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NORMAL SUBALGEBRAS IN GENERAL ALGEBRAS

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The concept of a normal subgroup, or an ideal of a ring, has no proper generalization for general algebraic systems. The leading idea is that these subalgebras are suitable to be kernels of homomorphisms, which means that they may be identified by a homomorphism or a congruence relation with a special element forming a one-element subalgebra. But when dealing with algebras in general we do not have special elements, and even when we do have one-element subalgebra it may happen that it is not unique.

The aim of this note is to propose a definition of normality for subalgebras, which generalizes the concept of normal subgroups and this of ideals, and, moreover, may be applied to such algebras as, e. g., semigroups, even without unit element. This definition makes use of the fact that both normal subgroups and ideals may be defined as subalgebras which are cosets of a certain congruence relation.

By an *algebra* we mean here a system $\langle A, F_1, F_2, \ldots\rangle$, where A is a non-void set and F_i are operations on A, i. e., functions of several variables from A to A. The whole algebra will be denoted by the same letter as its set.

By a *term* in an algebra A we mean an operation on A defined by composition of operations F_i. If every operation F_i is finite (i. e. it is a function of finitely many variables), then so is every term. Terms will be denoted by τ, the value of τ for $a_1, a_2, \ldots$ in A by $\tau(a_1, a_2, \ldots)$.

Let us remember that a congruence in an algebra A means an equivalence relation in the set A preserving all the operations F_i. It follows that a congruence preserves also all terms.

For a congruence $\sim$ in A and for an element a in A, by $a/\sim$ we denote the coset of a, i. e., the set of all b in A with $b \sim a$. The algebra defined on the class of cosets by

$$F_i(a_1/\sim, a_2/\sim, \ldots) = F_i(a_1, a_2, \ldots)/\sim$$

is called the *quotient algebra* and is denoted by $A/\sim$. Every term τ in A

has a correspondent meaning in $A/\sim$. From the preserving of terms by a congruence it follows that for every term τ and any elements a_i in A we have

$$\tau(a_1, a_2, \ldots)/\sim \; = \tau(a_1/\sim, a_2/\sim, \ldots).$$

Definition. A subalgebra B of A is called *normal* in A, iff there exists a congruence relation $\sim$ in A such that B is one of the cosets of $\sim$, i. e., if $b/\sim \; = B$ for b in B.

THEOREM. *If all operations of A are finite, then a subalgebra B of A is normal iff for every term τ and any elements $a_1, \ldots, a_k$ in A, and $b_1, \ldots, b_l$ in B, if $\tau(a_1, \ldots, a_k, b_1, \ldots, b_l)$ is in B, then for arbitrary $b'_1, \ldots, b'_l$ in B, $\tau(a_1, \ldots, a_k, b'_1, \ldots, b'_l)$ is in B.*

For the necessity, let $b/\sim \; = B$ and let $\tau(a_1, \ldots, a_k, b_1, \ldots, b_l)$ be in B, where a_i are in A and b_i in B. For arbitrary b'_i in B we have

$$\begin{aligned} B &= \tau(a_1, \ldots, a_k, b_1, \ldots, b_l)/\sim \; = \tau(a_1/\sim, \ldots, a_k/\sim, b_1/\sim \ldots, b_l/\sim) \\ &= \tau(a_1/\sim, \ldots, a_k/\sim, B, \ldots, B) = \tau(a_1/\sim, \ldots, a_k/\sim, b'_1/\sim, \ldots, b'_l/\sim) \\ &= \tau(a_1, \ldots, a_k, b'_1, \ldots, b'_l)/\sim \end{aligned}$$

which proves that $\tau(a_1, \ldots, a_k, b'_1, \ldots, b'_l)$ is in B.

It is easy to see that this proof can be carried over to the case of infinite operations in A.

To show that the condition is sufficient, let us define for a and b in A, $a \simeq b$ iff for some term τ and $a_1, \ldots, a_k$ in A, $b_1, \ldots, b_l, b'_1, \ldots, b'_l$ in B, we have $a = \tau(a_1, \ldots, a_k, b_1, \ldots, b_l)$ and $b = \tau(a_1, \ldots, a_k, b'_1, \ldots, b'_l)$.

The relation $\simeq$ has all properties of a congruence but transitivity. Moreover, it has two following properties:

1. If b is in B, a is in A and $a \simeq b$, then a is in B.

2. If a and b are in B, then $a \simeq b$.

The first follows from the assumption on B, the second one may be obtained by taking for τ the zero-composition of the operations, i. e., a single variable x.

From these properties of $\simeq$ it follows that the least transitive relation containing $\simeq$ (i. e., the relation defined as: $a \sim b$ iff there exist $c_1, \ldots, c_n$ in A with $a \simeq c_1 \simeq c_2 \simeq \ldots \simeq c_n \simeq b$) is a congruence relation with the required property.

Let us remark that generally for a normal subalgebra B there exists more than one congruence relation $\sim$ with $b/\sim \; = B$, but between them there is a least one and this indeed is defined in the proof of the theorem above. If we denote by A/B the quotient algebra of this congruence, we shall obtain an operation similar to this of forming a quotient group or a ring. In general, this operation does not enjoy all the properties of a quotient group or a quotient ring known from the groups or rings theory.

There are many problems related with the properties of normal subalgebras. Let us mention here one connected with our theorem. May this theorem (and in which form) be generalized for algebras with infinite operations (**P 466**)? The sufficiency proof cannot be carried over to this case, since the least transitive relation containing $\simeq$ may be, in this case, not a congruence.

Added in proof. E. Marczewski remarked that in the proof of the theorem no use was made of the assumption that B is a subalgebra of A. Indeed, since it is so, we may define the concept of a normal set in a natural way and then the condition stated in the theorem is necessary and sufficient for a subset B of A to be normal.

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