5 (iv) and the corresponding term on the right of (13) is zero. If we substitute (14) in (13) and use Lemma 6 (iii) and (iv), we obtain the result of the lemma.

The following considerations may help to make the form of the above result seem logical. We may write

$$(15)\quad \frac{f_m(s)}{f(s)} = \sum_{R_k \in \mathfrak{R}(m)} \{\varrho^*(R_k)\}^{-1} \prod_{i=1}^{k} \sum_{\substack{p \neq q \\ p \in S(r_i)}} P\big(p, \mu_p(r_i); s\big) + \\ + O\Big(\sum_{R_k \in \mathfrak{R}(m)} \{\varrho^*(R_k)\}^{-1} \sum_{\substack{p \neq q \\ p \in S(r_{j_1}) \\ p \in S(r_{j_2})}} P\big(p, \mu_p(r_{j_1}); s\big) \times \\ \times P\big(p, \mu_p(r_{j_2}); s\big) \prod_{\substack{i=1 \\ i \neq j_1, j_2}}^{k} \sum_{\substack{p \neq q \\ p \in S(r_i)}} P\big(p, \mu_p(r_i); s\big)\Big),$$

where the error term is of smaller order of magnitude than the first term, as will become apparent in § 5 when $m' \geqslant 1$, unless $|f_m(s)/f(s)| = O(1)$ for $\sigma > \frac{1}{2}$ which is so in § 4 and in § 5 when $m' = 0$. It can easily be shown that the main term on the right of (15) is the coefficient of x^m in the expansion of

$$\exp\Big\{ \sum_{r=1}^{\infty} x^r \sum_{\substack{p \neq q \\ p \in S(r)}} P\big(p, \mu_p(r); s\big) \Big\}.$$

4. Proofs of Theorems 1 (i) and 2 (i). In this section we shall assume that h is odd. It follows from (12) that μ_p cannot be even, so that $\mu_p \geqslant 3$, and hence $\mu_p(r) \geqslant 3$ for all $r \geqslant 1$. From the definition of $P\big(p, \mu_p(r); s\big)$,

$$\big|P\big(p, \mu_p(r); s\big)\big| \leqslant \frac{(1+p^{-\sigma})(1+p^{-\mu_p\sigma})(1+p^{-(q-1)\mu_p(r)\sigma})\, p^{-(\mu_p(r)-1)\sigma}}{(1-p^{-(\mu_p-1)\sigma})(1-p^{-\mu_p(r)\sigma})(1-p^{-q\mu_p(r)\sigma})} \\ \leqslant \Omega(\sigma) p^{-(\mu_p(r)-1)\sigma},$$

where $\Omega(\sigma)$, a function of $\sigma = \operatorname{Re} s$ only, is obtained by using the inequalities $p \geqslant 2$, $\mu_p \geqslant 3$, $\mu_p(r) \geqslant 3$. Hence

$$(16)\quad \Big| \sum_{\substack{p_i \neq q, p_1, \ldots, p_{i-1} \\ p_i \in S(r_i)}} P\big(p_i, \mu_{p_i}(r_i); s\big) \Big| \leqslant \sum_p \big|P\big(p, \mu_p(r_i); s\big)\big| \\ \leqslant \Omega(\sigma) \sum_p p^{-(\mu_p(r_i)-1)\sigma} \leqslant \Omega(\sigma) \sum_p p^{-2\sigma}$$

which is convergent for $\sigma > \frac{1}{2}$; thus the sum on the left is absolutely convergent for $\sigma > \frac{1}{2}$.

Since $\mu_p \geqslant 3$, the infinite product in the expression for $f(s)$, given in Lemma 7, is also absolutely convergent for $\sigma > \frac{1}{2}$. Hence it follows from Lemmas 7 and 9 and above that

$$f_m(s) = \zeta(s)g(s),$$

where $g(s)$ is holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \geqslant \frac{1}{2}+\delta$ $(\delta > 0)$. This completes the proof of Theorem 2 (i).

We now show that Theorem 1 (i) follows from Theorem 2 (i) and the Wiener-Ikehara Theorem which we state in

LEMMA 10. *If $\Phi(\tau)$ is a non-negative, non-decreasing function in $0 \leqslant \tau < \infty$ such that the integral*

$$F(s) = \int_0^\infty e^{-s\tau}\Phi(\tau)\,d\tau$$

converges for $\sigma > 1$, and if for some constant B and some function $G(t)$, where $t = \operatorname{Im} s$,

$$\lim_{\sigma\to 1+}\left\{F(s) - \frac{B}{s-1}\right\} = G(t)$$

uniformly in every finite interval $-a \leqslant t \leqslant a$, then

$$\lim_{\tau\to\infty}\Phi(\tau)e^{-\tau} = B.$$

This is given in §17 of Chapter V of Widder [4]. To deduce the result from this, let

$$sF(s) = f_m(s) = \sum_{n=1}^{\infty} a_m(n)n^{-s}, \qquad \Phi(\tau) = S(e^\tau) = \sum_{n=1}^{[e^\tau]} a_m(n), \qquad B = g(1);$$

then in order to prove Theorem 1 (i) we need to estimate $S(x)$, for

$$S(x) = \sum_{n=1}^{x} a_m(n) = D_m(\nu, q; x).$$

Clearly $f_m(s)$ is holomorphic for $\sigma > 1$, so that

$$f_m(s) = \sum_{n=1}^{\infty} a_m(n)n^{-s} = \sum_{n=1}^{\infty}\{S(n)-S(n-1)\}n^{-s} = \int_1^\infty y^{-s}\,dS(y)$$

$$= \int_0^\infty e^{-\tau s}\,dS(e^\tau) = s\int_0^\infty e^{-\tau s}S(e^\tau)\,d\tau = s\int_0^\infty e^{-\tau s}\Phi(\tau)\,d\tau$$

converges for $\sigma > 1$. Since $\zeta(s)-(s-1)^{-1}$ is holomorphic for $\sigma > 0$ (see Lemma 13 (i)) and $g(s)$ is holomorphic for $\sigma > \frac{1}{2}$, it follows that $f_m(s)s^{-1} - g(1)(s-1)^{-1}$ is holomorphic for $\sigma > \frac{1}{2}$, so that

$$\lim_{\sigma\to 1+}\{f_m(s)s^{-1} - g(1)(s-1)^{-1}\} = G(t)$$

uniformly in every finite interval $-a \leqslant t \leqslant a$. Hence the conditions of Lemma 10 are satisfied and an application of it yields

$$\lim_{\tau\to\infty}\Phi(\tau)e^{-\tau} = g(1),$$

whence

$$\lim_{x\to\infty} S(x)x^{-1} = g(1).$$

Thus as $x \to \infty$,

$$S(x) = \sum_{n=1}^{x} a_m(n) = D_m(\nu, q; x) \sim g(1)x,$$

which is Theorem 1 (i).

5. Proof of Theorem 2 (ii). We assume first that h is even and $m' = [m/(\gamma+1)] \geqslant 1$. Then it follows from the proof of Lemma 4 that for any positive integer r there exist primes p for which $\mu_p(r) = 2$. For such a prime p we have by the definition of $P(p, \mu_p(r); s)$ that

$$P(p, \mu_p(r); s) = P(p, 2; s) = \left\{\frac{1-p^{-2(q-1)s}}{1-p^{-2qs}}\right\}p^{-s} \tag{17}$$

$$= \left\{1 - \frac{p^{-2(q-1)s} - p^{-2qs}}{1-p^{-2qs}}\right\}p^{-s}.$$

Let $S_2(r)$ be the set of all primes p which satisfy $p \in S(r)$, $p \neq q$, $\mu_p(r) = 2$; we recall that $p \in S(r)$ if $\mu_p(r+1) = q\mu_p(r)$. Then

$$\sum_{\substack{p_i \neq q, p_1, \ldots, p_{i-1} \\ p_i \in S(r_i)}} P(p_i, \mu_{p_i}(r_i); s) = \sum_{p \in S_2(r_i)} p^{-s} + \psi_i(s), \tag{18}$$

where the sums on the left and right are non-empty if and only if $r_i \geqslant \gamma+1$ by Lemma 4, and, by the arguments used at the beginning of § 4, $\psi_i(s)$ is holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \geqslant \frac{1}{2}+\delta$ for any $\delta > 0$.

LEMMA 11. *Assume that $r \geqslant \gamma+1$. Let b_j, $j = 1, 2, \ldots, \varkappa$, where $\varkappa = \varphi(q^{\gamma+1})(\nu, q-1)$, be the distinct elements of a reduced residue system* (mod q^{r+1}) *which occur in the proof of Lemma 4, and let χ be a character and χ_0 the principal character* (mod q^{r+1}). *If $L(s, \chi)$ is the Dirichlet L-series associated with the character χ, and $G(s, \chi)$ is a certain function which is holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \geqslant \frac{1}{2}+\delta$ for any $\delta > 0$, then*

$$\sum_{p \in S_2(r)} p^{-s} = (\nu, q-1)q^{\gamma-r}\{\log\zeta(s) + \log(1-q^{-s})\} +$$

$$+ q^{-r}(q-1)^{-1}\sum_{j=1}^{\varkappa}\left\{\sum_{\chi\neq\chi_0}\frac{\log L(s,\chi)}{\chi(b_j)} + \sum_{\chi}\frac{G(s,\chi)}{\chi(b_j)}\right\},$$

where the sum over χ is over all characters χ (mod q^{r+1}) *except, when indicated, χ_0.*

Proof. By Lemma 4,

(19) $$\sum_{p\in S_2(r)} p^{-s} = \sum_{j=1}^{\varkappa} \sum_{p\equiv b_j (\mathrm{mod}\, q^{r+1})} p^{-s},$$

and the b_j can be determined from the proof of Lemma 4. It is well known that

$$\varphi(q^{r+1}) \sum_{p\equiv b_j (\mathrm{mod}\, q^{r+1})} p^{-s} = \sum_{\chi} \sum_{p} \frac{\chi(p)}{\chi(b_j)}\, p^{-s},$$

where the sum over χ is over all characters χ (mod q^{r+1}), and that

$$\sum_{p} \chi(p)p^{-s} = \log L(s,\chi) - \sum_{p}\sum_{u=2}^{\infty} \frac{1}{u}\cdot\frac{\chi(p^u)}{p^{us}} = \log L(s,\chi)+G(s,\chi)$$

(say), where $G(s,\chi)$ satisfies the conditions given in the statement of the lemma; these results appear, for example, in § 13 and § 14 of Hasse [5]. Since $L(s,\chi_0) = (1-q^{-s})\zeta(s)$, it follows that

(20) $$\sum_{p\equiv b_j (\mathrm{mod}\, q^{r+1})} p^{-s} = \{\varphi(q^{r+1})\}^{-1}\Big\{\log\{(1-q^{-s})\zeta(s)\} + \sum_{\chi\neq\chi_0}\frac{\log L(s,\chi)}{\chi(b_j)} + \sum_{\chi} \frac{G(s,\chi)}{\chi(b_j)}\Big\},$$

and hence the result of the lemma follows from this and (19).

LEMMA 12. *Let g be a primitive root* (mod q) *and let $\chi(n)$ be the character defined by*

$$\chi(n) = e(\beta/h) \quad \textit{for} \quad n\equiv g^{\beta} \pmod{q},$$

where $e(z) = \exp(2\pi i z)$. *Then*

$$f(s) = \zeta(s)\{F(s)\}^{1/h}\psi(s),$$

where $\psi(s)$ is holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \geqslant \frac{1}{2}+\delta$ for any $\delta>0$, and where

$$F(s) = \prod_{r=1}^{h} \{L(s,\chi^r)/L(s,\chi^{2r})\}$$
$$= \prod_{p\mid h}(1-p^{-s})^{-1}\{\zeta(s)\}^{-1}\prod_{r=1}^{h-1}L(s,\chi^r)\Big\{\prod_{r=1}^{\frac{1}{2}h-1} L(s,\chi^{2r})\Big\}^{-2}.$$

With the exception of the last representation for $F(s)$, this lemma is proved by Rankin [1] in the paragraphs leading up to equation (14) of his paper. To prove the last part, we observe that, since χ is a character (mod h),

$$F(s) = \prod_{r=1}^{h-1} L(s,\chi^r)\Big\{\prod_{\substack{r=1\\ r\neq\frac{1}{2}h}}^{h-1} L(s,\chi^{2r})\Big\}^{-1}\{L(s,\chi^{2h})\}^{-1}$$
$$= \prod_{p\mid h}(1-p^{-s})^{-1}\{\zeta(s)\}^{-1}\prod_{r=1}^{h-1}L(s,\chi^r)\Big\{\prod_{r=1}^{\frac{1}{2}h-1} L(s,\chi^{2r})\Big\}^{-2}.$$

We are now able to complete the proof of Theorem 2 (ii) when $m'\geqslant 1$. From (18) and Lemmas 9 and 11 we obtain

(21) $$f_m(s) = f(s)\sum_{R_k^*\in\Re(m)} \{\varrho^*(R_k^*)\}^{-1}\prod_{i=1}^{k} \Big\{(\nu,q-1)q^{\gamma-r_i}\{\log\zeta(s)+\log(1-q^{-s})\} + q^{-r_i}(q-1)^{-1}\sum_{j=1}^{\varkappa}\Big\{\sum_{\chi^{(r_i)}\neq\chi_0^{(r_i)}}\frac{\log L(s,\chi^{(r_i)})}{\chi^{(r_i)}(b_j)} + \sum_{\chi^{(r_i)}}\frac{G(s,\chi^{(r_i)})}{\chi^{(r_i)}(b_j)}\Big\} + \psi_i(s)\Big\},$$

where $\chi^{(r_i)}$ is a character (mod q^{r_i+1}), and R_k^* represents a set of positive integers $r_1, r_2, \ldots, r_k$ satisfying $\gamma+1\leqslant r_1\leqslant r_2\leqslant\ldots\leqslant r_k$ (so that the set of all R_k^* is a subset of the set of all R_k). Clearly the term on the right containing the highest power of $\log\zeta(s)$ will occur when the product contains its maximum number of terms which implies that k takes its maximum value. Now k will be greatest when the r_i are as near to the value $\gamma+1$ as possible, and hence the maximum value of k is

$$[m/(\gamma+1)] = m';$$

In this case $r_i = \gamma+1+r_i'$, $i=1,2,\ldots,m'$, where

$$0\leqslant r_i'\leqslant m-m'(\gamma+1) < \gamma+1 \quad\text{and}\quad \sum_{i=1}^{m'} r_i' = m-m'(\gamma+1).$$

Now $R_{m'}^*$ represents a set of the form $\gamma+1+r_1', \gamma+1+r_2', \ldots, \gamma+1+r_{m'}'$, and the number of sets $R_{m'}^*\in\Re(m)$ is $\hat{\varrho}(m,\gamma)$, where $\hat{\varrho}(m,\gamma)$ is the number of unrestricted partitions of $m-m'(\gamma+1)$ into at most m' parts. It follows that the term on the right of (21) which contains the highest power of $\log\zeta(s)$ is

(22) $$f(s)\,\varrho(m,\gamma)(\nu,q-1)^{m'}q^{m'\gamma-m}\{\log\zeta(s)\}^{m'},$$

where

$$\varrho(m,\gamma) = \sum_{R_{m'}^*\in\Re(m)} \{\varrho^*(R_{m'}^*)\}^{-1},$$

the sum having $\hat{\varrho}(m,\gamma)$ terms. The remaining terms will be of the form

$$f(s)\{\log\zeta(s)\}^u\prod_{i=1}^{v}\left\{\{\log L(s,\chi^{(r_i)})\}\eta_i(s)\right\}, \tag{23}$$

where $0\leqslant u<m'$, $0\leqslant v\leqslant m'-u$ and $r_i\leqslant m-u(\gamma+1)$, where the r_i are not necessarily all distinct, and $\chi^{(r_i)}$ is a non-principal character $(\operatorname{mod} q^{r_i+1})$, and where $\eta_i(s)$, a function of s and the characters occurring in (21), is holomorphic for $\sigma>\frac{1}{2}$ and bounded for $\sigma\geqslant\frac{1}{2}+\delta$ for any $\delta>0$. From Lemma 12 and (21) to (23), it follows that

$$f_m(s)=\{\zeta(s)\}^{1-1/h}\sum_{u=0}^{m'}\{\log\zeta(s)\}^u H_u(s),$$

where $H_u(s)$, $0\leqslant u\leqslant m'$, satisfies the conditions of Theorem 2 (ii), and $H_{m'}(s)$ can be obtained from (22) and Lemma 12.

In order to complete the proof of Theorem 2 (ii), we now assume that $m'=0$, and as before that h is even. Since $m'=0$, $m\leqslant\gamma$ so that $q^m\mid\nu$. If $r\leqslant m\leqslant\gamma$, then by (12) and Lemma 4,

$$\mu_p(r)=2\quad\text{ and }\quad\mu_p(r+1)=q\mu_p(r)$$

cannot both hold; for if $p^\nu\not\equiv 1\ (\operatorname{mod} q)$, $\mu_p(r)=\mu_p$ for $r\leqslant\gamma+1$, and if $p^\nu\equiv 1\ (\operatorname{mod} q)$, $\mu_p(r)=q^r\geqslant q\geqslant 3$. Hence, for $r_i\leqslant m$,

$$\sum_{\substack{p_i\neq q,p_1,\ldots,p_{i-1}\\ p_i\in S(r_i)}}P\left(p_i,\mu_{p_i}(r_i);s\right)$$

is absolutely convergent for $\sigma>\frac{1}{2}$ by the arguments which lead to (16). From Lemma 9 it follows that

$$f_m(s)=f(s)\eta(s),$$

where $\eta(s)$ is holomorphic for $\sigma>\frac{1}{2}$ and bounded for $\sigma\geqslant\frac{1}{2}+\delta$ for any $\delta>0$. Since h is even there exist primes p for which $\mu_p=2$, and hence by Lemma 12

$$f_m(s)=\{\zeta(s)\}^{1-1/h}H_0(s),$$

where $H_0(s)$ satisfies the conditions of Theorem 2 (ii).

III. Proof of Theorem 2 (iii)

We have proved Theorem 2 when q is an odd prime, and we shall now sketch the proof of part (iii) for which we assume that $q=2$. Where possible we shall refer the reader to part II of the paper; to facilitate this, when a lemma or equation has to be restated or a section replaced, it will be given the same number followed by $'$. The main differences occur in the first section (owing to the peculiarities of the prime 2), and sections 4 and 5 are replaced by a new section.

1′. A preliminary result. If $r\geqslant 3$, every odd prime p satisfies

$$p\equiv\pm 5^{c_p(r)}\ (\operatorname{mod} 2^r)\quad\text{ where }\quad 1\leqslant c_p(r)\leqslant 2^{r-2}, \tag{6'}$$

where we take the $+$ or $-$ sign according as $p\equiv 1$ or $3\ (\operatorname{mod} 4)$. Equations (7) and (8) become

$$c_p(r_2)\equiv c_p(r_1)\ (\operatorname{mod} 2^{r_1-2}), \tag{7'}$$

$$\varepsilon_p(r_1)=\min\left(\varepsilon_p(r_2),r_1-2\right), \tag{8'}$$

and equation (9) still holds; Lemma 1 is not relevant when $q=2$.

When $q=2$, $h=1$. We define t as before by $2^t\,\|\,(p^\nu-1)$. Furthermore if $p\equiv 3\ (\operatorname{mod} 4)$ we define $t'=t'(p)$ by

$$2^{t'}\,\|\,\left((-p)^\nu-1\right);$$

then $t'=t$ when ν is even, but $t'\geqslant 2$ and $t=1$ when ν is odd. Clearly $t\geqslant 1$ always. We assume that $r\geqslant t$ and $r\geqslant 3$.

LEMMA 2′. *If $r>t$ and $t\geqslant 3$, then the order of $p^\nu\ (\operatorname{mod} 2^r)$ is 2^{r-t}.*

This result can be deduced from

LEMMA 3′. *The order of $p^\nu\ (\operatorname{mod} 2^r)$ is $\lambda_p(r)$, where*

$$\lambda_p(r)=\begin{cases}2^{r-2-\gamma-\varepsilon_p(r)} & \text{if}\quad r-2-\gamma-\varepsilon_p(r)\geqslant 0,\\ 1 & \text{if}\quad r-2-\gamma-\varepsilon_p(r)\leqslant 0\end{cases}$$

except when ν is odd, $p\equiv 3\ (\operatorname{mod} 4)$ and $p^\nu\equiv -1\ (\operatorname{mod} 2^r)$, in which case

$$\lambda_p(r)=2.$$

Proof. When $p\equiv 1\ (\operatorname{mod} 4)$, the proof is similar to the proof of Lemma 3. Hence suppose that $p\equiv 3\ (\operatorname{mod} 4)$, so that $-p\equiv 1\ (\operatorname{mod} 4)$. If $r>t'$, the order of $(-p)^\nu\ (\operatorname{mod} 2^r)$ is $2^{r-2-\gamma-\varepsilon_p(r)}=\lambda_p(r)>1$. Thus, since $(-1)^{\nu\lambda_p(r)}=+1$, the order of $p^\nu\ (\operatorname{mod} 2^r)$ is $\lambda_p(r)$ in this case. If $r\leqslant t'$, $(-p)^\nu\equiv 1\ (\operatorname{mod} 2^r)$, and hence $p^\nu\equiv(-1)^\nu\ (\operatorname{mod} 2^r)$. We are assuming that $r\geqslant t$, and hence $r=t=t'$ when ν is even, and so the order of $p^\nu\ (\operatorname{mod} 2^r)$ is $\lambda_p(r)=1$. However when ν is odd, $p^\nu\equiv -1\ (\operatorname{mod} 2^r)$ and the order of $p^\nu\ (\operatorname{mod} 2^r)$ is 2.

When $t\geqslant 3$, so that $8\mid(p^\nu-1)$, Lemma 2′ gives us a simpler expression for the order of $p^\nu\ (\operatorname{mod} 2^r)$ than Lemma 3′; (the fact that these expressions are equivalent follows as in part II). We can also simplify Lemma 3′ when $t=1$ or 2 as we show in the following

Corollary. *If $t=2$ or $t=1$ and $p\equiv 3 \pmod 8$, then the order of $p^\nu \pmod{2^r}$ is 2^{r-2}. If $t=1$ and $p\equiv 7\pmod 8$, then the order of $p^\nu \pmod{2^r}$, that is $\lambda_p(r)$, is given by*

$$\lambda_p(r)=2 \quad \text{if} \quad 3\leqslant r\leqslant t' \quad \text{and} \quad \lambda_p(r)=2^{r-t'} \quad \text{if} \quad r>t'.$$

Proof. If $t=1$ or 2, then ν must be odd, so that $\gamma=0$; for if ν is even, $p^\nu\equiv 1\pmod 8$ and $t\geqslant 3$. If $t=2$, so that $4\,\|\,(p^\nu-1)$, then $p\equiv 5\pmod 8$, and if $t=1$ and $p\equiv 3\pmod 8$, then $p\equiv -5\pmod 8$; in either case $c_p(3)=1$, $\varepsilon_p(3)=0$ and $\lambda_p(3)=2$. By (7′)

$$c_p(r)\equiv c_p(3)\pmod 2$$

for $r\geqslant 3$, and hence $c_p(r)$ is odd, so that $\varepsilon_p(r)=0$. Thus by Lemma 3′, $\lambda_p(r)=2^{r-2}$.

If $t=1$, then $p\equiv 3\pmod 4$, and the only case left to consider is $p\equiv 7\pmod 8$. Since ν is odd and $2^{t'}\,\|\,(p^\nu+1)$, $2^{t'}\,\|\,(p+1)$, and hence $p\equiv -5^{2^{t'-2}}\pmod{2^{t'}}$, giving $c_p(t')=2^{t'-2}$ and $\varepsilon_p(t')=t'-2$. By the lemma, $\lambda_p(r)=2$ for $3\leqslant r\leqslant t'$. The argument used to deduce Lemma 2 from Lemma 3 can be used to show that $\lambda_p(t'+1)=2$, so that $\varepsilon_p(t'+1)=\varepsilon_p(t')=t'-2$ and hence if $r>t'$, $\varepsilon_p(r)=\varepsilon_p(t')=t'-2$. From Lemma 3′, it follows that

$$\lambda_p(r)=2^{r-t'} \quad \text{for} \quad r>t'.$$

We again define $\mu_p(r)$ to be the order of $p^\nu \pmod{2^{r+t}}$; then from above

$$(12')\qquad \mu_p(r)=\begin{cases} 2^r & \text{if} \quad r+t>t\geqslant 2,\\ 2^{r-1} & \text{if} \quad r\geqslant 2,\ t=1 \text{ and } p\equiv 3\pmod 8,\\ \lambda_p(r+1) & \text{if} \quad r\geqslant 2,\ t=1 \text{ and } p\equiv 7\pmod 8,\end{cases}$$

where $\lambda_p(r+1)$ is given by the Corollary. Note that if $t\geqslant 2$, $\mu_p=2$ and if $t=1$, $\mu_p(2)=2$; for completeness we define $\mu_p=2$ when $t=1$.

Lemma 4′. *If $t\geqslant 2$, then $\mu_p=2$ and for all $r\geqslant 1$, $\mu_p(r+1)=2\mu_p(r)$. If $t=1$ and $p\equiv 3\pmod 8$, then $\mu_p(2)=2$ and for all $r\geqslant 2$, $\mu_p(r+1)=2\mu_p(r)$. If $t=1$ and $p\equiv 7\pmod 8$, then for $2\leqslant r\leqslant t'$, $\mu_p(r)=2$ and for all $r\geqslant t'$, $\mu_p(r+1)=2\mu_p(r)$.*

This lemma follows from the definition of $\mu_p(r)$. We observe that when $t=1$ and $p\equiv 7\pmod 8$, $p\equiv 2^{t'}-1\pmod{2^{t'+1}}$ and $t'\geqslant 3$.

2′. The evaluation of $\sum_{a=1}^{\infty} a_r(p^a)$. Lemmas 5 and 6, parts (i) and (ii), hold without modification when $q=2$. The proof of Lemma 5 (iii) and (iv) is valid in most cases. However if $2\,\|\,(p^\nu-1)$, $a_1(p^a)=0$ for all a; for ν is odd and $p\equiv 3\pmod 4$ and so $\mu_p=\mu_p(2)=2$, and result follows from the proof of Lemma 5. Hence we have

Lemma 5′. (iii) *If $p\neq 2$, if $r\geqslant 2$ when $2\,\|\,(p^\nu-1)$ and $r\geqslant 1$ otherwise, and if $\mu_p(r+1)=2\mu_p(r)$, then $a_r(p^a)=1$ if and only if $a=u\mu_p(r)-1$ where $(u,2)=1$.*

(iv) *If $r\geqslant 1$ and if $p=2$ or $r=1$ when $2\,\|\,(p^\nu-1)$ or $\mu_p(r+1)=\mu_p(r)$, then $a_r(p^a)=0$ for all a.*

Lemma 6′. (iii) *If $p\neq 2$, if $r\geqslant 2$ when $2\,\|\,(p^\nu-1)$ and $r\geqslant 1$ otherwise, and if $\mu_p(r+1)=2\mu_p(r)$, then*

$$\sum_{a=1}^{\infty} a_r(p^a)p^{-as}=p^{-(\mu_p(r)-1)s}/(1-p^{-2\mu_p(r)s}).$$

(iv) *If $r\geqslant 1$ and if $p=2$ or $r=1$ when $2\,\|\,(p^\nu-1)$ or $\mu_p(r+1)=\mu_p(r)$, then*

$$\sum_{a=1}^{\infty} a_r(p^a)p^{-as}=0.$$

3′. The generating functions.

Lemma 7′.

$$f(s)=(1+2^{-s})\zeta(2s).$$

This follows from Lemma 7 since $\mu_p=2$. Lemma 8 continues to hold when $q=2$.

Lemma 9′.

$$f_m(s)=f(s)\sum_{R_k\in\mathfrak{R}(m)}\{\varrho^*(R_k)\}^{-1}\sum_{p_1}(Pp_1,\mu_{p_1}(r_1);s)\sum_{p_2}P(p_2,\mu_{p_2}(r_2);s)\ldots$$
$$\ldots\sum_{p_k}P(p_k,\mu_{p_k}(r_k);s),$$

where

$$P(p,\mu_p(r);s)=(1-p^{-2s})p^{-(\mu_p(r)-1)s}/(1-p^{-2\mu_p(r)s}),$$

and where the sum over p_i $(i=1,2,\ldots,k)$ is over all primes $p\in S(r_i)$ except $2,p_1,p_2,\ldots,p_{i-1}$ and, when ν is odd and $r_i=1$, those p satisfying $p\equiv 3\pmod 4$.

The notation is the same as in Lemma 9. The necessity of the additional condition needed when ν is odd in the sums involving $r_i=1$ arises from Lemma 6′ (iv); for if ν is odd and $p\equiv 3\pmod 4$, $2\,\|\,(p^\nu-1)$.

5′. Proof of Theorem 2 (iii). From Lemma 4′ we observe that

$$\mu_p(r) = 2 \quad \text{and} \quad \mu_p(r+1) = 2\mu_p(r)$$

do not both hold unless (i) ν is even and $r=1$, (ii) ν is odd, $r=1$ and $p \equiv 1 \pmod 4$, (iii) ν is odd, $r=2$ and $p \equiv 3 \pmod 8$ or (iv) ν is odd, $r \geqslant 3$ and $p \equiv 2^r - 1 \pmod{2^{r+1}}$; thus for every odd prime p there is exactly one value of r for which $p \in S_2(r)$. It follows that

$$(24) \qquad \sum_{p \in S_2(r)} p^{-s} = \begin{cases} \sum\limits_{p \neq 2} p^{-s} & \text{if } \nu \text{ is even and } r=1, \\ 0 & \text{if } \nu \text{ is even and } r>1, \\ \sum\limits_{p \equiv 2^r-1 \pmod{2^{r+1}}} p^{-s} & \text{if } \nu \text{ is odd and } r \geqslant 1. \end{cases}$$

As in § 5 of part II,

$$(18') \qquad \sum_{\substack{p_i \neq 2, p_1, \ldots, p_{i-1} \\ p_i \in S(r)}} P\big(p_i, \mu_{p_i}(r); s\big) = \sum_{p \in S_2(r)} p^{-s} + \psi_i(s),$$

where $\psi_i(s)$ is holomorphic for $\sigma > \frac12$ and bounded for $\sigma \geqslant \frac12 + \delta$ for any $\delta > 0$; we see from (24) that the sum on the right of (18′) may be empty, but it follows from Lemma 4′ that the sum on the left of (18′) is never empty.

Suppose that ν is even. It is well known that

$$\sum_{p \neq 2} p^{-s} = \log\zeta(s) - \sum_p \sum_{u=2}^{\infty} \frac{1}{up^{us}} - 2^{-s} = \log\zeta(s) + G(s)$$

(say), where $G(s)$ is holomorphic for $\sigma > \frac12$ and bounded for $\sigma \geqslant \frac12 + \delta$ for any $\delta > 0$. Hence from (24), (18′) and Lemmas 7′ and 9′ we obtain

$$f_m(s)$$
$$= (1+2^{-s})\zeta(2s) \sum_{j=0}^{m} \sum_{R_k^{(j)} \in \mathfrak{R}(m)} \{\varrho^*(R_k^{(j)})\}^{-1} \prod_{i=1}^{j} \{\log\zeta(s) + G(s) + \psi_i(s)\} \prod_{i=j+1}^{k} \psi_i(s),$$

where the set of $R_k^{(j)}$ is that subset of the set of R_k for which $r_i = 1$ for $1 \leqslant i \leqslant j$, and $r_i > 1$ for $j+1 \leqslant i \leqslant k$. It follows that

$$f_m(s) = \sum_{u=0}^{m} \{\log\zeta(s)\}^u H_u(s),$$

where $H_m(s) = \dfrac{1}{m!}(1+2^{-s})\zeta(2s)$, and where $H_u(s)$ $(0 \leqslant u \leqslant m)$ satisfies the conditions of Theorem 2 (iii).

Suppose now that ν is odd. Then by (20)

$$(20') \qquad \sum_{p \equiv 2^r-1 \pmod{2^{r+1}}} p^{-s}$$
$$= 2^{-r}\left\{\log\zeta(s) + \log(1-2^{-s}) + \sum_{\chi \neq \chi_0} \frac{\log L(s,\chi)}{\chi(2^r-1)} + \sum_{\chi} \frac{G(s,\chi)}{\chi(2^r-1)}\right\},$$

where χ runs through all characters $(\bmod\ 2^{r+1})$ except, when indicated, χ_0. Hence from (24), (18′) and (20′) and Lemmas 7′ and 9′,

$$f_m(s) = (1+2^{-s})\zeta(2s) \sum_{R_k \in \mathfrak{R}(m)} \{\varrho^*(R_k)\}^{-1} \prod_{i=1}^{k} \Bigg[2^{-r_i}\Bigg\{\log\zeta(s) + \log(1-2^{-s}) + $$
$$+ \sum_{\chi^{(r_i)} \neq \chi_0^{(r_i)}} \frac{\log L(s,\chi^{(r_i)})}{\chi^{(r_i)}(2^{r_i}-1)} + \sum_{\chi^{(r_i)}} \frac{G(s,\chi^{(r_i)})}{\chi^{(r_i)}(2^{r_i}-1)}\Bigg\} + \psi_i(s)\Bigg],$$

where $\chi^{(r_i)}$ is a character $(\bmod\ 2^{r_i+1})$. The maximum value of k is m and when $k=m$, $r_1 = r_2 = \ldots = r_m = 1$; thus the highest power of $\log\zeta(s)$ appearing on the right is $\{\log\zeta(s)\}^m$. Hence

$$f_m(s) = \sum_{u=0}^{m} \{\log\zeta(s)\}^u H_u(s),$$

where $H_m(s) = \dfrac{1}{m!} 2^{-m}(1+2^{-s})\zeta(2s)$, and $H_u(s)$ $(0 \leqslant u \leqslant m)$ satisfies the conditions of Theorem 2 (iii).

IV. Proof of Theorem 3

In part I we defined $h(s)$ to be a function which can be expressed both as an infinite sum of the form

$$h(s) = \sum_{n=1}^{\infty} b(n) n^{-s},$$

where $b(n) \geqslant 0$, and as a product of the form

$$h(s) = \{\zeta(s)\}^{1-\beta}\{\log\zeta(s)\}^u H(s),$$

where $0 < \beta \leqslant 1$, u is a non-negative integer, and $H(s)$ is a product of powers of Dirichlet L-functions associated with non-principal characters, non-negative powers of the logarithms of such functions, and a function holomorphic for $\sigma > \frac12$ and bounded for $\sigma \geqslant \frac12 + \delta$ $(\delta > 0)$. More precisely we can write $H(s)$ in the form

$$H(s) = \prod_{i=1}^{\lambda_1} \{\log L(s, \chi_1^{(i)})\}^{v_i} \prod_{i=1}^{\lambda_3} \{L(s, \chi_2^{(i)})\}^{w_i} \psi(s),$$

where the v_i, $i=1,2,\ldots,\lambda_1$, are non-negative integers, the w_i, $i=1,2,\ldots,\lambda_2$, are positive numbers, the w_i, $i=\lambda_2+1,\lambda_2+2,\ldots,\lambda_3$, are negative numbers, where the $\chi_j^{(i)}$, for $j=1,2$ and all i, are non-principal characters (mod $k_j^{(i)}$) and where $\psi(s)$ is holomorphic for $\sigma>\frac{1}{2}$ and bounded for $\sigma\geqslant\frac{1}{2}+\delta$ $(\delta>0)$.

The aim of part IV is to obtain an estimate for $\sum_{n=1}^{x}b(n)$. The method used to do this follows in principle the corresponding part of one of the methods used to prove the Prime Number Theorem (given, for example, in Landau [6]). Briefly, we integrate the function $x^s h(s)s^{-2}$ round a certain contour Γ inside and on which $h(s)$ is holomorphic in order to obtain, in Lemma 20, an estimate for

$$\sum_{n=1}^{x}b(n)\log(x/n).$$

The required result is deduced from this.

1. Preliminary lemmas. In the next two lemmas we shall state some properties of $\zeta(s)$ and of $L(s,\chi)$ which we shall need in order to determine the behaviour of $h(s)$. In these lemmas $c_1,c_2,\ldots$ denote positive constants, and in Lemma 14 these constants depend on the character χ occuring in the lemma.

LEMMA 13. (i) *$\zeta(s)-(s-1)^{-1}$ is holomorphic for $\sigma>0$.*

(ii) *There exists c_1 such that $\zeta(s)\neq 0$ for $\sigma\geqslant 1-c_1\{\log|t|\}^{-9}$, $|t|\geqslant 3$, and for $\sigma\geqslant 1-c_1\{\log 3\}^{-9}$, $|t|\leqslant 3$.*

(iii) *There exists c_2 such that*

$$|\zeta(s)|<c_2\log|t|$$

for $\sigma\geqslant 1-\{\log|t|\}^{-1}$, $|t|\geqslant 3$, and c_3 such that

$$|\log\zeta(s)|<c_3\{\log|t|\}^9$$

for $\sigma\geqslant 1-c_1\{\log|t|\}^{-9}$, $|t|\geqslant 3$.

(iv) *There exist c_4, c_5 and c_6 such that*

$$|\zeta(s)|<c_4 \quad \text{and} \quad |\log\zeta(s)|<c_5$$

for $1-c_1\{\log 3\}^{-9}\leqslant\sigma\leqslant 1-c_6<1$, $|t|\leqslant 3$.

The properties given in parts (i), (ii) and (iii) of the lemma are contained in § 42 to § 48 and § 64 of Landau [6]; part (iv) is an immediate consequence of the rest of the lemma.

LEMMA 14. *Let χ be a non-principal character* (mod k); *then:*

(i) *$L(s,\chi)$ is holomorphic for $\sigma>0$.*

(ii) *There exist c_7, c_8, c_9 and c_{10} such that*

$$|L(s,\chi)|<c_7\log|t|$$

for $\sigma\geqslant 1-\{\log|t|\}^{-1}$, $|t|\geqslant 3$, and

$$|L(s,\chi)|>c_8\{\log|t|\}^{-5} \quad \text{and} \quad |\log L(s,\chi)|<c_9\{\log|t|\}^7$$

for $\sigma\geqslant 1-c_{10}\{\log|t|\}^{-7}$, $|t|\geqslant 3$.

(iii) *There exist c_{11}, c_{12} and c_{13} such that*

$$0<c_{11}<|L(s,\chi)|<c_{12} \quad \text{and} \quad |\log L(s,\chi)|<c_{13}$$

for $1-c_{10}\{\log 3\}^{-7}\leqslant\sigma\leqslant 1$, $|t|\leqslant 3$.

With the exception of the bound for $|\log L(s,\chi)|$, the properties given in parts (i) and (ii) of the lemma are contained in § 114 to § 117 of Landau [6]. The bound for $|\log L(s,\chi)|$ can be deduced from that of $|L'(s,\chi)/L(s,\chi)|$ (which is $c_{14}\{\log|t|\}^7$, as is given in § 117 of Landau [6]) in the same way as the bound for $|\log\zeta(s)|$ is deduced from that of $|\zeta'(s)/\zeta(s)|$ in § 64 of Landau [6]. Part (iii) is an immediate consequence of the rest of the lemma. We observe that the lower bound for $|L(s,\chi)|$ implies that $L(s,\chi)\neq 0$. It is known that the powers of $\log|t|$ appearing in Lemmas 13 and 14 may be replaced by numerically smaller powers, but no advantage would be gained by using this development in this paper.

The next lemma follows immediately from Lemmas 13 and 14 and the definition of $h(s)$. We observe that, if $\sigma\geqslant\frac{1}{2}+\delta$ $(\delta>0)$, $|\psi(s)|<c_{15}$ since $\psi(s)$ is bounded. For suitable positive constants d_1, d_2, d_3, we have

LEMMA 15. (i) *The function $h(s)$ is holomorphic for $\sigma\geqslant 1-d_1\{\log|t|\}^{-9}$, $|t|\geqslant 3$ and for $\sigma\geqslant 1-d_1\{\log 3\}^{-9}$, $|t|\leqslant 3$ except for a singularity at $s=1$.*

(ii) *$|h(s)|<d_2\{\log|t|\}^k$ for $\sigma\geqslant 1-d_1\{\log|t|\}^{-9}$, $|t|\geqslant 3$, where $k>0$, and $|h(s)|<d_3$ for $\sigma=1-d_1\{\log 3\}^{-9}$, $|t|\leqslant 3$.*

It follows from Lemmas 13 and 14 and the definition of $h(s)$ that we may take

$$k=(1-\beta)+9u+\sum_{i=1}^{\lambda_1}7v_i+\sum_{i=1}^{\lambda_2}w_i+\sum_{i=\lambda_2+1}^{\lambda_3}5\,|w_i|,$$

and that the constants d_2 and d_3 are products of the constants c. The constant d_1 must be chosen so that all parts of Lemma 13 and, for all characters χ appearing in the definition of $h(s)$, all parts of Lemma 14 are applicable in the corresponding regions of Lemma 15.

LEMMA 16. *If* $|s-1| \leqslant d_1\{\log 3\}^{-9}$,

$$h(s)s^{-2}-H(1)(s-1)^{\beta-1}\{-\log(s-1)\}^u = \sum_{j=0}^{u}\binom{u}{j}\{-\log(s-1)\}^{u-j}(s-1)^{\beta}\sum_{k=1}^{\infty}\omega_{jk}(s-1)^{k-1},$$

where the ω_{jk} *are constants, and* $\sum_{k=1}^{\infty}\omega_{jk}(s-1)^{k-1}$ *is convergent for each* j.

Proof. By Lemma 13 (i), $(s-1)\zeta(s)$ is holomorphic for $\sigma > 0$, and

$$\lim_{s\to 1}(s-1)\zeta(s) = 1; \tag{25}$$

also, by Lemma 13 (ii), it is certainly true that

$$\zeta(s) \neq 0$$

if $|s-1| \leqslant d_1\{\log 3\}^{-9}$. Hence $K(s) = \log\{\zeta(s)(s-1)\}$ is holomorphic when $|s-1| \leqslant d_1\{\log 3\}^{-9}$, and

$$\lim_{s\to 1}K(s) = 0. \tag{26}$$

Now

$$h(s)s^{-2} = \{\zeta(s)\}^{1-\beta}\{\log\zeta(s)\}^u H(s)s^{-2},$$

where $H(s)s^{-2}$ is holomorphic and bounded when $|s-1| \leqslant d_1\{\log 3\}^{-9}$ by Lemma 14. From above we may write

$$h(s)s^{-2} = (s-1)^{\beta-1}\{-\log(s-1)+K(s)\}^u\{(s-1)\zeta(s)\}^{1-\beta}H(s)s^{-2}$$

$$= (s-1)^{\beta-1}\sum_{j=0}^{u}\binom{u}{j}\{-\log(s-1)\}^{u-j}\{K(s)\}^j\{(s-1)\zeta(s)\}^{1-\beta}H(s)s^{-2}.$$

For all j, $\{K(s)\}^j\{(s-1)\zeta(s)\}^{1-\beta}H(s)s^{-2}$ is holomorphic when $|s-1| \leqslant d_1\{\log 3\}^{-9}$, and hence it can be expanded as a (convergent) power series in the form

$$\sum_{k=0}^{\infty}\omega_{jk}(s-1)^k.$$

From (25) and (26) we have that

$$\omega_{j0} = \lim_{s\to 1}\{K(s)\}^j\{(s-1)\zeta(s)\}^{1-\beta}H(s)s^{-2} = 0$$

for all $j \geqslant 1$, and that

$$\omega_{00} = \lim_{s\to 1}\{(s-1)\zeta(s)\}^{1-\beta}H(s)s^{-2} = H(1).$$

Hence

$$h(s)s^{-2} = H(1)(s-1)^{\beta-1}\{-\log(s-1)\}^u + (s-1)^{\beta-1}\sum_{j=0}^{u}\binom{u}{j}\{-\log(s-1)\}^{u-j}\sum_{k=1}^{\infty}\omega_{jk}(s-1)^k,$$

which gives the result of the lemma.

2. An estimate for $\sum_{n=1}^{x} b(n)\log(x/n)$.

LEMMA 17.

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} y^s s^{-2}\,ds = \begin{cases} 0 & \text{if} \quad 0 < y \leqslant 1, \\ \log y & \text{if} \quad y \geqslant 1. \end{cases}$$

This is proved in § 49 of Landau [6].

LEMMA 18.

$$\sum_{n=1}^{x} b(n)\log(x/n) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^s h(s)s^{-2}\,ds.$$

Proof.

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^s h(s)s^{-2}\,ds = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} s^{-2}\sum_{n=1}^{\infty} b(n)(x/n)^s\,ds$$

$$= \frac{1}{2\pi i}\sum_{n=1}^{\infty} b(n)\int_{2-i\infty}^{2+i\infty}(x/n)^s s^{-2}\,ds = \sum_{n=1}^{x} b(n)\log(x/n)$$

by Lemma 17.

Our next aim is to estimate the integral appearing on the right in Lemma 18. To do this we cut the complex plane along the real axis from the point $s = 1$ to the left. Let Γ be the contour $\bar{A}ABCDE\bar{E}\bar{D}\bar{C}\bar{B}\bar{A}$, where the vertices above the real axis are defined by $A = 2+ix^2$, $B = 1-d_1\{\log x^2\}^{-9}+ix^2$, $C = 1-d_1\{\log 3\}^{-9}+3i$, $D = 1-d_1\{\log 3\}^{-9}$, $E = 1-\delta$ for a small, positive δ (which will tend to zero later), and where $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{E}$ are the complex conjugates of A, B, C, D, E; curves joining neighbouring vertices are straight lines except for BC which is the curve $\sigma = 1-d_1\{\log t\}^{-9}$ ($x^2 \geqslant t \geqslant 3$), $\bar{B}\bar{C}$ which is the image in the real axis of BC, and $E\bar{E}$ which is the circle $|s-1| = \delta$. The constant d_1 has been chosen so that $h(s)$ is holomorphic in the region bounded by Γ; this follows from Lemma 15 (i). Hence by Cauchy's Theorem

$$\int_{\Gamma} x^s h(s)s^{-2}\,ds = 0,$$

so that

(27) $$\int_{\bar{A}A} x^s h(s) s^{-2} ds = -\int_{ABCDE\bar{E}\bar{D}\bar{C}\bar{B}\bar{A}} x^s h(s) s^{-2} ds.$$

Let $\Sigma_1(x) = \sum_{n=1}^{x} b(n)\log(x/n)$.

LEMMA 19.
$$\Sigma_1(x) = -\frac{1}{2\pi i}\Big\{\int_{DE} + \int_{\bar{E}\bar{D}}\Big\} x^s h(s) s^{-2} ds + O(xe^{-(\log x)^{1/11}}).$$

Proof. From Lemma 18 and (27) we obtain

(28) $$\Sigma_1(x) = \frac{1}{2\pi i}\Big\{\int_{2-i\infty}^{2-ix^2} + \int_{\bar{A}A} + \int_{2+ix^2}^{2+i\infty}\Big\} x^s h(s) s^{-2} ds$$
$$= \frac{1}{2\pi i}\Big\{\int_{2-i\infty}^{2-ix^2} - \int_{ABCDE\bar{E}\bar{D}\bar{C}\bar{B}\bar{A}} + \int_{2+ix^2}^{2+i\infty}\Big\} x^s h(s) s^{-2} ds.$$

We now show that all the integrals on the right except those along DE and $\bar{E}\bar{D}$ are sufficiently small in absolute value to be included in the error term of the lemma.

(i) By Lemma 15 (ii),
$$\Big|\int_{2+ix^2}^{2+i\infty} x^s h(s) s^{-2} ds\Big| \leqslant \int_{x^2}^{\infty} x^2 |h(2+it)|\,|2+it|^{-2} dt < x^2 \int_{x^2}^{\infty} d_2 (\log t)^k t^{-2} dt$$
$$< x^2 \int_{x^2}^{\infty} d_2 t^{-2+\varepsilon} dt = O(x^{2\varepsilon})$$
for any small $\varepsilon > 0$.

(ii) Since $|AB| < 2$, we have by Lemma 15 (ii) that
$$\Big|\int_{AB} x^s h(s) s^{-2} ds\Big| < x^2 d_2 (\log x^2)^k x^{-4} 2 = O(x^{-2+\varepsilon})$$
for any small $\varepsilon > 0$.

(iii) On BC, $\sigma = 1 - d_1\{\log t\}^{-9}$ and $x^2 \geqslant t \geqslant 3$; hence by Lemma 15 (ii),
$$\Big|\int_{BC} x^s h(s) s^{-2} ds\Big| < \int_{3}^{x^2} x^{1-d_1(\log t)^{-9}} d_2 (\log t)^k t^{-2} |9d_1(\log t)^{-10} t^{-1} + i|\, dt$$
$$= O\Big(x(\log x)^k \int_{3}^{x^2} x^{-d_1(\log t)^{-9}} t^{-2} dt\Big).$$

Let $y = \exp\{(\log x)^{1/10}\}$; then
$$\int_{3}^{x^2} x^{-d_1(\log t)^{-9}} t^{-2} dt = \Big\{\int_{3}^{y} + \int_{y}^{x^2}\Big\} x^{-d_1(\log t)^{-9}} t^{-2} dt$$
$$= O(x^{-d_1(\log y)^{-9}}) + O\Big(\int_{y}^{x^2} t^{-2} dt\Big) = O(e^{-d_1(\log x)^{1/10}}) + O(y^{-1}).$$

Therefore
$$\Big|\int_{BC} x^s h(s) s^{-2} ds\Big| = O\big(x(\log x)^k \{e^{-d_1(\log x)^{1/10}} + e^{-(\log x)^{1/10}}\}\big) = O(xe^{-(\log x)^{1/11}}).$$

(iv) By Lemma 15 (ii)
$$\Big|\int_{CD} x^s h(s) s^{-2} ds\Big| = O(x^{1-d_1(\log 3)^{-9}}).$$

(v) By Lemma 16
$$\Big|\int_{E\bar{E}} x^s h(s) s^{-2} ds\Big| = O(x^{1+\delta} |\log\delta|^u \delta^{\beta-1} 2\pi\delta);$$
since $\beta > 0$,
$$\lim_{\delta\to 0}\{x^{1+\delta}|\log\delta|^u \delta^\beta\} = 0,$$
and thus
$$\lim_{\delta\to 0}\Big\{\int_{E\bar{E}} x^s h(s) s^{-2} ds\Big\} = 0.$$

By symmetry the bounds for the integrals along curves in the lower half plane are the same as the bounds for the corresponding integral (i), (ii), (iii) or (iv) in the upper half plane. The result of the lemma now follows from (28) and (i) to (v).

LEMMA 20. (i) *If* $0 < \beta < 1$ *and* $u \geqslant 1$, *then*
$$\Sigma_1(x) = \frac{H(1)}{\Gamma(1-\beta)} x(\log\log x)^u (\log x)^{-\beta} + O\big(x(\log\log x)^{u-1}(\log x)^{-\beta}\big).$$

(ii) *If* $0 < \beta < 1$ *and* $u = 0$, *then*
$$\Sigma_1(x) = \frac{H(1)}{\Gamma(1-\beta)} x(\log x)^{-\beta} + O\big(x(\log x)^{-1}\big).$$

(iii) *If* $\beta = 1$ *and* $u \geqslant 2$, *then*
$$\Sigma_1(x) = uH(1)x(\log\log x)^{u-1}(\log x)^{-1} + O\big(x(\log\log x)^{u-2}(\log x)^{-1}\big).$$

(iv) *If* $\beta = 1$ *and* $u = 1$, *then*
$$\Sigma_1(x) = H(1)x(\log x)^{-1} + O\big(x\log\log x(\log x)^{-2}\big).$$

(v) *If* $\beta = 1$ *and* $u = 0$, *then*

$$\Sigma_1(x) = O\big(x(\log x)^{-2}\big).$$

Proof. Let $\theta = 1-d_1(\log 3)^{-9}$. Suppose first that β and u satisfy the conditions of (i), (ii) or (iii). If $\theta \leqslant s \leqslant 1$, then by Lemma 16

$$\begin{aligned}|x^s h(s)s^{-2} - H(1)x^s\{-\log(s-1)\}^u(s-1)^{\beta-1}| &= O\Big(x^s\sum_{j=0}^{u}|\{\log(s-1)\}^{u-j}(s-1)^{\beta}|\,\Big|\sum_{k=1}^{\infty}\omega_{jk}(s-1)^{k-1}\Big|\Big)\\ &= O\Big(x^s\sum_{j=0}^{u}|\{\log(s-1)\}^{u-j}(s-1)^{\beta}|\Big)\end{aligned}$$

since $\sum_{k=1}^{\infty}\omega_{jk}(s-1)^{k-1}$ is convergent. When $\theta \leqslant s \leqslant 1$, $|\{\log(s-1)\}^{u-j}(s-1)^{\beta}| = O(1)$ since $\beta > 0$; hence

$$\int_{\theta}^{1} x^s\sum_{j=0}^{u}|\{\log(s-1)\}^{u-j}(s-1)^{\beta}|\,ds = O\Big(\int_{\theta}^{1}x^s\,ds\Big) = O\big(x(\log x)^{-1}\big).$$

Hence

$$\begin{aligned}\Big\{\int_{DE} + \int_{\bar{E}\bar{D}}\Big\}x^s h(s)s^{-2}\,ds &= \int_{\theta}^{1} H(1)x^{s^+}\{-\log(s^+-1)\}^u(s^+-1)^{\beta-1}\,ds^+ -\\ &\quad -\int_{\theta}^{1} H(1)x^{s^-}\{-\log(s^--1)\}^u(s^--1)^{\beta-1}\,ds^- + O\big(x(\log x)^{-1}\big),\end{aligned}$$

where s^+ and s^- indicate the upper edge and lower edge respectively of the cut. Since $(s^+-1) = (1-s^+)e^{\pi i}$ and $(s^--1) = (1-s^+)e^{-\pi i}$, it follows that

$$\begin{aligned}(29)\quad \Big\{\int_{DE} + \int_{\bar{E}\bar{D}}\Big\}x^s h(s)s^{-2}\,ds &= H(1)\int_{\theta}^{1} x^{s^+}(1-s^+)^{\beta-1}\big[\{-\log(1-s^+)-\pi i\}^u e^{\pi i(\beta-1)} -\\ &\quad -\{-\log(1-s^+)+\pi i\}^u e^{-\pi i(\beta-1)}\big]\,ds^+ + O\big(x(\log x)^{-1}\big)\\ &= H(1)\int_{\theta}^{1} x^s(1-s)^{\beta-1}\sum_{m=0}^{u}\binom{u}{m}\{-\log(1-s)\}^{u-m}\{(-\pi i)^m e^{\pi i(\beta-1)} -\\ &\quad -(\pi i)^m e^{-\pi i(\beta-1)}\}\,ds + O\big(x(\log x)^{-1}\big)\\ &= H(1)\sum_{m=0}^{u}\binom{u}{m}\{(-\pi i)^m e^{\pi i(\beta-1)} - (\pi i)^m e^{-\pi i(\beta-1)}\}\int_{\theta}^{1}x^s(1-s)^{\beta-1}\times\\ &\quad \times\{-\log(1-s)\}^{u-m}\,ds + O\big(x(\log x)^{-1}\big)\end{aligned}$$

on writing s for s^+.

Assume now that the conditions of (i) are satisfied, so that $0 < \beta < 1$ and $u \geqslant 1$, and consider the integral

$$I = \int_{\theta}^{1} x^s(1-s)^{\beta-1}\{-\log(1-s)\}^k\,ds$$

where $0 \leqslant k \leqslant u$. On using the substitution $s = 1-\dfrac{\eta}{\log x}$, we obtain

$$\begin{aligned}I &= x(\log x)^{-\beta}\int_{0}^{(1-\theta)\log x} e^{-\eta}\eta^{\beta-1}\{\log\log x - \log\eta\}^k\,d\eta\\ &= x(\log x)^{-\beta}\sum_{r=0}^{k}(-1)^r\binom{k}{r}(\log\log x)^{k-r}\int_{0}^{(1-\theta)\log x} e^{-\eta}\eta^{\beta-1}(\log\eta)^r\,d\eta.\end{aligned}$$

Now

$$\int_{0}^{(1-\theta)\log x} e^{-\eta}\eta^{\beta-1}(\log\eta)^r\,d\eta = \int_{0}^{\infty} e^{-\eta}\eta^{\beta-1}(\log\eta)^r\,d\eta + O\big((\log x)^{\beta-1+\varepsilon}x^{-1+\theta}\big)$$

for any ε satisfying $0 < \varepsilon < 1-\beta$. The integral on the right is absolutely convergent for all r, and in particular when $r = 0$ its value is $\Gamma(\beta)$. Hence

$$(30)\qquad I = \Gamma(\beta)x(\log\log x)^k(\log x)^{-\beta} + O\big(x(\log\log x)^{k-1}(\log x)^{-\beta}\big)$$

unless $k = 0$, in which case the error term is $O\big(x^{\theta}(\log x)^{-1}\big)$. Hence by (29)

$$\begin{aligned}&\Big\{\int_{DE} + \int_{\bar{E}\bar{D}}\Big\}x^s h(s)s^{-2}\,ds\\ &= H(1)\Gamma(\beta)2i\sin\pi(\beta-1)x(\log\log x)^u(\log x)^{-\beta} + O\big(x(\log\log x)^{u-1}(\log x)^{-\beta}\big)\\ &= \frac{-2\pi i H(1)}{\Gamma(1-\beta)}x(\log\log x)^u(\log x)^{-\beta} + O\big(x(\log\log x)^{u-1}(\log x)^{-\beta}\big)\end{aligned}$$

since $\Gamma(\beta)\Gamma(1-\beta) = \pi/\sin\pi\beta$. Part (i) of the lemma now follows from Lemma 19.

Similarly in case (ii), when $0 < \beta < 1$ and $u = 0$, the integral I is given by

$$I = \int_{\theta}^{1} x^s(1-s)^{\beta-1}\,ds = \Gamma(\beta)x(\log x)^{-\beta} + O\big(x^{\theta}(\log x)^{-1}\big).$$

As above it follows that

$$\Big\{\int_{DE} + \int_{\bar{E}\bar{D}}\Big\}x^s h(s)s^{-2}\,ds = \frac{-2\pi i H(1)}{\Gamma(1-\beta)}x(\log x)^{-\beta} + O\big(x(\log x)^{-1}\big),$$

and on using Lemma 19 we obtain the result of Lemma 20 (ii).

We turn now to case (iii), so that $\beta = 1$ and $u \geqslant 2$. Then (29) becomes

$$\Big\{\int_{DE} + \int_{\bar{E}\bar{D}}\Big\} x^s h(s) s^{-2} ds$$
$$= H(1) \sum_{m=0}^{u} \binom{u}{m} \{(-\pi i)^m - (\pi i)^m\} \int_0^1 x^s \{-\log(1-s)\}^{u-m} ds + O\big(x(\log x)^{-1}\big);$$

we observe that the term corresponding to $m = 0$ is zero. Now $\Gamma(1) = 1$, and hence by (30)

$$I = \int_0^1 x^s \{-\log(1-s)\}^k ds$$
$$= x(\log\log x)^k (\log x)^{-1} + O\big(x(\log\log x)^{k-1}(\log x)^{-1}\big)$$

unless $k = 0$, in which case

$$I = (x - x^0)/\log x.$$

Hence

$$\Big\{\int_{DE} + \int_{\bar{E}\bar{D}}\Big\} x^s h(s) s^{-2} ds$$
$$= -2\pi i u H(1) x (\log\log x)^{u-1} (\log x)^{-1} + O\big(x(\log\log x)^{u-2}(\log x)^{-1}\big),$$

and the result of part (iii) follows from Lemma 19.

Assume now that the conditions of case (iv) hold, so that $\beta = 1$ and $u = 1$. In this case the function to be integrated round Γ is the product of the holomorphic function $H(s)$ and $x^s s^{-2} \log\zeta(s)$, the function integrated round a similar contour in one proof (given in § 64 of Landau [6]) of the Prime Number Theorem. Proceeding as above, we obtain from Lemma 16

$$|x^s h(s) s^{-2} - H(1) x^s \{-\log(s-1)\}| = O\big(x^s |s-1| |\log(s-1)|\big).$$

Hence, since

$$\int_0^1 x^s |s-1| |\log(s-1)| ds = O\big(x \log\log x (\log x)^{-2}\big),$$

which is proved by using the substitution $s = 1 - \dfrac{\eta}{\log x}$,

$$\Big\{\int_{DE} + \int_{\bar{E}\bar{D}}\Big\} x^s h(s) s^{-2} ds = \int_0^1 H(1) x^{s^+} \{-\log(s^+ - 1)\} ds^+ -$$
$$- \int_0^1 H(1) x^{s^-} \{-\log(s^- - 1)\} ds^- + O\big(x \log\log x (\log x)^{-2}\big)$$
$$= -2\pi i H(1) x (\log x)^{-1} + O\big(x \log\log x (\log x)^{-2}\big).$$

The result of (iv) now follows from Lemma 19.

For part (v) of the lemma, $\beta = 1$ and $u = 0$ and so $h(s) = H(s)$. Hence $h(s)$ is holomorphic inside the contour $\bar{A}ABCD\bar{D}\bar{C}\bar{B}\bar{A}$, where the complex plane is no longer cut so that $\bar{D} = D$, and where the rest of the contour is the same shape as the corresponding part of Γ. Using the results of Lemma 19 and integrating round this contour we obtain

$$\Sigma_1(x) = O\big(x e^{-(\log x)^{1/11}}\big) = O\big(x(\log x)^{-2}\big)$$

which is (v). This completes the proof of Lemma 20.

3. Proof of Theorem 3. Let $\Sigma_2(x) = \sum_{n=1}^{x} b(n)$. Then we have

LEMMA 21. *Suppose that*

(31) $$\Sigma_1(x) = Bx^\alpha (\log\log x)^\gamma (\log x)^{-\beta} + O\big(x^\alpha (\log\log x)^{\gamma_1} (\log x)^{-\beta_1}\big),$$

where $B, \alpha, \beta, \gamma, \beta_1, \gamma_1$ *are non-negative constants and* $B \neq 0$, $\alpha \neq 0$, *and where either* $\gamma_1 < \gamma$ *and* $\beta \leqslant \beta_1 < \beta + 2$ *or* $\gamma_1 \geqslant \gamma$ *and* $\beta < \beta_1 \leqslant \beta + 2$. *Then*

$$\Sigma_2(x) = B\alpha x^\alpha (\log\log x)^\gamma (\log x)^{-\beta} + O\big(x^\alpha (\log\log x)^{\frac12(\gamma+\gamma_1)} (\log x)^{-\frac12(\beta+\beta_1)}\big).$$

Proof. Let $\delta = \delta(x) = o(1)$ be a positive function of x to be chosen later, and suppose that $x(1+\delta)$ is an integer. Then, since

$$\log x(1+\delta) = \log x + O(\delta) \quad \text{and} \quad \log\log x(1+\delta) = \log\log x + O\big(\delta(\log x)^{-1}\big),$$

(32) $$\Sigma_1\big(x(1+\delta)\big) = B(1+\delta)^\alpha x^\alpha (\log\log x)^\gamma (\log x)^{-\beta} \times$$
$$\times \{1 + O\big(\delta(\log x \log\log x)^{-1}\big) + O\big(\delta(\log x)^{-1}\big)\} +$$
$$+ O\big(x^\alpha (\log\log x)^{\gamma_1} (\log x)^{-\beta_1}\big)$$
$$= Bx^\alpha (\log\log x)^\gamma (\log x)^{-\beta} \{1 + \alpha\delta + O(\delta^2) + O\big(\delta(\log x)^{-1}\big)\} +$$
$$+ O\big(x^\alpha (\log\log x)^{\gamma_1} (\log x)^{-\beta_1}\big).$$

By definition

(33) $$\Sigma_1\big(x(1+\delta)\big) - \Sigma_1(x) = \sum_{n=1}^{x(1+\delta)} b(n) \log x(1+\delta)/n - \sum_{n=1}^{x} b(n) \log x/n$$
$$= \log(1+\delta) \sum_{n=1}^{x} b(n) + \sum_{n=x+1}^{x(1+\delta)} b(n) \log x(1+\delta)/n$$
$$\geqslant \log(1+\delta)\, \Sigma_2(x)$$

since the second sum is not negative. Similarly

(34) $$\Sigma_1\big(x(1+\delta)\big) - \Sigma_1(x) = \log(1+\delta) \sum_{n=1}^{x(1+\delta)} b(n) + \sum_{n=x+1}^{x(1+\delta)} b(n) \log x/n$$
$$\leqslant \log(1+\delta)\, \Sigma_2\big(x(1+\delta)\big)$$

since the second sum is not positive.

By (31), (32) and (33)

(35) $$\Sigma_2(x) \leqslant \{\Sigma_1(x(1+\delta))-\Sigma_1(x)\}/\log(1+\delta) = \{\Sigma_1(x(1+\delta))-\Sigma_1(x)\}\{1+O(\delta)\}\delta^{-1} = Bx^a(\log\log x)^\gamma(\log x)^{-\beta}\{a+O(\delta)+O((\log x)^{-1})+ +O((\log\log x)^{\gamma_1-\gamma}\delta^{-1}(\log x)^{-\beta_1+\beta})\}.$$

By (31), (32) and (34)

(36) $$\Sigma_2(x(1+\delta)) \geqslant \{\Sigma_1(x(1+\delta))-\Sigma_1(x)\}/\log(1+\delta) = Bx^a(\log\log x)^\gamma(\log x)^{-\beta}\{a+O(\delta)+O((\log x)^{-1})+ +O((\log\log x)^{\gamma_1-\gamma}\delta^{-1}(\log x)^{-\beta_1+\beta})\}.$$

If we replace x by $x/(1+\delta)$ in (36), we obtain

(37) $$\Sigma_2(x) \geqslant Bx^a(\log\log x)^\gamma(\log x)^{-\beta}\{a+O(\delta)+O((\log x)^{-1})+ +O((\log\log x)^{\gamma_1-\gamma}\delta^{-1}(\log x)^{-\beta_1+\beta})\}.$$

We now choose δ so that all the error terms of (35) and (37) are of a smaller order of magnitude than the first term; since $\beta \leqslant \beta_1 \leqslant \beta+2$, we can take $\delta = x^{-1}[x\delta']$, where

$$\delta' = (\log\log x)^{\frac{1}{2}(\gamma_1-\gamma)}(\log x)^{-\frac{1}{2}(\beta_1-\beta)},$$

and then the error terms of (35) and (37) are

$$O(x^a(\log\log x)^\gamma(\log x)^{-\beta}(\log\log x)^{\frac{1}{2}(\gamma_1-\gamma)}(\log x)^{-\frac{1}{2}(\beta_1-\beta)}) = O(x^a(\log\log x)^{\frac{1}{2}(\gamma+\gamma_1)}(\log x)^{-\frac{1}{2}(\beta+\beta_1)}).$$

Hence from (35) and (37) it follows that

$$\Sigma_2(x) = Bax^a(\log\log x)^\gamma(\log x)^{-\beta}+O(x^a(\log\log x)^{\frac{1}{2}(\gamma+\gamma_1)}(\log x)^{-\frac{1}{2}(\beta+\beta_1)}),$$

which is the result of the lemma.

We observe that if $\beta_1 > \beta+2$, the result of the lemma holds provided that we replace the error term by

$$O(x^a(\log\log x)^\gamma(\log x)^{-\beta-1}).$$

Corollary. *If* $\Sigma_1(x) = O(x(\log x)^{-2})$, *then* $\Sigma_2(x) = O(x(\log x)^{-3/2})$.

Proof. By the above method, we can show that

$$\Sigma_2(x) = O(x(\log x)^{-2}\delta^{-1}) = O(x(\log x)^{-3/2})$$

if we choose $\delta = x^{-1}[x\delta']$ where $\delta' = (\log x)^{-1/2}$.

We can now deduce Theorem 3 from Lemmas 20 and 21. If we take

(i) $a = 1,\ \gamma = u \geqslant 1,\ \gamma_1 = u-1,\ \beta_1 = \beta < 1,$

(ii) $a = 1,\ \gamma = \gamma_1 = u = 0,\ \beta < \beta_1 = 1,$

(iii) $a = 1,\ \gamma = u-1 \geqslant 1,\ \gamma_1 = u-2,\ \beta = \beta_1 = 1,$

(iv) $a = 1,\ \gamma = u-1 = 0,\ \gamma_1 = 1,\ \beta = 1,\ \beta_1 = 2$

in Lemma 21, the estimates for $\Sigma_1(x)$ appearing in the statement of this lemma being those given in the corresponding part of Lemma 20, then we obtain, in turn, the first four parts of Theorem 3. We obtain the last part of Theorem 3 from Lemma 20 (v) and the Corollary to Lemma 21.

V. Some deductions

1. Proof of Theorem 1 (ii) and (iii). If q is odd and h is even, then by Theorem 2 (ii),

$$f_m(s) = \{\zeta(s)\}^{1-1/h}\sum_{u=0}^{m'}\{\log\zeta(s)\}^u H_u(s),$$

where each $H_u(s)$ $(0 \leqslant u \leqslant m')$ is a sum of functions satisfying the conditions imposed on $H(s)$ in Theorem 3. Hence, if $m' \geqslant 1$, we have from Theorem 3 (i) and (ii) (with $\beta = 1/h < 1$) that

$$D_m(\nu, q; x) = \sum_{n=1}^{x} a_m(n) = \frac{H_{m'}(1)}{\Gamma(1-1/h)}x(\log\log x)^{m'}(\log x)^{-1/h}+ +O(x(\log\log x)^{m'-\frac{1}{2}}(\log x)^{-1/h}),$$

where the constant $H_{m'}(1)$ can be obtained from (22) and Lemma 12. Similarly if $m' = 0$ we have from Theorem 3 (ii) that

$$D_m(\nu, q; x) = \frac{H_0(1)}{\Gamma(1-1/h)}x(\log x)^{-1/h}+O(x(\log x)^{-\frac{1}{2}(1+1/h)}),$$

where $H_0(1)$ may be obtained from the end of § 5 and Lemma 12. This proves Theorem 1 (ii).

If $q = 2$, then by Theorem 2 (iii),

$$f_m(s) = \sum_{u=0}^{m}\{\log\zeta(s)\}^u H_u(s),$$

where each $H_u(s)$ $(0 \leqslant u \leqslant m)$ is a sum of functions satisfying the conditions imposed on $H(s)$ in Theorem 3. Hence if $m \geqslant 2$ we have from Theorem 3 (iii), (iv) and (v) that

$$D_m(\nu, 2; x) = mH_m(1)x(\log\log x)^{m-1}(\log x)^{-1}+O(x(\log\log x)^{m-3/2}(\log x)^{-1}),$$

and if $m = 1$ we have from Theorem 3 (iv) and (v) that

$$D_1(\nu, 2; x) = H_1(1)x(\log x)^{-1}+O(x(\log\log x)^{1/2}(\log x)^{-3/2});$$

in either case

$$H_m(1) = \begin{cases} \frac{1}{m!}(1+2^{-1})\zeta(2) = \frac{\pi^2}{4}\cdot\frac{1}{m!} & \text{if } \nu \text{ is even},\\ \frac{1}{2^m m!}(1+2^{-1})\zeta(2) = \frac{\pi^2}{4}\cdot\frac{1}{2^m m!} & \text{if } \nu \text{ is odd}.\end{cases}$$

The result of Theorem 1 (iii) now follows.

2. An asymptotic expression for $N(\nu, q^m; x)$. We have already seen in equation (3) that

$$N(\nu, q^m; x) = \sum_{r=0}^{m-1} D_r(\nu, q; x), \tag{38}$$

and that $D_0(\nu, q; x) = N(\nu, q; x)$ is given by (2). Let $l = [(m-1)/(\gamma+1)]$ and assume that $m \geqslant 2$; then we have

COROLLARY 1. *As $x \to \infty$*

$$N(\nu, q^m; x) \sim \begin{cases} B_1^{(m)} x & \textit{if } q \textit{ and } h \textit{ are both odd},\\ B_2^{(m)} x (\log\log x)^l (\log x)^{-1/h} & \textit{if } q \textit{ is odd and } h \textit{ is even},\\ B_3^{(m)} x (\log\log x)^{m-2} (\log x)^{-1} & \textit{if } q = 2,\end{cases}$$

where $B_1^{(m)} = \sum_{r=0}^{m-1} A_1^{(r)}$, $B_2^{(m)} = \sum_{r=l(\gamma+1)}^{m-1} A_2^{(r)}$, $B_3^{(m)} = A_3^{(m-1)}$.

Proof. If $q = 2$, or if q is odd, h is even and $q \nmid \nu$ (so that $\gamma = 0$), then from (38) and Theorem 1 (ii) and (iii), we obtain

$$N(\nu, q^m; x) \sim D_{m-1}(\nu, q; x),$$

and the result follows in these cases. If q and h are both odd, the result follows immediately from (38) and Theorem 1 (i). Finally suppose that q is odd, h is even and $q \mid \nu$ (so that $\gamma \geqslant 1$). Then it is clear from (38) and Theorem 1 (ii) that

$$N(\nu, q^m; x) \sim \sum_{r=0}^{m-1} A_2^{(r)} x (\log\log x)^{[r/(\gamma+1)]} (\log x)^{-1/h}.$$

Since $\max_{0\leqslant r\leqslant m-1} [r/(\gamma+1)] = [(m-1)/(\gamma+1)] = l$, and since $[r/(\gamma+1)] = l$ when $l(\gamma+1) \leqslant r \leqslant m-1$, the result of the Corollary follows.

3. Some results for $N(\nu, k; x)$. In this section we assume that k is divisible by at least two distinct primes and we shall deduce some estimates for $N(\nu, k; x)$. Some results in this direction have already been obtained by Rankin in § 4 of his paper [1], and these results are improvements on Watson's estimate for $N(\nu, k; x)$, stated in (4).

Let

$$k = \prod_{r=0}^{t} q_r^{m_r},$$

where $q_0, q_1, \ldots, q_t$ are primes and $2 = q_0 < q_1 < q_2 < \ldots < q_t$, and where $m_0 \geqslant 0$ and $m_r \geqslant 1$ for $1 \leqslant r \leqslant t$; since k is divisible by at least two primes, $t \geqslant 1$ if $m_0 > 0$ and $t \geqslant 2$ if $m_0 = 0$. As in (19) and (21) of [1],

$$\max_{0\leqslant r\leqslant t} N(\nu, q_r^{m_r}; x) \leqslant N(\nu, k; x) \leqslant \sum_{r=0}^{t} N(\nu, q_r^{m_r}; x); \tag{39}$$

for if $q_r^{m_r} \mid \sigma_\nu(n)$ for some r satisfying $0 \leqslant r \leqslant t$, it does not necessarily follow that $k \mid \sigma_\nu(n)$, but if $k \mid \sigma_\nu(n)$ then $q_r^{m_r} \mid \sigma_\nu(n)$ for all r satisfying $0 \leqslant r \leqslant t$. We observe that

$$N(\nu, 2^{m_0}; x) = o\big(N(\nu, q^m; x)\big)$$

for all odd primes q and all possible values of m_0 and m. If $m_0 = 0$, we take the value of $N(\nu, 2^{m_0}; x)$ in (39) to be zero.

For $1 \leqslant r \leqslant t$, define $h_r = (q_r-1)/(\nu, q_r-1)$ and γ_r by $q_r^{\gamma_r} \| \nu$; if all the h_r are even, define

$$\lambda = \max_{1\leqslant r\leqslant t} h_r \quad \text{and} \quad \mu = \max_{\substack{1\leqslant r\leqslant t\\ h_r=\lambda}} [(m_r-1)/(\gamma_r+1)].$$

Then we have

COROLLARY 2. (i) *If h_r is odd when $r = i$ for exactly one value of i satisfying $1 \leqslant i \leqslant t$, and h_r is even otherwise, then as $x \to \infty$*

$$N(\nu, k; x) \sim B_1^{(m_i)} x.$$

(ii) *If all the h_r are even, and if the relations*

$$h_r = \lambda \quad \textit{and} \quad [(m_r-1)/(\gamma_r+1)] = \mu$$

hold simultaneously when $r = i$ for exactly one value of i satisfying $1 \leqslant i \leqslant t$, and not otherwise, then as $x \to \infty$

$$N(\nu, k; x) \sim B_2^{(m_i)} x (\log\log x)^\mu (\log x)^{-1/\lambda}.$$

Proof. These results follow immediately from (39) and Corollary 1 since under the conditions stated

$$\sum_{r=0}^{t} N(\nu, q_r^{m_r}; x) \sim N(\nu, q_i^{m_i}; x) = \max_{0\leqslant r\leqslant t} N(\nu, q_r^{m_r}; x).$$

The constants $B_1^{(m_i)}$ and $B_2^{(m_i)}$ are given by Corollary 1.

COROLLARY 3. (i) *If h_r is odd for at least two integers r satisfying $1 \leqslant r \leqslant t$, then*

$$C_1 \leqslant \lim_{x\to\infty} x^{-1}N(\nu, k; x) \leqslant C_2,$$

where C_1 and C_2 are positive constants and $C_1 \neq C_2$.

(ii) *If all the h_r are even, and if the relations*

$$h_r = \lambda \quad \text{and} \quad [(m_r-1)/(\gamma_r+1)] = \mu$$

hold simultaneously for at least two integers r satisfying $1 \leqslant r \leqslant t$, then

$$C_3 \leqslant \lim_{x\to\infty} \{x^{-1}(\log\log x)^{-\mu}(\log x)^{1/\lambda}N(\nu, k; x)\} \leqslant C_4,$$

where C_3 and C_4 are positive constants and $C_3 \neq C_4$.

Proof. (i) Suppose that h_r is odd when $r = r_i$, $i = 1, 2, \ldots, j$, where $2 \leqslant j \leqslant t$, and h_r is even otherwise. Then by Corollary 1, as $x \to \infty$,

$$\sum_{r=0}^{t} N(\nu, q_r^{m_r}; x) \sim \Big\{\sum_{i=1}^{j} B_1^{(m_{r_i})}\Big\} x = C_2 x,$$

and

$$\max_{0\leqslant r\leqslant t} N(\nu, q_r^{m_r}; x) \sim \{\max_{1\leqslant i\leqslant j} B_1^{(m_{r_i})}\} x = C_1 x;$$

clearly $C_1 \neq C_2$. The result now follows from (39).

The proof of (ii) is similar.

4. Results for $d(n)$ and $\varphi(n)$. We have said that the proof of Theorem 1 can be adapted to prove results analogous to Theorem 1 for the functions $d(n)$ and $\varphi(n)$. In this section we state these results, and we consider $d(n)$ first.

If $n = \prod p^a$, then $d(n) = \prod_{p^a\|n} (a+1)$. Define

$$b_0(n) = \begin{cases} 1 & \text{if } q \nmid d(n), \\ 0 & \text{otherwise}, \end{cases} \qquad b_r(n) = \begin{cases} 1 & \text{if } q^r \,\|\, d(n), \\ 0 & \text{otherwise} \end{cases}$$

for $r \geqslant 1$, where q is a prime. Let

$$D_m(0, q; x) = \sum_{n=1}^{x} b_m(n)$$

for $m \geqslant 0$, and

$$g_m(s) = \sum_{n=1}^{\infty} b_m(n) n^{-s}.$$

Then we have

THEOREM 4. (i) *If $q \neq 2$ and $m \geqslant 0$,*

$$g_m(s) = \zeta(s)\psi(s),$$

where $\psi(s)$ is holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \geqslant \frac{1}{2}+\delta$ ($\delta > 0$). Hence

$$D_m(0, q; x) \sim \psi(1)x.$$

(ii) *If $q = 2$ and $m > 0$,*

$$g_m(s) = \sum_{u=0}^{m} \{\log\zeta(s)\}^u \psi_u(s),$$

where $\psi_u(s)$ is holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \geqslant \frac{1}{2}+\delta$ ($\delta > 0$), and $\psi_m(s) = \dfrac{1}{m!}\zeta(2s)$. Hence

$$D_m(0, 2; x) \sim \frac{1}{(m-1)!}\cdot\frac{\pi^2}{6} x(\log\log x)^{m-1}(\log x)^{-1}.$$

If $q = 2$ and $m = 0$,

$$g_0(s) = \zeta(2s),$$

and hence

$$D_0(0, 2; x) \sim x^{1/2}.$$

We observe that $D_0(0, q; x)$ represents the number of positive integers $n \leqslant x$ for which $q \nmid d(n)$. If $q \neq 2$, from the proof of Theorem 4 it can be deduced that

$$D_0(0, q; x) \sim \frac{\zeta(q)}{\zeta(q-1)} x.$$

This result is an immediate consequence of a result proved by L. G. Sathe in a paper [7] published in 1943. In fact from Sathe's result it follows that

$$D_0(0, q; x) = \frac{\zeta(q)}{\zeta(q-1)} x + O(x^{1/(q-1)}\log x).$$

We turn now to $\varphi(n)$. If $n = \prod p^a$, $\varphi(n) = \prod_{p^a\|n} p^{a-1}(p-1)$. Define

$$c_0(n) = \begin{cases} 1 & \text{if } q \nmid \varphi(n), \\ 0 & \text{otherwise}, \end{cases} \qquad c_r(n) = \begin{cases} 1 & \text{if } q^r \,\|\, \varphi(n), \\ 0 & \text{otherwise} \end{cases}$$

for $r \geqslant 1$, where q is a prime. Let

$$D_m^*(q; x) = \sum_{n=1}^{x} c_m(n)$$

for $m \geqslant 0$, and

$$h_m(s) = \sum_{n=1}^{\infty} c_m(n) n^{-s}.$$

Then we have

THEOREM 5. (i) *If* $q \neq 2$ *and* $m \geqslant 0$,

$$h_m(s) = \{\zeta(s)\}^{1-1/(q-1)} \sum_{u=0}^{m} \{\log\zeta(s)\}^u H_u(s),$$

where each $H_u(s)$ $(0 \leqslant u \leqslant m)$ *satisfies the conditions given in Theorem* 2 (ii). *Hence*

$$D_m^*(q; x) \sim \frac{H_m(1)}{\Gamma(1-1/(q-1))} x(\log\log x)^m (\log x)^{-1/(q-1)}.$$

(ii) *If* $q = 2$ *and* $m > 0$,

$$h_m(s) = \sum_{u=0}^{m} \{\log\zeta(s)\}^u H_u(s),$$

where $H_m(s) = \dfrac{1}{m!} 2^{-m}(1+2^{-s})$ *and each* $H_u(s)$ $(0 \leqslant u \leqslant m)$ *satisfies the conditions given in Theorem* 2 (ii). *Hence*

$$D_m^*(2; x) \sim \frac{3}{2} \cdot \frac{2^{-m}}{(m-1)!} x(\log\log x)^{m-1} (\log x)^{-1}.$$

If $q = 2$ *and* $m = 0$,

$$h_0(s) = 1 + 2^{-s}$$

and hence for all $x \geqslant 2$

$$D_0^*(2; x) = 2.$$

The proofs of Theorems 4 and 5 are more straightforward than the proof of Theorem 1. From the definitions we can immediately obtain the result analogous to Lemma 5, and no section analogous to § 1 of part II is necessary; Lemma 8 can again be used. The generating functions $g_m(s)$ and $h_m(s)$ can be obtained by following the method of part II. No variation of the method of proof is required to obtain $g_m(s)$. To obtain $h_m(s)$ the only additional fact required in the proof is an estimate of $\prod_{p \equiv 1 (\bmod q)} (1-p^{-s})$, and otherwise the method of proof is unaltered. Clearly we can use Theorem 3 or the Wiener-Ikehara Theorem, whichever is applicable, to deduce the asymptotic equations for $D_m(0, q; x)$ and $D_m^*(q; x)$ from $g_m(s)$ and $h_m(s)$ respectively.

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