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Some remarks on certain generalized Dedekind sums

by

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Dedicated to L. J. Mordell
on the occassion of his 75th birthday

1. In recent investigations concerning the functions

$$\sigma_{gh}(0,\tau) = \prod_{m=0}^{\infty}\left(1-e^{2\pi ih/f}e^{2\pi i\tau(m+g/f)}\right)\prod_{n=1}^{\infty}\left(1-e^{-2\pi ih/f}e^{2\pi i\tau(n-g/f)}\right) \tag{1.1}$$

Curt Meyer [1] and Ulrich Dieter [2] have introduced the following generalized Dedekind sums

$$s_{g,h}(a,c) = \sum_{\mu \bmod c}\left(\left(\frac{a\mu}{c}+\frac{ag+ch}{cf}\right)\right)\left(\left(\frac{\mu}{c}+\frac{g}{cf}\right)\right). \tag{1.2}$$

They derived a whole theory of these sums, including also a reciprocity theorem.

The functions (1.1) have been introduced by F. Klein [3], in the theory of "division" of σ-functions. C. Meyer uses these functions for investigations of class numbers of Abelian fields over quadratic ground fields, following the lead of Hecke. They appear, also, not recognized as Klein's σ_{gh} however, in papers by J. Lehner [4] and J. Livingood [5] where they play the role of conjugates of the generating function for certain partition numbers. In these papers one also finds certain generalizations of Dedekind sums, which turn out to be special cases of the sums (1.2).

I observe now that the rationality of g/f and h/f is completely irrelevant in the theory of the sums (1.2) and that the true generalization of the above kind is contained in the *definition*:

$$\begin{aligned} s(h,k;\,x,y) &= \sum_{\mu \bmod k}\left(\left(h\left(\frac{\mu+y}{k}+\frac{x}{h}\right)\right)\right)\left(\left(\frac{\mu+y}{k}\right)\right) \\ &= \sum_{\mu \bmod k}\left(\left(h\,\frac{\mu+y}{k}+x\right)\right)\left(\left(\frac{\mu+y}{k}\right)\right), \end{aligned} \tag{1.3}$$

where x and y are real numbers and h, k are coprime integers, $k > 0$. The symbol $((x))$ means here as usual the function

$$((x)) = \begin{cases} x-[x]-\frac{1}{2}, & x \text{ not integer}, \\ 0, & x \text{ integer}. \end{cases}$$

The definition (1.3) shows immediately that $s(h, k;\ x, y)$ has the period 1 in x as well as in y. We may therefore restrict our considerations to the range

$$0 \leqslant x < 1, \quad 0 \leqslant y < 1. \tag{1.4}$$

For $x = y = 0$ (and thus also for x, y both integers) the new sums are the classical Dedekind sums. The sums (1.2) are special cases of (1.3) with $h = a$, $k = c$, $x = h/f$, $y = g/f$.

2. We need further on the

THEOREM 1.

$$s(1, k; 0, 0) = s(1, k) = \frac{k}{12}+\frac{1}{6k}-\frac{1}{4}; \tag{2.1}$$

$$s(1, k;\ 0, y) = \frac{k}{12}+\frac{1}{k}B_2(y) \quad \textit{for} \quad 0 < y < 1, \tag{2.2}$$

where $B_2(y)$ *is the second Bernoulli function*, $B_2(y) = y^2-y+1/6$.

Proof. The case $y = 0$ is known from the theory of Dedekind sums and results from

$$s(1, k) = \sum_{\mu=1}^{k-1}\left(\left(\frac{\mu}{k}\right)\right)^2 = \sum_{\mu=1}^{k-1}\left(\frac{\mu}{k}-\frac{1}{2}\right)^2$$

by straightforward calculation.

The case $0 < y < 1$ requires

$$s(1, k; 0, y) = \sum_{\mu=0}^{k-1}\left(\left(\frac{\mu+y}{k}\right)\right)^2 = \sum_{\mu=0}^{k-1}\left(\frac{\mu+y}{k}-\frac{1}{2}\right)^2$$

from which (2.2) follows by simple calculation.

COROLLARY. *Because of the periodicity in x and y we have also*

$$s(1, k; 0, y) = \frac{k}{12}+\frac{1}{6k}-\frac{1}{4} \quad \textit{for } y \textit{ integer},$$

$$s(1, k; 0, y) = \frac{k}{12}+\frac{1}{k}\psi_2(y) \quad \textit{for } y \textit{ not integer},$$

where we use here and subsequently the abbreviation

$$\psi_2(y) = B_2(y-[y]). \tag{2.3}$$

For the establishment of the reciprocity formula we use a device by U. Dieter [2]. We need first the

LEMMA.

$$\sum_{\mu \bmod k}\left(\left(\frac{\mu+w}{k}\right)\right) = ((w)). \tag{2.4}$$

Proof. Let us investigate the difference

$$D(w) = \sum_{\mu \bmod k}\left(\left(\frac{\mu+w}{k}\right)\right)-((w)).$$

This difference has obviously the period 1 in w, since both terms have it. We need thus only consider $0 \leqslant w < 1$. For $w = 0$ we have

$$D(0) = \sum_{\mu \bmod k}\left(\left(\frac{\mu}{k}\right)\right) = 0,$$

and for $0 < w < 1$

$$\begin{aligned} D(w) &= \sum_{\mu=0}^{k-1}\left(\left(\frac{\mu+w}{k}\right)\right)-((w)) \\ &= \sum_{\mu=0}^{k-1}\left(\frac{\mu+w}{k}-\frac{1}{2}\right)-\left(w-\frac{1}{2}\right) \\ &= \frac{k-1}{2}+w-\frac{k}{2}-\left(w-\frac{1}{2}\right) = 0. \end{aligned}$$

Therefore $D(w) = 0$ for all w, which proves the lemma.

3. We study now, with $h > 0, k > 0, (h, k) = 1$,

$$S = s(h, k;\ x, y)+s(k, h;\ y, x) \tag{3.1}$$

$$= \sum_{\mu \bmod k}\left(\left(h\left(\frac{\mu+y}{k}+\frac{x}{h}\right)\right)\right)\left(\left(\frac{\mu+y}{k}\right)\right)+\sum_{\nu \bmod h}\left(\left(k\left(\frac{\nu+x}{h}+\frac{y}{k}\right)\right)\right)\left(\left(\frac{\nu+x}{h}\right)\right).$$

Applying now (2.4) to the two sums we obtain

$$\begin{aligned} S = {} & \sum_{\mu \bmod k}\sum_{\nu \bmod h}\left(\left(\frac{\nu}{h}+\frac{\mu+y}{k}+\frac{x}{h}\right)\right)\left(\left(\frac{\mu+y}{k}\right)\right)+ \\ & +\sum_{\nu \bmod h}\sum_{\mu \bmod k}\left(\left(\frac{\mu}{k}+\frac{\nu+x}{h}+\frac{y}{k}\right)\right)\left(\left(\frac{\nu+x}{h}\right)\right) \\ = {} & \sum_{\mu=0}^{k-1}\sum_{\nu=0}^{h-1}\left(\left(\frac{\mu+y}{k}+\frac{\nu+x}{h}\right)\right)\left\{\left(\left(\frac{\mu+y}{k}\right)\right)+\left(\left(\frac{\nu+x}{h}\right)\right)\right\}. \end{aligned} \tag{3.2}$$

We have to study only the range (1.4), with the exception of the case $x=0$, $y=0$, which is covered by the classical reciprocity formula for Dedekind sums. We have thus only

$$0<x+y<2.$$

It is preferable to postpone also the case $x=0$ *or* $y=0$ and to begin with

$$0<x<1, \qquad 0<y<1.$$

Then we get

$$S=\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{h-1}\left(\left(\frac{\mu+y}{k}+\frac{\nu+x}{h}\right)\right)\left(\frac{\mu+y}{k}+\frac{\nu+x}{h}-1\right). \tag{3.3}$$

Let us now look at the sum

$$\begin{aligned} T &= \sum_{\mu=0}^{k-1}\sum_{\nu=0}^{h-1}\left\{\left(\left(\frac{\mu+y}{k}+\frac{\nu+x}{h}\right)\right)-\left(\frac{\mu+y}{k}+\frac{\nu+x}{h}-1\right)\right\}^2 \\ &= \sum_{\mu=0}^{k-1}\sum_{\nu=0}^{h-1}\left(\left(\frac{\mu+y}{k}+\frac{\nu+x}{h}\right)\right)^2- \\ &\quad -2\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{h-1}\left(\left(\frac{\mu+y}{k}+\frac{\nu+x}{h}\right)\right)\left(\frac{\mu+y}{k}+\frac{\nu+x}{h}-1\right)+ \\ &\quad +\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{h-1}\left(\frac{\mu+y}{k}+\frac{\nu+x}{h}-1\right)^2 \end{aligned} \tag{3.4}$$

or, say,

$$T=S_1-2S_2+S_3. \tag{3.5}$$

We discuss these sums separately. In S_1 the variables μ and ν need only to be taken modulo k and h, respectively. Then we have

$$\begin{aligned} S_1 &= \sum_{\substack{\mu \bmod k\\ \nu \bmod h}}\left(\left(\frac{h\mu+k\nu}{hk}+\frac{hy+kx}{hk}\right)\right)^2 \\ &= \sum_{\varrho \bmod hk}\left(\left(\frac{\varrho+hy+kx}{hk}\right)\right)^2 = s(1,hk;0,hy+kx). \end{aligned}$$

4. Now we have to distinguish two cases:

I. $hy+kx$ *not integer*, II. $hy+kx$ *integer*.

We continue until further notice solely with Case I. Then we obtain from the Corollary to Theorem 1

$$S_1=\frac{hk}{12}+\frac{1}{hk}\,\psi_2(hy+kx). \tag{4.1}$$

Comparing (3.3), (3.4), (3.5) we recognize that

$$S_2=S. \tag{4.2}$$

Furthermore, we have

$$S_3=\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{h-1}\left(\frac{\mu+y}{k}+\frac{\nu+x}{h}-1\right)^2.$$

This sum is *continuous* in x and y. We obtain first

$$\begin{aligned} S_3 &= h\sum_{\mu=0}^{k-1}\left(\frac{\mu+y}{k}\right)^2+k\sum_{\nu=0}^{h-1}\left(\frac{\nu+x}{h}\right)^2+hk+ \\ &\quad +2\sum_{\mu=0}^{k-1}\frac{\mu+y}{k}\sum_{\nu=0}^{h-1}\frac{\nu+x}{h}-2h\sum_{\mu=0}^{k-1}\frac{\mu+y}{k}-2k\sum_{\nu=0}^{h-1}\frac{\nu+x}{h}. \end{aligned}$$

The computation yields finally

$$S_3=\frac{h}{k}B_2(y)+\frac{k}{h}B_2(x)+2\left(x-\frac12\right)\left(y-\frac12\right)+\frac{hk}{6}. \tag{4.3}$$

If on the other hand we use $((z))-z=-[z]-1/2$ in case z is not integer, then definition (3.4) for T gives

$$T=\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{h-1}\left\{-\left[\frac{\mu+y}{k}+\frac{\nu+x}{h}\right]+\frac12\right\}^2 \tag{4.4}$$

since in Case I none of the values

$$\frac{\mu+y}{k}+\frac{\nu+x}{h}, \qquad \mu=0,\ldots,k-1,\ \nu=0,\ldots,h-1,$$

can be an integer. We have thus

$$T=\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{h-1}\left[\frac{\mu+y}{k}+\frac{\nu+x}{h}\right]\cdot\left(\left[\frac{\mu+y}{k}+\frac{\nu+x}{h}\right]-1\right)+\frac{hk}{4}.$$

Since with $0<x<1$, $0<y<1$ and within the range of μ and ν the bracket $\left[\frac{\mu+y}{k}+\frac{\nu+x}{h}\right]$ can only have the values 0 or 1, the double sum is 0 and we obtain thus

$$T=hk/4. \tag{4.5}$$

Putting together now formulae (3.5) and (4.1) to (4.5) we have

$$S = \tfrac{1}{2}(S_1 - T + S_3) \tag{4.6}$$

with

$$S_1 - T = \frac{1}{hk}\psi_2(hy+kx) - \frac{hk}{6} \tag{4.7}$$

and thus, finally,

$$S = ((x))((y)) + \frac{1}{2}\left\{\frac{k}{h}\psi_2(x) + \frac{1}{hk}\psi_2(hy+kx) + \frac{h}{k}\psi_2(y)\right\}, \tag{4.8}$$

where we have replaced $B_2(x)$, $B_2(y)$ by $\psi_2(x)$, $\psi_2(y)$, which is permissible in the range (4.1) of x and y.

5. We come now to Case II, $hy+kx$ integer, but still $0<x<1$, $0<y<1$. Formula (4.6) remains true here also, from the definitions. However, in this case a few modifications of our argument are necessary. We notice that now $(\mu+y)/k+(\nu+x)/h$ may be an integer for certain μ and ν. Indeed, this would require

$$\mu h + \nu k \equiv -(hy+kx) \pmod{hk},$$

which has exactly one solution μ_0, ν_0 in the range of μ and ν since h and k are coprime. Equation (3.5) remains valid. But now we have, according to Theorem 1 and its Corollary,

(5.1)

$$S_1 = S(1, hk;\ 0, hy+kx) = \frac{hk}{12} + \frac{1}{6hk} - \frac{1}{4} = \frac{hk}{12} + \frac{1}{hk}\psi_2(hy+kx) - \frac{1}{4}.$$

The exiplicit result (4.5) for T will have to be revised since we have to remember the exceptional pair of values μ_0, ν_0. We have then from (3.4)

$$T = \sum_{\substack{\mu=0\\(\mu,\nu)\neq(\mu_0,\nu_0)}}^{k-1}\sum_{\nu=0}^{h-1}\left\{-\left[\frac{\mu+y}{k}+\frac{\nu+x}{h}\right]+\frac{1}{2}\right\}^2 + \left\{-\left[\frac{\mu_0+y}{k}+\frac{\nu_0+x}{h}\right]+1\right\}^2$$

$$= \sum_{\mu=0}^{k-1}\sum_{\nu=0}^{h-1}\left\{-\left[\frac{\mu+y}{k}+\frac{\nu+x}{h}\right]+\frac{1}{2}\right\}^2 + 2\left\{-\left[\frac{\mu_0+y}{k}+\frac{\nu_0+x}{h}\right]+\frac{1}{2}\right\}\cdot\frac{1}{2}+\frac{1}{4}.$$

But, because of $0\leqslant\mu_0<k$, $0\leqslant\nu_0<h$, $0<x<1$, $0<y<1$ the value of

$$\left[\frac{\mu_0+y}{k}+\frac{\nu_0+x}{h}\right]$$

can only be 1. We obtain therefore this time

$$T = \sum_{\mu=0}^{k-1}\sum_{\nu=0}^{h-1}\left\{-\left[\frac{\mu+y}{k}+\frac{\nu+x}{h}\right]+\frac{1}{2}\right\}^2 - \frac{1}{4} = \frac{hk}{4} - \frac{1}{4}.$$

We see here and in (5.1) that in Case II S_1 and T are both diminished by $\frac{1}{4}$, compared to Case I. Hence S_1-T retains the value (4.7) it had for Case I. Since, moreover, S_3 is continuous in x and y, it is not effected in the distinction of Cases I and II. Therefore, since (4.6) remains true here also, the result (4.8) is correct also for Case II.

6. There remains now only the case $x=0$, $0<y<1$ (or the symmetric one $0<x<1$, $y=0$). We start anew from (3.2) with

$$S = \sum_{\mu=0}^{k-1}\sum_{\nu=0}^{h-1}\left(\left(\frac{\mu+y}{k}+\frac{\nu}{h}\right)\right)\left\{\left(\left(\frac{\mu+y}{k}\right)\right)+\left(\left(\frac{\nu}{h}\right)\right)\right\}.$$

Here the case $\nu=0$ has to be specially treated.

$$S = \sum_{\mu=0}^{k-1}\sum_{\nu=1}^{h-1}\left(\left(\frac{\mu+y}{k}+\frac{\nu}{h}\right)\right)\left\{\frac{\mu+y}{k}+\frac{\nu}{h}-1\right\} + \sum_{\mu=0}^{k-1}\left(\left(\frac{\mu+y}{k}\right)\right)\left\{\frac{\mu+y}{k}-\frac{1}{2}\right\}$$

$$= \sum_{\mu=0}^{k-1}\sum_{\nu=0}^{h-1}\left(\left(\frac{\mu+y}{k}+\frac{\nu}{h}\right)\right)\left\{\frac{\mu+y}{k}+\frac{\nu}{h}-1\right\} + \frac{1}{2}\sum_{\mu=0}^{k-1}\left(\left(\frac{\mu+y}{k}\right)\right)$$

$$= S_2 + \frac{1}{2}((y))$$

according to (3.4), (3.5), and (2.4). Thus (3.5) yields

$$2S = S_1 - T + S_3 + ((y)). \tag{6.1}$$

We know that S_1-T does not depend on the arithmetical nature of $hy+k\cdot 0$ and is given by (4.7). Moreover, S_3 is continous. For $x=0$, $0<y<1$ it can be written as

$$S_3 = \frac{h}{k}\psi_2(y) + \frac{k}{h}\psi_2(0) - ((y)) + \frac{hk}{6}.$$

If we put this and (4.7) into (6.1) we obtain

$$S = \frac{1}{2}\left\{\frac{k}{h}\psi_2(0) + \frac{1}{hk}\psi_2(hy) + \frac{h}{k}\psi_2(y)\right\}.$$

Since $((0)) = 0$, this turns out to be the special case $x=0$ of (4.8).

We can now lift the restriction of the range (4.1) on x and y. Our result is

THEOREM 2. *For x, y both integers the classical formula*

$$s(h,k;\ x,y)+s(k,h;\ y,x) = s(h,k)+s(k,h) = -\frac{1}{4}+\frac{1}{12}\left(\frac{h}{k}+\frac{1}{hk}+\frac{k}{h}\right)$$

remains in force.

If x, y are not both integers then the reciprocity formula is

$$s(h,k;\ x,y)+s(k,h;\ y,x)$$
$$= ((x))((y))+\frac{1}{2}\left\{\frac{h}{k}\psi_2(y)+\frac{1}{hk}\psi_2(hy+kx)+\frac{k}{h}\psi_2(x)\right\}.$$

7. The reciprocity theorem can be used in case of the ordinary Dedekind sums to compute $s(h,k)$ by means of a Euclidean algorithm. This can be done here also, however only with the rule given by the following theorem.

THEOREM 3. *Let m be an integer. Then*

$$s(h,k;\ x,y) = s(h-mk,k;\ x+my,y). \tag{7.1}$$

Proof. We have from the definition (1.3)

$$s(h-k,k;\ x+y,y) = \sum_{\mu \bmod k}\left(\left((h-k)\left(\frac{\mu+y}{k}\right)+x+y\right)\right)\left(\left(\frac{\mu+y}{k}\right)\right)$$
$$= \sum_{\mu \bmod k}\left(\left(h\left(\frac{\mu+y}{k}\right)-\mu+x\right)\right)\left(\left(\frac{\mu+y}{k}\right)\right)$$
$$= \sum_{\mu \bmod k}\left(\left(h\left(\frac{\mu+y}{k}+\frac{x}{h}\right)\right)\right)\left(\left(\frac{\mu+y}{k}\right)\right)$$
$$= s(h,k;\ x,y).$$

This settles the case $m=1$. For any integer m the theorem follows now by iteration.

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