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SOME THEOREMS ON INTERPOLATION BY PERIODIC FUNCTIONS

BY

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A sequence $\{t_n\}$ of positive numbers is said to have *property* (P) in a class K of sequences of real numbers if for every $\{\varepsilon_n\}\in K$ there exists a continuous periodic function $f(t)$, $-\infty<t<\infty$, such that

$$f(t_n)=\varepsilon_n \quad \text{for} \quad n=1,2,\ldots$$

The problem is to find conditions on $\{t_n\}$ implying (P). Some results concerning this problem and related questions can be found in [1]-[4].

In particular Ryll-Nardzewski [3] has shown that the sequence $\{3^n\}$, but no sequence with $0<t_n\leqslant C\cdot 2^n$ (C any constant, $n=1,2,\ldots$), has the property (P) in the class K_2 of all sequences taking on values 0 and 1. He asked whether every sequence $\{t_n\}$ such that

$$t_{n+1}\geqslant(s+\beta)t_n \quad \text{for} \quad n=1,2,\ldots,$$

where $\beta>0$ and s is an arbitrary positive integer, has the property (P) in the class K_s of all sequences taking on s different values only. I have shown in [4] that this condition is not necessary. In this note some further results concerning Ryll-Nardzewski's last question will be presented.

In the proofs we shall use closed intervals $[a_n,b_n]$ ($a_n>0$, $b_n-a_n\leqslant\frac{1}{2}$) satisfying some of the following conditions:

$$(\mathrm{A}_n) \qquad \frac{t_n}{t_{n-1}}a_{n-1}\leqslant a_n<b_n\leqslant\frac{t_n}{t_{n-1}}b_{n-1} \quad (n>1),$$

$$(\mathrm{B}_n) \qquad [a_n,b_n]\subset\begin{cases}\left[0,\frac{1}{2}\right] \pmod 1, & \text{if} \quad \varepsilon_n=0,\\ \left[\frac{1}{2},1\right] \pmod 1, & \text{if} \quad \varepsilon_n=1,\end{cases}$$

$$(\mathrm{C}_n) \qquad b_n-a_n=\tfrac{1}{2}.$$

For any numbers denoted by b_n and b_{n+m} ($m=m(n)>0$ integer) and any $\gamma=\gamma(n)>0$

$$(\mathrm{D}^{\gamma}_{n,m}) \text{ means } b_{n+m}\leqslant\frac{t_{n+m}}{t_n}(b_n-\gamma).$$

We put

$$q_n = \frac{t_{n+1}}{t_n} \quad (n = 1, 2, \ldots).$$

The following theorem is a generalization of a result by Mycielski [2]:

THEOREM 1. *Every sequence* $\{t_n\}$ *satisfying the condition*

$$q_n = \frac{t_{n+1}}{t_n} \geqslant 3 \quad \textit{for} \quad n = 1, 2, \ldots$$

has the property (P) *in the class* K_2.

The proof of Theorem 1 is based on seven lemmas.

LEMMA 1. *In order that a sequence* $\{t_n\}$ *has property* (P) *in the class* K_2 *it is sufficient that there exists a sequence of intervals* $[a_n, b_n]$ *such that for every* $n = 2, 3, \ldots$ *condition* (A_n) *and for every* $n = 1, 2, \ldots$ *condition* (B_n) *hold and* $(\mathrm{D}^{\gamma}_{n,m})$ *is satisfied with an* $m = m_n$, *whereas* γ *is fixed.*

Lemma 1 is proved in [4] (see [4], Lemma 1).

LEMMA 2. *If*

$$q_k \geqslant \tfrac{25}{8} \tag{1}$$

and for a given interval $[a_k, b_k]$ *condition* (C_k) *holds, then there exists an interval* $[a_{k+1}, b_{k+1}]$ *such that we have* (A_{k+1}), (B_{k+1}), (C_{k+1}) and $(\mathrm{D}^{\gamma}_{k,1})$ *with* $\gamma = \frac{1}{50}$.

Proof. By (C_k) we have

$$d_k = q_k(b_k - a_k) = \frac{1}{2} q_k = \frac{3}{2} + \frac{q_k - 3}{2}.$$

Consequently there exists a closed interval $[a_{k+1}, b_{k+1}]$ fulfilling (A_{k+1}), (B_{k+1}) and (C_{k+1}) for which

$$b_{k+1} \leqslant q_k b_k - \frac{q_k - 3}{2} = q_k\left(b_k + \frac{3}{2q_k} - \frac{1}{2}\right).$$

Hence, according to (1), we otbain

$$b_{k+1} \leqslant \frac{t_{k+1}}{t_k}\left(b_k - \frac{1}{50}\right), \quad \text{q. e. d.}$$

We shall say that the interval $[a_n, b_n]$ has the *property* (Q_n) if there exists an interval $[a_{n+1}, b_{n+1}]$ for which (A_{n+1}), (B_{n+1}) and (C_{n+1}) hold and the inequality

$$b_{n+1} \leqslant q_n b_n - \tfrac{1}{16} \tag{E_{n+1}}$$

is fulfilled. In the opposite case we shall say that the interval $[a_n, b_n]$ has the *property* (Q'_n).

LEMMA 3. *If for a given interval* $[a_k, b_k]$ *conditions* (C_k) *and* (Q_k) *hold, then there exists an interval* $[a_{k+1}, b_{k+1}]$ *satisfying* (A_{k+1}), (B_{k+1}), (C_{k+1}) *and* $(D^{\gamma}_{k,1})$ *with* $\gamma = \frac{1}{50}$.

Proof. In virtue of Lemma 2 it is sufficient to consider the case

$$3 \leqslant q_k < \tfrac{25}{8}.$$

In this case we take an interval $[a_{k+1}, b_{k+1}]$ fulfilling (A_{k+1}), (B_{k+1}), (C_{k+1}) and (E_{k+1}). By (E_{k+1}) we have

$$b_{k+1} \leqslant q_k\left(b_k - \frac{1}{16q_k}\right) < q_k\left(b_k - \frac{8}{16\cdot 25}\right) = q_k\left(b_k - \frac{1}{50}\right), \quad \text{q. e. d.}$$

LEMMA 4. *If* $q_k \geqslant \frac{25}{8}$ *and, for a given interval* $[a_k, b_k]$, (C_k) *holds, then the interval* $[a_k, b_k]$ *has the property* (Q_k).

The proof is analogous to that of Lemma 2, since we have

$$d_k = q_k(b_k - a_k) \geqslant \tfrac{25}{8}\cdot\tfrac{1}{2} = \tfrac{3}{2} + \tfrac{1}{16}.$$

LEMMA 5. *Let us suppose that for a given interval* $[a_k, b_k]$ *conditions* (C_k) *and* (Q'_k) *hold and let* $[a_{k+1}, b_{k+1}]$ *denote an interval for which conditions* (A_{k+1}), (B_{k+1}) *and* (C_{k+1}) *are fulfilled. (The existence of such an interval follows immediately from* (C_k)*). If this interval has property* (Q_{k+1}), *then there exists an interval* $[a_{k+2}, b_{k+2}]$ *such that the conditions* (A_{k+2}), (B_{k+2}), (C_{k+2}) *and* $(D^{\gamma}_{k,2})$ *with* $\gamma = \frac{4}{625}$ *hold.*

Proof. In view of Lemma 3, by (C_{k+1}) and (Q_{k+1}), there exists an interval $[a_{k+2}, b_{k+2}]$ for which we have (A_{k+2}), (B_{k+2}), (C_{k+2}) and $(D^{\gamma}_{k+1,1})$ with $\gamma = \frac{1}{50}$. Since (C_k) and (Q'_k) hold, Lemma 4 implies

$$q_k < \tfrac{25}{8}. \tag{F_k}$$

Hence, according to $(D^{\gamma}_{k+1,1})$ and (A_{k+1}) we obtain

$$\begin{aligned} b_{k+2} &\leqslant \frac{t_{k+2}}{t_{k+1}}\left(b_{k+1} - \frac{1}{50}\right) \leqslant \frac{t_{k+2}}{t_{k+1}}\left(q_k b_k - \frac{1}{50}\right) \\ &= \frac{t_{k+2}}{t_k}\left(b_k - \frac{1}{50q_k}\right) < \frac{t_{k+2}}{t_k}\left(b_k - \frac{4}{625}\right), \quad \text{q. e. d.} \end{aligned}$$

In the same way we prove

LEMMA 6. *If any three given intervals* $[a_k, b_k]$, $[a_{k+1}, b_{k+1}]$, $[a_{k+2}, b_{k+2}]$ *have the following properties*:

1) *The interval* $[a_k, b_k]$ *fulfils conditions* (C_k) *and* (Q'_k),

2) *the interval* $[a_{k+1}, b_{k+1}]$ *fulfils conditions* (A_{k+1}), (B_{k+1}), (C_{k+1}) *and* (Q'_{k+1}),

3) *the interval* $[a_{k+2}, b_{k+2}]$ *fulfils conditions* (A_{k+2}), (B_{k+2}), (C_{k+2}) *and* (Q'_{k+2}),

then there exists an interval $[a_{k+3}, b_{k+3}]$ *for which we have* (A_{k+3}), (B_{k+3}), (C_{k+3}) *and* $(\mathrm{D}^{\gamma}_{k,3})$ *with* $\gamma = 2^5/5^6$.

Now we proceed to the main lemma.

LEMMA 7. *If for a given interval* $[a_k, b_k]$ *condition* (C_k) *holds, then at least one of the following possibilities takes place:*

1° *there exists an interval* $[a_{k+1}, b_{k+1}]$ *such that conditions* (A_{k+1}), (B_{k+1}), (C_{k+1}) *and* $(\mathrm{D}^{\gamma}_{k+1})$ *with* $\gamma \geqslant 2^5/5^6$ *are fulfilled,*

2° *there exist two intervals* $[a_{k+1}, b_{k+1}]$ *and* $[a_{k+2}, b_{k+2}]$ *such that conditions* (A_{k+1}), (B_{k+1}), (A_{k+2}), (B_{k+2}), (C_{k+2}) *and* $(\mathrm{D}^{\gamma}_{k,3})$ *with* $\gamma \geqslant 2^5/5^6$ *are fulfilled,*

3° *there exist three intervals* $[a_{k+l}, b_{k+l}]$, $l = 1, 2, 3$, *such that conditions* (A_{k+l}), (B_{k+l}), $l = 1, 2, 3$, (C_{k+3}) *and* $(\mathrm{D}^{\gamma}_{k,3})$ *with* $\gamma \geqslant 2^5/5^6$ *are fulfilled.*

Proof. If (Q_k) holds, then it follows from Lemma 3 that the first possibility occurs. Let us assume that (Q'_k) holds. Since $b_k - a_k = \frac{1}{2}$ and $q_k \geqslant 3$, there exists an interval $[u_{k+1}, v_{k+1}]$ satisfying (A_{k+1}), (B_{k+1}) and (C_{k+1}). Hence, by (Q'_k), we obtain

$$(\mathrm{E}'_{k+1}) \qquad v_{k+1} > q_k b_k - \tfrac{1}{16}.$$

We first show that

$$(2) \qquad (v_{k+1} - 1) - q_k a_k < \tfrac{1}{2}.$$

In fact, if the contrary is true, then according to (A_{k+1}) we obtain (see fig. 1)

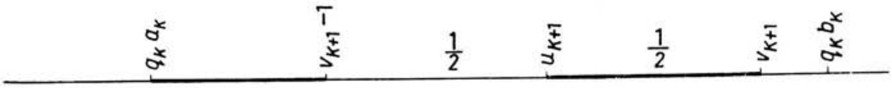

Fig. 1

$$q_k a_k \leqslant v_{k+1} - \tfrac{3}{2} < v_{k+1} - 1 \leqslant q_k b_k - 1.$$

Moreover,

$$[v_{k+1} - \tfrac{3}{2}, v_{k+1} - 1] = [u_{k+1}, v_{k+1}] \,(\mathrm{mod}\, 1).$$

Consequently the interval $[v_{k+1} - \frac{3}{2}, v_{k+1} - 1]$ satisfies (A_{k+1}), (B_{k+1}), (C_{k+1}) and (E_{k+1}), which contradicts (Q'_k).

We also note that

$$q_k a_k < v_{k+1} - 1. \tag{3}$$

This inequality immediately follows from (E'_{k+1}), $q_k \geqslant 3$ and $b_k = a_k + \frac{1}{2}$. By (2) and (3),

$$[q_k a_k, (v_{k+1} - 1)] \subset [u_{k+1}, v_{k+1}] \pmod 1, \tag{4}$$

i. e., the interval $[q_k a_k, (v_{k+1} - 1)]$ fulfills conditions (A_{k+1}) and (B_{k+1}) (but not (C_{k+1})).

If the interval $[u_{k+1}, v_{k+1}]$ has property (Q_{k+1}), then in virtue of Lemma 5 the second possibility is realized by putting $a_{k+1} = u_{k+1}$, $b_{k+1} = v_{k+1}$.

Now let us suppose that for the interval $[u_{k+1}, v_{k+1}]$ the property (Q'_{k+1}) holds. We shall denote by $[u_{k+2}, v_{k+2}]$ an interval satisfying conditions (A_{k+2}), (B_{k+2}), (C_{k+2}) and (E'_{k+2}). Repeating the former considerations we show that the interval $[q_{k+1} u_{k+1}, (v_{k+2} - 1)]$ also satisfies (A_{k+2}) and (B_{k+2}) (see fig. 2).

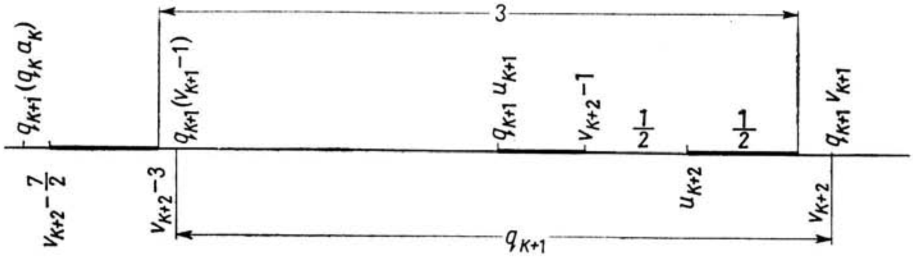

Fig. 2

We shall now prove that

$$q_{k+1}(q_k a_k) < v_{k+2} - \tfrac{7}{2} < q_{k+1}(v_{k+1} - 1). \tag{5}$$

We remind that for the intervals $[a_k, b_k]$, $[u_{k+1}, v_{k+1}]$, $[u_{k+2}, v_{k+2}]$ conditions (C_k), (A_{k+l}), (B_{k+l}), (C_{k+l}) $(l = 1, 2)$, (Q'_k) and (Q'_{k+1}) hold. Consequently, by the definition of (Q'_n), (E'_{k+1}) and (E'_{k+2}) are also true. Hence

$$\begin{aligned} v_{k+2} &> q_{k+1} v_{k+1} - \tfrac{1}{16} > q_{k+1}\left(q_k b_k - \tfrac{1}{16}\right) - \tfrac{1}{16} \\ &= q_{k+1} q_k \left(a_k + \tfrac{1}{2}\right) - \tfrac{1}{16}(q_{k+1} + 1) \geqslant q_{k+1}(q_k a_k) + \tfrac{9}{2} - \tfrac{1}{16}(q_{k+1} + 1). \end{aligned}$$

In virtue of Lemma 4 it follows from (C_k) and (Q'_{k+1}) that

$$q_{k+1} < \tfrac{25}{8}. \tag{F_{k+1}}$$

Therefore we obtain

$$v_{k+2} - \tfrac{7}{2} > q_{k+1}(q_k a_k) + 1 - \tfrac{33}{128} > q_{k+1}(q_k a_k). \tag{6}$$

From (A_{k+2}) and (F_{k+1}) it follows that

$$v_{k+2}-\tfrac{7}{2} \leqslant q_{k+1}v_{k+1}-\tfrac{7}{2} < q_{k+1}v_{k+1}-q_{k+1}.$$

Hence, by (6), inequality (5) holds.

If v_{k+2} satisfies the inequality

$$(G_{k+2}) \qquad v_{k+2}-3 \leqslant q_{k+1}(v_{k+1}-1)$$

(see fig. 2), then we shall put

$$(7) \qquad \begin{aligned} a_{k+1} &= q_k a_k, & a_{k+2} &= u_{k+2}-3, \\ b_{k+1} &= v_{k+1}-1, & b_{k+2} &= v_{k+2}-3. \end{aligned}$$

As we have already noticed, the interval $[q_k a_k, (v_{k+1}-1)]$ satisfies conditions (A_{k+1}) and (B_{k+1}) in virtue of (4). From (5) and (G_{k+2}) it follows that for the interval $[u_{k+2}-3, v_{k+2}-3]$ the conditions (A_{k+2}), (B_{k+2}) and (C_{k+2}) hold (see fig. 2). Moreover, by (A_{k+1}) and (F_k),

$$b_{k+1} = v_{k+1}-1 \leqslant q_k b_k - 1 = q_k\left(b_k - \frac{1}{q_k}\right) < \frac{t_{k+1}}{t_k}\left(b_k-\tfrac{8}{25}\right),$$

i. e., the condition $(D^{\gamma}_{k,1})$ with $\gamma = \frac{8}{25}$ is satisfied. Thus, the second condition of the assertion is realized in the case when (Q'_k), (Q'_{k+1}) and (G_{k+2}) hold.

Now let us assume that (Q'_k), (Q'_{k+1}) and

$$(G'_{k+2}) \qquad v_{k+2}-3 > q_{k+1}(v_{k+1}-1)$$

hold. In this case putting $a_{k+1} = q_k a_k$, $b_{k+1} = v_{k+1}-1$ we infer by (5) that the interval $[v_{k+2}-\frac{7}{2}, q_{k+1}(v_{k+1}-1)]$ satisfies the conditions (A_{k+2}) and (B_{k+2}) (see fig. 3). We shall consider the intervals with the index

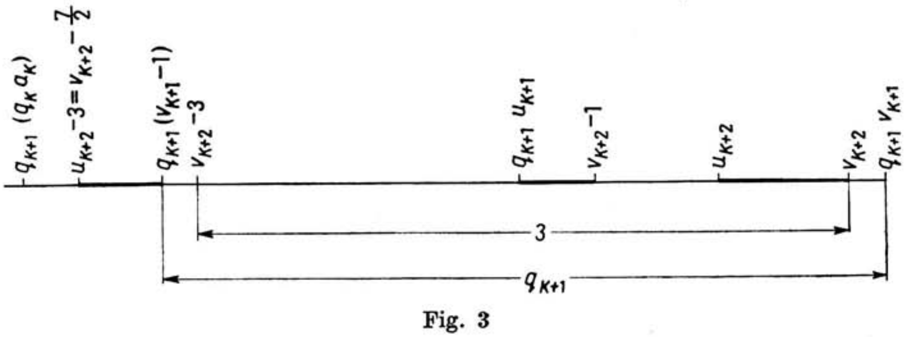

Fig. 3

$n = k+3$. If the interval $[u_{k+2}, v_{k+2}]$ has the property (Q_{k+2}), then the third condition of the assertion follows from Lemma 6. (In this case we put $a_{k+1} = u_{k+1}$, $b_{k+1} = v_{k+1}$, $a_{k+2} = u_{k+2}$, $b_{k+2} = v_{k+2}$.)

Let us suppose that for $[u_{k+2}, v_{k+2}]$ the property (Q'_{k+2}) holds. Denote by $[u_{k+3}, v_{k+3}]$ the interval satisfying conditions (B_{k+3}), (C_{k+3}) and

$$(A_{k+3}) \qquad q_{k+2}u_{k+2} \leqslant u_{k+3} < v_{k+3} \leqslant q_{k+2}v_{k+2}.$$

If (G_{k+3}) holds we put

$$(7') \qquad \begin{array}{lll} a_{k+1} = u_{k+1}, & a_{k+2} = q_{k+1}u_{k+1}, & a_{k+3} = u_{k+3}-3, \\ b_{k+1} = v_{k+1}, & b_{k+2} = v_{k+2}-1, & b_{k+3} = v_{k+3}-3. \end{array}$$

The interval $[a_{k+2}, b_{k+2}]$ fulfils (A_{k+2}) and (B_{k+2}) which can be shown in the same way as the conditions (A_{k+1}) and (B_{k+1}) have been checked for the interval $[q_k a_k, (v_{k+1}-1)]$ under assumption (Q'_k). Further, the interval $[a_{k+3}, b_{k+3}]$ satisfies (A_{k+3}). We obtain this from (Q'_{k+1}), (Q'_{k+2}) and (G_{k+3}) similarly as (A_{k+2}) has been proved for the interval $[u_{k+2}-3, v_{k+2}-3]$ under assumption (Q'_k), (Q'_{k+1}) and (G_{k+2}). Finally, $[a_{k+3}, b_{k+3}]$ satisfies (B_{k+3}) and (C_{k+3}), since $[u_{k+3}, v_{k+3}]$ does. Taking into account that the intervals $[u_{k+1}, v_{k+1}]$ and $[u_{k+2}, v_{k+2}]$ satisfy (A_{k+1}) and (A_{k+2}), we obtain from (F_k) and (F_{k+1})

$$\begin{aligned} b_{k+2} &= v_{k+2}-1 \leqslant q_{k+1}v_{k+1}-1 \leqslant q_{k+1}q_k b_k-1 \\ &\leqslant q_{k+1}q_k\left(b_k-\frac{64}{625}\right) = \frac{t_{k+2}}{t_k}\left(b_k-\frac{64}{625}\right), \end{aligned}$$

i. e., the condition $(D^{\gamma}_{k,2})$ with $\gamma = \frac{64}{625}$ is also fulfilled. Hence, in the case that (Q'_k), (Q'_{k+1}), (Q'_{k+2}) and (G_{k+3}) hold, the third possibility of the assertion is again realized.

Now let us suppose that

$$(G'_{k+3}) \qquad v_{k+3}-3 > q_{k+2}(v_{k+2}-1).$$

We shall prove that the inequality

$$(8) \qquad q_{k+2}\left(v_{k+2}-\frac{7}{2}\right) \leqslant v_{k+3}-\frac{21}{2} < v_{k+3}-10 \leqslant q_{k+2}q_{k+1}(v_{k+1}-1)$$

follows from (Q'_k), (Q'_{k+1}), (Q'_{k+2}) and (G'_{k+3}). Taking (G'_{k+3}) into account we obtain

$$\begin{aligned} v_{k+3}-\frac{21}{2} &> q_{k+2}(v_{k+2}-1)-\frac{15}{2} \\ &\geqslant q_{k+2}(v_{k+2}-1)-\frac{q_{k+2}}{3}\cdot\frac{15}{2} = q_{k+2}\left(v_{k+2}-\frac{7}{2}\right). \end{aligned}$$

Thus the first inequality in (8) is proved. Since (C_{k+2}) and (Q'_{k+2}) hold, we have, by Lemma 4,

$$(F_{k+2}) \qquad q_{k+2} < \tfrac{25}{8}.$$

From (A_{k+3}), (A_{k+2}), (F_{k+1}) and (F_{k+2}) it follows that

$$v_{k+3}-10 \leqslant q_{k+2}v_{k+2}-10 \leqslant q_{k+2}q_{k+1}v_{k+1}-10$$
$$< q_{k+2}q_{k+1}v_{k+1}-q_{k+2}q_{k+1} = q_{k+2}q_{k+1}(v_{k+1}-1),$$

i. e., the third inequality in (8) is also true. Hence, if we put

$$(9)\qquad \begin{aligned} &a_{k+1} = q_k a_k, && a_{k+2} = v_{k+2}-\tfrac{7}{2}, && a_{k+3} = u_{k+3}-10,\\ &b_{k+1} = v_{k+1}-1, && b_{k+2} = q_{k+1}(v_{k+1}-1), && b_{k+3} = v_{k+3}-10, \end{aligned}$$

the intervals $[a_{k+l}, b_{k+l}]$, $l = 1, 2, 3$, satisfy conditions (A_{k+l}), (B_{k+l}), $l = 1, 2, 3$, respectively and $[a_{k+3}, b_{k+3}]$ fulfils (C_{k+3}). Moreover, by (A_{k+1}) and (F_k) we obtain

$$b_{k+1} = v_{k+1}-1 \leqslant q_k b_k-1 \leqslant \frac{t_{k+1}}{t_k}\left(b_k-\frac{8}{25}\right).$$

Therefore $(D^{\gamma}_{k,1})$ with $\gamma > 2^5/5^6$ is also satisfied.

The reader may note that the following cases have been considered in the above-mentioned proof:

1) (Q_k);

2) (Q'_k), (Q_{k+1});

3) (Q'_k), (Q'_{k+1}), (G_{k+2});

4) (Q'_k), (Q'_{k+1}), (G'_{k+2}), (Q_{k+2});

5) (Q'_k), (Q'_{k+1}), (G'_{k+2}), (Q'_{k+2}), (G_{k+3});

6) (Q'_k), (Q'_{k+1}), (G'_{k+2}), (Q'_{k+2}), (G'_{k+3}).

Thus all possibilities have been exhausted and Lemma 7 is proved.

Proof of Theorem 1. Let us put $a_1 = 1$, $b_1 = \frac{3}{2}$, if $\varepsilon_1 = 0$ and $a_1 = \frac{1}{2}$, $b_1 = 1$, if $\varepsilon_1 = 1$ and build the sequence of intervals $[a_n, b_n]$, $n = 1, 2, \ldots$, by induction as follows: if $b_k-a_k = \frac{1}{2}$, then we choose subsequent intervals according to Lemma 7 till we get an interval of length $\frac{1}{2}$. On account of Lemma 1 in order to prove Theorem 1 it is sufficient to show that condition $(D^{\gamma}_{n,m})$ is satisfied for every n with a $\gamma > 0$ independent of n, e. g. with $\gamma = 2^{11}/5^{10}$.

1) If $b_n-a_n = \frac{1}{2}$, then we obtain $(D^{\gamma}_{n,m})$ $(\gamma \geqslant 2^5/5^6$, $m = 1$, 2 or 3) as a direct consequence of Lemma 7 putting $n = k$.

2) If $b_n-a_n < \frac{1}{2}$ then, by Lemma 7, it is sufficient to distinguish the following cases:

a) Case (Q'_{n-1}), (Q'_n), (G_{n+1}). Then the intervals $[a_n, b_n]$ and $[a_{n+1}, b_{n+1}]$ satisfy formulae (7) with $n = k+1$ or (7′) with $n = k+2$. Since $b_{n+1}-a_{n+1} = \frac{1}{2}$ there exists, by Lemma 7, a positive integer m

($m = 1, 2$ or 3) such that

$$(\mathrm{D}^{\gamma}_{n+1,m}) \qquad b_{n+1+m} \leqslant \frac{t_{n+1+m}}{t_{n+1}}\left(b_{n+1} - \frac{2^5}{5^6}\right).$$

Consequently, according to (A_{n+1}) and (F_n) we obtain

$$b_{n+1+m} \leqslant \frac{t_{n+1+m}}{t_{n+1}}\left(q_n b_n - \frac{2^5}{5^6}\right) \leqslant \frac{t_{n+1+m}}{t_n}\left(b_n - \frac{2^8}{5^8}\right).$$

Thus $(\mathrm{D}^{\gamma}_{n,m+1})$ with $\gamma > 2^{11}/5^{10}$, $m = 1, 2$ or 3 holds.

b) Case (Q'_{n-1}), (Q'_n), (G'_{n+1}), (Q'_{n+1}), (G'_{n+2}). The intervals $[a_n, b_n]$, $[a_{n+1}, b_{n+1}]$ and $[a_{n+2}, b_{n+2}]$ satisfy formulae (9) with $n = k+1$. In this case $b_n - a_n < \frac{1}{2}$, $q_n < \frac{25}{8}$, $b_{n+1} - a_{n+1} < \frac{1}{2}$, $q_{n+1} < \frac{25}{8}$, $b_{n+2} - a_{n+2} = \frac{1}{2}$. Reasoning like in the previous case we get the inequality

$$b_{n+m} < \frac{t_{n+m}}{t_n}\left(b_n - \frac{2^{11}}{5^{10}}\right)$$

for a positive integer m $(2 \leqslant m \leqslant 5)$.

c) Case (Q'_{n-2}), (Q'_{n-1}), (G'_n), (Q'_n), (G'_{n+1}). The intervals $[a_n, b_n]$ and $[a_{n+1}, b_{n+1}]$ fulfill formulae (9) with $n = k+2$. In this case $b_n - a_n < \frac{1}{2}$, $q_n < \frac{25}{8}$, $b_{n+1} - a_{n+1} = \frac{1}{2}$. Therefore, by Lemma 7, there exists a positive integer m $(1 \leqslant m \leqslant 4)$ such that

$$b_{n+m} < \frac{t_{n+m}}{t_n}\left(b_n - \frac{2^8}{5^8}\right).$$

Theorem 1 is thus proved.

In a similar way we can prove the following theorems.

THEOREM 2. *The sequence*

$$t_n = \left(3 - \frac{1}{s+1}\right)^n \quad (n = 1, 2, 3, \ldots),$$

where s is any positive integer, has the property P *in the class K_2.*

Remark. The case $\{(5/2)^n\}$ must be considered separately.

THEOREM 3. *If the q_n's assume only two integer values m or $m+1$ $(m \geqslant 2)$ and the number of consecutive terms $q_n = m$ does not exceed a finite upper bound, then the sequence $\{t_n\}$ has the property* P *in the class K_m.*

Remark. In the case of $m = 2$ the period δ of the function $f(t)$ for which $f(t_n) = \varepsilon_n$ $(\varepsilon_n = 0, 1)$ can be obtained by the formula

$$\frac{1}{\delta} = \sum_{n=1}^{\infty} \frac{a_n}{t_n},$$

where $a_1 = \frac{1}{2}\varepsilon_1$ and for $n = 1, 2, \ldots$

$$a_{n+1} = \begin{cases} \frac{1}{2}\varepsilon_{n+1}, & \text{if} \quad q_n = 2, \\ \frac{1}{2}|\varepsilon_{n+1} - \varepsilon_n|, & \text{if} \quad q_n = 3. \end{cases}$$

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