Let h be a homeomorphism which maps F_{Hn+i} into F_C according to (56), and let $F = h(F_{Hn+i})$. Putting $k_1 = n-i$, i.e. $F^*_{Pk_1} = F^*_{Pn-i}$, and $k_2 = n-j$, i.e. $F^*_{Pk_2} = F^*_{Pn-j}$, we have identically

$$F \cup F^*_{Pk_1} = h(F_{Hn+i}) \cup F^*_{Pn-i}. \tag{66}$$

Further, let h_1 be a homeomorphism of $F_{Hn+i} \cup F^*_{Pn-i}$ defined as follows:

$$h_1(p) = \begin{cases} h(p) & \text{when} \quad p \in F_{Hn+i}, \\ p & \text{when} \quad p \in F^*_{Pn-i}. \end{cases}$$

Therefore we have by (66) $F \cup F^*_{Pk_1} = h_1(F_{Hn+i} \cup F^*_{Pn-i})$, whence $F \cup F^*_{Pk_1} = h_1(D_i)$. Thus $D_i = h_1^{-1}(F \cup F^*_{Pk_1})$ and, by (65), $gh_1^{-1}(F \cup F^*_{Pk_1}) = F^*_{Pk_2}$. In consequence of P2 we then have $k_2 \leqslant k_1$, i.e. $n-j \leqslant n-i$, contrary to (64).

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INSTYTUT MATEMATYCZNY UNIWERSYTETU WROCŁAWSKIEGO
MATHEMATICAL INSTITUTE OF THE WROCŁAW UNIVERSITY

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On the axiom of determinateness *

by

Jan Mycielski (Wrocław)

1. Introduction. It is the purpose of this paper to study the consequences of a mathematical proposition introduced by H. Steinhaus and the author in [17]. This proposition, called *axiom of determinateness* and denoted by (A), is inconsistent with the axiom of choice but has many interesting implications. Several of them are opposite to the 'sad facts' following from the axiom of choice such as, e.g., paradoxical decompositions of the sphere. The actual state of knowledge permits to conjecture that replacing in the Zermelo-Fraenkel-Skolem set theory (ZFS) the axiom of choice by (A) we obtain a consistent theory.

The failure of the axiom of choice in this new theory is considered as a 'sad fact' by the author. He believes that the natural models of ZFS (see e.g. [14]) are real enough to prove the consistency of ZFS. The new theory does not present any such evidence of consistency. We can only hope that some submodels of the natural models of ZFS are models of the new theory (1). In that case (A) may be considered as a limitation of the notion of a set excluding some 'patological' ZFS-sets (2). From such a point of view (A) seems to be very successful.

Most of the results of this paper have a clear game-theoretical meaning. They are claiming that such and such constructions permit constructions of such and such not-determined infinite games with perfect information (3). Of course their validity does not depend on the consistency of (A).

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(1) I.e. a subclass of the class of all sets with the same membership relation. It would be still more pleasant if such a submodel contains all the real numbers of the natural model.

(2) Some other general remarks on (A) are given in [17]. Very few alternations to the axiom of choice were considered in literature. Some propositions of that kind are given by Church [3] and Specker [30]. Tarski has considered the proposition 'the set of real numbers is a denumerable union of denumerable sets' (by the results given below this is inconsistent with (A)). But none of them seem to have so many interesting consequences as (A).

(3) The definition and theory of such games is exposed in [15].

In Section 2 some general notations used throughout this paper are given. In Section 3 there is a table of the logical relations of (A) and several other mathematical propositions, in Section 4 different formulations of (A) and its relations to the theory of games, in Section 5 some results connected with the problems of independence and consistency of (A), in Section 6 some proofs of the theorems formulated in the preceding sections. In Section 7 we investigate some propositions generalizing (A). Finally, an appendix contains the proofs of some facts which are easy to formulate or essentially known but difficult to find in literature or unpublished.

2. Notation and general definitions. An ordinal number is identified with the set of preceding ordinals, e.g.

$$1 = \{0\},\ 2 = \{0, 1\},\ \ldots,\ n = \{0, 1, \ldots, n-1\}, \qquad \omega = \{0, 1, \ldots\}, \ldots;$$

ω_α denotes the αth initial ordinal ($\omega_0 = \omega$).

If f is a function, i.e. $f = \{(x, f(x)): x \in Df\}$, where Df is the domain of f, then $f|X = \{(x, f(x)): x \in X \cap Df\}$ for any set X.

For every sets X and Y, Y^X denote the set of all functions $f\colon X \to Y$.

$\mathcal{T}$ denotes the set theory ZFS without the axiom of choice (see [14]).

$|X|$ denotes the cardinal number of the set X (see [14] for a treatment of cardinal numbers in $\mathcal{T}$).

A set of sequences X^ω is often treated as a topological space with the Tychonoff product topology, X being treated as a discrete Hausdorff space. Hence $2^\omega = \{0, 1\}^\omega$ is the Cantor discontinuum and ω^ω is the space of irrational numbers (4).

For every $X \neq 0$ and $P \subseteq X^\omega$, $G_X(P)$ denotes the following infinite positional game with perfect information (3): There are two players I and II. They choose alternatively consecutive terms of a sequence $x_0, x_1, \ldots$ ($x_i \in X$, x_{2i} are the choices of I and x_{2i+1} the choices of II). Each player knows $X, P, x_0, \ldots, x_{n-1}$ when he is choosing x_n. Player I wins if $(x_0, x_1, \ldots) \in P$ and player II wins if $(x_0, x_1, \ldots) \notin P$.

$G^*_X(P)$ and $G^{**}_X(P)$ are the following modifications of the above game: It is no longer supposed that x_{2i} are chosen by I and x_{2i+1} by II, but the first choice is still that of player I. In G^* player I in each of his choices can give any finite (may be empty) sequence of elements of X; in G^{**} both players can choose finite non empty sequences of elements of X. The sequence $(x_0, x_1, \ldots)$ is a concatenation of the consecutive choices (it is always infinite).

$\mathscr{A}_X(P)$ stands for '*the game* $G_X(P)$ *is determined* (i.e. *one of the players has a winning strategy*)'. $\mathscr{A}^*_X(P)$ and $\mathscr{A}^{**}_X(P)$ have the same meaning pertaining to the games $G^*_X(P)$ and $G^{**}_X(P)$, respectively.

(4) Ordinal exponentiation will not be used in that paper.

$\mathscr{A}_X$ stands for '$\mathscr{A}_X(P)$ *holds for every* $P \subseteq X^\omega$'. $\mathscr{A}^*_X$ and $\mathscr{A}^{**}_X$ have the same meaning concerning $\mathscr{A}^*_X(P)$ and $\mathscr{A}^{**}_X(P)$, respectively.

(A) denotes the proposition $\mathscr{A}_2$.

$\mathcal{M}$ stands for '*every subset of the real line is Lebesgue-measurable*'. (Let us recall (for the proof see [18]) that $\mathcal{M}$ implies (5) the following self-refinement: *for every finite denumerably additive measure* μ *over the field of Borel subsets of a separable metric space* X *and every* $Y \subseteq X$ *there are Borel sets* $B_1 \subseteq Y \subseteq B_2$ *such that* $\mu(B_1) = \mu(B_2)$.)

$\mathcal{B}$ stands for '*every subset of the Cantor discontinuum has the property of Baire* (i.e. *is of the form* $G \cup K_1 - K_2$ *where* G *is open and* K_1 *and* K_2 *are of the first category*)'. (Let us recall that $\mathcal{B}$ implies (5) that *every subset of a separable metric space has the property of Baire*. The proof is analogous to that of the above mentioned generalization of $\mathcal{M}$ (6).)

$\mathcal{P}$ stands for '*every non-denumerable subset of the Cantor discontinuum has a perfect subset*'. ($\mathcal{P}$ also implies (5) that *every non-denumerable separable metric space contains a compact perfect set*.)

$\mathcal{C}$ stands for the following weak form of the axiom of choice: *for every family of sets* $\boldsymbol{F}$, *such that* $0 \notin \boldsymbol{F}$, $|\boldsymbol{F}| \leqslant \aleph_0$ *and* $|\bigcup_{X \in \boldsymbol{F}} X| \leqslant 2^{\aleph_0}$ *there exists a choice set* (7).

3. The main consequences of (A). The following implications and equivalences can be proved in $\mathcal{T}$

$$\begin{tikzcd}
 & \mathscr{A}_2 \arrow[r, leftrightarrow] & \mathscr{A}_\omega \arrow[r] \arrow[dl] \arrow[dr] & \mathscr{A}^*_\omega \arrow[r] \arrow[d] & \mathscr{A}^{**}_\omega \arrow[r, leftrightarrow] & \mathscr{A}^{**}_2 \arrow[d, leftrightarrow] \\
 & \mathcal{C} & \mathcal{M} & \mathscr{A}^*_2 \arrow[d, leftrightarrow] & & \mathcal{B} \\
 & & & \mathcal{P} & &
\end{tikzcd}$$

Some of these implications between the propositions $\mathscr{A}$ follow from the following obvious theorem of $\mathcal{T}$

(3.1) *If* $Y \subseteq X$ *then* $\mathscr{A}_X \to \mathscr{A}_Y$, $\mathscr{A}^*_X \to \mathscr{A}^*_Y$ *and* $\mathscr{A}^{**}_X \to \mathscr{A}^{**}_Y$.

The remaining implications between the propositions $\mathscr{A}$ will be proved in Section 6. In view of (3.1) and the first and last implications of the first line of the above table we get also in $\mathcal{T}$

(3.2) $\mathscr{A}_2 \leftrightarrow \mathscr{A}_n$ *and* $\mathscr{A}^{**}_2 \leftrightarrow \mathscr{A}^{**}_n$ *for every* $2 \leqslant n \leqslant \omega$.

$\mathscr{A}_\omega \to \mathcal{C}$ will be proved in Section 7 (8).

(5) In the theory $\mathcal{T}$ & $\mathcal{C}$ (see below for the meaning of $\mathcal{C}$).

(6) For a detailed study of the property of Baire see [8].

(7) The axiom of choice for denumerable families of sets, which implies $\mathcal{C}$, is often used in literature, see [31], [32].

(8) This implication was found by S. Świerczkowski and independently by D. Scott and the author.

$\mathcal{A}_\omega \to \mathcal{M}$ is proved in [18].

$\mathcal{A}_2^* \leftrightarrow \mathcal{P}$ follows clearly from the following theorem of Morton Davis [4] (valid in $\mathfrak{T}$):

(3.3) *$G_2^*(P)$ is a win for I iff P has a perfect subset* [9] *and is a win for II iff P is at most denumerable.*

$\mathcal{A}_2^{**} \leftrightarrow \mathcal{B}$ follows from the following theorem essentially due to Banach and Mazur [10]:

(3.4) *$G_2^{**}(P)$ is a win for I iff P is residual in some non empty open set in 2^ω and is a win for II iff P is of the first category.*

Indeed for the implication $\mathcal{B} \to \mathcal{A}_2^{**}$ this is obvious. An easy construction which gives a set P which is nowhere residual nor of the first category, starting from any set without the property of Baire, yields $\mathcal{A}_2^{**} \to \mathcal{B}$.

It would be interesting to get any other implications between the propositions of that table, but this is probably impossible by the methods used in this paper. E.g. $\mathcal{A}_2^* \to \mathcal{A}_2^{**}$ would give $\mathcal{P} \to \mathcal{B}$.

The above table shows what results can be obtained in the theory $\mathfrak{T}$ & (A). In particular, $\mathcal{C}$ permits to get most of the applications of the axiom of choice in analysis, e.g., it implies the denumerable additivity of the Lebesgue measure and of sets of the first category [11], the equivalence of Cauchy and Heine definitions of continuity of functions (see [25]), etc. [12] (Let us recall that the compactness of the Hilbert cube I^ω, the Hahn-Banach theorem for separable Banach spaces [13] and the existence of the Haar measure for all locally compact groups (see [2]) can be obtained without any use of the axiom of choice unlike in the usual proofs by the Tychonoff theorem (which is of course inconsistent in $\mathfrak{T}$ & (A)).)

For any set X, $\boldsymbol{B}(X)$ denotes the Boolean algebra of all subsets of X and $\mathfrak{k}$ the cardinal of the factor algebra $\boldsymbol{B}(\omega)\backslash\boldsymbol{I}$, where $\boldsymbol{I}$ is the ideal of finite subsets of ω.

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[9] $P \subseteq 2^\omega$ and we have the topology introduced in Section 2.

[10] The first published proof with a natural generalization was given by Oxtoby [21]; the game was invented by S. Mazur and the theorem conjectured in the *Scottish Book*; a solution of S. Banach was announced there in 1935. In our case the proof of [21] must be modified to avoid the axiom of choice, but this is easy since 2^ω is a separable space.

[11] The proofs are given in the Appendix.

[12] For all these results the supposition of separability of the involved spaces is necessary. Cf. [25].

[13] The Hahn-Banach theorem in its general form implies the existence of a non-negative and non-zero finitely additive measure over every Boolean algebra (and actually is equivalent in $\mathfrak{T}$ to this theorem — see [23]). I do not know if these statements are consistent with (A).

Each of the propositions $\mathcal{B}$ and $\mathcal{M}$ implies the following statements

(o) *There is no maximal non-principal ideal in $\boldsymbol{B}(\omega)$* [14].

(i) *There are sets X such that there is a maximal non-principal denumerably additive ideal in $\boldsymbol{B}(X)$, e.g., if $|X| = \mathfrak{k}$* [15].

(ii) *For every $2 \leqslant n \leqslant \omega$ there is a family of sets $\boldsymbol{F}_n$ such that $|\bigcup_{X \in F_n} X| = \mathfrak{k}$ each member of $\boldsymbol{F}_n$ is of potency $|n|$ and there is no choice set for $\boldsymbol{F}_n$* [16].

The proposition '$\mathcal{B}$ and $\mathcal{M}$' imply

(iii) *If G is a metric topological group dense in itself which is not of the first category onto itself or has a left invariant non-zero finite Borel measure vanishing on points and H a denumerable subgroup of G dense in G then the family G/H has no choice set and the set G/H satisfies* (i). *If moreover H is normal and for every $a \in G$, $a^2 \in H$ implies $a \in H$ then the set of unordered pairs $\{\{aH, a^{-1}H\}\colon\ a \in G - H\}$ has no choice set* [17].

The proposition $\mathcal{P}$ implies obviously the following statemens

(iv) *There is no cardinal $\mathfrak{n}$ such that $\aleph_0 < \mathfrak{n} < 2^{\aleph_0}$* [18].

(v) *There is no choice set for the Lebesgue decomposition of the real line* [19].

Each of the propositions $\mathcal{B}$, $\mathcal{M}$, $\mathcal{P}$ implies (this is well known)

(vi) *There is no well ordering of the real numbers.*

Each of the propositions (ii) with $n < \omega$ obviously implies

(vii) *A set of potency $\mathfrak{k}$ can not be ordered.*

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[14] $\mathcal{M} \to$(o) has been proved by Sierpiński [28], and $\mathcal{B} \to$(o) can be obtained in an analogous way.

[15] A proof is given in the Appendix. If X satisfies (i) then of course every image $Y = f(X)$, with $|f^{-1}(y)| \leqslant \aleph_0$ for every $y \in Y$, satisfies (i). The axiom of choice implies that if a set satisfying (i) exists its cardinal is much larger than the first inaccessible cardinal (this is a recent improvement of a classical result of Ulam [35] obtained by Tarski and his school [7], [34]) and the axiom of constructibility implies that there is no such set at all (Scott [24]).

[16] This is essentially known (see [26], [27]), but a proof is given in the Appendix.

[17] The first part is essentially due to Vitali; for the second part, see Sierpiński [26], [27]. We supply a proof in the Appendix. It can be seen from further results that (A) implies that $|G| = 2^{\aleph_0}$, but I do not know if $|G/H|$ is constant (sometimes $|G/H| = \mathfrak{k}$—see Appendix).

[18] Hence $\mathcal{P}$ implies the negation of the Cantor hypothesis $2^{\aleph_0} = \aleph_1$ and of the Lusin hypothesis $2^{\aleph_0} = 2^{\aleph_1}$.

[19] This decomposition is given in the Appendix.

Let *in* denote the relation of incomparability of cardinals, i.e. $\mathfrak{m}$ *in* $\mathfrak{n} \leftrightarrow \mathrm{non}(\mathfrak{m} \leqslant \mathfrak{n} \vee \mathfrak{n} \leqslant \mathfrak{m})$. The conjunction of (iv), (vi) and (vii) implies [20] the following inequalities and incomparabilities

$$2^{\aleph_0} < \mathfrak{k};$$

$$\aleph_1 < \aleph_1 + 2^{\aleph_0} < \aleph_1 + \mathfrak{k};$$

$$2^{\aleph_0} < 2^{\aleph_0} + \aleph_1 < 2^{\aleph_1} < 2^{\aleph_1} + \mathfrak{k} < 2^{\mathfrak{k}} = 2^{2^{\aleph_0}};$$

$$2^{\aleph_0} \text{ in } \aleph_1;$$

$$2^{\aleph_1} < \mathfrak{k} \quad \text{or} \quad (\mathfrak{k} \text{ in } 2^{\aleph_1} \text{ in } \aleph_1 + \mathfrak{k} < 2^{\aleph_1} + \mathfrak{k}).$$

These relations show that there are much less one-to-one mappings if the axiom of choice is replaced by (A). The first inequality shows that the image of a set can have a cardinality larger than this set. The third line shows that, in contrast to (iv), there are at least three cardinal numbers between $2^{\aleph_0}$ and $2^{2^{\aleph_0}}$. Therefore this theory of cardinal numbers does not present this interest which the classical one has [21].

4. Some equivalent forms of (A). Let us consider the product $X^\omega \times X^\omega$, where X is a non-empty set. Denote by X_0^ω and X_1^ω the first and second coordinate of this Cartesian square. Let F_0 denote the set of all mappings $f\colon X_0^\omega \to X_1^\omega$ which are of the form

(4.1) $$f(x) = \big(f_0(x|0), f_1(x|1), f_2(x|2), \ldots\big) \qquad (x \in X_0^\omega)$$

and F_1 the set of all mappings $f\colon X_1^\omega \to X_0^\omega$ which are of the form

(4.2) $$f(x) = \big(f_0(x|1), f_1(x|2), f_2(x|3), \ldots\big) \qquad (x \in X_1^\omega).$$

THEOREM 1 [22]. *The proposition $\mathcal{A}_X$ is equivalent to the following one*

$\mathcal{F}_X$. *Every set $Q \subseteq X^\omega \times X^\omega$ contains by inclusion a function $f \in F_0$ or its complement $X^\omega \times X^\omega - Q$ contains by inclusion a function $f \in F_1$.*

Proof. Let us consider the mapping $\varphi\colon X^\omega \times X^\omega \to X^\omega$ defined by

$$\varphi\big((x_0, x_1, \ldots), (y_0, y_1, \ldots)\big) = (x_0, y_0, x_1, y_1, \ldots)$$

and the game $G_X(P)$, where $P = \varphi(Q)$.

The members of F_0 represent strategies of player I (the condition (4.1) means that the nth choice of I takes into account only the first $n-1$ choices of II) and the members of F_1 represent strategies of II ((4.2) means that the nth choice of II takes into account only the first n choices of I). Of course $f \in F_0$ is a winning strategy for I iff $f \subseteq Q$ and $f \in F_1$ is a winning strategy for II iff $f \subseteq X^\omega \times X^\omega - Q$. q.e.d.

COROLLARY. $(A) \leftrightarrow \mathcal{F}_n$ *for* $2 \leqslant n \leqslant \omega$.

Proof. By (3.2) and Theorem 1.

Remarks. 1. All functions satisfying (4.1) or (4.2) are continuous (for the topology see Section 2).

2. Let us consider the usual metrisation of the Cantor discontinuum 2^ω given by the identification with the set of real numbers $\{\sum_1^\infty 2c_i/3^i\colon c_i \in \{0,1\}\}$. Then, for $X = 2$, the condition (4.1) is equivalent to the Lipschitz condition with constant 1/3 and (4.2) to the Lipschitz condition with constant 1.

3. It is well known that the existence of a well ordering of the reals implies the existence of a set $P_0 \subseteq 2^\omega \times 2^\omega$ such that neither P_0 nor $2^\omega \times 2^\omega - P_0$ contains a perfect subset. Since the graph of a continuous function over 2^ω is perfect, then $\mathcal{F}_2$ fails and we get non-$\mathcal{A}_2$.

4. Theorem 1 and Remark 1 suggest the study of the following proposition weaker than $\mathcal{F}_X$:

$\mathcal{F}'_X$. *Every set $Q \subseteq X^\omega \times X^\omega$ contains by inclusion a continuous function $f\colon X_0^\omega \to X_1^\omega$ or its complement $X^\omega \times X^\omega - Q$ contains by inclusions a continuous function $f\colon X_1^\omega \to X_0^\omega$.*

But I do not know any interesting consequence of $\mathcal{F}'_X$ except that $\mathcal{F}'_2$ excludes decompositions of 2^ω into pairs of totally imperfect sets.

THEOREM 2 [23]. *The proposition $\mathcal{A}_\omega$ is equivalent to the following one*

$\mathcal{G}$. *Every infinite positional game $\Gamma(\Phi) = (M, S, J, \Phi)$* (see [15]), *where M is at most denumerable, is determined.*

Proof. The implication $\mathcal{G} \to \mathcal{A}_\omega$ is obvious since $G_\omega(P)$ are games of the form $\Gamma(\Phi)$, where $M = \omega$, $S = M^\omega$,

(*) $$J(s|n) = \begin{cases} I & \text{if } n \text{ is even}, \\ II & \text{if } n \text{ is odd}, \end{cases}$$

and Φ is the characteristic function of P over S.

For proving $\mathcal{A}_\omega \to \mathcal{G}$ it is enough to show on account of $\mathcal{A}_\omega$ that for every real number r the game $\Gamma(\Phi_r)$ is determined, where Φ_r is the characteristic function of the set $X_r = \{s\colon \Phi(s) \geqslant r\}$.

In the case (*) does not hold it easy to modify $\Gamma(\Phi_r)$ in such a way that (*) is satisfied and the modified game is equivalent [24] to the original-

[20] See Appendix.

[21] On the other hand let us recall that many interesting and deep investigations on cardinal numbers become trivial if the axiom of choice is accepted. See e.g. [12], [31], [32], [33] and [29].

[22] From now on everything is done in $\mathcal{C}$.

[23] The formulation and proof of this theorem uses notions introduced in [15].

[24] In a natural sense of these words.

nal one (by adding an extra element μ to M and by a suitable modification of S and X_r forcing the players to choose μ whenever it was not originally their turn to move).

Now assume $(*)$ and let f be a one-to-one mapping of M into ω (it exists since M is at most denumerable). For every $s \in S$ and $n \in \omega$ let $T_{s|n}$ be the set of all elements of M possible as choices after the sequence of choices $s|n$, i.e. $T_{s|n} = \{x\colon x \in M$ and there is an $s' \in S$ such that $s'|n = s|n$ and $s'_n = x\}$. The following rules will define a game of the form $G_\omega(P)$: (1) the first player who chooses a natural number x_k such that $f^{-1}(x_k) \notin T(f^{-1}(x_0), \ldots, f^{-1}(x_{k-1}))$, where $(x_0, \ldots, x_{k-1})$ is the sequence of previous choices, looses; (2) if (1) does not apply then player I wins if $(f^{-1}(x_0), f^{-1}(x_1), \ldots) \in X_r$ and player II wins if $(f^{-1}(x_0), f^{-1}(x_1), \ldots) \notin X_r$. Clearly this game $G_\omega(P)$ is equivalent to $\Gamma(\Phi_r)$ and by $\mathcal{A}_\omega$ we get that it is determined, q.e.d.

COROLLARY [25]. (A) *implies that* $\Gamma(\Phi) = (M, S, J, \Phi)$ *is determined whenever* M *is at most denumerable.*

Proof. By the equivalence $(A) \leftrightarrow A_\omega$ and Theorem 2.

5. The problem of independence and consistency of (A). Of course the independence of (A) from the axioms of $\mathcal{T}$ follows from the consistency of the axiom of choice with these axioms (see [6]). But using more refined results of Gödel-Addison-Novikov [5], [1], [20] we get more information, e.g., that the proposition '$\mathcal{A}_2(P)$ holds for every analytic set $P \subseteq 2^{\omega}$' is independent. The axiom of constructibility ($\mathcal{AC}$) [26] being consistent with $\mathcal{T}$ (see [5]), our best result of that kind is the following:

THEOREM 3. $\mathcal{AC}$ *implies for every* $2 \leqslant n \leqslant \omega$ *that*

(i) $\mathcal{A}_n(P)$ *fails for some sets* $P \in \boldsymbol{A}$ [27];

(ii) $\mathcal{A}_n(P)$ *and* $\mathcal{A}_n^*(P)$ *fail for some sets* $P \in \boldsymbol{CA}$;

(iii) $\mathcal{A}_n^{**}(P)$ *fails for some sets* $P \in \boldsymbol{PCA} \cap \boldsymbol{CPCA}$.

This theorem is proved in the next section.

The problem of consistency of (A) with $\mathcal{T}$ is very difficult since it would give the independence of the axiom of choice, but until today there is no result indicating the independence of $\mathcal{AC}$ in $\mathcal{T}$.

On the other hand one can try to prove in $\mathcal{T}$ that a possibly large class of sets $P \subseteq 2^\omega$ satisfies $\mathcal{A}_2(P)$. This has been done for a long time and by several authors. The best result actually known is due to Morton Davis [4]. An easy modification of his argument gives the generalization

(25) Stated without proof in [15] (proposition (B*)).

(26) See [24] for a short formulation of $\mathcal{AC}$.

(27) $P \subseteq n^\omega$; $\boldsymbol{A}$ denotes the class of analytic sets, $\boldsymbol{CA}$—complement of analytic, etc. (see [8]).

of his theorem mentioned in [15] (proposition (A*)). If we suppose moreover that M (a set involved in (A*)) is well ordered, then the axiom of choice is not needed in the proof. Finally (A*) with this supposition gives the following theorem of $\mathcal{T}$.

THEOREM 4. *If there exists a well ordering of* X (e.g. $X = \omega$) *then* $\mathcal{A}_X(P)$, $\mathcal{A}_X^*(P)$ *and* $\mathcal{A}_X^{**}(P)$ *hold for every* $P \in \boldsymbol{F}_{\sigma\delta} \cup \boldsymbol{G}_{\delta\sigma}$ [28].

Remarks. 1. On account of Theorem 3 the best possible class of sets $P \subseteq 2^\omega$ for which $\mathcal{A}_2(P)$ could be proved is the class of Borelian sets (and this is perhaps the main open problem related to the subject of this paper).

2. Analogous conjectures about $\mathcal{A}_n(P)$ $(2 \leqslant n \leqslant \omega)$ are all equivalent [29], and they would imply the same about $\mathcal{A}_n^*(P)$ and $\mathcal{A}_n^{**}(P)$.

3. I do not know if $\mathcal{AC}$ implies the negation of proposition $\mathcal{F}'_n$ with a set $Q \in \boldsymbol{PCA} \cap \boldsymbol{CPCA}$ [30]. Such an example for some n would give the same for every $2 \leqslant n \leqslant \omega$.

We add here the connections of (A) and other propositions contradicting the axiom of choice introduced and studied by Church [3]. Then a result of E. Specker [30] is applied to show some corollaries, which are relevant with respect to the problem of consistency of (A).

Church [3] has introduced three necessary but mutually exclusive possibilities A, B and C which present themselves in the set theory $\mathcal{T}$ without the axiom of choice. These possibilities can be stated as follows.

A. There is a choice set for the Lebesgue decomposition of the real line [19].

B. There is no choice set for the Lebesgue decomposition of the real line but ω_1 is a regular ordinal.

C. ω_1 is not regular.

We know already that (A) implies $\aleph_1 \nleqslant 2^{\aleph_0}$ and a weak form of the axiom of choice $\mathcal{C}$. Now $\mathcal{C}$ implies the regularity of ω_1 (see Appendix). Hence

(5.1) (A) *implies* B.

As shown by Specker ([30], § 2.32)

(5.2) B *implies that* ω_1 *is constructibly inaccessible* (*i.e.* ω_1 *is a regular initial-limit ordinal in the universe of constructible sets*).

(28) I.e. $P = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} F_{ij}$ or $P = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} G_{ij}$, where all F_{ij} are closed and all G_{ij} are open, the topology in X^ω being that of Section 2. The axiom of choice being not supposed we assume that this double sequence $\{F_{ij}\}$ or $\{G_{ij}\}$ exists.

(29) The proof of these equivalentnces can be obtained by the methods used in the next section.

(30) Cf. Section 4, Remark 4, and Section 6, Theorems (6.2) and (6.3).

Therefore it is impossible to prove the consistency of the theory $\mathcal{T}$ & B or $\mathcal{T}$ & (A) by a relative interpretation of these theories in $\mathcal{T}$ as Gödel did in [6] for $\mathcal{T}$ & (the axiom of choice).

Let $\mathcal{I}$ denote the axiom of existence of inaccessible ordinals (the fact which follows would be also valid for $\mathcal{I}$ being the axiom of existence of strongly inaccessible cardinals). Let us prove the following corollary to Specker's result.

(5.3) *If $\mathcal{T}$ & $\mathcal{I}$ has a well founded model (i.e. a model in which the order of ordinals is a well ordering relation) then it has such a well founded model which satisfies $\mathcal{AC}$ and no submodel of which satisfies $\mathcal{T}$ & B.*

Suppose that (5.3) fails. Then we get an infinite descending sequence of well founded models

$$\mathfrak{M}_1 \supset \mathfrak{M}_1^* \supset \mathfrak{M}_2 \supset \mathfrak{M}_2^* \supset \mathfrak{M}_3 \supset \mathfrak{M}_3^* \supset \ldots$$

where all $\mathfrak{M}_i$ satisfy $\mathcal{T}$ & $\mathcal{I}$ and all $\mathfrak{M}_i^*$ satisfy $\mathcal{T}$ & B ($\mathfrak{M}_{i+1}$ is always the maximal submodel of $\mathfrak{M}_i^*$ which satisfies $\mathcal{AC}$). By (5.2) the ω_1 of $\mathfrak{M}_i$ is always larger than the ω_1 of $\mathfrak{M}_{i+1}$; hence there would be an infinite descending sequence of ordinals in $\mathfrak{M}_1$, which contradicts the well foundedness of this model.

6. The remaining proofs. On account of (3.1) only the following implications between the propositions $\mathcal{A}$ of the table of Section 3 remain to be proved

(6.1) $$\mathcal{A}_2 \to \mathcal{A}_\omega \to \mathcal{A}_\omega^* \to \mathcal{A}_\omega^{**} \leftarrow \mathcal{A}_2^{**}\,.$$

Proof. $\mathcal{A}_2 \to \mathcal{A}_\omega$. It is enough to show that for each game $G_\omega(P)$ there exists an equivalent [24] game of the form $G_2(Q)$. Let Q be defined by the following three conditions:

(1) If player I chooses 0 only finitely many times then the choosen sequence $s \notin Q$.

(2) If (1) does not apply and player II chooses 0 only finitely many times then $s \in Q$.

(3) If (1) and (2) do not apply then we define a function $f_1(s) = (n_0, n_1, \ldots) \in \omega^\omega$, where n_0 is the number of consecutive choices of I which are 1's (at the begining of s); n_1 is the number of consecutive 1's choosen by II after the first 0 choosen by I; n_2 is the number of consecutive 1's choosen by I after the first 0 choosen by II following the first 0 choosen by I; etc. Let be $f_1^{-1}(P) \subseteq Q$ and $f_1^{-1}(\omega^\omega - P) \cap Q = 0$.

This completes the definition of Q and it is obvious that $G_\omega(P)$ and $G_2(Q)$ are equivalent, q.e.d.

$\mathcal{A}_\omega \to \mathcal{A}_\omega^*$. It is enough to show that for each game $G_\omega^*(P)$ there exists an equivalent game $G_\omega(Q)$. Let $a_0, a_1, \ldots$ be a sequence of all finite sequences of natural numbers (with the empty sequence), let $\frown$ be the operation of concatenation of sequences and (n) denotes the sequence with a single term n. We put

$$f_2(n_0, n_1, \ldots) = a_{n_0} \frown (n_1) \frown a_{n_2} \frown (n_3) \frown \ldots \qquad (n_i \in \omega)$$

and

$$Q = f_2^{-1}(P)\,.$$

It is obvious that $G_\omega^*(P)$ and $G_\omega(Q)$ are equivalent, q.e.d.

$\mathcal{A}_\omega^* \to \mathcal{A}_\omega^{**}$. It is enough to show that for each game $G_\omega^{**}(P)$ there exists an equivalent game $G_\omega^*(Q)$. Let $a_0, a_1, \ldots$ be a sequence of all finite sequences of natural numbers (without the empty sequence). We put

$$f_3(n_0, n_1, \ldots) = a_{n_0} \frown a_{n_1} \frown a_{n_2} \frown \ldots \qquad (n_i \in \omega)$$

and

$$Q = f_3^{-1}(P)\,.$$

Clearly $G_\omega^{**}(P)$ and $G_\omega^*(Q)$ are equivalent, q.e.d.

$\mathcal{A}_2^{**} \to \mathcal{A}_\omega^{**}$. It is enough to show that for each game $G_\omega^{**}(P)$ there is an equivalent game $G_2^{**}(Q)$. Let Q contain all sequences $s \in 2^\omega$ which have only finitely many 0's. If s has infinitely many 0's let $f_4(s) = (n_0, n_1, \ldots)$, where n_0 is the number of consecutive 1's at the beginning of s, n_1 the number of consecutive 1's after the first 0 in s, n_2 the number of consecutive 1's after the second 0 in s, etc.

Let be

$$f_4^{-1}(P) \subseteq Q \quad \text{ and } \quad f_4^{-1}(\omega^\omega - P) \cap Q = 0\,.$$

This completes the definition of Q and it is easy to see that $G_\omega^{**}(P)$ and $G_2^{**}(Q)$ are equivalent, q.e.d.

Proof of Theorem 3. The conclusions (i) and (ii) are based on the following theorem announced by Gödel [5] [31].

(6.2) *$\mathcal{AC}$ implies the existence of a set $P \subseteq 2^\omega$, $P \in CA$ of potency $2^{\aleph_0}$ without perfect subsets.*

Now (ii) follows from (3.3), (6.2), the fact that the mappings f_1 and f_2 in the proof of (6.1) are continuous and an analogous analysis of the proofs of (3.1) and (3.2). (i) can be proved by means of the part of (ii) pertaining to $\mathcal{A}_n(P)$. It is enough to use the fact that in the games $G_n(P)$ the role of the players is almost symmetric (we modify the example constructed for the proof of (ii) in such a way that after the first choice of player I the position of II is the same as was the position of I at the beginning in the original game).

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[31] The first published proof is due to Novikov [20]. In the original form P was supposed to be a set of reals, but on account of that it is easy to get $P \subseteq 2^\omega$.

For proving (iii) we apply the following theorem essentially due to Gödel [5] ([32]).

(6.3) *$\mathcal{AC}$ implies the existence of a set $P \subseteq 2^\omega$, $P \in \boldsymbol{PCA} \cap \boldsymbol{CPCA}$, which is not residual in any open set nor of the first category.*

Then (iii) follows from (3.4), (6.3), the fact that the mapping f_4 in the proof of (6.1) is continuous and an analogous analysis of the proof of (3.1) and (3.2).

7. Miscellaneous propositions generalizing (A). Let $G_X^\alpha(P)$, where α is any ordinal and $P \subseteq X^\alpha$, be a modification of $G_X(P)$, where the two players are alternatively choosing the terms of an α-sequence $s \in X^\alpha$, the limit choices being done by player I (hence $G_X(P) = G_X^\omega(P)$). And let $\mathcal{A}_X^\alpha(P)$ and $\mathcal{A}_X^\alpha$ denote the propositions corresponding to $\mathcal{A}_X(P)$ and $\mathcal{A}_X$ respectively (hence $\mathcal{A}_X(P) \leftrightarrow \mathcal{A}_X^\omega(P)$ and $\mathcal{A}_X \leftrightarrow \mathcal{A}_X^\omega$).

(7.1) *The proposition '$\mathcal{A}_X^2$ holds for every set X' is equivalent to the axiom of choice.*

Proof. Let $\boldsymbol{F}$ be a family of sets and $0 \notin \boldsymbol{F} \neq 0$. Let G be the following game: I chooses any $S \in \boldsymbol{F}$, then II chooses any $t \in \bigcup_{S \in \boldsymbol{F}} S$. Player I wins if $t \notin S$ and player II wins if $t \in S$. It is clear that there is no winning strategy for player I in this game. It is easy to construct a game of the form $G_X^2(P)$ equivalent to G. Hence $\mathcal{A}_X^2$ implies the existence of a winning strategy for player II in G. Clearly this is a function f such that $Df = \boldsymbol{F}$ and $f(S) \in S$ for every $S \in \boldsymbol{F}$, i.e. a choice function for $\boldsymbol{F}$.

On the other hand the following tautology which is a consequence of the rule of de Morgan

$$\bigvee_{u_0 \in X} \bigwedge_{u_1 \in X} (u_0, u_1) \in P \vee \bigwedge_{u_1 \in X} \bigvee_{u_0 \in X} (u_0, u_1) \notin P$$

and the axiom of choice clearly imply $\mathcal{A}_X^2(P)$, q.e.d.

Remark. In the same way as in the second part of the above proof the following consequence of the rule of de Morgan

$$\underset{u_0 \in X_0}{Q^0} \; \underset{u_1 \in X_{(u_0)}}{Q^1} \cdots \underset{u_n \in X_{(u_0, \dots, u_{n-1})}}{Q^n} (u_0, \dots, u_n) \in P \vee$$
$$\vee \underset{u_0 \in X_0}{\tilde{Q}^0} \; \underset{u_1 \in X_{(u_0)}}{\tilde{Q}^1} \cdots \underset{u_n \in X_{(u_0, \dots, u_{n-1})}}{\tilde{Q}^n} (u_0, \dots, u_n) \notin P$$

($Q^0, \dots, Q^n$ is any sequence of quantifiers and $\tilde{\bigvee} = \bigwedge$, $\tilde{\bigwedge} = \bigvee$) and the axiom of choice imply the theorem of Zermelo-von Neumann [37], [19]

([32]) P is also non-measurable with respect to the product measure in 2^ω. In the original form P was supposed to be a set of reals but on account of that we easily get $P \subseteq 2^\omega$. For convenience of the reader we give in the Appendix a derivation of (6.3) from the results of Gödel-Addison [1], based on an idea of Kuratowski which is briefly mentioned in [1] (see also Kuratowski, Sierpiński [9]).

on the strict determinateness of finite games with perfect information ([33]).

The last implication of the table of Section 3 which remains to be proved ([34]) is the following

(7.2) $\mathcal{A}_\omega^\omega \rightarrow \mathcal{C}$.

Proof. Let be $\boldsymbol{F} = \{X_1, X_2, \dots\}$ ($\boldsymbol{F}$ is the family appearing in $\mathcal{C}$). We can suppose without loss of generality that $\bigcup_{n=0}^{\infty} X_n \subseteq \omega^\omega$ and $X_n \neq 0$ for $n \in \omega$. Let $G_\omega^\omega(P)$ be defined as follows. Player II is winning if and only if, n_0 being the first choice of I and $(n_1, n_3, \dots)$ being the sequence of choices of II, we have $(n_1, n_3, \dots) \in X_{n_0}$. It is clear that the sets X_n being non empty there is no winning strategy for player I and that a winning strategy for player II is a choice function for $\boldsymbol{F}$, q.e.d.

(7.3) *Each of the propositions $\mathcal{A}_{\omega_1}^\omega$ and $\mathcal{A}_2^{\omega_1}$ is inconsistent.*

Proof. Concerning the first part it is clear that $\mathcal{A}_{\omega_1}^\omega \rightarrow$ (A). We will define a game of type $G_{\omega_1}(P)$ such that there is no winning strategy for I, but the existence of a winning strategy for II implies the existence of a set of real numbers of power $\aleph_1$. This (by Section 3, $2^{\aleph_0}$ *in* $\aleph_1$) will give an inconsistency with (A).

Let be $2^\omega = \bigcup_{\alpha < \omega_1} X_\alpha$, where X_α are disjoint and non empty ([35]). Suppose that player II is winning if and only if, α_0 being the first choice of I and $(\alpha_1, \alpha_3, \dots)$ being the sequence of choices of II, we have $(\alpha_1, \alpha_3, \dots) \in X_{\alpha_0}$. It is clear that, X_α being non empty, player I has no winning strategy, and the existence of a winning strategy for player II implies the existence of a choice set for the family $\{X_\alpha\}_{\alpha < \omega_1}$ ([36]).

Concerning the second part it is clear that $\mathcal{A}_2^{\omega_1} \rightarrow$ (A). We have also $\mathcal{A}_2^{\omega_1} \rightarrow \mathcal{A}_{2^\omega}^{\omega_1}$ due to the fact that the ordinal product $\omega \cdot \omega_1 = \omega_1$. Now we define a game $G_{2^\omega}^{\omega_1}(P)$. According to (A) every sequence $s = (s_\alpha)_{\alpha < \omega_1}$ ($s_\alpha \in 2^\omega$) has repetitions; let s_{α_0} be the first repeating term in s. Let be $s \in P$ if and only if α_0 is odd. The situation of the two players in this game is essentially symmetric, i.e. the existence of a winning strategy for one of them would imply the same for the other, which is inconsistent, q.e.d.

Remarks. 1. In contrast to (7.3) it seems that the proposition $\bigwedge_{\alpha < \omega_1} \mathcal{A}_2^\alpha$ may be consistent in $\mathcal{C}$. This proposition implies $\bigwedge_{\alpha < \omega_1} \mathcal{A}_R^\alpha$, where

([33]) The connections of positional games and formulae with a long prefix of interchanging quantifiers has been observed by Ulam [36], pp. 24-25 (see also p. 23 for other remarks of which the present paper may be considered as a development).

([34]) See footnote ([8]).

([35]) This is the Lebesgue decomposition—see Appendix.

([36]) This proof is a modification of an argument of D. Scott quoted at the end of this section.

$|R| = 2^{\aleph_0}$, and a strengthening of $\mathcal{C}$ where the condition $|F| \leqslant \aleph_0$ is replaced by $|F| \leqslant 2^{\aleph_0}$ (the proof of the last fact would be almost the same as that of (7.2)).

2. The statement $\mathcal{A}_X^\omega$, where $|X| = \mathfrak{k}$ is inconsistent in $\mathcal{T}$ (37).

The first idea of the axiom of determinateness proposed by prof. Steinhaus (see [17], footnote on p. 1) was 'every infinite positional game with perfect information (3) is determined'. By (7.1) this proposition is inconsistent. I have tried to modify the proposition $\mathcal{A}_X$ in such a way to obtain an infinitistic rule of de Morgan which would not imply the axiom of choice. But this also gave only inconsistencies. Nevertheless let me mention these rules and the counter-examples.

Let $\Phi(E, U, \prec, X, P)$, where $E \cap U = 0 \neq E \cup U$, $\prec$ is an ordering of $E \cup U$, $X \neq 0$ and $P \subseteq X^{E \cup U}$, denote the following formula

$$\bigvee_F \bigwedge_{f \in X^{E \cup U}} \Big\{ \bigwedge_{a \in E} F(f|a) \neq 0 \wedge \Big[\Big(\bigwedge_{a \in E} f(a) \in F(f|a)\Big) \rightarrow f \in P\Big]\Big\},$$

where $f|a$ is an abbreviation for $f|\{x\colon x \in E \cup U \wedge x \prec a\}$ and F runs over arbitrary functions with $DF = \{f|a\colon f \in X^{E \cup U} \wedge a \in E\}$ and whose values are subsets of X.

The above definition is justified by the following theorem of $\mathcal{T}$

(7.4) *If $E \cup U = \{0, \ldots, n-1\}$ and $i \prec j \leftrightarrow i < j$ for $i, j < n$ then*

$$\Phi(E, U, \prec, X, P) \leftrightarrow \underset{u_0 \in X}{Q^0} \ldots \underset{u_{n-1} \in X}{Q^{n-1}} (u_0, \ldots, u_{n-1}) \in P,$$

where $Q^i = \bigvee$ if $i \in E$ and $Q^i = \bigwedge$ if $i \in U$.

Hence Φ is a generalization of a formula with a prefix of quantifiers which are relativised to X and determined by $(E, U, \prec)$ (38). The following theorem of $\mathcal{T}$ shows the connection of these formulas with games of type $G_X^a(P)$.

(7.5) *If there exists a well ordering of X and $E \cup U = a$, E is the set of even and U of odd members of a and $a \prec b \leftrightarrow a < b$, then $\Phi(E, U, \prec, X, P)$ and $\Phi(U, E, \prec, X, X^{E \cup U} - P)$ are equivalent to the existence of a winning strategy for player I and for player II respectively in the game $G_X^a(P)$.*

Proof. $\Phi(E, U, \prec, X, P)$ is equivalent to

$$(*) \qquad \bigvee_g \bigwedge_{f \in X^{E \cup U}} \Big\{\Big(\bigwedge_{a \in E} f(a) = g(f|a)\Big) \rightarrow f \in P\Big\},$$

(37) The proof is given in [16] (it is an argument of D. Scott mentioned above).

(38) Some generalizations and properties of such formulae are studied in [16].

where g runs over functions with $Dg = \{f|a\colon f \in X^{E \cup U} \wedge a \in E\}$ and with values in X. In fact we can define g putting

$$g(f|a) = \min F(f|a) \quad \text{(min with respect to a fixed well ordering of } X).$$

Clearly $(*)$ expresses the existence of a winning strategy for player I.

The proof of the other case is analogous, q.e.d.

Let $\Psi(E, U, \prec, X, P)$ denote the following generalization of the rule of de Morgan

$$\sim\Phi(E, U, \prec, X, P) \leftrightarrow \Phi(U, E, \prec, X, X^{E \cup U} - P).$$

Let $\Psi_{\leftarrow}$ and $\Psi_{\rightarrow}$ denote the left and right implications of this equivalence.

On account of (7.5) and from the fact that $(*)$ implies Φ, we get the following theorem of $\mathcal{T}$

(7.6) *On the supposition of* (7.5) *we have*

$$\Psi_{\rightarrow}(E, U, \prec, X, P) \leftrightarrow \mathcal{A}_X^a(P)$$

and the left implication of this equivalence is valid without supposing the existence of a well ordering of X.

Let us mention that if $\prec$ is a well ordering of type a then the generalization of $\Psi_{\leftarrow}$ with respect to its free variables is equivalent to a principle of dependent choices of order a (39). Hence for $a = \omega_1$ this is already inconsistent with (A). On the other hand there are instances of $\Psi_{\leftarrow}$ which may be disproved in $\mathcal{T}$ (already in the case when the order type of $\prec$ is ω^* (40)).

There are also instances of $\Psi_{\rightarrow}$ which may be disproved in $\mathcal{T}$. This follows from (7.3) and (7.6). Another such example, historically the first, was given by D. Scott. In his example $|X| = \mathfrak{k}$ and $\prec$ has the order type of ω (41).

Appendix

1. *$\mathcal{C}$ implies the denumerable additivity of the Lebesgue measure and sets of the first category* (analogous results are valid for separable spaces and regular measures).

Proof. The standard definition of the Lebesgue measure, the measurability of open sets and the proof of the lemma *if $U_0, U_1, \ldots$ is a sequence of open sets then* $\operatorname{mes}(\bigcup_{k=0}^{\infty} U_k) \leqslant \sum_{k=0}^{\infty} \operatorname{mes}(U_k)$ *and equality holds if $U_i \cap U_j = 0$ for all $i \neq j$* are done without using the axiom of choice

(39) Such a principle of order ω is formulated in [31], [13].

(40) I.e. that of the negative integers; see [16].

(41) See [16].

(for an analogous construction in general locally compact groups, see H. Cartan [2]). In the remaining part of the theory the axiom of choice is applied several times but always in the following form *given a sequence $X_0, X_1, \ldots$ of non empty families of open sets there exists a sequence $V_0, V_1, \ldots$ with $V_i \in X_i$ for each $i \in \omega$*. For a space S having a denumerable basis of open sets $\{G_0, G_1, \ldots\}$ this follows from $\mathcal{C}$. In fact to each open set U corresponds a unique subset of ω namely $\{i\colon G_i \subseteq U\}$. Therefore the class of open sets has potency $\leqslant 2^{\aleph_0}$ and we can apply $\mathcal{C}$.

Let be $X = \bigcup_{n=0}^{\infty} X_n$ where X_n are of the first category, i.e. for every X_n there is a sequence of closed nowhere dense sets $(Y_{ni})_{i\in\omega}$ such that $X_n \subseteq \bigcup_{i=0}^{\infty} Y_{ni}$. Hence for every X_n there exists a double sequence $(m_{ij})_{i,j\in\omega}$ such that for every $i \in \omega$ the set $\bigcup_{j=0}^{\infty} G_{m_{ij}}$ is dense in space S and $X_n \subseteq \bigcup_{i=0}^{\infty} (S - \bigcup_{j=0}^{\infty} G_{m_{ij}})$. Let R_n denote the set of all such double sequences. Then, by $\mathcal{C}$, there exists a sequence $(s_0, s_1, \ldots)$, where $s_k = (m_{ij}^k)_{i,j\in\omega} \in R_k$ $(k \in \omega)$. It is clear that

$$X \subseteq \bigcup_{k=0}^{\infty}\bigcup_{i=0}^{\infty} (S - \bigcup_{j=0}^{\infty} G_{m_{ij}^k}),$$

and X is of the first category, q.e.d.

2. Proof of the statement (i) of Section 3. Let C_2 be the cyclic group of order $|2|$ and C_2^{ω} the complete direct sum of $\aleph_0$ coppies of C_2 and $C_2^{(\omega)}$ the weak direct sum of $\aleph_0$ coppies of C_2. Then we consider C_2^{ω} as a topological group (with the natural compact product topology). It is obvious that

$$(*) \qquad |C_2^{\omega}/C_2^{(\omega)}| = \mathfrak{k}.$$

Let f be the natural mapping $f\colon C_2^{\omega} \to C_2^{\omega}/C_2^{(\omega)}$. Clearly for every set $X \subseteq C_2^{\omega}/C_2^{(\omega)}$ the reciprocal image $f^{-1}(X)$ is $C_2^{(\omega)}$-invariant in C_2^{ω}. Hence by the 01-law (see [22]) every set of the form $f^{-1}(X)$ which has the property of Baire is residual or of the first category. Let be $\boldsymbol{I} = \{X\colon X \subseteq C_2/C_2$ and $f^{-1}(X)$ is of the first category$\}$. It is clear that $\boldsymbol{I}$ is a denumerably additive non principal ideal in the algebra $\boldsymbol{B}(C_2^{\omega}/C_2^{(\omega)})$. Clearly $\mathcal{B}$ implies that $\boldsymbol{I}$ is maximal.

Using the product measure in C_2^{ω}, an analogous 01-law for measures and the property 'mes$(f^{-1}(X)) = 0$' in place of '$f^{-1}(X)$ is of the first category' we can get on account of $\mathcal{M}$ the same result.

3. Proof of the statement (ii) of Section 3. Let us define in an analogous way C_n^{ω} and $C_2^{(\omega)}$, where C_n is the cyclic group of order $|n|$ $(2 \leqslant n \leqslant \omega)$. Let $G_n = C_n^{\omega}/C_2^{(\omega)}$ and f be the natural mapping $f\colon C_n^{\omega} \to G_n$. Let c be a generator of C_n, $a_0 = (c, c, \ldots) \in C_n^{\omega}$ and A_n the cyclic sub-

group of G_n generated by $f(a_0)$. Let $\boldsymbol{F}_n$ be the factor group G_n/A_n [42]. Hence each member of $\boldsymbol{F}_n$ is of potency $|n|$. It is easy to see that

$$|G_2| \leqslant |G_n| \leqslant |G_\omega|.$$

Every $b \in C_\omega^{\omega}$ is a subset of $\omega \times C_\omega$ and this easely gives $|G_\omega| \leqslant \mathfrak{k}$. By $(*)$ we have $|G_2| = \mathfrak{k}$ and by the Cantor-Bernstein theorem $|G_n| = \mathfrak{k}$. Therefore

$$|\bigcup_{X\in \boldsymbol{F}_n} X| = \mathfrak{k} \quad \text{for every } 2 \leqslant n \leqslant \omega.$$

Now suppose a contrario that there exists a choice set Z for $\boldsymbol{F}_n$. Then clearly

$$\bigcup_{a\in A_n} f^{-1}(aZ) = C_n^{\omega},$$

where $f^{-1}(a_1Z) \cap f^{-1}(a_2Z) = 0$ for $a_1, a_2 \in A_n$, $a_1 \neq a_2$. Moreover the sets $f^{-1}(aZ)$ are $C_n^{(\omega)}$-invariant and congruent in C_n^{ω}. Hence by $\mathcal{B}$ and the 01-law they are all of the first category of all residual in C_n^{ω}. But this is inconsistent since A_n is at most denumerable.

An analogous proof would give the same on account of $\mathcal{M}$, q.e.d.

4. Proof of the statement (iii) of Section 3. Suppose that Z is a choice set for G/H, then $G = \bigcup_{h\in H} hZ$. But this is inconsistent with the measurability or the property of Baire of Z respectively.

A proof that G/H satisfies (i) is analogous to the argument of point 2 of this Appendix.

For proving the second part let us remark that $aH \cap a^{-1}H = 0$ for every $a \in G - H$; indeed otherwise we get $ah_1 = a^{-1}h_2$ with $h_1, h_2 \in H$ and $a^2 = h_2h_1^{-1} \in H$ which contradicts the supposition. Let Z^* be a choice set for the family $\{\{aH, a^{-1}H\}\colon a \in G - H\}$ and $Z = \bigcup_{X\in Z^*} X$. Hence $Z \cap Z^{-1} = 0$ and $Z \cup Z^{-1} = G - H$. It is clear that Z and Z^{-1} are H-invariant and hence, by $\mathcal{B}$, both of the first category or both residual in G, and all this is inconsistent if G is not of the first category onto itself. An analogous argument would give the same on account of the measure supposition and $\mathcal{M}$, q.e.d.

5. The Lebesgue decomposition (see [10]). We make any 1-1 identification of the reals with all subsets of the set of rationals. S is the set of reals which correspond to such sets of rationals which are not well ordered (by $<$); S_α $(\alpha < \omega_1)$ is the set of reals which correspond to well ordered sets of rationals of type α. Clearly this is a partition of the whole real line into $\aleph_1$ non-empty sets.

(Note that the sets S_α are Borelian and the set S is analytic—see [11] or [8], § 35, VIII.)

[42] I do not know if $|\boldsymbol{F}_n|$ depends on n, but $\boldsymbol{F}_n$ is an image of a set X with $|X| = \mathfrak{k}$ by a mapping satisfying the condition formulated in footnote [15].

6. *$\mathcal{C}$ implies the regularity of ω_1* (see [3]). Suppose that ω_1 is not regular, i.e. there exists a denumerable set $A \subset \omega_1$ such that $l.u.b.\,A = \omega_1$. By $\mathcal{C}$ there exists a choice set for the family $\{S_a\}_{a \in A}$. Given a point $p_a \in S_a$ we get a 1-1 mapping f_a of the ordinal a into the set of rationals. We take a decomposition of ω_0 into denumerably many disjoint infinite sets $\{T_a\}_{a \in A}$ and we modify each f_a to get a 1-1 mapping $f_a^*\colon a \to T_a$. Let be $f = \bigcup_{a \in A} \big(f_a^* | (Df_a^* - \bigcup_{\xi \in A \cap a} Df_\xi^*)\big)$; clearly f is a 1-1 mapping of ω_1 into ω_0 which is a contradiction.

7. $2^{\aleph_0} < \mathfrak{k}$. The set of real numbers being orderable we get by (vii) that $2^{\aleph_0} \neq \mathfrak{k}$. Taking a family of potency $2^{\aleph_0}$ of 'almost disjoint' elements of $\boldsymbol{B}(\omega)$ (43) we get $2^{\aleph_0} \leqslant \mathfrak{k}$.

$\aleph_1 < \aleph_1 + 2^{\aleph_0}$. By (vi).

$\aleph_1 + 2^{\aleph_0} < \aleph_1 + \mathfrak{k}$. Since $2^{\aleph_0} \leqslant \mathfrak{k}$ and by (vii).

$2^{\aleph_0} < 2^{\aleph_0} + \aleph_1$. By (iv).

$2^{\aleph_0} + \aleph_1 < 2^{\aleph_1}$. By (iv) we have $2^{\aleph_0} < 2^{\aleph_1}$. Now it is enough to apply the following theorem of Tarski (44) which is valid in $\mathcal{T}$:

(T) *If* $\mathfrak{m} + 1 = \mathfrak{m}$ *then* $2^{\mathfrak{m}} - \mathfrak{m}$ *exists and* $2^{\mathfrak{m}} - \mathfrak{m} = 2^{\mathfrak{m}}$.

$2^{\aleph_1} < 2^{\aleph_1} + \mathfrak{k}$. Since a set of potency $2^{\aleph_1}$ can be ordered and by (vii).

$2^{\mathfrak{k}} = 2^{2^{\aleph_0}}$. By $2^{\aleph_0} \leqslant \mathfrak{k}$ we get $2^{\mathfrak{k}} \geqslant 2^{2^{\aleph_0}}$. Since $\mathfrak{k}$ is the cardinal of a family of disjoint sets of real numbers we get $2^{\mathfrak{k}} \leqslant 2^{2^{\aleph_0}}$.

$2^{\aleph_1} + \mathfrak{k} < 2^{\mathfrak{k}}$. By (vii) we have $2^{\aleph_1} \neq 2^{\mathfrak{k}}$. By the Lebesgue decomposition we have $2^{\aleph_1} \leqslant 2^{2^{\aleph_0}}$ $(= 2^{\mathfrak{k}})$. Since $2^{\aleph_0} \leqslant \mathfrak{k}$ we have $\mathfrak{k} + 1 = \mathfrak{k}$ and by (T) $2^{\mathfrak{k}} - \mathfrak{k} = 2^{\mathfrak{k}}$. These facts already imply the required inequality.

$2^{\aleph_0}$ in $\aleph_1$. By (iv) and (vi).

$2^{\aleph_1} < \mathfrak{k}$ or ($\mathfrak{k}$ *in* $2^{\aleph_1}$ *in* $\aleph_1 + \mathfrak{k} < 2^{\aleph_1} + \mathfrak{k}$). Suppose that $2^{\aleph_1} \not< \mathfrak{k}$. Then $\mathfrak{k}$ *in* $2^{\aleph_1}$ and $2^{\aleph_1} \not\geqslant \aleph_1 + \mathfrak{k}$ by (vii). Suppose that $2^{\aleph_1} < \aleph_1 + \mathfrak{k}$; then we get by (T) and $2^{\aleph_1} \neq \mathfrak{k}$ that $2^{\aleph_1} < \mathfrak{k}$ which contradicts the main supposition. Hence we have also $2^{\aleph_1}$ *in* $\aleph_1 + \mathfrak{k}$. Of course $\aleph_1 + \mathfrak{k} \leqslant 2^{\aleph_1} + \mathfrak{k}$. Suppose $\aleph_1 + \mathfrak{k} = 2^{\aleph_1} + \mathfrak{k}$; then we get by (T) that $\mathfrak{k} = 2^{\aleph_1} + \mathfrak{k}$ and once more $2^{\aleph_1} < \mathfrak{k}$, q.e.d.

Remark. The equalities $\mathfrak{k}^2 = \mathfrak{k}$, $2^{\mathfrak{k}\aleph_0} = 2^{\mathfrak{k}}$ and $|V| = \mathfrak{k}$, where V is the factor group of the reals by the rationals (i.e. the decomposition of Vitali), are theorems of $\mathcal{T}$ (45).

8. Proof of (6.3) (46). It is proved in [1] that $\mathcal{AC}$ implies the existence of a well ordering W of the real numbers which is of type ω_1 and such that the set $W = \{(x, y)\colon xWy\} \in \boldsymbol{PCA}$, in the usual topology of the plane R^2. Let D be the diagonal of R^2. Since $R^2 - W = \{(x, y)\colon yWx\} \cup D$, it follows that $W \in \boldsymbol{PCA} \cap \boldsymbol{CPCA}$. Now every line in R^2 parallel to the axis x contains only denumerably many points of W and every line parallel to the axis y contains only denumerably many points of $R^2 - W$. By these properties and an analogue of Fubini's theorem (see [8], § 24, VI) we get that W is nowhere residual nor of the first category. The same argument is valid if the plane R^2 is replaced by $2^\omega \times 2^\omega$ (which is homeomorphic to 2^ω), q.e.d.

(43) See e.g. [29], p. 77.

(44) The first published proof is due to Sierpiński; see e.g. [29].

(45) Cf. footnote (17).

(46) See footnote (32).

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INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

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