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From accessible to inaccessible cardinals

Results holding for all accessible cardinal numbers and the problem of their extension to inaccessible ones

by

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Introduction (⁰). In the recent literature (¹) various mathematical problems in set theory and related domains have been discussed which exhibit the following pattern. Each of the problems consists in determining all infinite cardinals α which possess a given property P. It has been known for some time that the properties involved fail for the smallest infinite cardinal, ω (or $\aleph_0$), while they hold for all accessible cardinals (i.e., roughly speaking, for all those non-denumerable cardinals which can be obtained from smaller ones by means of ordinary arithmetical operations). The question whether these properties apply to any or all inaccessible cardinals different from ω had been open; it even seemed plausible that this question could not be answered in either direction on the basis of the familiar axiomatic foundations of set theory. Quite recently, however, an answer to this question has been found for large classes of non-denumerable inaccessible cardinals. It has turned out, contrary to expectations, that all cardinals in these classes possess the properties involved, and thus behave not like the smallest inaccessible cardinal ω, but like all the accessible cardinals. Actually we do not know at present any non-denumerable cardinal which is capable of a "constructive characterization" (in some very general and rather loose sense

(⁰) The present paper is an outgrowth of two short articles of the authors [15] and [48]. (The numbers in square brackets refer throughout to the bibliography at the end of the paper.) The paper has been prepared for publication during the period when Tarski, and for a short time also Keisler, were working at the University of California at Berkeley on a research project in the foundations of mathematics sponsored by the U. S. National Science Foundation (Grants G14006 and G19673). The authors express their sincere appreciation to Haragauri N. Gupta who, working on the same project, rendered considerable help in preparing the manuscript.

(¹) See, for example, [4], [23], and [48], where references to the earlier literature can also be found. In cases important for our discussion we shall restate here results and occasionally proofs known from the literature.

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of the term) and for which we could not prove that it possesses the properties discussed. Nevertheless, the straightforward question whether *all* cardinals larger than ω possess these properties is still open, and it does not seem very likely that the methods now available will bring forth an answer to this question.

The results we have mentioned concerning large classes of inaccessible cardinals were originally obtained with the essential help of metamathematical (model-theoretical) methods (2). These methods still provide the intuitively and deductively simplest approach of the topic in its full generality. In our opinion this circumstance provides new and significant evidence of the power of metamathematics as a tool in purely mathematical research, and at the same time does not detract in the least from the value of results obtained. Nevertheless we have decided to undertake in this paper an exhaustive purely mathematical treatment of the whole topic avoiding any use of metamathematical notions and methods. We have been motivated by the realization of the practical fact that the knowledge of metamathematics is not sufficiently widespread and may be defective among mathematicians who would otherwise be intensely interested in the topics discussed, and to a certain extent also by some (irrational) inclination toward puritanism in methods. As will be seen from some remarks below, we do not feel that we have been entirely successful in our undertaking. The final judgement must be left to the reader.

The authors are planning to publish one or more papers in the future containing the discussion of metamathematical problems which exhibit the same pattern as the problems treated in this work, and embodying all the related results the authors have obtained (3).

The problems which will be discussed here are formulated in terms of several branches of mathematics: general set theory, theory of Boolean algebras, theory of measure, group theory, topology, and functional analysis. A closer study shows, however, that one can single out a small number of problems such that each of the remaining ones is equivalent to one of them. Actually we have found it possible to concentrate upon three properties of infinite cardinals expressed by the following conditions (the exact meaning of the terms involved will be explained below):

(a) There exists an α-complete field of sets with at most α generators in which some α-complete proper ideal cannot be extended to an α-complete prime ideal.

(2) In this connection see [48], where the original proofs are sketched and the underlying metamathematical results, of Hanf and one of the authors, are briefly discussed; cf. also [50]. The results of Hanf just mentioned are now presented in full in [9].

(3) The metamathematical results obtained by the authors in this direction are stated without proof in [15], [17], [18], [48], [49], [50].

(b) In the field of all subsets of a set of power α every α-complete prime ideal is principal.

(c) There exists an α-complete field of sets in which some α-complete proper ideal cannot be extended to an α-complete prime ideal.

The classes of all cardinals satisfying (a), (b), and (c) will respectively be denoted by $\boldsymbol{C}_0$, $\boldsymbol{C}_1$, and $\boldsymbol{C}_2$ (4). It will be seen that $\boldsymbol{C}_0$ is included in $\boldsymbol{C}_1$ and that $\boldsymbol{C}_1$ is included in $\boldsymbol{C}_2$. It is known that ω does not belong to any of these classes and that each of them contains all accessible cardinals, but the problems are open whether any two of these three classes coincide, and whether any of these classes contain all non-denumerable cardinals. Among these classes, $\boldsymbol{C}_1$ proves to be more closely tied up with mathematical problems outside of general set theory and more readily handled by mathematical methods. On the other hand, $\boldsymbol{C}_0$ and $\boldsymbol{C}_2$, in addition to their mathematical content, admit of simple metamathematical characterizations, and their study is relevant for metamathematical purposes.

The paper is divided into six sections, numbered § 0 to § 5. § 0 contains an account of terminology, notation, and some basic results which are for the most part known from the literature and are stated (with one exception) without proofs. The main task of § 1 is to provide as much information as practical concerning the extent of the class $\boldsymbol{C}_1$. The main results of this section have the following form: a given "constructively characterized" class $\boldsymbol{X}$ of cardinals is included in $\boldsymbol{C}_1$. To give an example, assume that all the inaccessible cardinals are arranged in a strictly increasing sequence $\theta_0, \theta_1, \theta_2, \ldots, \theta_\xi, \ldots$, and let $\boldsymbol{L}$ be the class obtained from that of all accessible cardinals by adjoining all those θ_ξ's which are larger than their index ξ. Then a relatively simple and rather weak result implied by our discussion is that $\boldsymbol{L}$ is included in $\boldsymbol{C}_1$.

With more detail, our procedure in § 1 can loosely be described as follows. We single out a certain family of subclasses of $\boldsymbol{C}_1$ to which we refer as *normal classes*. It turns out that the class of all accessible cardinals is normal, while on the other hand the class $\boldsymbol{C}_1$ proves to be

(4) In a different terminology, these classes have been discussed in [4]. In fact, $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ respectively coincide with the class of all cardinals having the properties $\boldsymbol{P}_3$ and $\boldsymbol{P}_4$ of [4]; it will be seen in § 4 of this paper that the class $\boldsymbol{C}_0$ is just the class of all those cardinals which either are accessible or have the property $\boldsymbol{Q}$. In [23] the infinite cardinals not belonging to $\boldsymbol{C}_1$ are referred to as Ulam numbers. The infinite cardinals not belonging to $\boldsymbol{C}_2$ prove to coincide with those which are referred to there as Stone numbers and also with those referred to as Tarski numbers. (The definitions of Stone numbers and Tarski numbers are different, and the author of [23] did not realize that they are equivalent.) As regards metamathematical characterization, $\boldsymbol{C}_2$ coincides with the class of all incompact cardinals in the sense of [48]; $\boldsymbol{C}_0$ consists of all accessible cardinals and those inaccessible cardinals which are strongly incompact, and under the generalized continuum hypothesis simply coincides with the class of all strongly incompact cardinals.

normal if and only if it contains all non-denumerable cardinals and hence, in view of what we have mentioned before, the normality of C_1 is an open problem. Furthermore, we establish several induction principles to the effect that certain "constructively characterized" operations, when performed on normal classes (or sequences of normal classes), yield new normal classes. These operations are of the kind that they lead from a class X to a new class Y which includes X. Actually, under some weak assumptions on X, e.g. if X contains all accessible cardinals and is properly included in C_1, the resulting class Y properly includes X and is still properly included in C_1. Hence, by taking the class of accessible cardinals as the point of departure and by performing repeatedly the operations which preserve normality, we obtain more and more comprehensive classes of cardinals which are included in C_1. Unless we are able to show (by a different method) that C_1 contains all non-denumerable cardinals, this process will never end, and thus the methods which we use will never produce a largest "constructively characterized" class included in C_1.

In the development of § 1 some traces of its metamathematical origin can undoubtedly be discovered. We believe nevertheless that this development shows a sufficient degree of mathematical simplicity to enable the reader, not only to follow the discussion formally, but also to assimilate it intuitively without depending on a familiarity with the ideas underlying the origin of the discussion. The notion of normal classes (which in our development functions as a tool rather than as a goal) may become an object of interest in its own right, and it may well turn out that the theory of these classes is better suited to a mathematical than to a metamathematical treatment.

In § 2 we establish a variety of characteristic properties of cardinals belonging to C_1; in other words, we consider various problems we show to be equivalent to the problem of determining the extent of the class C_1, and to which, therefore, the results obtained in § 1 automatically extend. Actually, we succeed in getting more general results by considering, instead of individual cardinals α in C_1, couples of cardinals α, β such that both α and β as well as all cardinals between them belong to C_1. We begin with a detailed discussion of problems formulated in general set theory and abstract Boolean algebras. In the later part of the section we give an account of related results from other branches of mathematics. Two of these results are new; they provide a topological and an algebraic characterization of the class C_1. The remaining results in this account can be found in the literature, and will therefore be discussed only briefly. Strictly speaking, these results concern, not C_1, but a related class C_1': if C_1 contains all non-denumerable cardinals, then C_1' coincides with the class of all infinite cardinals; otherwise C_1' is the initial segment in the class of all infinite cardinals determined by the first non-denumerable cardinal not belonging to C_1. The mathematical importance of C_1' appears clearly from the fact that C_1' consists just of all infinite cardinals α such that on no set of power α does there exist a non-trivial countably additive two-valued measure.

The discussion in § 3 essentially parallels that in § 1, but concerns the class C_0 instead of C_1. The aim is to describe as fully as possible the extent of C_0. We first establish certain upper bounds for the extent of this class. Not only do we show that C_0 is a part of C_1, but we prove that the class C_0 is normal. Hence C_0 is a proper part of C_1 (unless the latter contains all non-denumerable cardinals) (5), and of course the same applies to all larger classes obtained from C_0 by performing operations which preserve normality. We then extend to C_0 the main results established in § 1 for C_1 (i.e. we show that various "constructively defined" classes of cardinals are included in C_0); since C_0 is a part of C_1, these extensions are actually improvements of the corresponding results in § 1. Unfortunately, it turns out that the arguments to be used for obtaining these extensions, although basically similar to those applied in the case of C_1, are technically much more complicated and can hardly be regarded as satisfactory from the viewpoint of formal simplicity and intuitive clarity. For these reasons we have decided to apply the following procedure, which seemed to us to be the best way out of our predicament. We establish in full detail one rather weak result of the kind in which we are interested—a result which at any rate implies that the class L (mentioned above in connection with C_1) is included in C_0. From then on the development assumes a very sketchy character: we simply formulate in proper order the main results with necessary definitions and lemmas, omitting all the proofs. Our advice for the reader is to skip part of § 3 at first reading, making only a mental note of the main results. If afterwards he wishes to fully convince himself of the truth of these results, he may attempt to reconstruct the missing proofs by using our outline and following the lines of the arguments in § 1; we believe that this reconstruction may require time and patience but presents no fundamental difficulties nor new ideas. Instead, and this may be simpler, he may use some sources available in the literature to reconstruct metamathematical proofs of the main results (6). Or, finally, he may wait for the proposed paper or papers of these authors which are to contain detailed metamathematical discussion of the results involved.

(5) The fact that C_0 is a proper part of C_1 (but not that C_0 is normal) was first stated in Hanf-Scott [11]. The result that C_0 is normal was obtained by one of the authors while proving a closely related metamathematical result, announced as Theorem 2 in [18].

(6) We believe that the recent article [9] of Hanf together with [48] will provide sufficient tools for the reconstruction of the proofs.

The discussion in § 4 and § 5 is parallel to that of § 2. The main topic is the study of characteristic properties of the class $\boldsymbol{C}_0$ in § 4 and of the class $\boldsymbol{C}_2$ in § 5. We have found it advantageous, however, to consider, in addition to the classes $\boldsymbol{C}_0$ and $\boldsymbol{C}_2$, a binary relation $\boldsymbol{R}$ between cardinals with the property that $\alpha \in \boldsymbol{C}_0$ if and only if $\alpha \boldsymbol{R} \alpha$, and $\alpha \in \boldsymbol{C}_2$ if and only if there is a β such that $\alpha \boldsymbol{R} \beta$. In this way various theorems on $\boldsymbol{C}_0$ and $\boldsymbol{C}_2$ are obtained as immediate corollaries of more general results concerning $\boldsymbol{R}$. In the two sections we establish numerous necessary and sufficient conditions for $\alpha \boldsymbol{R} \beta$, $\alpha \in \boldsymbol{C}_0$, and $\alpha \in \boldsymbol{C}_2$ formulated in terms of general set theory, theory of Boolean algebras, and point-set topology.

We do not know and hence do not give any specific results concerning the extent of the class $\boldsymbol{C}_2$ which would be analogous to those stated in § 1 for the class $\boldsymbol{C}_1$ and in § 3 for the class $\boldsymbol{C}_0$. Since, however, $\boldsymbol{C}_2$ includes $\boldsymbol{C}_1$, the main results of § 1 automatically extend to $\boldsymbol{C}_2$. Recall that in § 3, in addition to establishing the inclusion between $\boldsymbol{C}_0$ and $\boldsymbol{C}_1$, we give some stronger results concerning the relationship between these two classes; the problem whether analogous results can also be established concerning the relationship between $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ is still fully open.

§ 0. Preliminaries. We are not committed in this paper to any definite formalization or axiomatization of set theory. For most of our purposes the axiomatic system of Bernays is adequate. As is well known, Bernays set theory can be treated as an axiomatic theory formalized within first order predicate logic, with the membership symbol ϵ as the only undefined non-logical constant; we distinguish in this theory between classes and sets, defining sets as those classes which are members of other classes. In some portions of our paper however, we discuss notions which, in the Bernays set theory, either cannot be formalized in a natural way or cannot be formalized at all; e.g. such notions as sequences of classes indexed by arbitrary ordinals, operations (functions) defined for arbitrary classes and assuming arbitrary classes as values, and even operations on such operations.

Several methods are available which would provide these portions of our discussion with adequate deductive foundations. Thus we could replace Bernays set theory by a stronger system which would admit, in addition to sets and classes of sets, also classes of classes, classes of classes of classes, etc. Or else, we could formalize Bernays set theory, not within first order predicate logic, but within a higher order logic, e.g. within the simple theory of types. Or, finally, we can regard certain formal statements occurring in our discussion not as definitions and theorems but as definition schemata and theorem schemata in which variables representing operations on classes, etc., are to be replaced in each instance by well-defined constants.

We shall usually use letters $\boldsymbol{X}, \boldsymbol{Y}, \ldots$ to represent classes which are not assumed to be sets. We employ the usual set-theoretic terminology and symbolism. Thus, for example, $\subseteq$, ϵ, 0, $\cup$, $\bigcup$ denote respectively the relations of inclusion and membership, the empty set, and the operations of forming unions of two classes and of arbitrarily many classes. $X \sim Y$ is the set-theoretic difference of X and Y. $S(X)$ is the set of all subsets of X. A symbolic expression of the form $\{x\colon \Phi\}$, where Φ is to be replaced by any formula containing x (as a free variable), denotes the class of all x which satisfy this formula. $\{x\}$ is the set whose only element is x, $\{x, y\}$ is the unordered pair with the elements x and y, and $\langle x, y\rangle$ is the ordered pair with x as the first term and y as the second term. If f is a function and x is an element of the domain D of f, then the value of f at x is denoted by $f(x)$ or sometimes by f_x; thus f is the set of all ordered pairs $\langle x, f(x)\rangle$ where $x \in D$. We denote by ${}^{Y}\boldsymbol{X}$ the class of all functions on Y into the class $\boldsymbol{X}$, i.e., all functions with domain Y and range included in $\boldsymbol{X}$ (7). If $f \in {}^{Y}\boldsymbol{X}$ and $Z \subseteq \boldsymbol{X}$, we shall write $f^{-1}(Z)$ for the set $\{y\colon f(y) \in Z\}$.

We assume that *ordinals* have been introduced in such a way that every ordinal coincides with the set of all smaller ordinals. Consequently the intersection of all members of a non-empty class $\boldsymbol{X}$ of ordinals is again an ordinal, and in fact is the smallest ordinal belonging to $\boldsymbol{X}$. Moreover, the union of any set of ordinals is again an ordinal; in particular, $\bigcup 0 = 0$. We shall denote the class of all ordinals by $\boldsymbol{OR}$, and shall use the letters $\xi, \zeta, \eta, \varrho, \mu, \nu$ to represent arbitrary ordinals. The formulas $\xi < \zeta$, $\zeta > \xi$, and $\xi \in \zeta$ are equivalent and will be used interchangeably. The operation $+$ of addition on ordinals is assumed to be known. If a function φ has an ordinal ξ as its domain, it is sometimes called a *ξ-termed sequence*; in that case its values are written in the form φ_ζ for $\zeta < \xi$.

By a *cardinal* we shall mean an initial ordinal, i.e. an ordinal whose power exceeds the power of each smaller ordinal. The power, or cardinality, of a set A is denoted by $|A|$. We shall use the letters α, β, γ, sometimes with subscripts, to represent arbitrary infinite cardinals, and we shall reserve the letters $\delta, \varepsilon, \varkappa$, to represent arbitrary (finite or infinite) cardinals. The smallest infinite cardinal is denoted by ω. We shall denote by $\boldsymbol{C}$ the class $\{\beta\colon \beta > \omega\}$ of all cardinals greater than ω. For each cardinal δ the symbol δ^+ denotes the least cardinal greater than δ. By *the closed interval of cardinals between α and β*, denoted by $[\alpha, \beta]$, we mean the set of all cardinals γ such that $\alpha \leqslant \gamma \leqslant \beta$. Thus, in case $\alpha > \omega$, we have

$$[\alpha, \beta] = \boldsymbol{C} \cap (\beta^+ \sim \alpha)\,.$$

(7) We use the notation ${}^{Y}\boldsymbol{X}$ instead of the customary notation $\boldsymbol{X}^{Y}$ in order to avoid confusion with other types of exponentiation, e.g. exponentiation of cardinals.

We shall denote the half-open interval $\{\gamma\colon \alpha \leqslant \gamma < \beta\}$ by $[\alpha, \beta)$. Thus, in case $\alpha > \omega$, we have

$$[\alpha, \beta) = C \cap (\beta \sim \alpha)\,.$$

We let $S_\delta(X)$ denote the set of all subsets of the set X which have power $<\delta$, i.e.

$$S_\delta(X) = \{y \in S(X)\colon\ |y| < \delta\}\,.$$

ε and $\varkappa$ being arbitrary cardinals, we shall denote by $\varkappa^\varepsilon$ the cardinal power with base $\varkappa$ and exponent ε; thus $\varkappa^\varepsilon$ is the power of the set ${}^\varepsilon\varkappa$. By the weak cardinal power, denoted by $\varkappa^{\underline{\varepsilon}}$, we mean the cardinal $\bigcup_{\delta<\varepsilon}\varkappa^\delta$. The intersection of any non-empty class of cardinals, as well as the union of any set of cardinals, is again a cardinal.

An ordinal ζ is said to be *confinal* with an ordinal ξ if there is a sequence $\varphi \in {}^\xi\zeta$ such that $\varphi_\eta < \varphi_\varrho$ whenever $\eta < \varrho < \xi$, and

$$\zeta = \bigcup_{\eta<\xi}(\varphi_\eta + 1)\,.$$

The *confinality index* of ζ, denoted by $cf(\zeta)$, is the smallest ordinal ξ such that ζ is confinal with ξ. For every ordinal ξ, $cf(\zeta)$ is a cardinal and $cf(\zeta) \leqslant \zeta$. We shall say that ζ is a *limit ordinal* if $\zeta = \bigcup\zeta$, or equivalently if there is no ordinal η such that $\zeta = \eta+1$. Thus ζ is a non-limit ordinal iff $cf(\zeta) = 1$ ("iff" is used in this paper as an abbreviation for "if and only if"). An infinite cardinal α is said to be a *limit cardinal* if $\beta < \alpha$ implies $\beta^+ < \alpha$, a *strong limit cardinal* if $\beta < \alpha$ implies $2^\beta < \alpha$, a *singular cardinal* if $cf(\alpha) < \alpha$, and a *regular cardinal* if $cf(\alpha) = \alpha$. We shall denote by $\boldsymbol{SN}$ the class of all singular cardinals. Any singular cardinal is a limit cardinal, and any infinite non-limit cardinal is regular. α is said to be *strongly inaccessible*, or simply *inaccessible*, if it is a regular strong limit cardinal; otherwise α is said to be *accessible*. We denote by $\boldsymbol{AC}$ the class of all (infinite) accessible cardinals. The smallest inaccessible cardinal is obviously ω, and we have

$$\boldsymbol{SN} \subseteq \boldsymbol{AC} \subseteq \boldsymbol{C} \subseteq \boldsymbol{OR}\,.$$

We assume that the inaccessible cardinals have been arranged in a strictly increasing sequence $\theta_0, \theta_1, \ldots, \theta_\xi, \ldots$ Thus $\theta_0 = \omega$ and θ_1 is the first non-denumerable inaccessible number. It should be emphasized that the existence of θ_1, and *a fortiori* of any non-denumerable inaccessible cardinal, cannot be established on the basis of Bernays' axioms (or any other familiar system of axiomatic set theory). This question, however, has no influence on the present work. All our results and observations involving inaccessible cardinals are assumed to be provided if needed with a premise to the effect that the cardinals involved actually exist.

By the continuum hypothesis we mean the hypothesis that $\omega^+ = 2^\omega$. By the generalized continuum hypothesis we mean the hypothesis that $\alpha^+ = 2^\alpha$ whenever $\omega \leqslant \alpha$.

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A set B of sets is called a *field of sets* if $\bigcup B \in B$ and, for all $x, y \in B$, we have $x \sim y \in B$ and $x \cup y \in B$. It follows that, for all $x, y \in B$, $x \cap y \in B$. B is said to be a *field of subsets of* u if B is a field of sets whose unit set, $\bigcup B$, coincides with u. A set x is an *atom* of a field B of sets if $x \in B$, $x \neq 0$, and there is no set $y \in B$ such that $y \neq 0$, $y \neq x$, and $y \subseteq x$. I is an *ideal* in the field B of sets if $I \neq 0$, $I \subseteq B$, and, for all $x, y \in I$ and $z \in B$, we have $x \cup y \in I$ and $x \cap z \in I$. An ideal I in a field B of sets is a *proper ideal* if $I \neq B$, a *principal ideal* if $\bigcup I \in I$, a *prime ideal* if I is proper and, for each $x \in B$, either $x \in I$ or $\bigcup B \sim x \in I$, and an *α-complete ideal* if $\bigcup X \in I$ whenever $X \in S_\alpha(I)$. Thus every ideal is ω-complete. A field of sets B is said to be α-complete if, regarded as the non-proper ideal, it is α-complete. It is clear that a principal ideal in a field of sets is a prime ideal if and only if it is proper and contains the complement $\bigcup B \sim x$ of an atom x. A field of sets B is said to be *α-generated* by the set X if B is α-complete, $X \subseteq B$, and there is no α-complete field of subsets of $\bigcup B$ which includes X and is properly included in B. Thus, for any set y and any subset $X \subseteq S(y)$, there exists exactly one α-complete field of subsets of y which is α-generated by X, namely the intersection of all α-complete fields of subsets of y which include X. B is said to be an *α-complete field of sets with β generators* if B is α-generated by a set of power β but B is not α-generated by a set of power $<\beta$. Note that an α-complete field B is α-generated by some set of power β if and only if it is of power $\geqslant\beta$ and has at most β generators. It is easily seen that, for any set x, $S(x)$ is an α-complete field of sets for every cardinal α.

A set X of sets is said to be *disjointed* if $x, y \in X$ and $x \neq y$ implies $x \cap y = 0$. An ideal I in a field B of sets is said to be *δ-saturated* if every disjointed subset of $B \sim I$ has power $<\delta$. Notice that an ideal is never 0-saturated, is 1-saturated if and only if it is not proper, and is 2-saturated if and only if it is either prime or not proper. If $\delta < \varepsilon$, then every δ-saturated ideal is obviously ε-saturated.

An algebraic structure will in general be denoted by a capital German letter, and it will always be understood that the corresponding capital Roman letter denotes the set of elements of the algebraic structure; for example, the set of elements of the algebraic structure $\mathfrak{A}$ is denoted by A. The direct product of a sequence $\mathfrak{A}_\zeta$, $\zeta < \xi$, of algebraic structures is denoted by $\mathbf{P}_{\zeta<\xi}\mathfrak{A}_\zeta$, and the direct power of a structure $\mathfrak{A}$ indexed by a set X will be denoted by ${}^X\mathfrak{A}$. For detailed definitions of general algebraic notions such as algebraic structure, direct product, and homomorphism, see (for example) [46].

The notion of a *Boolean algebra* and those of an atom, an ideal, a prime ideal, a principal ideal, and a subalgebra of a Boolean algebra are assumed to be familiar to the reader; the same applies to the notion of a quotient algebra $\mathfrak{B}/I$, where $\mathfrak{B}$ is a Boolean algebra and I is one of its ideals. We use the symbols $+$, $\cdot$, and $-$ to denote the fundamental Boolean-algebraic operations of addition (join), multiplication (meet) and complementation, respectively; $\leqslant$ denotes the Boolean-algebraic inclusion, while $\sum$ and $\prod$ denote the usual infinite generalizations of $+$ and $\cdot$; thus $\prod B$, $\sum B$ are the zero and unit elements of $\mathfrak{B}$, respectively. Whenever a Boolean algebra is represented by a capital German letter, say $\mathfrak{B}$, we stipulate that $\mathfrak{B}$ is an ordered quadruple formed by the set B and the fundamental operations $+$, $\cdot$, and $-$.

An ideal I in a Boolean algebra $\mathfrak{B}$ is said to be α-complete if $\sum_{x\in X}x$ exists and is in I whenever $X \in S_\alpha(I)$; hence every ideal is ω-complete. In case the ideal $I = B$ is α-complete, the Boolean algebra $\mathfrak{B}$ itself is said to be α-complete. The algebra $\mathfrak{B}$ is called *absolutely complete* or (simply) *complete* if it is α-complete for every cardinal α. A Boolean algebra $\mathfrak{C}$ is said to be an *α-subalgebra* of the Boolean algebra $\mathfrak{B}$ if $\mathfrak{C}$ is a subalgebra of $\mathfrak{B}$, $\mathfrak{C}$ is an α-complete Boolean algebra, and any set $Y \subseteq C$ with power $<\alpha$ has the same sum in $\mathfrak{C}$ as in $\mathfrak{B}$. A Boolean algebra $\mathfrak{B}$ is said to be *α-generated* by a set $X \subseteq B$ if $\mathfrak{B}$ is α-complete and there is no α-subalgebra $\mathfrak{C}$ of $\mathfrak{B}$ such that $X \subseteq C$ and $C \neq B$. A Boolean algebra $\mathfrak{B}$ is said to be an *α-complete Boolean algebra with β generators* if $\mathfrak{B}$ is α-generated by a set of power β but is not α-generated by any set of power $<\beta$. A set X of elements of a Boolean algebra $\mathfrak{B}$ is said to be *disjointed* if $x, y \in X$ and $x \neq y$ implies $x\cdot y = 0$. An ideal I in a Boolean algebra $\mathfrak{B}$ is said to be *δ-saturated* if every disjointed subset of $B \sim I$ has power $<\delta$ (8).

A Boolean algebra $\mathfrak{B}$ is said to be *α-distributive* if it is α-complete and satisfies the following condition: let I be any set with $|I| < \alpha$, J be any function which assigns a set J_i with $|J_i| < \alpha$ to every element $i \in I$, and K be the Cartesian product $P_{i\in I}J_i$; let x be a function which assigns an element $x_{i,j}$ to every ordered pair $\langle i, j\rangle$ with $i \in I$ and $j \in J_i$; under these assumptions $\sum_{f\in K}\prod_{i\in I}x_{i,f(i)}$ exists and equals $\prod_{i\in I}\sum_{j\in J_i}x_{i,j}$. In the present paper we do not extend the notion of α-distributivity to Boolean algebras which are not α-complete, although such an extension often proves useful.

A *set algebra* is a Boolean algebra $\mathfrak{B}$ in which B is a field of sets, and the fundamental operations $+$, $\cdot$, and $-$ coincide with the set-theoretic operations $\cup$, $\cap$, and complementation with respect to $\bigcup B$. A set algebra $\mathfrak{B}$ is referred to as an *α-complete set algebra* if B is an α-complete field

(8) In the literature a slightly different notion of δ-saturated ideals has been used; namely, an ideal I is said to be δ-saturated if every disjointed set of non-zero elements in the quotient algebra $\mathfrak{B}/I$ has power $<\delta$. (See, for example, [43].) An ideal which is δ-saturated in the second sense is also δ-saturated in our sense. Moreover, if the Boolean algebra $\mathfrak{B}$ is α-complete and $\delta \leqslant \alpha$, then the two notions of δ-saturated ideals coincide; it is this case in which we shall be primarily interested.

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of sets. Notice that any α-complete set algebra is α-complete as a Boolean algebra, but a set algebra may be α-complete as a Boolean algebra without being an α-complete set algebra in our sense. We denote by $\mathfrak{S}(X)$ the set algebra $\mathfrak{B}$ such that $B = S(X)$. Thus $\mathfrak{S}(X)$ is always α-complete for every cardinal α.

A Boolean algebra is said to be (strongly) *α-representable* if it is isomorphic to an α-complete set algebra, and is said to be *weakly α-representable* if it is isomorphic to the quotient algebra of some α-complete set algebra with respect to an α-complete ideal. Thus any α-representable Boolean algebra is weakly α-representable, and any weakly α-representable Boolean algebra is α-complete.

We assume a familiarity with the notion of a topological space and such related notions as a discrete space, a continuous function, and an accumulation point (cf. [19]). By the product of a sequence T_ζ, $\zeta < \xi$, of topological spaces we mean the least topology T on the Cartesian product set such that, for all $\zeta < \xi$, the projection function from T onto T_ζ is continuous.

We shall now state a series of elementary and known results from the theory of Boolean algebras. We give a proof of one of them, namely 0.8, since with this generality it cannot be found in the literature. Concerning the remaining results, consult [4] and [38], where references to earlier papers can also be found.

THEOREM 0.1. *Let $\mathfrak{B}$ be an arbitrary Boolean algebra, let X, $Y \subseteq B$, and suppose that $\sum X$, $\sum Y$ exist in B. Then*

$$\sum\{x\cdot y\colon\ x \in X,\ y \in Y\}$$

exists and equals

$$(\sum X)\cdot(\sum Y)\,.$$

THEOREM 0.2. *In every Boolean algebra every proper ideal can be extended to a prime ideal, and every non-principal proper ideal can be extended to a non-principal prime ideal.*

It may be noticed that the second part of this theorem easily follows from the first. In fact, a non-principal ideal I in a Boolean algebra $\mathfrak{B}$ can first be extended to a proper ideal I' which contains all atoms of $\mathfrak{B}$; I' can be defined, for instance, as the set of all elements of the form $x+y$ where $x \in I$ and y satisfies the condition: $y\cdot z = 0$ for every $z \in I$. By applying to I' the first part of 0.2, we obtain a prime ideal J which includes I and contains all atoms of $\mathfrak{B}$, and hence is non-principal.

THEOREM 0.3. *Every Boolean algebra is ω-representable.*

THEOREM 0.4. *If α is singular, then every α-complete set algebra, Boolean algebra, or ideal, is an α^+-complete set algebra, Boolean algebra, or ideal, respectively. If α is a singular strong limit cardinal, then every α-distributive Boolean algebra is α^+-distributive.*

THEOREM 0.5. *A Boolean algebra is α-representable iff it is α-complete and every principal ideal can be extended to an α-complete prime ideal.*

THEOREM 0.6. *Every α-representable Boolean algebra is α-distributive.*

THEOREM 0.7. *If $\mathfrak{B}$ is a weakly α-representable Boolean algebra and I is an α-complete ideal in $\mathfrak{B}$, then $\mathfrak{B}/I$ is also weakly α-representable.*

THEOREM 0.8. *If α is either regular or is a strong limit cardinal, then every α-distributive Boolean algebra is weakly α-representable* (9).

Proof. By 0.4, we may assume that α is regular. Let $\mathfrak{B}$ be an α-distributive Boolean algebra. Let P denote the set of all prime ideals in $\mathfrak{B}$. We define a function g on B into $S(P)$ by the condition that, for all $x \in B$,

$$g(x) = \{I\colon\ I \in P \text{ and } x \notin I\}\,. \tag{1}$$

Let C be the α-complete field of subsets of P which is α-generated by the range of g. Furthermore, let

$$J = \{\textstyle\bigcap_{x \in X} g(x)\colon\ X \in S_\alpha(B) \text{ and } \prod X = 0\} \tag{2}$$

and let K be the α-complete ideal generated by J in C. We shall show that the function f such that

$$f(x) = g(x)/K \quad \text{for all} \quad x \in B$$

is an isomorphism of $\mathfrak{B}$ onto $\mathfrak{C}/K$, and thus that $\mathfrak{B}$ is weakly α-representable.

(9) For the case in which α is not a limit cardinal this result is stated in [37]; a proof for that case is carried out in detail in [1], and our present proof uses essentially the same argument as [1]. In [33] an explicit equational characterization of those Boolean algebras which are weakly α-representable (for α regular) is announced without proof, and Theorem 0.8 above is an easy consequence of that characterization. This opportunity is taken to correct a mistake and fill a gap in [4]. It is stated there on p. 66 that every α-distributive Boolean algebra is weakly α-representable; for the proof the reader is referred to [37]. As mentioned in the text, however, it seems dubious whether the statement in question can be proved for all singular cardinals (without assuming the generalized continuum hypothesis), and at any rate the proof in [37] does not apply to this case. The statement in its whole generality is used on p. 69 in the proof of Theorem 3.3, part (iv), to show that a certain property of an infinite cardinal λ, $\boldsymbol{P}_4^{(3)}$, implies another property, $\boldsymbol{P}_4^{(4)}$. In consequence the proof turns out to be defective in case λ is singular. Fortunately, however, the gap can easily be filled since it can be shown directly that every singular cardinal has the property $\boldsymbol{P}_4^{(4)}$. In fact, $\boldsymbol{P}_4^{(4)}$ applies to a cardinal λ iff there is a λ-complete set algebra $\mathfrak{A}$ and a λ-complete ideal I in $\mathfrak{A}$ such that $\mathfrak{A}/I$ is not isomorphic to any λ-complete set algebra. Let now λ be a singular cardinal, $\mathfrak{A} = \mathfrak{S}(\lambda^+)$, and $I = S_{\lambda^+}(\lambda^+)$. Clearly, $\mathfrak{A}$ and I are λ^+-complete and *a fortiori* λ-complete. Since λ^+ is accessible, every λ^+-complete prime ideal in $\mathfrak{A}$ is principal (see [4], Theorem 2.2, p. 61). Hence, as is easily seen, $\mathfrak{A}/I$ has no λ^+-complete prime ideals and therefore is not isomorphic to any λ^+-complete set algebra; since, however, λ is singular, the λ-complete and the λ^+-complete set algebras coincide.

It is clear that the range of f α-generates $\mathfrak{C}/K$. For all $x \in B$,

$$g(\bar{x}) = P \sim g(x)\,, \tag{3}$$

and therefore, in $\mathfrak{C}/K$,

$$f(\bar{x}) = \overline{f(x)}\,. \tag{4}$$

Let $X \in S_\alpha(B)$ and $y = \sum X$. By De Morgan's law we have

$$y \cdot \left(\textstyle\prod_{x \in X} \bar{x}\right) = 0\,, \quad \text{and hence} \quad g(y) \cap \textstyle\bigcap_{x \in X} g(\bar{x}) \in K\,.$$

Since by (3)

$$\textstyle\bigcap_{x \in X} g(\bar{x}) = \bigcap_{x \in X}\big(P \sim g(x)\big) = P \sim \bigcup_{x \in X} g(x)\,,$$

we have

$$g(y) \sim \textstyle\bigcup_{x \in X} g(x) \in K\,. \tag{5}$$

On the other hand, by (2),

$$g(x) \cap g(\bar{y}) \in K$$

for each $x \in X$, and since K is α-complete it follows that

$$\left(\textstyle\bigcup_{x \in X} g(x)\right) \sim g(y) \in K\,. \tag{6}$$

We see from (5), (6), and the α-completeness of K that

$$f(y) = \left(\textstyle\bigcup_{x \in X} g(x)\right)\big/K = \sum_{x \in X} f(x)\,. \tag{7}$$

From (4), (7), and the fact that $\mathfrak{C}/K$ is α-generated by the range of f, it follows that f is a homomorphism of $\mathfrak{B}$ onto $\mathfrak{C}/K$.

It remains to prove that f is one-one, or equivalently that $f(x) = 0$ implies $x = 0$. Suppose that $x \in B$ and $f(x) = 0$. Then $g(x) \in K$, and since α is regular, it follows that, for some $\beta < \alpha$ and some $z \in {}^{\beta\times\beta}B$, we have

$$\textstyle\prod_{\eta<\beta} z(\xi, \eta) = 0 \quad \text{for all} \quad \xi < \beta\,, \tag{8}$$

and

$$g(x) \subseteq \textstyle\bigcup_{\xi<\beta} \bigcap_{\eta<\beta} g\big(z(\xi, \eta)\big)\,.$$

By the set-theoretic distributive law,

$$\textstyle\bigcup_{\xi<\beta} \bigcap_{\eta<\beta} g\big(z(\xi, \eta)\big) = \bigcap_{\varphi \in {}^\beta\beta} \bigcup_{\xi<\beta} g\Big(z\big(\xi, \varphi(\xi)\big)\Big),$$

and therefore

$$g(x) \subseteq \textstyle\bigcup_{\xi<\beta} g\Big(z\big(\xi, \varphi(\xi)\big)\Big) \quad \text{for all} \quad \varphi \in {}^\beta\beta\,. \tag{9}$$

From (1) and (9) we see that, for each $\varphi \in {}^\beta\beta$ and $I \in P$, if $z\big(\xi, \varphi(\xi)\big) \in I$ for all $\xi < \beta$, then $x \in I$. Then, in view of 0.2,

$$x \leqslant \textstyle\sum_{\xi<\beta} z\big(\xi, \varphi(\xi)\big) \quad \text{for all} \quad \varphi \in {}^\beta\beta\,. \tag{10}$$

Since $\mathfrak{B}$ is α-distributive and $\beta < \alpha$, we have

$$\prod_{\varphi \in {}^{\beta}\beta} \sum_{\xi<\beta} z(\xi, \varphi(\xi)) = \sum_{\xi<\beta} \prod_{\eta<\beta} z(\xi, \eta) \tag{11}$$

and product on the left exists. (8) and (11) imply

$$\prod_{\varphi \in {}^{\beta}\beta} \sum_{\xi<\beta} z(\xi, \varphi(\xi)) = 0\,. \tag{12}$$

From (10) and (12) we conclude that $x = 0$, and thus f is one-one.

Under the generalized continuum hypothesis, every cardinal α has the property that each α-distributive Boolean algebra is weakly α-representable, because every singular cardinal is a strong limit cardinal. We do not know whether that property can be established for every cardinal α (and in particular for singular non-strong limit cardinals α) without the generalized continuum hypothesis.

THEOREM 0.9. *Every weakly $(2^\alpha)^+$-representable Boolean algebra is α^+-distributive.*

THEOREM 0.10. *If $\alpha \notin \boldsymbol{AC}$, then a Boolean algebra is weakly α-representable iff it is α-distributive.*

§ 1. Extent of the class C_1. It is well known that in any field of sets every proper ideal is included in a prime ideal (cf. 0.2). By applying this result to the ideal generated by the set of atoms (which may be empty) of a given field of sets, we see that every infinite field of sets has a non-principal prime ideal. The situation, however, changes radically when we turn to α-complete fields of sets and α-complete prime ideals in these fields with $\alpha > \omega$.

In this section we shall concentrate on the question of whether there exist non-principal α-complete prime ideals in the field $S(\alpha)$ of all subsets of α. This question has not yet been completely answered. By what has been said before, the answer is affirmative if $\alpha = \omega$. On the other hand, it has been known for many years that the cardinals $\alpha > \omega$ for which the answer is affirmative are very exceptional (if they exist at all), and in fact are all included among the inaccessible cardinals. (See [4], [51].) The results of the present section will constitute a further step toward a complete solution of the problem by showing that those hypothetical cardinals are very rare even among inaccessible cardinals.

To obtain a convenient notation for the formulation of the relevant results, we introduce the following

DEFINITION 1.1. *By $\boldsymbol{C}_1$ we denote the class of all infinite cardinals α such that every α-complete prime ideal in $S(\alpha)$ is principal.*

The main purpose of this section is to provide as much information as possible concerning the extent of the class $\boldsymbol{C}_1$.

First of all, the known results mentioned above may be restated as follows.

THEOREM 1.2. (i) $\boldsymbol{C}_1 \subseteq \boldsymbol{C}$ (*i.e.* $\omega \notin \boldsymbol{C}_1$).

(ii) $\boldsymbol{AC} \subseteq \boldsymbol{C}_1$.

We now turn our attention to inaccessible cardinals. We shall carry through a series of constructions which will enable us to prove that various classes which are much more comprehensive than the class of accessible cardinals are also included in $\boldsymbol{C}_1$.

Our arguments will have the following structure. We shall define a certain property of classes of cardinals which we shall call the property of being *normal*. It will be clear that any normal class is included in $\boldsymbol{C}_1$ and that any subclass of a normal class is also normal. We shall next establish a series of induction principles which enable us to conclude that, if a given class of cardinals is normal, then certain larger classes of cardinals are also normal, and consequently are also included in $\boldsymbol{C}_1$. Furthermore (partly with the help of the induction principles) we shall prove that the class $\boldsymbol{AC}$ of infinite accessible cardinals is normal. By combining these results we have a method of obtaining larger and larger classes of cardinals which are normal and thus included in $\boldsymbol{C}_1$. On the other hand, we shall see in Theorem 1.14 below that the class $\boldsymbol{C}_1$ itself is normal if and only if $\boldsymbol{C}_1 = \boldsymbol{C}$.

LEMMA 1.3. *Let $\alpha \in \boldsymbol{C}$ and let I be an ω^+-complete prime ideal in the field $S(\alpha)$. Then there exists a unique ordinal η and a unique function φ on ${}^{\alpha}\alpha$ onto η such that:*

(i) *for all $f, g \in {}^{\alpha}\alpha$,*

$$\varphi(f) = \varphi(g) \quad \textit{iff} \quad \{\xi\colon\ \xi < \alpha,\ f(\xi) = g(\xi)\} \notin I\,;$$

(ii) *for all $f, g \in {}^{\alpha}\alpha$,*

$$\varphi(f) \leqslant \varphi(g) \quad \textit{iff} \quad \{\xi\colon\ \xi < \alpha,\ f(\xi) \leqslant g(\xi)\} \notin I.$$

Proof. Since any two distinct ordinals have distinct order types, it is clear that there exist at most one pair η, φ satisfying (i) and (ii).

The existence of η and φ will be established by means of the prime reduced product construction developed in [6]. For any $f, g \in {}^{\alpha}\alpha$, we write $f \equiv g$ if

$$\{\xi\colon\ \xi < \alpha,\ f(\xi) = g(\xi)\} \notin I\,.$$

As is easily seen, $\equiv$ is an equivalence relation on ${}^{\alpha}\alpha$. For each $f \in {}^{\alpha}\alpha$, we shall write

$$(f) = \{g\colon\ g \in {}^{\alpha}\alpha,\ f \equiv g\}\,;$$

thus (f) denotes the equivalence class of f with respect to the equivalence relation $\equiv$. Now let

$$H = \{(f)\colon\ f \in {}^{\alpha}\alpha\}\,.$$

For any $a, b \in H$, we shall write $a \ll b$ if there exist $f \in a$, $g \in b$ such that

$$\{\xi\colon\ \xi < \alpha, f(\xi) \leqslant g(\xi)\} \notin I\,.$$

It is easily shown that, for all $f, g \in {}^{\alpha}\alpha$, $(f) \leqslant (g)$ iff

$$\{\xi\colon\ \xi < \alpha, f(\xi) \leqslant g(\xi)\} \notin I .$$

To complete the proof of our lemma it is sufficient to show that the set H is well-ordered by the relation $\leqslant$.

The fact that I is prime implies that for all $f, g \in {}^{\alpha}\alpha$, either $(f) = (g)$ or exactly one of the two conditions $(f) \leqslant (g)$, $(g) \leqslant (f)$ holds. Moreover, if $f, g, h \in {}^{\alpha}\alpha$, $(f) \leqslant (g)$, and $(g) \leqslant (h)$, then

$$\{\xi\colon\ \xi < \alpha, f(\xi) \leqslant g(\xi)\} \cap \{\xi\colon\ \xi < \alpha, g(\xi) \leqslant h(\xi)\} \notin I ,$$

so

$$\{\xi\colon\ \xi < \alpha, f(\xi) \leqslant h(\xi)\} \notin I ,$$

and thus $(f) \leqslant (h)$. Therefore $\leqslant$ is a simple ordering.

Suppose $f_0, f_1, f_2, \ldots \in {}^{\alpha}\alpha$ and for each $\eta \in \omega$, we have $(f_{\eta+1}) \leqslant (f_\eta)$ and $(f_{\eta+1}) \neq (f_\eta)$. Since I is ω^{+}-complete, we have

$$X = \{\xi\colon\ \xi < \alpha, f_{\eta+1}(\xi) < f_\eta(\xi) \text{ for all } \eta \in \omega\} \notin I .$$

Therefore the set X is non-empty, and there is an infinite decreasing sequence of ordinals, a contradiction. It follows that H has no infinite strictly decreasing sequence with respect to $\leqslant$, and hence $\leqslant$ well orders H.

It is helpful to think of the members of I as "small sets", and to read

$$\text{“}\{\xi\colon\ \xi < \alpha, f(\xi) = g(\xi)\} \notin I\text{”}$$

as "f and g are equal almost everywhere". Similarly,

$$\text{“}\{\xi\colon\ \xi < \alpha, f(\xi) < g(\xi)\} \notin I\text{”}$$

may be read "f is less than g almost everywhere", and in general any statement about an arbitrary element $\xi \in \alpha$ which holds except when ξ belongs to some fixed member of I may be thought of as holding "almost everywhere".

DEFINITION 1.4. *Let $\alpha \in \boldsymbol{C}$ and let I be an ω^{+}-complete prime ideal in the field $S(\alpha)$. We shall denote by α_I and τ_I the unique ordinal η and the unique function φ, respectively, which satisfy conditions* 1.3 (i) *and* 1.3 (ii).

The equation $\tau_I(f) = \xi$ may be read "the type of f with respect to I is ξ".

The simplest situation in which the hypotheses of 1.3 are satisfied is when I is a principal prime ideal in $S(\alpha)$ which is generated, say, by the set $\alpha \sim \{\xi\}$. In this case it is easily seen that $\alpha_I = \alpha$ and, for each $f \in {}^{\alpha}\alpha$, $\tau_I(f) = f(\xi)$ (cf. [6], p. 201). We shall be chiefly interested, however, in the case in which I is non-principal.

LEMMA 1.5. *Suppose that $\alpha \in \boldsymbol{C}$, $\xi < \alpha$, $f \in {}^{\alpha}\alpha$, and I is an α-complete prime ideal in $S(\alpha)$. We then have*

(i) *$f \in {}^{\alpha}\xi$ implies $\tau_I(f) < \xi$;*

(ii) *$f \in {}^{\alpha}\{\xi\}$ implies $\tau_I(f) = \xi$;*

(iii) *$\tau_I(f) < \xi$ iff $f^{-1}(\xi) \notin I$;*

(iv) *$\tau_I(f) = \xi$ iff $f^{-1}(\{\xi\}) \notin I$.*

Proof. It is clear that (iii) implies (i) and that (iv) implies (ii). Let $g_\xi \equiv \alpha \times \{\xi\}$. We shall prove by transfinite induction that $\tau_I(g_\xi) = \xi$ for all $\xi < \alpha$. This clearly implies (iii) and (iv).

Suppose that $\zeta < \alpha$ and $\tau_I(g_\xi) = \xi$ holds whenever $\xi < \zeta$. Then $\tau_I(g_\xi) \neq \xi$ for each $\xi < \zeta$, whence $\tau_I(g_\zeta) \geqslant \zeta$. If $\tau_I(f) < \tau_I(g_\zeta)$ then $f^{-1}(\zeta) \notin I$, and by α-completeness of I, $f^{-1}(\{\xi\}) \notin I$ for some $\xi < \zeta$. Hence $\tau_I(f) = \tau_I(g_\xi) = \xi < \zeta$. It follows that $\tau_I(g_\zeta) = \zeta$.

LEMMA 1.6. *Suppose $\alpha \in \boldsymbol{C}$ and I is an α-complete non-principal prime ideal in $S(\alpha)$. Then $\alpha < \alpha_I$ and*

$$\alpha \leqslant \tau_I(\{\langle\xi, \xi\rangle\colon\ \xi < \alpha\}) .$$

Proof. Let f be the identity function $\{\langle\xi, \xi\rangle\colon\ \xi < \alpha\}$ on α. Then, for each $\xi < \alpha$, we have

$$f^{-1}(\{\xi\}) = \{\xi\}$$

and, since I is non-principal, $\{\xi\} \in I$. Therefore, by the α-completeness of I and by 1.5, $\tau_I(f) \neq \xi$ for each $\xi < \alpha$, and thus $\tau_I(f) \geqslant \alpha$. Since $\tau_I(f) < \alpha_I$, we have $\alpha < \alpha_I$.

DEFINITION 1.7. *We shall say that α is representable by f iff $\alpha \in \boldsymbol{C}$, $f \in {}^{\alpha}\alpha$, and there exists an α-complete prime ideal I in $S(\alpha)$ such that $\tau_I(f) = \alpha$.*

A part of the next theorem is due to Dana Scott, who has formulated 1.8 (iii) and proved its equivalence to 1.8 (ii).

THEOREM 1.8. *The following four conditions are equivalent:*

(i) $\alpha \in \boldsymbol{C} \sim \boldsymbol{C}_1$;

(ii) *α is representable by some function $g \in {}^{\alpha}\alpha$;*

(iii) *α is representable by the identity function $\{\langle\xi, \xi\rangle\colon\ \xi < \alpha\}$;*

(iv) *α is representable by some function $h \in {}^{\alpha}(\boldsymbol{C} \cap \alpha)$.*

Proof. The fact that (i) implies (ii) follows easily from 1.4 and 1.6. We shall prove in turn the implications (ii)$\Rightarrow$(iii), (iii)$\Rightarrow$(iv), and (iv)$\Rightarrow$(i).

Assume (ii), and let I be an α-complete non-principal prime ideal in $S(\alpha)$ such that $\tau_I(g) = \alpha$. Let f be the identity function on α. Let

$$J = \{x\colon\ x \subseteq \alpha, g^{-1}(x) \in I\} .$$

It is easily seen that J is an α-complete prime ideal in $S(\alpha)$. For every $\xi<\alpha$ we have $\tau_I(g)\neq\xi$, so, by 1.5, $g^{-1}(\{\xi\})\in I$, and hence $\{\xi\}\in J$. Therefore J is non-principal. By 1.6 we have $\tau_J(f)\geqslant\alpha$. Suppose now that $\tau_J(f')<\tau_J(f)$, and let g' be defined by the condition $g'(\zeta)=f'(g(\zeta))$ for each $\zeta<\alpha$. Then

$$\{\xi:\ \xi<\alpha,\ f'(\xi)<f(\xi)\}\notin J,$$

so

$$g^{-1}(\{\xi:\ \xi<\alpha,\ f'(\xi)<f(\xi)\})\notin I$$

and

$$\{\zeta:\ \zeta<\alpha,\ g'(\zeta)<g(\zeta)\}\notin I;$$

therefore by 1.4 we have $\tau_I(g')<\tau_I(g)$. Let $\eta=\tau_I(g')$; thus $\eta<\alpha$. By 1.5, $g'^{-1}(\{\eta\})\notin I$. It follows that $g^{-1}(f'^{-1}(\{\eta\}))\notin I$, and hence $f'^{-1}(\{\eta\})\notin J$. Then, by 1.5, $\tau_J(f')=\eta$. This shows that $\tau_J(f)\leqslant\alpha$, hence $\tau_J(f)=\alpha$, and α is representable by f. We have proved that (ii) implies (iii).

Now assume (iii), and let J be an α-complete non-principal prime ideal in $S(\alpha)$ such that

$$\tau_J(\{\langle\xi,\xi\rangle:\ \xi<\alpha\})=\alpha.$$

Let h be the function defined by the condition

$$h(\xi)=\begin{cases}\omega^+ & \text{if}\quad \xi<\omega^+,\\ |\xi| & \text{if}\quad \omega^+\leqslant\xi<\alpha.\end{cases}$$

By 1.2 (ii) and 1.7, we must have $\alpha>\omega$ and $\alpha\notin\boldsymbol{AC}$, whence $\alpha>\omega^+$. Therefore $h\in{}^{\alpha}(\boldsymbol{C}\cap\alpha)$. Moreover, since $\omega^+\in J$ and $h(\xi)\leqslant\xi$ whenever $\xi\in\alpha\sim\omega^+$, we have $\tau_J(h)\leqslant\alpha$. For each $\xi<\alpha$, $h^{-1}(\{\xi\})\subseteq|\xi|^+$. Since $\alpha\notin\boldsymbol{AC}$, we have $|\xi|^+<\alpha$ and $|\xi|^+\in J$ whenever $\xi<\alpha$. Therefore $h^{-1}(\{\xi\})\in J$ for each $\xi\in\alpha$. By 1.5, $\tau_J(h)\geqslant\alpha$. Thus $\tau_J(h)=\alpha$, and α is representable by h. This verifies (iv).

It is obvious from 1.1 and 1.7 that (iv) implies (i). Our proof is complete.

LEMMA 1.9. *If $\xi<\alpha$ and $f\in{}^{\alpha}\xi$, then α is not representable by f.*

Proof. By Lemma 1.5.

DEFINITION 1.10. *$\boldsymbol{X}$ is said to be a normal class iff $\boldsymbol{X}\subseteq\boldsymbol{C}_1$ and no cardinal α is representable by a function $f\in{}^{\alpha}(\boldsymbol{X}\cap\alpha)$.*

The term "normal class" was chosen chiefly for lack of a better term, and a more meaningful name would be highly desirable. In what follows, however, we shall see that normal classes are normal in the sense that subclasses of $\boldsymbol{C}_1$ which are not normal are of a rather exceptional character and that it is not even known whether "abnormal" classes actually exist.

THEOREM 1.11. *If $\boldsymbol{X}$ is a normal class and $\boldsymbol{Y}\subseteq\boldsymbol{X}$, then $\boldsymbol{Y}$ is normal.*

Proof. Immediate from 1.10.

THEOREM 1.12. *A sufficient condition for $\boldsymbol{X}$ to be normal is that the union of any non-empty subset of $\boldsymbol{X}$ belongs to $\boldsymbol{C}_1$.*

Proof. If $\alpha\in\boldsymbol{C}_1$, then, by 1.8, α is not representable by any function. On the other hand, let $\alpha\in\boldsymbol{C}\sim\boldsymbol{C}_1$. Then $\bigcup(\boldsymbol{X}\cap\alpha)<\alpha$, so by putting $\xi=(\bigcup(\boldsymbol{X}\cap\alpha))+1$ we have $\xi<\alpha$ and ${}^{\alpha}(\boldsymbol{X}\cap\alpha)\subseteq{}^{\alpha}\xi$. But then, by 1.9, α is not representable by any function $f\in{}^{\alpha}(\boldsymbol{X}\cap\alpha)$. It follows by 1.10 that $\boldsymbol{X}$ is normal.

We shall see from 1.33 that, if $\boldsymbol{C}_1\neq\boldsymbol{C}$, then the condition in 1.12 is not necessary for $\boldsymbol{X}$ to be normal.

COROLLARY 1.13. (i) *If $\beta>\omega$ and $\boldsymbol{X}\subseteq[\beta,\alpha]\subseteq\boldsymbol{C}_1$, then $\boldsymbol{X}$ is normal.*

(ii) *If $\boldsymbol{X}$ is finite and $\boldsymbol{X}\subseteq\boldsymbol{C}_1$, then $\boldsymbol{X}$ is normal.*

(iii) *$[\omega^+,\alpha]$ is normal iff $[\omega^+,\alpha]\subseteq\boldsymbol{C}_1$.*

Proof. (i) and (ii) follow directly from 1.12. (iii) follows from (i) and 1.10.

THEOREM 1.14. *The following three conditions are equivalent:*

(i) *$\boldsymbol{C}_1=\boldsymbol{C}$;*

(ii) *$\boldsymbol{C}_1$ is normal;*

(iii) *the class $\{\alpha:\ [\omega^+,\alpha]\subseteq\boldsymbol{C}_1\}$ is normal.*

Proof. Assume (i). Then (ii) and (iii) follow by 1.12. Now assume that (i) is false, and let $\beta=\bigcap(\boldsymbol{C}\sim\boldsymbol{C}_1)$. By 1.8 (iv), β is representable by some function $g\in{}^{\beta}(\boldsymbol{C}\cap\beta)$. Therefore $\boldsymbol{C}\cap\beta$ is not normal. However, it is clear that

$$\boldsymbol{C}\cap\beta=\boldsymbol{C}\cap\{\alpha:\ [\omega^+,\alpha]\subseteq\boldsymbol{C}_1\}.$$

Hence (iii) fails. Finally,

$$\boldsymbol{C}\cap\{\alpha:\ [\omega^+,\alpha]\subseteq\boldsymbol{C}_1\}\subseteq\boldsymbol{C}_1,$$

so, by 1.11, (ii) also fails.

We thus see that, in case $\boldsymbol{C}_1=\boldsymbol{C}$, every subclass of $\boldsymbol{C}_1$ is normal; on the other hand, in case $\boldsymbol{C}_1\neq\boldsymbol{C}$, 1.14 (ii), (iii) give us examples of subclasses of $\boldsymbol{C}_1$ which are not normal. At present, however, it is not known which of the two cases actually holds.

We shall now digress for a time in order to give an alternate definition of normal classes which is essentially due to Scott. It will be stated below as Theorem 1.18, and is included with his permission. The formulation of this alternative definition is simple and throws some light on the intuitive content of the notion of a normal class. However, the definition is not as convenient for our purposes as the original one.

DEFINITION 1.15. *An ideal I in the field $S(\beta)$ is said to be strongly α-complete iff I is α-complete and, whenever $X\in{}^{\alpha}I$, we have*

$$\{\xi:\ \xi<\beta,\ \xi\in\textstyle\bigcup_{\zeta<\xi}X_\zeta\}\in I.$$

Notice that any principal ideal in $S(\beta)$ is strongly α-complete.

LEMMA 1.16. *Suppose that $\alpha \in C$ and I is a non-principal α-complete prime ideal in $S(\alpha)$. Then I is strongly α-complete iff*

$$\tau_I(\{\langle\xi,\xi\rangle:\ \xi<\alpha\})=\alpha .$$

Proof. Let f denote the identity function $\{\langle\xi,\xi\rangle:\ \xi<\alpha\}$ on α.

Suppose that I is strongly α-complete. By 1.6 we have $\tau_I(f)\geqslant\alpha$. Let $\tau_I(g)<\tau_I(f)$. Define the function $X\in{}^{\alpha}S(\alpha)$ by the condition

$$X_\zeta=g^{-1}(\{\zeta\})\quad\text{for each}\quad \zeta<\alpha .$$

Let

$$Y=\{\xi:\ \xi<\alpha,\ \xi\in \textstyle\bigcup_{\zeta<\xi}X_\zeta\} .$$

Then

$$Y=\{\xi:\ \xi<\alpha,\ g(\xi)<f(\xi)\} .$$

Since $\tau_I(g)<\tau_I(f)$, it follows from 1.4 that $Y\notin I$. Therefore, since I is strongly α-complete, we conclude by 1.14 that $X\notin{}^{\alpha}I$. For some $\eta\in\alpha$, we have $X_\eta\notin I$, so $g^{-1}(\{\eta\})\notin I$ and, by 1.5, $\tau_I(g)=\eta$. Then $\tau_I(g)<\alpha$ and hence $\tau_I(f)\leqslant\alpha$. Thus we have shown that $\tau_I(f)=\alpha$.

Now suppose that $\tau_I(f)=\alpha$ and let $X\in{}^{\alpha}I$. Let

$$Y=\{\xi:\ \xi<\alpha,\ \xi\in\textstyle\bigcup_{\zeta<\xi}X_\zeta\}.$$

We let g be the function of α into α defined by the condition:

$$g(\zeta)=\bigcap\{\zeta:\ \zeta<\alpha,\ \xi\in X_\zeta\cup\{\zeta\}\}$$

and for each $\zeta<\alpha$ we have

$$g^{-1}(\{\zeta\})\subseteq X_\zeta\cup\{\zeta\} .$$

Since I is non-principal, it follows that, for each $\zeta<\alpha$, $g^{-1}(\{\zeta\})\in I$. Therefore, by 1.5, $\tau_I(g)\neq\zeta$ for each $\zeta<\alpha$, and hence $\tau_I(g)\geqslant\tau_I(f)$. Then, by 1.5, we conclude that $Y=\{\xi:\ \xi<\alpha,\ g(\xi)<\xi\}\in I$. By 1.14, I is strongly α-complete.

It is easily seen that every strongly ω-complete prime ideal in $S(\omega)$ is principal. On the other hand, we have the following result:

COROLLARY 1.17. *Suppose $\alpha\in C$. Then $\alpha\in C_1$ iff every strongly α-complete prime ideal in $S(\alpha)$ is principal.*

Proof. Clearly, by 1.1 and 1.15, $\alpha\in C_1$ implies that every strongly α-complete prime ideal in $S(\alpha)$ is principal.

Suppose that every strongly α-complete prime ideal in $S(\alpha)$ is principal. Then, by 1.16, α is not representable by the function $\{\langle\xi,\xi\rangle:\ \xi<\alpha\}$. By 1.8 we conclude that $\alpha\in C_1$.

THEOREM 1.18 (Scott). *A necessary and sufficient condition for $\boldsymbol{X}$ to be a normal class is that $\boldsymbol{X}\subseteq C_1$ and, for each cardinal α and each non-principal strongly α-complete prime ideal J in $S(\alpha)$, we have $\boldsymbol{X}\cap\alpha\in J$.*

Proof. Suppose that $\boldsymbol{X}$ is a normal class. Then, by 1.10, $\boldsymbol{X}\subseteq C_1$. Let J be a non-principal strongly α-complete prime ideal in $S(\alpha)$ and let

$$f=\{\langle\xi,\xi\rangle:\ \xi<\alpha\} .$$

By 1.16 we have $\tau_J(f)=\alpha$. If $\boldsymbol{X}\cap\alpha=0$, then $\boldsymbol{X}\cap\alpha\in J$, as we wish to show. Suppose that $\boldsymbol{X}\cap\alpha\neq 0$, and let $\beta\in\boldsymbol{X}\cap\alpha$. Let $g\in{}^{\alpha}\alpha$ be defined by:

$$g(\xi)=\begin{cases}\xi & \text{if}\quad \xi\in\boldsymbol{X}\cap\alpha ,\\ \beta & \text{if}\quad \xi\in\alpha\sim\boldsymbol{X} .\end{cases}$$

Then $g\in{}^{\alpha}(\boldsymbol{X}\cap\alpha)$. Since $\boldsymbol{X}$ is normal, α is not representable by g. Therefore, by 1.7, $\tau_J(g)\neq\tau_J(f)$ and, by 1.4,

$$\{\xi:\ \xi<\alpha,\ g(\xi)=f(\xi)\}\in J .$$

But

$$\boldsymbol{X}\cap\alpha\subseteq\{\xi:\ \xi<\alpha,\ g(\xi)=f(\xi)\} ,$$

so $\boldsymbol{X}\cap\alpha\in J$.

To prove the converse, suppose that $\boldsymbol{X}\subseteq C_1$ and, for each non-principal strongly α-complete prime ideal J in $S(\alpha)$, $\boldsymbol{X}\cap\alpha\in J$. Let I be a non-principal α-complete prime ideal in $S(\alpha)$, let $g\in{}^{\alpha}(\boldsymbol{X}\cap\alpha)$, and suppose that $\tau_I(g)=\alpha$. We shall arrive at a contradiction. Define

$$J=\{x:\ x\subseteq\alpha,\ g^{-1}(x)\in I\}$$

and let

$$f=\{\langle\xi,\xi\rangle:\ \xi<\alpha\} .$$

We have already shown, while proving Theorem 1.8, that J is a non-principal α-complete prime ideal in $S(\alpha)$ and that $\tau_J(f)=\alpha$. It follows from 1.16 that J is strongly α-complete. Hence, by our hypothesis, $\boldsymbol{X}\cap\alpha\in J$. However, since $g\in{}^{\alpha}(\boldsymbol{X}\cap\alpha)$, we have $g^{-1}(\boldsymbol{X}\cap\alpha)=\alpha$, and so we arrive at the impossible conclusion that $\alpha\in I$. It follows that α is not representable by any function $g\in{}^{\alpha}(\boldsymbol{X}\cap\alpha)$. Hence, by 1.10, $\boldsymbol{X}$ is a normal class.

We now return to our main line of development.

THEOREM 1.19 (FIRST INDUCTION PRINCIPLE). *Suppose that, for each ordinal μ, $\boldsymbol{X}_\mu$ is normal. Then*

$$\{\beta:\ \beta\in\textstyle\bigcup_{\mu<\beta}\boldsymbol{X}_\mu\}\ \textit{is normal.}$$

Proof. Let

$$\boldsymbol{X}=\{\beta:\ \beta\in\textstyle\bigcup_{\mu<\beta}\boldsymbol{X}_\mu\} .$$

Since $\boldsymbol{X}_\mu\subseteq C_1$ for each μ, we have $\boldsymbol{X}\subseteq C_1$.

Suppose $\alpha\in C$, I is a non-principal α-complete prime ideal on α, $f\in{}^{\alpha}(X\cap\alpha)$, and $\tau_I(f)=\alpha$. We shall arrive at a contradiction. Let $g\in{}^{\alpha}\alpha$ be the function defined by the following condition:

$$g(\xi)=\bigcap\{\mu:\ f(\xi)\in\boldsymbol{X}_\mu\}\quad\text{for each}\quad\xi<\alpha .$$

By the definitions of X and g, since $f \in {}^{a}(X \cap a)$, we have $g(\xi) < f(\xi)$ for each $\xi < a$. Therefore $\tau_I(g) < \tau_I(f) = a$, say, $\tau_I(g) = \nu$. Then by 1.5 we have $g^{-1}(\{\nu\}) \notin I$. We may choose function $h \in {}^{a}(X_\nu \cap a)$ such that $h(\xi) = f(\xi)$ for all $\xi \in g^{-1}(\{\nu\})$. By 1.4, we have $\tau_I(f) = \tau_I(h)$. It follows that $\tau_I(h) = a$, so a is representable by h, contradicting the fact that X_ν is normal.

COROLLARY 1.20. *Suppose that, for each $\mu < \nu$, X_μ is normal. Then $(\bigcup_{\mu<\nu} X_\mu) \sim \nu$ is normal.*

Proof. For each $\mu < \nu$, let $Y_\mu = X_\mu \sim \nu$; whenever $\nu \leqslant \mu$, let $Y_\mu = 0$. Then each Y_μ is normal. Moreover

$$\{\beta\colon \beta \in \bigcup_{\mu<\beta} Y_\mu = (\bigcup_{\mu<\nu} X_\mu) \sim \nu\,.$$

The conclusion follows by 1.19.

COROLLARY 1.21. (i) *If X and Y are normal, then $X \cup Y$ is normal*

(ii) *If $[\omega^+, a] \subseteq C_1$ and, for each $\mu < a$, X_μ is normal, then $\bigcup_{\mu<a} X_\mu$ is normal.*

Proof. (i) follows from 1.20 with $\nu = 2$.

To prove (ii), we note that, by 1.20, $(\bigcup_{\mu<a} X_\mu) \sim a$ is normal and, by 1.13, $[\omega^+, a]$ is normal. Then, in view of (i), $\bigcup_{\mu<a} X_\mu$ is normal.

COROLLARY 1.22. *Suppose $C \cap a \subseteq C_1$. Then the set*

$$\{X\colon X \in S(\beta),\ C \cap X \text{ is normal}\}$$

is a strongly a-complete ideal in $S(\beta)$.

Proof. By 1.11, 1.13, 1.21, 1.19.

COROLLARY 1.23. *Suppose $C_1 \neq C$. Then there is no maximal normal class, i.e. any normal class is properly included in another normal class.*

Proof. Suppose that X is normal. By 1.14, $X \neq C_1$. Let $a \in C_1 \sim X$. Then, by 1.13, $\{a\}$ is normal and, by 1.21, $X \cup \{a\}$ is a normal class which properly includes X.

Let F be any (unary) operation which associates a subclass $F(X)$ of C to each subclass $X \subseteq C$.

DEFINITION 1.24. *For each ordinal ν we define the operation $F^{(\nu)}$ recursively, on subclasses X of C, as follows:*

$F^{(0)}(X) = X$;

whenever $\nu = \mu+1$, $F^{(\nu)}(X) = F(F^{(\mu)}(X)) \sim \nu$;

for each ordinal ν such that $0 < \nu = \bigcup \nu$,

$$F^{(\nu)}(X) = \bigcup_{\mu<\nu} F^{(\mu)}(X) \sim \nu\,.$$

Moreover, we define the operation $F^{(\infty)}$ by the equation:

$$F^{(\infty)}(X) = \bigcup_{\mu \in OR} F^{(\mu)}(X)\,.$$

COROLLARY 1.25. *If $X \subseteq C$, then, for each cardinal a, the following three conditions are equivalent:*

(i) $a \in F^{(\infty)}(X)$;

(ii) $a \in F^{(a)}(X)$;

(iii) $a \in \bigcup_{\mu<a} F^{(\mu)}(X)$.

DEFINITION 1.26. *F is said to preserve normality iff $F(X)$ is normal whenever X is normal.*

THEOREM 1.27 (SECOND INDUCTION PRINCIPLE). *If F preserves normality, then $F^{(\infty)}$ preserves normality.*

Proof. Suppose that X is normal. We shall show by transfinite induction that, for each ordinal μ, $F^{(\mu)}(X)$ is normal. Suppose $F^{(\mu)}(X)$ is normal whenever $\mu < \nu$. We shall show that $F^{(\nu)}(X)$ is normal. First, $F^{(0)}(X)$ is normal, because $F^{(0)}(X) = X$. Now let $\nu = \mu+1$. Then

$$F^{(\nu)}(X) = F(F^{(\mu)}(X)) \sim \nu\,,$$

whence by 1.11 and in view of the fact that F preserves normality, $F^{(\nu)}(X)$ is normal. Finally, suppose $0 < \nu = \bigcup \nu$. By 1.24,

$$F^{(\nu)}(X) = \bigcup_{\mu<\nu} F^{(\mu)}(X) \sim \nu\,,$$

and hence by 1.20, $F^{(\nu)}(X)$ is normal. Therefore, by the first induction principle 1.19 and by 1.24, it follows that $F^{(\infty)}(X)$ is normal.

A simple operation which is easily seen to preserve normality is the operation L defined by:

$$L(X) = \{\bigcup y\colon y \subseteq X \text{ and } y \cup \{\bigcup y\} \text{ is a closed interval included in } C\}.$$

It is not difficult to see that the class $L(AC)$ consists of the accessible cardinals together with all inaccessible cardinals θ_ξ such that $0 < \xi < \theta_\xi$; thus $C \sim L(AC)$ is just the class of all fixed points of the sequence θ.

It follows from the second induction principle that the operation $L^{(\infty)}$ also preserves normality.

We shall now introduce an operation M which is stronger than the operations L and $L^{(\infty)}$ in the sense that we always have $M(X) \supseteq L^{(\infty)}(X) \supseteq L(X)$, but which still preserves normality.

DEFINITION 1.28. *A subclass Y of C is said to be closed in C iff the union of any non-empty subset of Y belongs to Y.*

1.28 defines the usual order topology on C; it is easily seen that any intersection of closed classes is closed and so is any finite union of closed classes. The following are simple examples of closed classes: finite subsets of C; sets of the form $[\beta, a]$; classes of the form $C \sim a$. Notice that, by 1.12, any subclass of C_1 which is closed in C is normal.

DEFINITION 1.29. *For every* $X \subseteq C$, *we set*

$$M(X) = \{\bigcup y\colon\ y \subseteq X \text{ and } y \cup \{\bigcup y\} \text{ is a closed subset of } C\}.$$

The operation M on classes was first suggested by the work of Mahlo in [26], and may well be referred to as *Mahlo's operation*. If X is the class of infinite cardinals which are either singular or non-limit cardinals, then $M(X)$ is the class of infinite cardinals which are not ϱ_0 numbers in the sense of Mahlo. An operation related to M (and actually dual to M) in its application to the class of accessible cardinals was implicitly studied by Lévy in [21]. Thus $M(AC)$ is the class of infinite cardinals which are hyper-accessible of type 1, i.e. which are not hyper-inaccessible of type 1 in the sense of Lévy. More generally, $M^{(\nu)}(AC)$ is the class of cardinals $\alpha > \nu$ which are hyper-accessible of type ν; finally $M^{(\infty)}(AC)$ is the class of cardinals $\alpha \in C$ which are hyper-accessible of type α.

THEOREM 1.30. *Suppose* $X \subseteq C$. *Then*

(i) $X \subseteq M(X) \subseteq C$;

(ii) *if* $\mu < \nu$, *then* $M^{(\mu)}(X) \sim \nu \subseteq M^{(\nu)}(X) \subseteq M^{(\infty)}(X)$;

(iii) *if* $Y \subseteq X$, *then* $M(Y) \subseteq M(X)$;

(iv) $X = M(X)$ *iff* X *is closed.*

Proof. The proofs of (i)-(iii) are obvious. It follows at once from 1.28, 1.29 that, if X is closed, then $M(X) \subseteq X$ and hence $X = M(X)$. Suppose X is not closed. Then there is a least $\alpha \in C \sim X$ such that $\alpha = \bigcup(X \cap \alpha)$. Moreover, $(X \cap \alpha) \cup \{\alpha\}$ is closed, because of our choice of α. Therefore, by 1.29, we have $\alpha \in M(X)$.

Notice that $M(X)$ and even $M^{(\infty)}(X)$ need not coincide with the closure $\overline{X}$ of X (in the natural order topology of C). For example, we have $\overline{AC} = C$, while we can hardly expect to prove that $M(AC) = C$, or even that $M^{(\infty)}(AC) = C$.

On the other hand, it may be interesting to notice that M is a topological operation since it can be defined entirely in terms of closure; in fact, it can easily be shown that

$$M(X) = \{\alpha\colon \text{ for some } y \subseteq X,\ \overline{y} = y \cup \{\alpha\}\}.$$

THEOREM 1.31. (THIRD INDUCTION PRINCIPLE). *The operation* M *preserves normality.*

Proof. Suppose that X is normal. We shall first prove that $M(X) \subseteq C_1$. Suppose that $\alpha \in M(X) \sim X$, I is a non-principal α-complete prime ideal in $S(\alpha)$, $f \in {}^{\alpha}\alpha$, and $\tau_I(f) = \alpha$. We shall arrive at a contradiction. Let y be such that $y \subseteq X$, $\alpha = \bigcup y$, and $y \cup \{\alpha\}$ is closed. Let $g \in {}^{\alpha}\alpha$ be the function defined by the condition:

$$(1)\qquad g(\xi) = \bigcup\{\beta\colon\ \beta \in y,\ \beta \leqslant f(\xi)\} \quad \text{for each} \quad \xi < \alpha.$$

Since $y \cup \{\alpha\}$ is closed, we have $g \in {}^{\alpha}(\{0\} \cup y)$. It is clear that $g(\xi) \leqslant f(\xi)$ for each $\xi < \alpha$. Hence, by 1.4, $\tau_I(g) \leqslant \tau_I(f) = \alpha$. On the other hand, for each $\beta \in y$ we have $\beta < \alpha$, whence by 1.5

$$\{\xi\colon\ \xi < \alpha,\ \beta \leqslant f(\xi)\} \notin I.$$

By (1), $\beta \leqslant g(\xi)$ whenever $\beta \leqslant f(\xi)$, so

$$\{\xi\colon\ \xi < \alpha,\ \beta \leqslant g(\xi)\} \notin I.$$

Therefore, by 1.4, $\beta \leqslant \tau_I(g)$. Since $\alpha = \bigcup y$ and $\alpha \notin y$, $\alpha \leqslant \tau_I(g)$ and hence $\alpha = \tau_I(g)$. But since $y \subseteq X \cap \alpha$ and $g \in {}^{\alpha}(\{0\} \cup y)$, there exists a function $h \in {}^{\alpha}(X \cap \alpha)$ such that $\tau_I(h) = \tau_I(g) = \alpha$. This contradicts the fact that X is normal, and we have verified that $M(X) \subseteq C_1$.

Now suppose that $\alpha \in C$, I is a non-principal α-complete prime ideal in $S(\alpha)$, $f \in {}^{\alpha}(M(X) \cap \alpha)$, and $\tau_I(f) = \alpha$. Again we shall reach a contradiction.

For each $\xi < \alpha$, let y_ξ be such that $y_\xi \subseteq X$, $f(\xi) = \bigcup y_\xi$, and $y_\xi \cup \{f(\xi)\}$ is closed. If

$$\{\xi\colon\ \xi < \alpha,\ f(\xi) \in y_\xi\} \notin I,$$

then $f(\xi) \in X$ for almost all $\xi < \alpha$. Hence $\tau_I(h) = \tau_I(f) = \alpha$ for some $h \in {}^{\alpha}(X \cap \alpha)$, which contradicts the fact that X is normal. Therefore

$$\{\xi\colon\ \xi < \alpha,\ y_\xi \subseteq f(\xi)\} \notin I.$$

Let

$$y = \{\zeta\colon\ \zeta < \alpha,\ \{\xi\colon\ \xi < \alpha,\ \zeta \in y_\xi\} \notin I\},$$

that is, let y be the set of all $\zeta < \alpha$ which belong to "almost every" y_ξ. Since each $y_\xi \subseteq X \cap \alpha$, we have $y \subseteq X \cap \alpha$. We next prove that $\bigcup y = \alpha$. Obviously $\bigcup y \leqslant \alpha$. Suppose $\zeta < \alpha$. Let $g \in {}^{\alpha}\alpha$ be such that $g(\xi) \in y_\xi$ for all $\xi < \alpha$, and $g(\xi) > \zeta$ whenever $f(\xi) > \zeta$. Then $g(\xi) < f(\xi)$ for almost all $\xi < \alpha$. So, by 1.4, $\tau_I(g) < \tau_I(f)$, and hence $\tau_I(g) < \alpha$. Let $\eta = \tau_I(g)$. By 1.5, $g^{-1}(\{\eta\}) \notin I$; then, since $\eta \in y_\xi$ whenever $g(\xi) = \eta$, we have $\eta \in y$. On the other hand, by 1.5,

$$\{\xi\colon\ \xi < \alpha,\ \zeta < f(\xi)\} \notin I,$$

so

$$\{\xi\colon\ \xi < \alpha,\ \zeta < g(\xi)\} \notin I$$

and $\zeta < \eta$. It follows that $\bigcup y = \alpha$.

It remains to prove that $y \cup \{\alpha\}$ is closed. Let z be a non-empty subset of y. If $\bigcup z = \alpha$, then $\bigcup z \in y \cup \{\alpha\}$. Suppose $\bigcup z < \alpha$. Then $|z| < \alpha$. It follows from the α-completeness of I that

$$\{\xi\colon\ \xi < \alpha,\ z \subseteq y_\xi\} \notin I.$$

Since y_ξ is closed for each $\xi < \alpha$, we have

$$\{\xi\colon\ \xi < \alpha,\ \bigcup z \in y_\xi\} \notin I\,,$$

and thus $\bigcup z \in y$.

We now conclude, in view of 1.29, that $\alpha \in M(X)$. But then $\alpha \in C_1$, which contradicts the existence of I.

COROLLARY 1.32. *The operations* $M^{(\infty)}$, $(M^{(\infty)})^{(\infty)}$, *etc. preserve normality.*

Proof. By 1.27, 1.31.

THEOREM 1.33. *The class* AC *is normal.*

Proof. Let

$$X = \{\beta\colon \text{ for some cardinal } \alpha, \text{ we have } \omega \leqslant \alpha < \beta \leqslant 2^\alpha\}\,.$$

For each cardinal $\delta \geqslant \omega$, let

$$X_\delta = [\delta^+, 2^\delta]$$

For each ordinal μ, let

$$X_\mu = \begin{cases} 0 & \text{if} \quad \mu < \omega\,, \\ X_{|\mu|} & \text{if} \quad \mu \geqslant \omega\,. \end{cases}$$

We then have

$$X = \{\beta\colon\ \beta \in \bigcup_{\mu<\beta} X_\mu\}\,.$$

Since $X_\alpha \subseteq AC$ for every α, it follows from 1.2, 1.12 that X_α is normal for every α, and thus X_μ is normal for every μ. Therefore, by the first induction principle 1.19, X is normal.

We shall now show that

$$AC \subseteq M^{(\infty)}(X)\,.$$

If $\alpha \in AC$ and α is regular, then $\alpha \in X$ and so, by 1.30, $\alpha \in M^{(\infty)}(X)$.

It remains to prove that every singular cardinal belongs to $M^{(\infty)}(X)$. It is sufficient to prove that each ordinal μ has the following property:

(1) $\qquad$ if $\alpha \in C$ and $cf(\alpha) < \mu \leqslant \alpha$, then $\alpha \in M^{(\mu)}(X)$;

once this is verified, it follows at once that $\alpha \in M^{(\alpha)}(X)$ whenever $\alpha \in C$ and $cf(\alpha) < \alpha$.

Suppose that (1) holds for each ordinal $\mu < \nu$, let $\alpha \in C$, and let $cf(\alpha) < \nu \leqslant \alpha$. There exists a strictly increasing function $\varphi \in {}^{cf(\alpha)}C$ such that

$$\alpha = \bigcup_{\zeta<cf(\alpha)} \varphi(\zeta)\,.$$

Since $\alpha \in C$, $cf(\alpha)$ must be a limit ordinal, and it follows that

$$\alpha = \bigcup_{\zeta<cf(\alpha)} \varphi(\zeta)^+\,.$$

Clearly $\varphi(\zeta)^+ \in X$ for each $\zeta < cf(\alpha)$. Let

$$y = \{\gamma\colon\ cf(\alpha) \leqslant \gamma \text{ and for some positive } \eta < cf(\alpha),\ \gamma = \bigcup_{\zeta<\eta} \varphi(\zeta)^+\}\,.$$

Then $\alpha = \bigcup y$ and $y \cup \{\alpha\}$ is closed in C. If $\gamma \in y$, then either $\gamma \in X$ or $cf(\gamma) < cf(\alpha) \leqslant \gamma$, so $\gamma \in M^{(cf(\alpha))}(X)$. Thus

$$y \subseteq M^{(cf(\alpha))}(X)\,.$$

By 1.29 we have

$$\alpha \in M\big(M^{(cf(\alpha))}(X)\big)$$

and, since $cf(\alpha) < \nu \leqslant \alpha$,

$$\alpha \in M^{(\nu)}(X)\,.$$

This verifies (1), and thus $AC \subseteq M^{(\infty)}(X)$.

From the second and third induction principles, or from 1.32, it follows that $M^{(\infty)}(X)$ is normal. Therefore, by 1.11, AC is normal.

The class AC is an example of a normal class which, assuming $C_1 \neq C$, does not satisfy the simple sufficient condition for normality given in 1.12; in fact any $\alpha \in C$ is expressible as the union of some subset of AC. The same holds for any normal class which includes AC.

From the point of view of mathematical elegance it would be desirable to prove Theorem 1.33 directly and before the induction principles. In principle this would be possible, but the direct proof which is known to us is rather involved and duplicates some portions of the proofs of the induction principles.

By combining our induction principles with the fact that AC is normal, we are able to show that larger and larger classes of cardinals are included in C_1. For example, we see at once that

THEOREM 1.34.

(i) $M(AC) \subseteq C_1$;

(ii) $M^{(\infty)}(AC) \subseteq C_1$;

(iii) $M^{(\infty)}\big(M^{(\infty)}(AC)\big) \subseteq C_1$;

(iv) $(M^{(\infty)})^{(\infty)}(AC) \subseteq C_1$.

To understand the scope of formula (i), the reader should compare the remarks which precede Definition 1.28 and concern the extend of the classes $L(AC)$ and $L^{(\infty)}(AC)$, and the fact that $L(AC) \subseteq L^{(\infty)}(AC) \subseteq M(AC)$. Precisely, (i) states that the class C_1 contains all hyper-accessible cardinals of type 1. By (ii), C_1 contains every cardinal α which is hyper-accessible of type α; (iii) and (iv) go even beyond this.

The process can clearly be continued, and in a sense will never end. Indeed we have seen in 1.24 that, if $C_1 \neq C$, there is no maximal normal class. Moreover, if $C \neq C_1$ and $AC \subseteq X \subseteq C_1$, then X cannot be closed, and hence, by 1.30 (iv), $X \neq M(X)$; thus, if X is normal, then $M(X)$ is again a normal class and properly includes X.

It is possible to formulate even stronger induction principles than the three we have given. As a further step, one can formulate principles involving arbitrary operations which, like the operation $^{(\infty)}$, pass from one operation of the type $\boldsymbol{F}$ in 1.23 to another.

We have not yet been able to define "constructively" any cardinal $\alpha \in \boldsymbol{C}$ for which we cannot prove $\alpha \in \boldsymbol{C}_1$. It has been shown by Scott in [32] that Gödel's axiom of constructibility (see [8]) actually implies $\boldsymbol{C} = \boldsymbol{C}_1$. Thus the axiom of constructibility and the hypothesis $\boldsymbol{C} \neq \boldsymbol{C}_1$ lead to two incompatible set theories.

The question whether the hypothesis $\boldsymbol{C} \neq \boldsymbol{C}_1$ is compatible with the usual axiom systems of set theory remains open. On the other hand, from the results in this section it is easily seen that the opposite hypothesis, $\boldsymbol{C} = \boldsymbol{C}_1$, which may be called the *prime ideal hypothesis*, is certainly compatible with the usual axiom systems and remains compatible if these axiom systems are enriched by various "strong infinity axioms" which guarantee the existence of very large cardinals. (We assume here, of course, that the axiom systems involved are consistent and remain consistent when enriched by the "strong infinity axioms".) It is also seen that the hypothesis $\boldsymbol{C} = \boldsymbol{C}_1$ becomes a provable statement if the usual axiom systems are enriched by the negations of various "strong infinity axioms". Thus, for instance, the prime ideal hypothesis is compatible with the Bernays axiom system and remains compatible if this system is enriched by an axiom to the effect that there are inaccessible cardinals $> \omega$, or that every cardinal is smaller than some inaccessible cardinal, i.e., that θ_ξ exists for every ordinal ξ. On the other hand, this hypothesis becomes a theorem if the Bernays system is enriched by an axiom stating that there are no inaccessible cardinals $> \omega$, or that there are no inaccessible cardinals α such that $\theta_\alpha = \alpha$. We see thus that the inclusion of the prime ideal hypothesis in the usual axiom systems of set theory does not lead to contradictions and does not prevent us from further expanding these axiom systems by including various "strong infinity axioms" or their negations.

All these remarks apply also to two related hypotheses, $\boldsymbol{C} = \boldsymbol{C}_0$ and $\boldsymbol{C} = \boldsymbol{C}_2$, which will be involved in our later discussion.

§ 2. Characteristic properties of cardinals in the class $\boldsymbol{C}_1$.

In this section we shall discuss a number of problems from various branches of mathematics which arise in connection with the class $\boldsymbol{C}_1$. It will be seen from our discussion that it is often more natural and advantageous, instead of dealing with individual cardinals α belonging to $\boldsymbol{C}_1$, to deal with closed intervals $[\alpha, \beta]$ included in $\boldsymbol{C}_1$. We shall begin with properties which are of general set-theoretical nature, and then pass to problems which arise in more specialized contexts.

THEOREM 2.1. *For any cardinals α and β, the following five conditions are equivalent:*

(i) $[\alpha, \beta] \subseteq \boldsymbol{C}_1$;

(ii) *every α-complete prime ideal in $S(\beta)$ is principal;*

(iii) *there exists an α-complete field of sets of power 2^β in which every α-complete prime ideal is principal;*

(iv) *there exists an α-complete field of sets of power $\geqslant \beta$ in which every α-complete prime ideal is principal;*

(v) *in every β^+-complete field of sets every α-complete prime ideal is β^+-complete.*

Proof. It is obvious that (ii) implies (iii) and that (iii) implies (iv). We shall complete the proof of the theorem by proving successively the implications (ii)$\Rightarrow$(v), (v)$\Rightarrow$(i), (i)$\Rightarrow$(ii), and (iv)$\Rightarrow$(i).

Suppose that (ii) holds, let B be a β^+-complete field of sets, and let I be an α-complete prime ideal in B. Let $x \in {}^\beta I$. We shall show that $\bigcup_{\xi<\beta} x(\beta) \in I$. For each $\xi < \beta$ let $y(\xi) = x(\xi) \sim \bigcup_{\zeta<\xi} x(\zeta)$. Then, since B is β^+-complete, $y \in {}^\beta I$. It is clear that $\bigcup_{\xi<\beta} y(\xi) = \bigcup_{\xi<\beta} x(\xi)$; moreover, whenever $\zeta < \xi < \beta$, we have $y(\xi) \cap y(\zeta) = 0$. We now define $J = \{z\colon z \in S(\beta), \bigcup_{\xi \in z} y(\xi) \in I\}$. Then J is an α-complete ideal in $S(\beta)$. Since I is prime, we have either $z \in J$ or $\beta \sim z \in J$ whenever $z \in S(\beta)$. Therefore J is 2-saturated. By (ii), J is principal. If $\zeta < \beta$, then $y(\zeta) \in I$, so $\{\zeta\} \in J$. It follows that $J = S(\beta)$. This verifies (v).

Now assume (v) and let $\gamma \in [\alpha, \beta]$. Let I be a γ-complete prime ideal in $S(\gamma)$. Then I is α-complete. Also, $S(\gamma)$ is β^+-complete. It follows from (v) that I is β^+-complete. Since $\gamma \leqslant \beta$ and $\bigcup I = \{\xi\colon \{\xi\} \in I\}$, we have $\bigcup I \in I$, and thus I is principal. Therefore $\gamma \in \boldsymbol{C}_1$ and (i) holds.

Assume (i) and let I be an α-complete prime ideal in $S(\beta)$. Let γ_0 be the least cardinal γ such that either $\gamma = \beta^+$ or I is not γ-complete. Then $\alpha < \gamma_0$ and $\gamma_0 \leqslant \beta^+$; γ_0 obviously cannot be a limit cardinal, so there exists $\delta \in [\alpha, \beta]$ such that $\gamma_0 = \delta^+$. Since $\delta \in \boldsymbol{C}_1$, (ii) holds when $\alpha = \beta = \delta$. Therefore by what we have proved above, (v) holds when $\alpha = \beta = \delta$. Since $S(\beta)$ is δ^+-complete and I is δ-complete, it follows by (v) with $\alpha = \beta = \delta$ that I is δ^+-complete, i.e., I is γ_0-complete. Therefore we must have $\gamma_0 = \beta^+$. Since I is β^+-complete, it is principal, and (ii) follows.

Assume that (iv) holds, and let B be an α-complete field of sets of power $\geqslant \beta$ in which every α-complete prime ideal is principal. We may assume without loss of generality that $\bigcup B$ is an infinite cardinal, say $\bigcup B = \beta_0$, and furthermore that every atom of B is a singleton. Then $B \subseteq S(\beta_0)$, every atom of B is an atom of $S(\beta_0)$, and β_0 is the unit element of B. Observe that, for each $\zeta \in \beta_0$, the set

$$\zeta^* = \{x \in B\colon \zeta \notin x\}$$

is an α-complete prime ideal in B. Thus, for every $\zeta \in \beta_0$, ζ^* is a principal ideal in B, and consequently $\{\zeta\} \in B$. Let I be an α-complete prime ideal in $S(\beta_0)$. Set $J = I \cap B$. Since B is α-complete, J is an α-complete prime ideal in B. Therefore J is a principal ideal in B, that is, $\bigcup J \in J$. Hence $\bigcup J \in I$. On the other hand, we have

$$\bigcup I = \{\zeta\colon \{\zeta\} \in I\} = \{\zeta\colon \{\zeta\} \in J\} = \bigcup J\,,$$

and thus I is principal. It follows that (ii) holds when $\beta = \beta_0$, and, by what we have shown before, we have $[\alpha, \beta_0] \subseteq \boldsymbol{C}_1$. Since $|B| \geqslant \beta$ and $2^{\beta_0} \geqslant |B|$, we have $2^{\beta_0} \geqslant \beta$. By Theorem 1.2, $[\beta_0^+, 2^{\beta_0}] \subseteq \boldsymbol{C}_1$. Therefore $[\alpha, \beta] \subseteq \boldsymbol{C}_1$. This completes our proof.

Notice that conditions (iii) and (iv) in 2.1 are existential statements, while condition (v) is a universal statement. Condition (ii) is a statement about the particular field $S(\beta)$ and permits us to pass from the existential conditions to the universal one. It is obvious that the universal condition (v) implies (ii), (iii), and (iv).

It follows from 2.1 that conditions (i)-(v) always hold when $\beta < \alpha$, for in this case $[\alpha, \beta]$ is empty and is thus included in $\boldsymbol{C}_1$. When $\beta = \alpha$, condition (ii) reduces to the condition $\alpha \in \boldsymbol{C}_1$ as stated in 1.1, while (iii), (iv), and (v) give necessary and sufficient conditions for $\alpha \in \boldsymbol{C}_1$. In case $\alpha = \omega$, it is easily seen, e.g. by 0.2, that each of the conditions 2.1 (i)-2.1 (v) fails.

Corollary 2.2. *For any α and γ the following two conditions are equivalent:*

(i) $[\alpha, \gamma) \subseteq \boldsymbol{C}_1$;

(ii) *in every γ-complete field of sets every α-complete prime ideal is γ-complete.*

Proof. Assume (i). Then condition 2.1 (i) holds whenever $\beta < \gamma$. Hence, by 2.1, condition 2.1 (v) holds whenever $\beta < \gamma$, and (ii) follows immediately.

If we assume (ii), then, as is easily seen, 2.1 (ii) holds whenever $\beta < \gamma$. Hence 2.1 (i) holds for all $\beta < \gamma$, and thus (i) holds.

Corollary 2.2 is a partial improvement of Theorem 2.1: if we substitute β^+ for γ in 2.2 (i) and 2.2 (ii), we immediately obtain 2.1 (i) and 2.1 (v), respectively.

Definition 2.3. *F is said to be a set function on a field of sets B if the domain of F is B and the range of F is a set of sets. In case B is an α-complete field of sets, a set function F on B is called α-additive, or α-multiplicative, if, whenever $X \subseteq B$ and $0 < |X| < \alpha$,*

$$F(\bigcup X) = \bigcup_{x \in X} F(x)\,,$$

or

$$F(\bigcap X) = \bigcap_{x \in X} F(x)\,,$$

respectively. In case B is complete, F is called completely additive, or completely multiplicative, if it is α-additive for every $\alpha \in \boldsymbol{C}$, or α-multiplicative for every $\alpha \in \boldsymbol{C}$, respectively.

It is known (and can easily be shown) that, for every set A, F is a completely additive and completely multiplicative set function on $S(A)$ iff there is a set C and a function g with the range included in A such that

$$F(X) = C \cup g^{-1}(X) \quad \text{for every} \quad X \subseteq A\,.$$

Theorem 2.4. *For any cardinals α and β the following three conditions are equivalent:*

(i) $[\alpha, \beta] \subseteq \boldsymbol{C}_1$;

(ii) *every α-additive and α-multiplicative set function on $S(\beta)$ is completely additive and completely multiplicative;*

(iii) *every α-additive and α-multiplicative set function on any β^+-complete field of sets is β^+-additive and β^+-multiplicative.*

Proof. We refer to Theorem 2.4 in [4] for a proof that condition 2.4 (ii) above is equivalent to condition 2.1 (ii) in the special case when $\alpha = \beta$. With obvious changes the same argument can be used to prove that conditions 2.1 (ii) and 2.4 (ii) are equivalent for any cardinals α and β, and that the same applies to conditions 2.1 (v) and 2.4 (iii). Hence, by referring to 2.1, we conclude that the three conditions of our theorem are actually equivalent.

From this theorem we can easily derive an analogue of Corollary 2.2, i.e. we can show that condition 2.4 (iii) with β^+ replaced by an arbitrary cardinal γ is equivalent to 2.2 (i).

We shall now pass from the question of the existence of non-principal α-complete prime ideals to the more general question of the existence of non-principal α-complete and δ-saturated ideals.

Definition 2.5. *We denote by $\boldsymbol{C}^{[\delta]}$ the class of all α such that every α-complete and δ-saturated ideal in $S(\alpha)$ is principal.*

Theorem 2.6.

(i) $\boldsymbol{C}^{[0]} = \boldsymbol{C}^{[1]} = \boldsymbol{C} \cup \{\omega\}$;

(ii) $\boldsymbol{C}^{[2]} = \boldsymbol{C}_1$;

(iii) *if $\delta \geqslant 2$, then $\boldsymbol{C}^{[\delta]} \subseteq \boldsymbol{C}_1$;*

(iv) *if $\delta \leqslant \delta'$, then $\boldsymbol{C}^{[\delta']} \subseteq \boldsymbol{C}^{[\delta]}$;*

(v) *if $\omega^+ < \gamma$, then $\boldsymbol{C}^{[\delta]} \neq \boldsymbol{C}_1$.*

Proof. (i), (ii), (iii), and (iv) are obvious. By 1.2 we have $\omega^+ \in \boldsymbol{C}_1$, hence to obtain (v) it suffices to show that $\omega^+ \notin \boldsymbol{C}^{[\gamma]}$. The set $I = S_{\omega^+}(\omega^+)$

is an ω^+-complete ideal in $S(\omega^+)$. Moreover, I is γ-saturated because any disjointed subset of $S(\omega^+)$ has power $\leqslant \omega^+$ and thus has power $<\gamma$. But I is not principal, so $\omega^+ \notin \boldsymbol{C}^{[\gamma]}$.

THEOREM 2.7. *$\boldsymbol{SN} \subseteq \boldsymbol{C}^{[\delta]}$ for all δ.*

Proof. Assume $\alpha \in \boldsymbol{SN}$ and consider any α-complete ideal I in $S(\alpha)$ (whether δ-saturated or not). By 0.4, I is α^+-complete and therefore

$$\bigcup I = \bigcup \{\{\xi\}: \{\xi\} \in I\} \in I .$$

Hence I is principal. Consequently, by 2.5, $\alpha \in \boldsymbol{C}^{[\delta]}$, and the proof is complete.

THEOREM 2.8. *If $\alpha < \delta$, then $\alpha \in \boldsymbol{C}^{[\delta]}$ iff $\alpha \in \boldsymbol{SN}$.*

Proof. If α is singular, then $\alpha \in \boldsymbol{C}^{[\delta]}$ by 2.7. Suppose that α is regular and let $I = S_\alpha(\alpha)$. Then I is a non-principal α-complete ideal in $S(\alpha)$. I is δ-saturated because any disjointed subset of $S(\alpha)$ has power $\leqslant \alpha$, and thus has power $<\delta$. Therefore $\alpha \notin \boldsymbol{C}^{[\delta]}$.

Our next theorem is a special case of [43], Theorem 4.15, p. 54, and the proof is indicated there. For convenience, however, we shall give a proof here.

THEOREM 2.9. *If α is not a regular limit cardinal and $\delta \leqslant \alpha$, then $\alpha \in \boldsymbol{C}^{[\delta]}$.*

Proof. If α is singular, then, by 2.7, $\alpha \in \boldsymbol{C}^{[\delta]}$.

Suppose that $\alpha = \gamma^+$ and $\delta \leqslant \alpha$. Let I be an α-complete δ-saturated ideal in $S(\alpha)$. We shall prove that I is principal. For each $\xi < \alpha$, let f_ξ be a one-one function on ξ into γ. For each $\zeta < \alpha$ and $\eta \in \gamma$, let

$$x(\zeta, \eta) = \{\xi: \zeta < \xi < \alpha, f_\xi(\zeta) = \eta\} .$$

Then, whenever $\zeta, \zeta' < \alpha$, $\zeta \neq \zeta'$, and $\eta < \gamma$, we have

$$x(\zeta, \eta) \cap x(\zeta', \eta) = 0 .$$

For each $\eta < \gamma$, let

$$y(\eta) = \{\zeta: \zeta < \alpha, x(\zeta, \eta) \notin I\} .$$

Since I is δ-saturated, we have $|y(\eta)| < \delta$, and thus $|y(\eta)| \leqslant \gamma$, for each $\eta < \gamma$. Therefore

$$|\bigcup_{\eta<\gamma} y(\eta)| \leqslant \gamma .$$

Let

$$\zeta_0 \in \alpha \sim \bigcup_{\eta<\gamma} y(\eta) .$$

Since I is α-complete and $x(\zeta_0, \eta) \in I$ for each $\eta < \gamma$, we have

$$\bigcup_{\eta<\gamma} x(\zeta_0, \eta) \in I .$$

But, by the definition of $x(\zeta, \eta)$,

$$\bigcup_{\eta<\gamma} x(\zeta_0, \eta) = \alpha \sim (\zeta_0+1) ,$$

so

$$\alpha \sim (\zeta_0+1) \in I .$$

Let

$$z = \{\xi < \alpha: \{\xi\} \notin I\} .$$

Then $z \subseteq \zeta_0+1$. Since I is α-complete, $\alpha \sim (\zeta_0+1) \in I$, and $|\zeta_0+1| < \alpha$, we have $\alpha \sim z \in I$. But then $\alpha \sim z = \bigcup I$, and thus I is principal. It follows that $\alpha \in \boldsymbol{C}^{[\delta]}$, and our proof is complete.

THEOREM 2.10. *For all α, β, and δ, the following three conditions are equivalent:*

(i) $[\alpha, \beta] \subseteq \boldsymbol{C}^{[\delta]}$;

(ii) *every α-complete and δ-saturated ideal in $S(\beta)$ is principal;*

(iii) *in every β^+-complete field of sets, every α-complete and δ-saturated ideal is β^+-complete.*

Proof. It is sufficient to prove that (ii) implies (iii), that (iii) implies (i), and that (i) implies (ii). The proofs of these implications are entirely analogous to the arguments given in the proof of 2.1 for the implications 2.1 (ii) $\Rightarrow$ 2.1 (v), 2.1 (v) $\Rightarrow$ 2.1 (i), and 2.1 (i) $\Rightarrow$ 2.1 (ii), respectively, with the notion of a δ-saturated ideal everywhere replacing that of a prime ideal. Note that (i), (ii), and (iii) are obviously true when $\delta \leqslant 1$.

COROLLARY 2.11. *A necessary and sufficient condition for $[\alpha, \gamma) \subseteq \boldsymbol{C}^{[\delta]}$ is that:*

(i) *in every γ-complete field of sets, every α-complete and δ-saturated ideal is γ-complete.*

Proof. The proof is analogous to that of 2.2, but refers to 2.10 instead of 2.1.

Notice that the conditions 2.10 (i), (ii), and (iii) are the analogues of the conditions 2.1 (i), (ii), and (v), respectively, which arise when we pass from the notion of a prime ideal to the more general notion of a δ-saturated ideal. It is natural to ask whether the analogues of 2.1 (iii) and 2.1 (iv) (which will be explicitly stated below as 2.12 (ii) and 2.12 (iv), respectively) are necessary and sufficient for $[\alpha, \beta] \subseteq \boldsymbol{C}^{[\delta]}$. It is obvious that 2.10 (ii) implies the analogue of 2.1 (iii), which in turn implies the analogue of 2.1 (iv); hence both of those analogues are at least necessary for $[\alpha, \beta] \subseteq \boldsymbol{C}^{[\delta]}$. On the other hand, by examining our proof of the implication 2.1 (iv) $\Rightarrow$ 2.1 (ii), we obtain the following

THEOREM 2.12. *Let*

$$\beta_0 = \bigcap\{\gamma: 2^\gamma = 2^\beta\} \quad \textit{and} \quad \beta_1 = \bigcap\{\gamma: \beta \leqslant 2^\gamma\} .$$

Then the following two conditions are equivalent:

(i) $[\alpha, \beta_0] \subseteq \boldsymbol{C}^{[\delta]}$;

(ii) *there exists an α-complete field of sets of power 2^β in which every α-complete and δ-saturated ideal is principal.*

Similarly, the following conditions are equivalent:

(iii) $[\alpha, \beta_1] \subseteq C^{[\delta]}$;

(iv) *there exists an α-complete field of sets of power $\geqslant \beta$ in which every α-complete and δ-saturated ideal is principal.*

In the above theorem note that we always have $\beta_1 \leqslant \beta_0 \leqslant \beta$.

COROLLARY 2.13. *If β is a strong limit cardinal, or in particular if $\beta \in C \sim AC$, then each of the conditions* 2.12 (ii), 2.12 (iv) *is necessary and sufficient for $[\alpha, \beta] \subseteq C^{[\delta]}$.*

Proof. We have $\beta = \beta_0 = \beta_1$ in 2.12.

COROLLARY 2.14. *Assume that the generalized continuum hypothesis holds. Then* 2.12 (ii) *is necessary and sufficient for $[\alpha, \beta] \subseteq C^{[\delta]}$.*

Proof. We have $\beta = \beta_0$ in 2.12.

COROLLARY 2.15. *Assume that the generalized continuum hypothesis holds or at least that $[\beta_1^+, \beta]$ contains no regular limit cardinals where $\beta_1 = \bigcap\{\gamma\colon \beta \leqslant 2^\gamma\}$. Then, in case $\delta \leqslant \beta_1^+$, each of the conditions* 2.12 (ii), 2.12 (iv) *is necessary and sufficient for $[\alpha, \beta] \subseteq C^{[\delta]}$.*

Proof. By 2.9, $[\beta_1^+, \beta] \subseteq C^{[\delta]}$. Therefore $[\alpha, \beta_1] \subseteq C^{[\delta]}$ implies that $[\alpha, \beta] \subseteq C^{[\delta]}$. The result follows by 2.12.

COROLLARY 2.16. *Suppose that β is not a limit cardinal and $\alpha = \beta < \delta$. Then* 2.12 (iv) *holds but $[\alpha, \beta] \subseteq C^{[\delta]}$ fails.*

Proof. If β_1 is defined as in 2.12, then, since β is of the form γ^+, we have $\beta_1 < \beta$. Therefore $[\alpha, \beta_1] = 0$, and so, by 2.12, 2.12 (iv) holds. However, β is regular and hence, by 2.8, $\beta \notin C^{[\delta]}$. Thus $[\alpha, \beta]$ is not included in $C^{[\delta]}$.

THEOREM 2.17. *Suppose that α is regular. Then the following condition* (i) *is necessary and sufficient for $[\alpha, \beta] \subseteq C^{[\delta]}$:*

(i) *if X is any set of sets which covers β, i.e. $\beta \subseteq \bigcup X$, and for which $S(\beta) \sim X$ has no disjointed subset of power δ, then there is a set $Y \in S_\alpha(X)$ which also covers β.*

Proof. We shall prove that 2.17 (i) is equivalent to 2.10 (ii). Assume 2.17 (i). Let I be an α-complete δ-saturated ideal in $S(\beta)$. Let

$$X = I \cup \{\beta \sim \bigcup I\}.$$

Then $\bigcup X = \beta$. By 2.17 (i), there exists $Y \in S_\alpha(X)$ such that $\bigcup Y = \beta$. But then

$$\bigcup (Y \cap I) = \bigcup I.$$

Since I is α-complete, we have $\bigcup I \in I$, so I is principal. This verifies 2.10 (ii).

Now assume 2.10 (ii). Let X be a set of sets such that $\beta \subseteq \bigcup X$ and $S(\beta) \sim X$ has no disjointed subset of power δ. Let

$$I = \{x\colon x \in S(\beta) \text{ and there exists } Y \in S_\alpha(X) \text{ such that } x \subseteq \bigcup Y\}.$$

From the fact that α is regular it follows that I is an α-complete ideal in $S(\beta)$. Since

$$S(\beta) \sim I \subseteq S(\beta) \sim X,$$

I is δ-saturated. Because $\beta \subseteq \bigcup X$, every singleton of $S(\beta)$ belongs to I, and hence $\beta = \bigcup I$. It follows from 2.10 (ii) that I is principal, and thus $\beta \in I$. Therefore there exists $Y \in S_\alpha(X)$ such that $\beta \subseteq \bigcup Y$. This verifies 2.17 (i).

The condition 2.17 (i) has been dealt with in the "covering theorems" in [40]. It has the pleasant feature that it is formulated entirely in basic set-theoretical terms, without involving the notion of an ideal.

The hypothesis that α is regular cannot be removed from 2.17. In fact, if α is singular, $\alpha = \beta$, and $\alpha < \delta$, then $[\alpha, \beta] \subseteq C^{[\delta]}$ holds, but 2.17 (i) fails, e.g. for X equal to the set of all singletons of $S(\alpha)$.

LEMMA 2.18. *Suppose $2^{\delta} < \alpha$. Then any α-complete non-principal δ-saturated ideal in $S(\beta)$ may be extended to an α-complete non-principal prime ideal in $S(\beta)$.*

2.18 is simply a restatement of [43], Theorem 4.12, p. 53, to which we refer for a proof. We shall see that 2.18 is also a direct consequence of Lemma 4.23 which will be proved in § 4.

THEOREM 2.19. (i) *If $\delta > 1$, then $C^{[\delta]} \sim (2^{\delta})^+ = C_1 \sim (2^{\delta})^+$.*

(ii) *If $\delta > 1$ and $2^{\delta} < \alpha$, then the formulas $[\alpha, \beta] \subseteq C^{[\delta]}$ and $[\alpha, \beta] \subseteq C_1$ are equivalent for every cardinal β, and so are the formulas $[\alpha, \gamma) \subseteq C^{[\delta]}$ and $[\alpha, \gamma) \subseteq C_1$ for every cardinal γ.*

Proof. (i) By 2.6 (iii) we have $C^{[\delta]} \subseteq C_1$, so $C^{[\delta]} \sim (2^{\delta})^+ \subseteq C_1 \sim (2^{\delta})^+$. Suppose $\alpha \in C_1 \sim (2^{\delta})^+$. Then, by 2.18, every α-complete non-principal δ-saturated ideal in $S(\alpha)$ can be extended to an α-complete non-principal prime ideal. Since $\alpha \in C_1$, every α-complete prime ideal in $S(\alpha)$ is principal. It follows that every α-complete δ-saturated ideal in $S(\alpha)$ is principal, and hence $\alpha \in C^{[\delta]}$.

(ii) follows directly from (i).

By applying 2.19 to various results which have been stated above or will be stated below in this section, we immediately obtain conclusions of the type in which we are primarily interested here, namely theorems providing necessary and sufficient conditions for $[\alpha, \beta] \subseteq C_1$ or $[\alpha, \gamma) \subseteq C_1$. For instance, 2.10 and 2.19 (ii) imply at once that, under the assumptions $1 < \delta$ and $2^{\delta} < \alpha$, the formula $[\alpha, \beta] \subseteq C_1$ is equivalent to each of the conditions 2.10 (ii) and 2.10 (iii).

COROLLARY 2.20. *If* $1 < \delta \leqslant \omega$, *then* $C^{[\delta]} = C_1$.

Proof. Since $\delta \leqslant \omega$, we also have $2^\delta \leqslant \omega$, and hence $C \sim (2^\delta)^+ = C$. The result follows by 2.19.

COROLLARY 2.21. (i) *If* $\delta > 1$ *and* $[\delta, 2^\delta] \subseteq C^{[\delta]}$, *then* $C^{[\delta]} \sim \delta = C_1 \sim \delta$.
(ii) *If* $[\omega^+, 2^\omega] \subseteq C^{[\omega^+]}$, *then* $C^{[\omega^+]} = C_1$.

COROLLARY 2.22. (i) *Assume that δ is not a regular limit cardinal and the generalized continuum hypothesis holds, or at least that $[\delta, 2^\delta]$ contains no regular limit cardinals. Then, if $\delta > 1$,* $C^{[\delta]} \sim \delta = C_1 \sim \delta$.

(ii) *Assume that the continuum hypothesis holds or at least that $[\omega^+, 2^\omega]$ contains no regular limit cardinals. Then* $C^{[\omega^+]} = C_1$.

Proof. By 2.9 and 2.21.

The results 2.8, 2.9, and 2.19-2.22 may be summed up as follows. Assume that δ is any cardinal greater than 1. In case $\delta \leqslant \omega$, we have $\alpha \in C^{[\delta]}$ iff $\alpha \in C_1$. Also, in case $2^\delta < \alpha$, we have $\alpha \in C^{[\delta]}$ iff $\alpha \in C_1$. On the other hand, in case $\alpha < \delta$, we have $\alpha \in C^{[\delta]}$ iff $\alpha \in SN$. In case α is not a regular limit cardinal and $\delta \leqslant \alpha$, we have $\alpha \in C^{[\delta]}$.

Thus the only situation in which we have not settled the question of whether a given cardinal α belonging to C_1 also belongs to $C^{[\delta]}$ arises when $\delta \leqslant \alpha \leqslant 2^\delta$ and α is a regular limit cardinal. The following are among the most natural open problems in this direction:

(1) *Is* $2^\omega \in C^{[\omega^+]}$?

(2) *Is* $[\omega^+, 2^\omega] \subseteq C^{[\omega^+]}$?

(3) *If α is the first uncountable regular limit cardinal, is* $\alpha \in C^{[\alpha]}$?

(4) *Is* $\theta_1 \in C^{[\theta_1]}$?

(5) *Is* $\alpha \in C^{[\alpha]}$ *for every* $\alpha \in AC$?

(6) *Is* $2^\alpha \in C^{[\alpha]}$ *for every* $\alpha \in AC$?

(7) *Is* $\alpha \in C^{[\alpha]}$ *for every* $\alpha \in C_1$?

Of course, by 2.22 the continuum hypothesis implies that $2^\omega \in C^{[\omega^+]}$ and $[\omega^+, 2^\omega] \subseteq C^{[\omega^+]}$, while the generalized continuum hypothesis implies that $\alpha \in C^{[\alpha]}$ and $2^\alpha \in C^{[\alpha]}$ for every $\alpha \in AC$.

However, even if we assume the generalized continuum hypothesis, the questions (3), (4), and (7) remain open, although in this case the questions (3) and (4) clearly become equivalent to each other.

We shall now consider some problems which arise in the theory of Boolean algebras.

THEOREM 2.23. *For any cardinals α and β, the following condition* (i) *is necessary and sufficient for* $[\alpha, \beta] \subseteq C_1$:

(i) *in every β^+-complete Boolean algebra, every α-complete prime ideal is β^+-complete.*

Proof. By 2.1, the condition 2.1 (ii) is equivalent to $[\alpha, \beta] \subseteq C_1$. It is obvious that 2.23 (i) implies 2.1 (ii); we shall show that, conversely 2.1 (ii) implies 2.23 (i).

Our proof of the implication 2.1 (ii) $\Rightarrow$ 2.23 (i) will closely resemble the proof we have previously given of the implication 2.1 (ii) $\Rightarrow$ 2.1 (v). Let $\mathfrak{B}$ be a β^+-complete Boolean algebra and let I be an α-complete prime ideal in $\mathfrak{B}$. Let $x \in {}^\beta I$. For each $\xi < \beta$, let

$$y(\xi) = x(\xi) - \sum_{\zeta < \xi} x(\zeta).$$

It follows from 0.1 that

$$\sum_{\xi < \beta} y(\xi) = \sum_{\xi < \beta} x(\xi).$$

Let

$$J = \{z\colon\ z \in S(\beta),\ \sum_{\xi \in z} y(\xi) \in I\}.$$

Then J is an α-complete ideal in $S(\beta)$. If $z, z' \in S(\beta)$ and $z \cap z' = 0$, then, by 0.1,

$$\Big(\sum_{\xi \in z} y(\xi)\Big) \cdot \Big(\sum_{\xi \in z'} y(\xi)\Big) = \sum_{\xi \in z, \zeta \in z'} \big(y(\xi) \cdot y(\zeta)\big) = 0.$$

Therefore J is 2-saturated. By 2.1 (ii), J is principal. If $\zeta < \beta$, then $y(\zeta) \in I$, so $\{\zeta\} \in J$. Therefore $J = S(\beta)$. This verifies that I is β^+-complete, and thus 2.23 (i) holds.

COROLLARY 2.24. *The following condition* (i) *is necessary and sufficient for* $[\alpha, \gamma) \subseteq C_1$:

(i) *in every γ-complete Boolean algebra every α-complete prime ideal is γ-complete.*

Proof. The proof is entirely analogous to that of 2.2.

THEOREM 2.25. *For any cardinals α and β, the condition* (i) *below is equivalent to the inclusion* $[\alpha, \beta] \subseteq C_1$:

(i) *in every complete Boolean algebra of power $\leqslant 2^\beta$ every α-complete prime ideal is principal.*

Proof. It is seen at once that 2.25 (i) implies 2.1 (ii), in view of the fact that the set algebra $\mathfrak{S}(\beta)$ is a complete Boolean algebra of power $\leqslant 2^\beta$. By 2.1, it follows that 2.25 (i) implies $[\alpha, \beta] \subseteq C_1$.

Assume that $[\alpha, \beta] \subseteq C_1$. By 1.2, we have $[\beta^+, 2^\beta] \subseteq C_1$, and therefore $[\alpha, 2^\beta] \subseteq C_1$. Applying 2.23, we see that in every $(2^\beta)^+$-complete Boolean algebra every α-complete prime ideal is $(2^\beta)^+$-complete. In each complete Boolean algebra of power $\leqslant 2^\beta$ every $(2^\beta)^+$-complete ideal I contains its sum $\sum I$, and thus is principal. Therefore 2.25 (i) holds.

THEOREM 2.26. *For any cardinals α and β, the following three conditions are equivalent*:

(i) $[\alpha, \beta] \subseteq C_1$;

(ii) *every α-complete set algebra which is β^+-complete as a Boolean algebra is a β^+-complete set algebra*;

(iii) *in every α-complete set algebra which is a complete atomistic Boolean algebra with β atoms the unit element is the union of all atoms.*

Proof. The implication (ii) $\Rightarrow$ (iii) is obvious. We shall prove that (i) implies (ii) and that (iii) implies (i).

In order to establish the implication (i) $\Rightarrow$ (ii), it suffices, by 2.23, to prove that 2.23 (i) implies (ii). Let $\mathfrak{B}$ be an α-complete set algebra which is a β^+-complete Boolean algebra. Suppose that $X \subseteq B$ and $|X| < \beta$. We must show that $\sum X \subseteq \bigcup X$. Suppose $a \in \sum X$. Then the set

$$I = \{x\colon\ x \in B,\ a \notin x\}$$

is an α-complete prime ideal in $\mathfrak{B}$, since $\mathfrak{B}$ is an α-complete set algebra. It follows from 2.23 (i) that I is β^+-complete. If $X \subseteq I$, then $\sum X \in I$, contradicting $a \in \sum X$. Therefore it is not true that $X \subseteq I$. It follows that $a \in \bigcup X$, hence $\mathfrak{B}$ is a β^+-complete set algebra, and 2.26 (ii) holds.

To show that (iii) implies (i) we may, by 2.1, assume (iii) and derive 2.1 (ii). Let I be an α-complete non-principal prime ideal in $S(\beta)$. We shall arrive at a contradiction. Let F be the set function on β defined by the conditions:

$$F(x) = \begin{cases} x & \text{if} \quad x \in I; \\ x \cup \{\beta\} & \text{if} \quad x \notin I. \end{cases}$$

Let B be the range of F. From the fact that I is a prime ideal it follows that B is a field of subsets of $\beta+1$. Since I is α-complete, B is an α-complete field of sets. Let $\mathfrak{B}$ be the set algebra determined by B. $\mathfrak{B}$ is an atomistic Boolean algebra, and since I is non-principal, the atoms of $\mathfrak{B}$ are exactly the atoms of $S(\beta)$, and hence are β in number. Therefore, the unit element, $\beta+1$, of β is not equal to the union, β, of all atoms of $\mathfrak{B}$. Furthermore, it is easily seen that $\mathfrak{B}$ is a complete Boolean algebra, because

$$F(\bigcup X) = \textstyle\sum_{x \in X} F(x)$$

for each $X \subseteq S(\beta)$. This contradicts (iii), and our proof is complete.

Theorem 2.26 leads naturally to the interesting question: *is the condition that every α-representable and β^+-complete Boolean algebra is β^+-representable necessary and sufficient for* $[\alpha, \beta] \subseteq C_1$? This question is still open, although it is easily seen from 2.26 that the condition is necessary for $[\alpha, \beta] \subseteq C_1$.

COROLLARY 2.27. *The following three conditions are equivalent for any cardinals α and γ*:

(i) $[\alpha, \gamma) \subseteq C_1$;

(ii) *every α-complete set algebra which is γ-complete as a Boolean algebra is a γ-complete set algebra*;

(iii) *in every α-complete set algebra which is a complete atomistic Boolean algebra with fewer than γ atoms the unit element is the union of all atoms.*

Proof. We see at once from 2.26 that implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i) hold.

THEOREM 2.28. *The following condition is necessary and sufficient for* $[\alpha, \beta] \subseteq C^{[\delta]}$:

(i) *in every β^+-complete Boolean algebra every α-complete and δ-saturated ideal is β^+-complete.*

Proof. It is obvious that 2.28 (i) implies 2.10 (ii), and hence, by 2.10, $[\alpha, \beta] \subseteq C^{[\delta]}$ follows.

By a straightforward modification of our proof of 2.23, with the notion of a δ-saturated ideal everywhere replacing that of a prime ideal, we show that 2.10 (ii) implies 2.28 (i). Thus, by 2.10, $[\alpha, \beta] \subseteq C^{[\delta]}$ implies 2.28 (i).

COROLLARY 2.29. *A necessary and sufficient condition for the inclusion* $[\alpha, \gamma) \subseteq C^{[\delta]}$ *is*:

(i) *in every γ-complete Boolean algebra every α-complete and δ-saturated ideal is γ-complete.*

Proof. The proof is analogous to that of 2.2.

THEOREM 2.30. *The following condition is necessary and sufficient for* $[\alpha, \beta] \subseteq C_1$:

(i) *for some sequence of discrete topological spaces, each with $< \alpha$ points, there exists a set of β points in the product space which has no accumulation point.*

Proof. Assume $[\alpha, \beta] \subseteq C_1$, and consider the set

$$A = \textstyle\bigcup_{\delta < \alpha} {}^{\beta}\delta\,.$$

We choose a sequence f whose range is A; let ξ be the domain of f. For each $\zeta < \xi$, let T_ζ be the discrete topological space whose set of points is the range of the function f_ζ; thus each of the spaces T_ζ has fewer than α points. Let T be the product space of the sequence T_ζ, $\zeta < \xi$. Whenever $\eta < \beta$, let t_η be the point of T defined by the condition: $t_\eta(\zeta) = f_\zeta(\eta)$ for all $\zeta < \xi$. We shall show that the set

$$U = \{t_\eta\colon\ \eta < \beta\}$$

of β points of T has no accumulation point in T. Suppose, on the contrary, that U does have an accumulation point u in T. Let

$$D = \{f_\xi^{-1}(\{u(\zeta)\}):\ \zeta < \xi\}$$

or, in other words,

$$D = \{x \in S(\beta):\ \text{for some } \zeta < \xi,\ x = \{\eta < \beta:\ t_\eta(\zeta) = u(\zeta)\}\}\,.$$

Then, since u is an accumulation point of U, each finite subset of D has an infinite intersection. Moreover, for any $x \in S(\beta)$, either $x \in D$ or $\beta \sim x \in D$, because the function

$$(x \times \{0\}) \cup ((\beta \sim x) \times \{1\})$$

belongs to ${}^\beta 2$ and thus belongs to A and occurs in the sequence f. It follows that the set

$$I = S(\beta) \sim D$$

is a non-principal prime ideal in the field $S(\beta)$. If $X \in S_\alpha(S(\beta))$ and $\bigcup X = \beta$, then there exists $\zeta < \xi$ such that

$$X = \{f_\zeta^{-1}(\{\varrho\}):\ \varrho \in \text{range of } f_\zeta\}\,;$$

consequently some member of X belongs to D, and hence X is not included in I. This shows that I is α-complete and, by 2.1, contradicts our assumption $[\alpha, \beta] \subseteq C_1$. Therefore U has no accumulation point, and (i) holds.

If, on the other hand, $[\alpha, \beta]$ is not included in C_1 then by 2.1 there exists an α-complete non-principal prime ideal I in $S(\beta)$. Let T_ζ, $\zeta < \xi$, be an arbitrary sequence of discrete topological spaces each with $< \alpha$ points, and let U be a set of β points in the product space T of this sequence. We wish to show that U has an accumulation point in T. Choose a one-one β-termed sequence t whose range is U. Then, for each $\zeta < \xi$, there is exactly one point ϱ or T_ζ such that

(1) $$\{\eta < \beta:\ t_\eta(\zeta) = \varrho\} \notin I\,.$$

Consider the point u of T such that, for each $\zeta < \xi$, $u(\zeta)$ is the unique point ϱ of T_ζ such that (1) holds. Then, since I is non-principal and prime, the following condition holds for all $x \in S_\omega(\xi)$: the set

$$\{\eta < \beta:\ t_\eta(\zeta) = u(\zeta) \text{ for all } \zeta \in x\}$$

is infinite. Hence u is an accumulation point of U, and (i) fails. Our proof is complete.

We wish now to consider the algebraic notion of a field, i.e. a commutative division ring (not to be confused with the notion of a field of sets).

Definition 2.31. *$\mathfrak{G}$ is said to be a quotient field of the field $\mathfrak{F}$ if $\mathfrak{G} = ({}^X\mathfrak{F})/M$ for some non-empty set X and some maximal ideal M in the ring ${}^X\mathfrak{F}$. We shall say that the quotient field $\mathfrak{G} = ({}^X\mathfrak{F})/M$ leaves the subfield $\mathfrak{F}_0$ of $\mathfrak{F}$ fixed if each function $h \in {}^XF_0$ is equivalent modulo M to some constant function $c \in {}^XF_0$.*

Lemma 2.32. *For any field $\mathfrak{F}$ and non-empty set X, there is a one-one correspondence I from the set of maximal ideals in the ring ${}^X\mathfrak{F}$ onto the set of prime ideals in $S(X)$ such that, for each maximal ideal M in ${}^X\mathfrak{F}$, we have*

$$M = \{f \in {}^XF:\ f^{-1}(\{0\}) \in I(M)\}\,,$$

where 0 is the zero of $\mathfrak{F}$.

The proof of 2.32 is given in [13], Theorem 36.

Theorem 2.33. *The following conditions are equivalent:*

(i) $[\alpha, \beta] \subseteq C_1$;

(ii) *$\mathfrak{F}$ being any field of power β, every quotient field of $\mathfrak{F}$ which leaves each subfield of $\mathfrak{F}$ of power $< \alpha$ fixed also leaves $\mathfrak{F}$ fixed;*

(iii) *there is a field $\mathfrak{F}$ of power β such that every quotient field of $\mathfrak{F}$ of the form $({}^\beta\mathfrak{F})/M$ which leaves each subfield of $\mathfrak{F}$ of power $< \alpha$ fixed also leaves $\mathfrak{F}$ fixed.*

Proof. It is obvious that (ii) implies (iii).

In what follows we consider a correspondence I between the maximal ideals of ${}^X\mathfrak{F}$ and the prime ideals of $S(X)$ which satisfies the condition in Lemma 2.32. Assume (i), let $\mathfrak{F}$ be a field of power β, and let $\mathfrak{G} = ({}^X\mathfrak{F})/M$ be a quotient field which leaves each subfield of $\mathfrak{F}$ of power $< \alpha$ fixed. We shall show that the ideal $I(M)$ in $S(X)$ is principal; from this it will follow at once that $\mathfrak{G}$ leaves $\mathfrak{F}$ fixed and hence that (ii) holds. To show that $I(M)$ is principal, it suffices by (i) to prove that it is α-complete. Let Y be an infinite disjointed subset of the ideal $I(M)$ of power $< \alpha$, and choose a subfield $\mathfrak{F}_0$ of $\mathfrak{F}$ of power $|Y|$. We may then choose a function $f \in {}^XF_0$ with the property that

(1) $$f^{-1}(\{0\}) = X \sim \bigcup Y$$

and, for each $r \in F_0 \sim \{0\}$,

(2) $$f^{-1}(\{r\}) \in Y\,.$$

By (2), f cannot be equivalent modulo M to any non-zero constant function in ${}^XF_0$; since $\mathfrak{G}$ leaves $\mathfrak{F}_0$ fixed, we must have f equivalent to the zero function modulo M, and hence, by (1),

$$X \sim \bigcup Y = f^{-1}(\{0\}) \notin I(M) \quad \text{and} \quad \bigcup Y \in I(M)\,.$$

This shows that $I(M)$ is α-complete.

Assuming (iii), we consider an arbitrary α-complete prime ideal J in $S(\beta)$. Let M be the maximal ideal in ${}^{\beta}\mathfrak{F}$ such that $J = I(M)$, and let $\mathfrak{G}$ be the quotient field $({}^{\beta}\mathfrak{F})/M$. For any subfield $\mathfrak{F}_0$ of $\mathfrak{F}$ of power $<\alpha$ and any function $f \in {}^{\beta}F_0$, the set

$$\{f^{-1}(\{r\}) \colon r \in F_0\}$$

is a disjointed set of fewer than α subsets of β whose union is β; hence, for some $r_0 \in F_0$, we have

$$f^{-1}(\{r_0\}) \notin J$$

and therefore f is equivalent modulo M to the constant function $c \in {}^{\beta}F_0$ whose range is $\{r_0\}$. We have verified that $\mathfrak{G}$ leaves fixed every subfield $\mathfrak{F}_0$ of $\mathfrak{F}$ of power $<\alpha$. Therefore $\mathfrak{G}$ leaves $\mathfrak{F}$ fixed. Let us choose a one-one function g on β onto F. By (iii), g is equivalent modulo M to some constant function on β into F; let ξ be the unique ordinal $<\beta$ such that $\{g(\xi)\}$ is the range of this constant function. We then have

$$\{\xi\} = g^{-1}\{g(\xi)\} \notin J\,,$$

and therefore J is principal.

DEFINITION 2.34. *We denote by* C_1' *the class* $\{\beta\colon [\omega^+, \beta] \subseteq C_1\}$.

COROLLARY 2.35. (i) *If* $C_1 = C$, *then* C_1' *is the class of all infinite cardinals, i.e.*

$$C_1' = C \cup \{\omega\}\,.$$

(ii) *If* $C_1 \neq C$ *and* $\beta = \bigcap(C \sim C_1)$, *then*

$$C_1' = (C \cap \beta) \cup \{\omega\} = [\omega, \beta)\,.$$

Proof. By 2.34.

COROLLARY 2.36. (i) $\omega \in C_1'$;

(ii) $C_1' \subseteq C_1 \cup \{\omega\}$;

(iii) *if* $\beta \in C_1'$, *then* $[\omega, \beta] \subseteq C_1'$.

Proof. By 2.34.

Notice that, when we set $\alpha = \omega^+$, the results 2.1, 2.4, 2.20-2.23, 2.25, 2.26, 2.30, and 2.33 yield necessary and sufficient conditions for $\beta \in C_1'$. Moreover, we have

COROLLARY 2.37. *Suppose* $C_1' = [\omega, \beta)$. *Then*

(i) *every* ω^+*-complete set algebra which is* β*-complete as a Boolean algebra is a* β*-complete set algebra*;

(ii) *in every* β*-complete Boolean algebra every* ω^+*-complete prime ideal is* β*-complete.*

Proof. By 2.27, 2.29, and 2.36.

The class C_1' arises frequently in various branches of mathematics. We shall give a few examples here.

DEFINITION 2.38. m *is said to be a (normalized countably additive non-trivial) measure on a set* X *if the following four conditions are satisfied:*

(i) m *is a function on* $S(X)$ *into the non-negative real numbers;*

(ii) $m(X) = 1$;

(iii) *if* Y *is a disjointed subset of* $S(X)$ *and* $|Y| \leqslant \omega$, *then* $m(\bigcup Y) = \sum_{y \in Y} m(y)$;

(iv) $m(\{x\}) = 0$ *for every* $x \in X$.

If in addition the range of m *is* $\{0, 1\}$, m *is said to be two-valued.*

THEOREM 2.39. *A necessary and sufficient condition for* $\beta \in C_1'$ *is that* β *have no two-valued measure* [10].

Proof. If m is a two-valued measure on β, then $m^{-1}(\{0\})$ is a non-principal ω^+-complete prime ideal in $S(\beta)$. Conversely, if I is a non-principal ω^+-complete prime ideal in $S(\beta)$, then the function $m = (I \times \{0\}) \cup \big((S(\beta) \sim I) \times \{1\}\big)$ is a two-valued measure on β. The result follows by 2.1.

It may be mentioned that we could formulate conditions stated in terms of measure which are necessary and sufficient for a cardinal β to belong to the whole class C_1, and not only to the subclass C_1'. For this purpose we would have to use the not quite natural notion of α-additive measure (cf. [51]).

By the *abstract measure problem* we mean the following problem: *determine the cardinals* α *such that every set of power* α *has a measure.* We may restrict our attention to those sets which are themselves infinite cardinals. By 2.39, if $\alpha \notin C_1'$, then α has a measure. It is not known whether there is any cardinal $\alpha \in C_1'$ which has a measure.

The more specific question: *does there exist a measure on the set of real numbers?* is also open. However, in view 1.2, we know that $2^{\omega} \in C_1'$ and thus, by 2.39, there is no two-valued measure on the set of real numbers.

THEOREM 2.40. *If* β *has a measure, then* $\beta \in C \sim C^{[\omega^+]}$.

We refer to [43], p. 60, for the proof of 2.40. Notice that in particular 2.40 tells us that ω has no measure.

THEOREM 2.41. *Suppose that* 2^{ω} *has no measure. Then the following three conditions are equivalent:*

(i) $\beta \in C_1'$;

(ii) β *has no two valued measure*;

(iii) β *has no measure.*

---

(10) Theorem 2.39 above is established in [41], page 164.

Proof. The conditions (i) and (ii) are equivalent by 2.39. The implication (iii)$\to$(ii) is obvious. The proof that (ii) implies (iii) is due to Ulam and is given in [51], Theorem 3.4, p. 149.

THEOREM 2.42. *Suppose that* $[\omega^+, 2^\omega] \subseteq C^{[\omega^+]}$. *Then the conditions* 2.41 (i)-(iii) *are each equivalent to the following condition:*

(iv) $[\omega^+, \beta] \subseteq C^{[\omega^+]}$.

Proof. By 2.40, the hypothesis implies that 2^ω has no measure. Hence, by 2.41, conditions 2.41 (i)-(iii) are equivalent. The equivalence of 2.41 (i) with 2.42 (iv) follows from 2.21 (ii).

In connection with this theorem we recall that, by 2.10, the formula 2.42 (iv) is equivalent to the condition that every ω^+-complete and ω^+-saturated ideal in $S(\beta)$ is principal. By 2.40, this formula implies 2.41 (iii) for every β (and hence, in particular, the hypothesis of 2.42 implies that of 2.41). The problem is open whether 2.41 (iii) and 2.42 (iv) can be shown to be equivalent without any additional hypothesis. From 2.42 we easily conclude that, if 2.41 (iii) and 2.42 (iv) are equivalent for $\beta = 2^\omega$, they are also equivalent for every $\beta > 2^\omega$.

COROLLARY 2.43. *Assume that the continuum hypothesis holds, or at least that* $[\omega^+, 2^\omega]$ *contains no regular limit cardinals. Then the conditions* 2.41 (i), 2.41 (ii), 2.41 (iii), *and* 2.42 (iv) *are all equivalent.*

Proof. By 1.2, 2.22, and 2.42.

COROLLARY 2.44. *Assume either of the hypotheses of* 2.43. *Then there is no measure on the set of real numbers.*

Proof. By 2.43.

This result is again due to Ulam; see [51], Theorem 3.4, p. 149.

From 2.43 we see that, under either of the hypotheses of 2.43, the abstract measure problem is closely related to the problem of whether $C_1 = C$.

We shall now mention briefly various other mathematical conditions which are known from the literature to be necessary and sufficient for $\beta \in C_1'$.

Let R denote the set of real numbers. We consider the vector lattice ${}^\beta R$ of all real-valued functions on β with the operations of pointwise addition, scalar multiplication, pointwise maximum, and pointwise minimum. The terms *linear functional* and *operator homomorphism* on ${}^\beta R$ into R may be used interchangeably. A linear functional F is said to be *positive* if $F(f) \geqslant 0$ whenever $f \in {}^\beta R$ and $f \geqslant 0$. A linear functional is said to be *bounded* if it maps any bounded subset of ${}^\beta R$ into a bounded subset of R. The *projections* Π_ξ on ${}^\beta R$ into R, defined for each $\xi < \beta$ by

$$\Pi_\xi(f) = f(\xi)\,,$$

are clearly bounded linear functionals.

THEOREM 2.45. *The following three conditions are all equivalent:*

(i) $\beta \in C_1'$;

(ii) *every positive linear functional on* ${}^\beta R$ *into* R *is a finite linear combination of projections;*

(iii) *every bounded linear functional on* ${}^\beta R$ *into* R *is a finite linear combination of projections.*

Theorem 2.45 is established in [25]. The equivalence of 2.45 (ii) and 2.45 (iii) follows from the fact that a linear functional on ${}^\beta R$ into R is bounded if and only if it can be represented as the difference of two positive linear functionals.

We assume familiarity with the basic notions in the theory of (additively written) Abelian groups. Let $\mathfrak{C}$ be an infinite cyclic group, e.g. the additive group of integers. By the weak direct product of a β-termed sequence $\mathfrak{G}$ of Abelian groups, in symbols $\mathbf{P}^{(w)}_{\xi<\beta}\mathfrak{G}_\xi$, we mean the subgroup of the direct product $\mathbf{P}_{\xi<\beta}\mathfrak{G}_\xi$ whose elements are just those functions f which satisy the formula $f(\xi) = 0$ for all but finitely many $\xi < \beta$. In case all the groups $\mathfrak{G}_\xi$ coincide with a given group $\mathfrak{H}$ the weak direct product $\mathbf{P}^{(w)}_{\xi<\beta}\mathfrak{G}_\xi$ is called the *β-th weak direct power of* $\mathfrak{H}$ and will be denoted by ${}^\beta\mathfrak{H}^{(w)}$. (In additive terminology direct products are sometimes called direct sums and are denoted differently; see e.g. [7].)

THEOREM 2.46. *The following condition is necessary and sufficient for* $\beta \in C_1'$:

(i) *every homomorphism on* ${}^\beta\mathfrak{C}$ *into* $\mathfrak{C}$ *is a finite linear combination or projections.*

This result is stated in [5].

$\mathfrak{G}$ is said to be a *slender group* if $\mathfrak{G}$ is a torsion-free Abelian group with the property that every homomorphism on the direct product ${}^\omega\mathfrak{C}$ into $\mathfrak{G}$ maps all but finitely many factors of the product into the zero of $\mathfrak{G}$ (for more details see [7]).

It is known that the group $\mathfrak{C}$ is slender. In fact, a torsion-free Abelian group is slender if and only if it does not have a subgroup isomorphic either to the additive group of rationals or to the group ${}^\omega\mathfrak{C}$ or, finally, to the group of p-adic integers for any prime p (see [29]).

THEOREM 2.47. *For any non-trivial slender group* $\mathfrak{G}$ *and any* β, *the following three conditions are equivalent:*

(i) $\beta \in C_1'$;

(ii) *the only homomorphism on* ${}^\beta\mathfrak{C}$ *into* $\mathfrak{G}$ *which maps* ${}^\beta\mathfrak{C}^{(w)}$ *into the zero of* $\mathfrak{G}$ *is the zero homomorphism;*

(iii) *for every* β*-termed sequence* $\mathfrak{H}$ *of torsion-free Abelian groups, the only homomorphism on* $\mathbf{P}_{\xi<\beta}\mathfrak{H}_\xi$ *into* $\mathfrak{G}$ *which maps* $\mathbf{P}^{(w)}_{\xi<\beta}\mathfrak{H}_\xi$ *into the zero of* $\mathfrak{G}$ *is the zero homomorphism.*

The notion of a slender group and Theorem 2.47 are due to Łoś. Theorem 2.46 is an easy consequence of 2.47, while its formulation is analogous to that of 2.45. For historical references concerning 2.46, 2.47 and for a proof of 2.47 see [7].

If T is a topological space, we denote by $C(T, R)$ the ring of all real-valued continuous functions on T with the operations of pointwise addition and pointwise multiplication.

THEOREM 2.48. *The following condition is necessary and sufficient for* $\beta \in \boldsymbol{C}_1'$:

(i) *the only completely regular spaces T such that $C(T, R)$ is isomorphic to the ring ${}^{\beta}R$ are the discrete spaces with exactly β points.*

Theorem 2.48 is established in [13], Theorem 68 (subject to Remark 1, p. 175, in [14]).

Let $\hat{2}$ denote the discrete topology with the two points 0 and 1, and for any a let ${}^{a}\hat{2}$ denote the product topology of an a-termed sequence of copies of the space $\hat{2}$. Let $\hat{R}$ denote the usual topology on the set of real numbers. A function f on a space T_1 into a space T_2 is said to be *sequentially continuous* if, whenever the denumerable sequence $x_0, x_1, x_2, \ldots$ of points of T_1 converges in T_1 to the limit x, then the sequence $f(x_0)$, $f(x_1), f(x_2), \ldots$ converges in T_2 to the limit $f(x)$.

THEOREM 2.49. *The following condition* (i) *is necessary for* $[\omega^+, \beta]$ *to contain no regular limit cardinal and is sufficient for β to have no measure*:

(i) *every sequentially continuous function on ${}^{\beta}\hat{2}$ into $\hat{R}$ is continuous.*

Furthermore, the following condition (ii) *is necessary for* $[\omega^+, \beta]$ *to contain no regular limit cardinal and is sufficient for β to belong to* $\boldsymbol{C}_1'$:

(ii) *every sequentially continuous function on ${}^{\beta}\hat{2}$ into $\hat{2}$ is continuous.*

Theorem 2.49 follows from results which are formulated in a much more general and abstract context in [27]. In both parts of 2.49 the sufficiency of the condition is easily seen; it is also obvious that 2.49 (i) implies 2.49 (ii). The essential part of 2.49 is the necessity of (i).

A number of problems naturally arise in connection with 2.49: is 2.49 (i) equivalent to the condition that β has no measure? is 2.49 (ii) equivalent to $\beta \in \boldsymbol{C}_1'$? does 2.49 hold whenever $[\omega^+, \beta] \subseteq \boldsymbol{AC}$? can results analogous to those of § 1 be obtained for the class of cardinals which satisfy 2.49 (ii)? All these problems are open.

To conclude this section we return to the prime ideal hypothesis $\boldsymbol{C} = \boldsymbol{C}_1$. In view of the remarks made at the end of § 1 it may be interesting to sum up the main implications of this hypothesis in general set theory and related domains. This will be done in the next three theorems, which are simple corollaries of various results previously stated in this section. All the consequences of the hypothesis $\boldsymbol{C} = \boldsymbol{C}_1$ formulated below are actually equivalent to this hypothesis. Since we are interested primarily in implications in one direction, we state these consequences in the strongest and most general form which we know (omitting various weaker and more special formulations which are still equivalent to the hypothesis discussed). In formulating the next theorems we use the terms "countably complete", "countably additive", etc., instead of "ω^+-complete", "ω^+-additive", etc.

THEOREM 2.50. *The hypothesis* $\boldsymbol{C} = \boldsymbol{C}_1$ *implies* (*and is implied by*) *each of the following statements*:

(i) *In every β-complete Boolean algebra every countably complete prime ideal is β-complete.*

(i′) *In every complete Boolean algebra every countably complete prime ideal is principal.*

(ii) [*and* (ii′)] *In every β-complete* [*or complete*] *Boolean algebra every $(2^a)^+$-complete and a-saturated ideal is β-complete* [*or principal, respectively*].

(iii) [*and* (iii′)] *Every countably complete set algebra which is β-complete* [*or complete*] *as a Boolean algebra is a β-complete set algebra* [*or a complete set algebra, respectively*].

(iv) [*and* (iv′)] *Every countably additive and countably multiplicative set function on a β-complete* [*or complete*] *field of sets is β-additive and β-multiplicative* [*or completely additive and completely multiplicative, respectively*].

(v) *If a set X of sets covers a set b (i.e., $b \subseteq \bigcup X$) and every disjointed subset of $S(b) \sim X$ has power less than a, then there is a set $Y \subseteq X$ of power at most $2^{\underline{a}}$ which also covers b.*

(vi) *There is no two-valued* (*countably additive non-trivial*) *measure on any set.*

(vii) [*and* (vii′)] *Every positive* [*or bounded*] *linear functional on ${}^{\beta}R$ into R* (*where R is the set of real numbers*) *is a finite linear combination of projections.*

(viii) *The only completely regular topological spaces T such that the ring of all real-valued continuous functions on T is isomorphic to ${}^{\beta}R$* (*where R is the ring of real numbers*) *are the discrete spaces with exactly β points.*

(ix) *For every β there exists a sequence of discrete topological spaces, each with countably many points, such that not every set of β points in the product space of this sequence has an accumulation point.*

(x) *Every homomorphism on ${}^{\beta}\mathfrak{C}$ into $\mathfrak{C}$* (*where $\mathfrak{C}$ is an infinite cyclic group*) *is a linear combination of projections.*

(xi) *If $\mathfrak{G}$ is a slender group and $\mathfrak{H}$ is a β-termed sequence of torsion-free Abelian groups, then the only homomorphism on $\mathbf{P}_{\xi<\beta}\mathfrak{H}_\xi$ into $\mathfrak{G}$ which maps $\mathbf{P}^{(w)}_{\xi<\beta}\mathfrak{H}_\xi$ into the zero of $\mathfrak{G}$ is the zero homomorphism.*

(xii) *If a quotient field of a field $\mathfrak{F}$ leaves every denumerable subfield of $\mathfrak{F}$ fixed, then it leaves $\mathfrak{F}$ fixed as well.*

THEOREM 2.51. *Assume that the continuum hypothesis holds or at least that $[\omega^+, 2^\omega]$ contains no regular limit cardinal. Then the hypothesis $\boldsymbol{C} = \boldsymbol{C}_1$ implies (and is implied by) each of the following statements:*

(i) *[and (i′)] In every β-complete [or complete] Boolean algebra every countably complete and countably saturated ideal is β-complete [or principal, respectively].*

(ii) *If a set X of sets covers a set b and every disjointed subset of $S(b) \sim X$ is at most denumerable, then there is an at most denumerable set $Y \subseteq X$ which also covers b.*

(iii) *There is no (countably additive non-trivial) measure on any set.*

This theorem holds under weaker assumptions: the hypothesis $\boldsymbol{C} = \boldsymbol{C}_1$ is equivalent to each of the statements 2.51 (i)-(iii) under the assumption that every countably complete countably saturated ideal in $S(2^\omega)$ is principal; it is equivalent to 2.51 (iii) under the assumption that 2^ω has no measure.

THEOREM 2.52. *Assume that the generalized continuum hypothesis holds or at least that every regular limit cardinal is inaccessible. Then the hypothesis $\boldsymbol{C} = \boldsymbol{C}_1$ implies (and is implied by) each of the following statements:*

(i) *[and (i′)] If a is an accessible cardinal, then in every β-complete [or complete] Boolean algebra every a-complete and a-saturated ideal is β-complete [or principal, respectively].*

(ii) *If a set X of sets covers a set b and every disjointed subset of $S(b) \sim X$ has power less than an accessible cardinal a, then there is a set $Y \subseteq X$ of power less than a which also covers b.*

While many portions of 2.50-2.52 follow directly from our earlier results, the derivation of the remaining portions also presents no difficulty.

Among papers in the literature containing other results which concern in some way the classes $\boldsymbol{C}_1$ and $\boldsymbol{C}_1'$ we mention [3] (set theory), [31] (group theory), [22] (group theory and topology), [14], [34], [52] (functional analysis), [35] (Boolean algebra and topology), and [2], [16] (theory of models). The classes $\boldsymbol{C}_1$ and $\boldsymbol{C}_1'$ also play a role in a number of the metamathematical papers which we have already cited, specifically in [11], [15], [17], [18], [32], [48]. The results stated in these papers yield in particular various metamathematical statements which are equivalent to the prime ideal hypothesis (or to one of the related hypothesis, $\boldsymbol{C} = \boldsymbol{C}_0$ and $\boldsymbol{C} = \boldsymbol{C}_2$, which will be discussed in later sections).

3. Extent of the class $\boldsymbol{C}_0$. In § 1 we considered a problem which was motivated by the well-known result that, in any field of sets, every proper ideal can be extended to a prime ideal. In this section we shall

concentrate on another question which originates with the same result. As is seen from the results of § 1, most cardinals a have the following property: (i) there is an a-complete field B of sets in which some a-complete proper ideal I cannot be extended to an a-complete prime ideal in B. In fact, every cardinal a which belongs to $\boldsymbol{C}_1$ has this property; to show this we take $S(a^+)$ for B and $S_{a^+}(a^+)$ for I. We shall show in this section that for most cardinals a we can find an a-complete field of sets B which satisfies (i) and has no more than a generators. (Instead of considering the numbers of generators, we could consider the total number of elements of a field of sets; this would compel us, however, either to state some theorems in a less general and more complicated form or to make their validity dependent on the generalized continuum hypothesis.)

DEFINITION 3.1. *We shall denote by $\boldsymbol{C}_0$ the class of all cardinals a such that there is an a-complete field of sets with at most a generators in which some a-complete proper ideal cannot be extended to an a-complete prime ideal.*

The main purpose of this section is to provide as much information as possible concerning the extent of the class $\boldsymbol{C}_0$. First we have, by Theorem 1.3,

THEOREM 3.2. $\boldsymbol{C}_0 \subseteq \boldsymbol{C}$ *(i.e. $\omega \notin \boldsymbol{C}_0$).*

A more interesting limitation on the extent of $\boldsymbol{C}_0$ is given in the following

THEOREM 3.3. $\boldsymbol{C}_0 \subseteq \boldsymbol{C}_1$.

Proof. Suppose $a \in \boldsymbol{C}_0$. By 3.2, $a \in \boldsymbol{C}$. If $a \in \boldsymbol{AC}$, then $a \in \boldsymbol{C}_1$ by 1.2. Suppose $a \in \boldsymbol{C} \sim \boldsymbol{AC}$. Then there exists an a-complete set algebra $\mathfrak{B}$ with at most a generators which has an a-complete proper ideal I not included in any a-complete prime ideal. By 0.3, $\mathfrak{B}/I$ is an a-distributive Boolean algebra. Moreover, it is easily seen that $\mathfrak{B}/I$ is a-generated by a set of power $\leqslant a$, and that $\mathfrak{B}/I$ has no a-complete prime ideals. By 0.5, it follows that $\mathfrak{B}/I$ is not a-representable, since $\{0\}$ is a proper principal ideal. Thus we have shown the following: (1) there exists an a-distributive Boolean algebra which is a-generated by a set of power $\leqslant a$ but is not a-representable. It is shown in [4], Theorem 4.4, that (1) implies $a \in \boldsymbol{C}_1$, and with this reference the proof of 3.14 is complete.

THEOREM 3.4. $\boldsymbol{C}_0$ *is normal.*

Proof. Suppose that $f \in {}^a(\boldsymbol{C}_0 \cap a)$ and a is representable by f. Let I be a non-principal a-complete prime ideal in $S(a)$ such that $\tau_I(f) = a$. Since $\boldsymbol{AC}$ is normal by 1.33, there exists a function f' such that

$$f' \in {}^a\big((\boldsymbol{C}_0 \sim \boldsymbol{AC}) \cap a\big) \quad \text{and} \quad \tau_I(f') = \tau_I(f) = a,$$

and we may as well assume that $f = f'$.

If $\beta \in C \sim AC$, then, as is easily seen, every β-complete field of sets with at most β generators has power $\leqslant \beta$. Hence, for each $\xi < \alpha$, we may choose an $f(\xi)$-complete field of sets B_ξ of power $\leqslant f(\xi)$ and an $f(\xi)$-complete proper ideal K_ξ in B_ξ such that K_ξ cannot be extended to an $f(\xi)$-complete prime ideal in B_ξ. For each $\xi < \alpha$, let F_ξ be a function on $f(\xi)$ onto the set B_ξ; let G_ξ be a function on $f(\xi)$ into $\bigcup B_\xi$ such that, whenever $\zeta < f(\xi)$ and $F_\xi(\zeta) \neq 0$, we have $G_\xi(\zeta) \in F_\xi(\zeta)$. For each $\zeta < \alpha$, let

$$x_\zeta = \{\xi < \alpha \colon f(\xi) > \zeta\} .$$

Then, in view of 1.4,

(1) $$x_\zeta \notin I \quad \text{whenever} \quad \zeta < \alpha .$$

For each $\zeta < \alpha$, let

(2) $$F_\alpha(\zeta) = \{\eta < \alpha \colon \{\xi \in x_\eta \cap x_\zeta \colon G_\xi(\eta) \in F_\xi(\zeta)\} \notin I\} ;$$

intuitively $\eta \in F_\alpha(\zeta)$ means that $G_\xi(\eta) \in F_\xi(\zeta)$ for almost all $\xi < \alpha$. It follows that, for all $\eta < \alpha$, either $F_\alpha(\eta) = 0$ or $\eta \in F_\alpha(\eta)$.

Let B_α be the range of F_α. Clearly we have $|B_\alpha| \leqslant \alpha$. We wish to show that B_α is an α-complete field of subsets of α. Let $\zeta < \alpha$ and $\varphi \in {}^\zeta\alpha$. We shall show that

$$\bigcup_{\eta<\zeta} F_\alpha(\varphi_\eta) \in B_\alpha .$$

We put

$$\varrho = \zeta \cup \bigcup_{\eta<\zeta} \varphi_\eta .$$

Then, whenever $\xi \in x_\varrho$, we have

$$\bigcup_{\eta<\zeta} F_\xi(\varphi_\eta) \in B_\xi .$$

Choose a function $g \in {}^\alpha\alpha$ such that, whenever $\xi \in x_\varrho$, $g(\xi) < f(\xi)$ and

(3) $$F_\xi(g(\xi)) = \bigcup_{\eta<\zeta} F_\xi(\varphi_\eta) .$$

Then it is easily seen, in view of (1)-(3) and the α-completeness of I, that $\tau_I(g) < \alpha$ and

$$F_\alpha(\tau_I(g)) = \bigcup_{\eta<\zeta} F_\alpha(\varphi_\eta) ,$$

whence

$$\bigcup_{\eta<\zeta} F_\alpha(\varphi_\eta) \in B_\alpha .$$

By a similar but simpler argument we show that $y \in B_\alpha$ implies $\alpha \sim y \in B_\alpha$, and that $\alpha \in B_\alpha$. Consequently B_α is an α-complete field of subsets of α with $\leqslant \alpha$ generators.

Let

$$K_\alpha = \{F_\alpha(\zeta) \colon \zeta < \alpha \text{ and } \{\xi < \alpha \colon F_\xi(\zeta) \in K_\xi\} \notin I\} ;$$

$F_\alpha(\zeta) \in K_\alpha$ expresses the fact that $F_\xi(\zeta) \in K_\xi$ for almost all $\xi < \alpha$. It can be shown (by an argument similar to that used above to prove that

B_α is an α-complete field of sets) that K_α is an α-complete ideal in B_α. We shall omit the details. To show that K_α is proper in B_α, it suffices to prove that $\alpha \notin K_\alpha$. Suppose $\zeta < \alpha$ and $F_\alpha(\zeta) \in K_\alpha$. Then, since K_ξ is a proper ideal in B_ξ for all $\xi < \alpha$, and since

$$\{\xi < \alpha \colon F_\xi(\zeta) \in K_\xi\} \notin I ,$$

we have

$$\{\xi < \alpha \colon F_\xi(\zeta) \neq \bigcup B_\xi\} \notin I .$$

Let $t \in {}^\alpha\alpha$ be such that, for each $\xi \in x_\zeta$, we have

$$t(\xi) < f(\xi) \quad \text{and} \quad F_\xi(t(\xi)) = \bigcup B_\xi \sim F_\xi(\zeta) .$$

Then $\tau_I(t) < \alpha$. Furthermore, whenever $\xi \in x_\zeta$ and $F_\xi(\zeta) \neq \bigcup B_\xi$, we have

$$G_\xi(t(\xi)) \in \bigcup B_\xi \sim F_\xi(\zeta) .$$

Therefore

$$\{\xi < \alpha \colon G_\xi(t(\xi)) \notin F_\xi(\zeta)\} \notin I$$

and

$$\{\xi < \alpha \colon G_\xi(\tau_I(t)) \notin F_\xi(\zeta)\} \notin I ,$$

and consequently

$$\tau_I(t) \notin F_\alpha(\zeta) .$$

Thus $F_\alpha(\zeta) \neq \alpha$, and $\alpha \notin K_\alpha$.

By 1.8, $\alpha \in C \sim C_1$, and thus by 3.3 we have $\alpha \in C \sim C_0$. Therefore, by 3.1, K_α can be extended to an α-complete prime ideal J_α in B_α. For each $\xi < \alpha$, let

(4) $$J_\xi = \{F_\xi(\zeta) \colon \zeta < f(\xi) \text{ and } F_\alpha(\zeta) \in J_\alpha\} .$$

We shall show that

$$\{\xi < \alpha \colon J_\xi \text{ is an } f(\xi)\text{-complete prime ideal in } B_\xi \text{ and } K_\xi \subseteq J_\xi\} \notin I ,$$

and thereby, since $f \in {}^\alpha(C_0 \cap \alpha)$ and $0 \in I$, we shall arrive at a contradiction. It is sufficient to verify each of the following conditions:

(5) $$\{\xi < \alpha \colon K_\xi \subseteq J_\xi\} \notin I ;$$

(6) $$\{\xi < \alpha \colon y \in J_\xi,\ z \subseteq y, \text{ and } z \in B_\xi \text{ imply } z \in J_\xi\} \notin I ;$$

(7) $$\{\xi < \alpha \colon \text{for all } y \in B_\xi, \text{ either } y \in J_\xi \text{ or } \bigcup B_\xi \sim y \in J_\xi\} \notin I ;$$

(8) $$\{\xi < \alpha \colon \text{for all } X \in S_{f(\xi)}(J_\xi),\ \bigcup X \in J_\xi\} \notin I .$$

We shall carry through the proof only for (8). (The other three conditions can be established in a similar manner, but the proofs are somewhat easier.) Let

$$y = \{\xi < \alpha \colon \text{for some } X \in S_{f(\xi)}(J_\xi),\ \bigcup X \notin J_\xi\} .$$

Suppose that (8) is false, i.e. that $y \notin I$. For each $\xi \in y$, choose a set $X_\xi \in S_{f(\xi)}(J_\xi)$ such that $\bigcup X_\xi \notin J_\xi$. Choose a function $g \in {}^\alpha\alpha$ such that, for each $\xi \in y$,

$$g(\xi) = |X_\xi| .$$

Then, by 1.4, $\tau_I(g) < \alpha$. Let

$$z = \{\xi \in y\colon\ g(\xi) = \tau_I(g)\} .$$

By 1.4, we have $z \notin I$. We may now choose a function h with domain $\tau_I(g) \times z$ such that, for all $\xi \in z$, we have

$$X_\xi = \{F_\xi(h(\zeta, \xi))\colon\ \zeta < \tau_I(g)\} .$$

For each $\zeta < \tau_I(g)$, we may choose a function $h_\zeta \in {}^\alpha\alpha$ such that, for all $\xi \in z$, we have

$$h_\zeta(\xi) = h(\zeta, \xi) .$$

Since $F_\xi^{-1}(X_\xi) \subseteq f(\xi)$ whenever $\xi \in z$, it follows by 1.4 that, for all $\zeta < \tau_I(g)$,

$$\tau_I(h_\zeta) < \alpha .$$

Since I is α-complete, the set

$$w = \{\xi \in z\colon \text{ for all } \zeta < \tau_I(g),\ h_\zeta(\xi) = \tau_I(h_\zeta)\}$$

does not belong to I.

Since, for all $\xi \in w$ and $\zeta < \tau_I(g)$, we have

$$F_\xi(\tau_I(h_\zeta)) \in X_\xi \subseteq J_\xi ,$$

it follows by (4) that, for all $\zeta < \tau_I(g)$,

$$F_\alpha(\tau_I(h_\zeta)) \in J_\alpha .$$

Let ϱ be the unique element of α such that

$$F_\alpha(\varrho) = \bigcup_{\zeta<\tau_I(g)} F_\alpha(\tau_I(h_\zeta)) .$$

Since J_α is α-complete, we have

$$F_\alpha(\varrho) \in J_\alpha .$$

It is easily seen from the α-completeness of I that

$$\{\xi \in w \cap x_\varrho\colon\ F_\xi(\varrho) = \bigcup_{\zeta<\tau_I(g)} F_\xi(h_\zeta(\xi))\} \in I ,$$

and that consequently the set

$$u = \{\xi \in w \cap x_\varrho\colon\ F_\xi(\varrho) = \bigcup X_\xi\}$$

does not belong to I. However, since $F_\alpha(\varrho) \in J_\alpha$, we see from (4) that, for all $\xi \in x_\varrho$,

$$F_\xi(\varrho) \in J_\xi .$$

Since $u \notin I$, we may choose $\xi_0 \in u$, and we have

$$F_{\xi_0}(\varrho) = \bigcup X_{\xi_0} \in J_{\xi_0} .$$

This is a contradiction, which completes the proof of (8) and hence also the whole proof of our theorem.

Theorem 3.4 implies 3.3 and is indeed stronger, for it leads to the conclusion that, in case $\boldsymbol{C}_0 \neq \boldsymbol{C}$, the least cardinal in $\boldsymbol{C} \sim \boldsymbol{C}_0$ belongs to $\boldsymbol{C}_1$ and, in case $\boldsymbol{C}_1 \neq \boldsymbol{C}$, there are many other cardinals which also belong to $\boldsymbol{C}_1$ but not to $\boldsymbol{C}_0$. These conclusions were first obtained using a different method by Hanf and Scott and were stated in [11].

To derive the conclusions from our results we combine 3.4 with the induction principles of § 1 and begin an inductive process taking the class $\boldsymbol{C}_0$ instead of $\boldsymbol{AC}$ as a starting point. In this way we obtain the following improvement of 1.34:

THEOREM 3.5.

(i) $\boldsymbol{M}(\boldsymbol{C}_0) \subseteq \boldsymbol{C}_1$;

(ii) $\boldsymbol{M}^{(\infty)}(\boldsymbol{C}_0) \subseteq \boldsymbol{C}_1$;

(iii) $\boldsymbol{M}^{(\infty)}(\boldsymbol{M}^{(\infty)}(\boldsymbol{C}_0)) \subseteq \boldsymbol{C}_1$;

(iv) $(\boldsymbol{M}^{(\infty)})^{(\infty)}(\boldsymbol{C}_0) \subseteq \boldsymbol{C}_1$.

Proof. By 3.4 and 1.32 [11].

The remaining theorems in this section will show that certain classes of cardinals are included in $\boldsymbol{C}_0$.

LEMMA 3.6. *Assume that B is a γ^+-complete field of sets and there exists $x \in {}^{\gamma\times\gamma}B$ such that the following two conditions hold:*

(i) *for each $\xi < \gamma$, $\bigcup_{\zeta<\gamma} x(\xi, \zeta) = \bigcup B$;*

(ii) *for each $\varphi \in {}^\gamma\gamma$, $\bigcap_{\xi<\gamma} x(\xi, \varphi(\xi))$ is either empty or an atom of B.*

Then every γ^+-complete prime ideal in B is principal [12].

---

[11] Theorem 3.5, which improves a result stated in abstract [11] and which is derived here in a purely mathematical way, can also be obtained by means of meta-mathematical methods used by the authors of that abstract. On the other hand, 3.5 is a direct consequence of two metamathematical theorems recently announced in [18]. (It should be pointed out, however, that Theorem 1 of [18] can also be derived directly from the results stated in [11].)

[12] This lemma is established in [41], Theorem 2.16. It follows directly from Lemma 3.6 that, by means of the simple construction used in the proof of Theorem 3.7 below, we obtain for each regular accessible α a weakly α-representable Boolean algebra with at most α generators which is not α-representable. Cf. [41], Theorem 7.13, and also Theorem 4.16 below.

Proof. Let I be any γ^+-complete prime ideal in B. Since I is proper, it follows that, for each $\xi < \gamma$, there exists $\zeta < \gamma$ such that $x(\xi, \zeta) \notin I$. We may therefore choose a function $\varphi \in {}^\gamma\gamma$ such that, for all $\xi < \gamma$,

$$x(\xi, \varphi(\xi)) \notin I .$$

Then we have

$$y = \bigcap_{\xi<\gamma} x(\xi, \varphi(\xi)) \notin I .$$

But, by (ii), y is either an atom of B or the empty set and, since $0 \in I$, y must be an atom of B. It follows that I is principal, and the proof is complete.

THEOREM 3.7. $\boldsymbol{AC} \subseteq \boldsymbol{C}_0$.

Proof. Suppose first that $\alpha \in \boldsymbol{AC} \sim \boldsymbol{SN}$. We may then choose a cardinal $\beta < \alpha$ such that $\alpha \leqslant 2^\beta$. We shall construct an α-complete Boolean algebra $\mathfrak{B}$ with at most β generators in which some α-complete proper ideal cannot be extended to an α-complete prime ideal, and thus establish that $\alpha \in \boldsymbol{C}_0$.

For each $\xi, \eta < \beta$, let

$$x(\xi, \eta) = \{\varphi\colon \varphi \in {}^\beta\beta \text{ and } \varphi(\xi) = \eta\} .$$

Let B be the α-complete field of subsets of ${}^\beta\beta$ which is α-generated by the set

$$\{x(\xi, \eta)\colon \xi, \eta < \beta\}$$

of power β. Then B is a β^+-complete field of sets such that conditions 3.6 (i) and 3.6 (ii) are satisfied when $\gamma = \beta$. By 3.6, every β^+-complete prime ideal, and hence every α-complete prime ideal in B is principal. Moreover, since $\beta < \alpha \leqslant 2^\beta$ and α is regular, the set $S_\alpha({}^\beta\beta)$ is included in B and in fact is a proper α-complete ideal in B. $S_\alpha({}^\beta\beta)$ cannot be extended to an α-complete prime ideal in B, because it cannot be extended to any proper principal ideal.

If $\alpha \in \boldsymbol{SN}$, then $\alpha^+ \in \boldsymbol{AC} \sim \boldsymbol{SN}$, and $\alpha < \alpha^+ \leqslant 2^\alpha$. The above construction gives an α^+-complete field B of sets, with at most α generators, in which some α^+-complete proper ideal cannot be extended to an α^+-complete prime ideal. But, by 0.4, every α-complete ideal is α^+-complete. Hence we have again $\alpha \in \boldsymbol{C}_0$, and our proof is complete.

In view of Theorem 3.7, Theorems 3.4 and 3.5 are improvements of Theorems 1.33 and 1.34, respectively.

The entire development of § 1 can be modified to obtain, in place of results which state that certain classes of cardinals are included in $\boldsymbol{C}_1$, stronger results stating that the same classes of cardinals are included in $\boldsymbol{C}_0$.

We shall establish in detail the result that $\boldsymbol{M}(\boldsymbol{AC}) \subseteq \boldsymbol{C}_0$; this will at least show that many inaccessible cardinals belong to $\boldsymbol{C}_0$. It will be evident that our methods are similar to, but technically more complicated than, the methods of § 1.

DEFINITION 3.8. *For any non-empty subset A of ${}^\alpha\alpha$, let $B(A)$ be the α-complete field of subsets of α which is α-generated by the set*

$$\{C_{f,g}\colon f, g \in A\} ,$$

where $C_{f,g} = \{\xi < \alpha\colon f(\xi) < g(\xi)\}$ for any $f, g \in {}^\alpha\alpha$.

Notice that the cardinal α is determined by any non-empty subset A of ${}^\alpha\alpha$, so that $B(A)$ depends only on the set A.

If $0 \neq A \subseteq {}^\alpha\alpha$, then, for any $f, g \in A$, the sets $\{\xi < \alpha\colon f(\xi) = g(\xi)\}$, $\{\xi < \alpha\colon f(\xi) \neq g(\xi)\}$, and $\{\xi < \alpha\colon f(\xi) \leqslant g(\xi)\}$ clearly belong to $B(A)$, and the unit of $B(A)$ is α. If $0 \neq A \subseteq A' \subseteq {}^\alpha\alpha$, then $B(A) \subseteq B(A')$. $B(A)$ is an α-complete field of sets with at most $|A| \cup \omega$ generators.

LEMMA 3.9. *Suppose that $\alpha \in \boldsymbol{C}$, $0 \neq A \subseteq {}^\alpha\alpha$, and I is an ω^+-complete prime ideal in $B(A)$. Then there exists a unique ordinal η and a unique function φ on A onto η such that:*

(i) *for $f, g \in A$, $\varphi(f) = \varphi(g)$ iff*

$$\{\xi\colon \xi < \alpha, f(\xi) = g(\xi)\} \notin I ;$$

(ii) *for $f, g \in A$, $\varphi(f) \leqslant \varphi(g)$ iff*

$$\{\xi\colon \xi < \alpha, f(\xi) \leqslant g(\xi)\} \notin I .$$

Proof. The proof of 3.9 is the same, word for word, as the proof of 1.3, except that we everywhere replace ${}^\alpha\alpha$ by A.

DEFINITION 3.10. *Assume the hypotheses of 3.9. We shall denote by $\alpha_{A,I}$ and $\tau_{A,I}$ the unique ordinal η and the unique function φ, respectively, which satisfy conditions* 3.9 (i) *and* 3.9 (ii).

Notice that if $A = {}^\alpha\alpha$, then $\alpha_{A,I} = \alpha_I$ and $\tau_{A,I} = \tau_I$, as defined in § 1.

We are principally interested in subsets $A \subseteq {}^\alpha\alpha$ and ideals I in $B(A)$ which satisfy certain very stringent conditions, which we shall now consider.

DEFINITION 3.11. *By the formula $\boldsymbol{P}(\alpha, A, I)$ we shall express the fact that the following conditions* (i)-(vii) *are satisfied:*

(i) $\alpha > \omega$;

(ii) $0 \neq A \subseteq {}^\alpha\alpha$;

(iii) *I is a prime ideal in $B(A)$;*

(iv) *I is α-complete;*

(v) *I is non-principal;*

(vi) *for each* $\xi < a$, *the constant function* $a \times \{\xi\}$ *belongs to* A;

(vii) *the identity function* $\{\langle \xi, \xi \rangle : \xi < a\}$ *belongs to* A.

The next two lemmas, 3.12 and 3.13, have exactly the same proofs as Lemmas 1.5 and 1.6, except that the symbol τ_I is everywhere replaced by $\tau_{A,I}$, and the references to 1.4 are replaced by references to 3.10.

LEMMA 3.12. *Assume* $\boldsymbol{P}(a, A, I)$ *and suppose that* $\xi < a$ *and* $f \in A$. *Then*

(i) $f \in {}^a\xi$ *implies* $\tau_{A,I}(f) < \xi$;

(ii) $f \in {}^a\{\xi\}$ *implies* $\tau_{A,I}(f) = \xi$;

(iii) $\tau_{A,I}(f) < \xi$ *iff* $f^{-1}(\xi) \notin I$;

(iv) $\tau_{A,I}(f) = \xi$ *iff* $f^{-1}(\{\xi\}) \notin I$.

LEMMA 3.13. *Assume* $\boldsymbol{P}(a, A, I)$. *Then* $a < a_{A,I}$ *and*

$$a \leqslant \tau_{A,I}(\{\langle \xi, \xi \rangle : \xi < a\}).$$

LEMMA 3.14. $\boldsymbol{M}(\boldsymbol{AC}) = \boldsymbol{AC} \cup \boldsymbol{M}(\boldsymbol{SN})$.

Proof. Clearly,

$$\boldsymbol{M}(\boldsymbol{AC}) \supseteq \boldsymbol{AC} \cup \boldsymbol{M}(\boldsymbol{SN}).$$

Let a belong to $\boldsymbol{M}(\boldsymbol{AC})$ but not to $\boldsymbol{AC}$. We shall show that a belongs to $\boldsymbol{M}(\boldsymbol{SN})$. There exists a subset y of $\boldsymbol{AC}$ such that $a = \bigcup y$ and $y \cup \{a\}$ is closed. For each cardinal β, let β' be the least strong limit cardinal which is not less than β; thus $\beta' = \beta$ in case β is a strong limit cardinal, and β' is the union of $\beta, 2^\beta, 2^{2^\beta}$, etc. otherwise. Furthermore, let

$$y' = \{\beta' : \beta \in y\}.$$

Clearly, $\beta \in \boldsymbol{AC}$ implies $\beta' \in \boldsymbol{SN}$. Then, since $a \notin \boldsymbol{AC}$,

$$a = \bigcup y' \quad \text{and} \quad y' \subset \boldsymbol{SN}.$$

Finally, making use of the fact that the union of strong limit cardinals is a strong limit cardinal, we show that $y' \cup \{a\}$ is closed.

THEOREM 3.15. $\boldsymbol{M}(\boldsymbol{AC}) \subseteq \boldsymbol{C}_0$.

Proof. By 3.14 and 3.7, it is sufficient to prove that

$$\boldsymbol{M}(\boldsymbol{SN}) \subseteq \boldsymbol{C}_0.$$

In view of 3.7 we have

$$\boldsymbol{SN} \subseteq \boldsymbol{C}_0.$$

Accordingly, suppose

$$a \in \boldsymbol{M}(\boldsymbol{SN}) \sim \boldsymbol{SN}.$$

In order to show that $a \in \boldsymbol{C}_0$, we shall construct a particular subset A of ${}^a a$ of power a for which we shall be able to establish that $B(A)$ cannot possibly have any non-principal a-complete prime ideals. From this fact it will clearly follow that the set $S_a(a)$ is an a-complete proper ideal in $B(A)$ which cannot be extended to an a-complete prime ideal; hence we will have shown that $a \in \boldsymbol{C}_0$.

To begin with, we shall introduce some auxiliary functions.

Consider first a set y such that $y \subseteq \boldsymbol{SN}$, $a = \bigcup y$, and $y \cup \{a\}$ is closed; such a set y exists in view of 1.29. For each $\xi < a$, let

$$\varphi(\xi) = \bigcup \big(y \cap (\xi+1)\big) \cup \bigcap y.$$

Then we have

$$\varphi \in {}^a y,$$

and indeed $\varphi(\xi)$ is always the greatest member of y which is $\leqslant \xi \cup \bigcap y$. Next, let us choose, for each $\xi < a$, a function ψ_ξ on $cf(\xi)$ into ξ such that

$$\xi = \bigcup_{\zeta < cf(\xi)} \big(\psi_\xi(\zeta) + 1\big).$$

Finally, for all $\xi, \zeta < a$, we put

$$\theta(\xi, \zeta) = \begin{cases} \xi, & \text{if} \quad \xi \leqslant \zeta; \\ \bigcap \{\eta : \eta < cf(\xi) \text{ and } \zeta \leqslant \psi_\xi(\eta)\}, & \text{if} \quad \zeta < \xi. \end{cases}$$

Thus, whenever $\zeta < \xi < a$, $\theta(\xi, \zeta)$ is a member of $cf(\xi)$ such that $\zeta \leqslant \psi_\xi(\theta(\xi, \zeta))$.

We now take for A the least subset A' of ${}^a a$ which satisfies, for all $\zeta < a$ and $f \in A'$, the following conditions:

(1) $a \times \{\zeta\} \in A'$;

(2) $\{\langle \xi, \xi \rangle : \xi < a\} \in A'$;

(3) if $g_1(\xi) = \varphi(f(\xi))$ for all $\xi < a$, then $g_1 \in A'$;

(4) if $g_2(\xi) = cf(f(\xi))$ for all $\xi < a$, then $g_2 \in A'$;

(5) if, for all $\xi < a$,

$$g_{3,\zeta}(\xi) = \begin{cases} \psi_{f(\xi)}(\zeta) & \text{whenever} \quad cf(f(\xi)) > \zeta, \\ f(\xi) & \text{whenever} \quad cf(f(\xi)) \leqslant \zeta, \end{cases}$$

then $g_{3,\zeta} \in A'$;

(6) if $g_{4,\zeta}(\xi) = \theta(f(\xi), \zeta)$ for all $\xi < a$, then $g_{4,\zeta} \in A'$.

It is clear that $|A| = a$. Therefore $B(A)$ is an a-complete field of sets with at most a generators. Moreover, by (1), (2), and 3.8, every one-element subset of a belongs to $B(A)$. Since a is regular, it follows that the set $S_a(a)$ is a proper a-complete ideal in $B(A)$. Suppose that $S_a(a)$ can be extended to an a-complete prime ideal I in $B(A)$. We shall arrive at a contradiction, and thereby show that $a \in \boldsymbol{C}_0$.

Since $S_a(a) \subseteq I$, I is non-principal, and thus, in view of (1) and (2), we see that $\boldsymbol{P}(\alpha, A, I)$ holds. By 3.13 we have $\alpha < \alpha_{A,I}$ and hence we may choose a function $f \in A$ such that

$$\tau_{A,I}(f) = \alpha .$$

Consider the function g_1 defined in (3). We have $g_1 \in {}^{\alpha}y$, $g_1 \in A$, and

$$\tau_{A,I}(g_1) \leqslant \tau_{A,I}(f) .$$

Moreover, for each $\zeta \in y$ and $\xi < \alpha$,

$$\zeta \leqslant g_1(\xi) \quad \text{iff} \quad \zeta \leqslant f(\xi) \cup \bigcap y .$$

It follows by 3.9 that $\zeta \leqslant \tau_{A,I}(g_1)$ whenever $\zeta \in y$, and therefore $\alpha \leqslant \tau_{A,I}(g_1)$. Hence

$$\tau_{A,I}(g_1) = \tau_{A,I}(f) ,$$

and thus, by 3.9,

$$\{\xi < \alpha \colon f(\xi) = g_1(\xi)\} \notin I .$$

Since $g_1 \in {}^{\alpha}y$ and $y \subseteq \boldsymbol{SN}$, we have

$$\{\xi < \alpha \colon f(\xi) \in \boldsymbol{SN}\} \notin I .$$

Consider now the "confinality function" g_2 defined in (4). By (4), $g_2 \in A$. Moreover, since $g_2(\xi) < f(\xi)$ whenever $\xi \in \boldsymbol{SN}$, we have

$$\tau_{A,I}(g_2) < \alpha .$$

Let

$$\beta = \tau_{A,I}(g_2) .$$

Then by 3.12,

$$\{\xi < \alpha \colon cf(f(\xi)) = \beta\} \notin I .$$

It follows from (5) that, whenever $\zeta < \beta$, the function $g_{3,\zeta}$ defined in (5) belongs to A and satisfies the condition

$$\{\xi < \alpha \colon g_{3,\zeta}(\xi) = \psi_{f(\xi)}(\zeta)\} \notin I .$$

Then, for all $\zeta < \beta$, we have $\tau_{A,I}(g_{3,\zeta}) < \alpha$ and, by letting $\psi_\alpha(\zeta) = \tau_{A,I}(g_{3,\zeta})$, we obtain $\psi_\alpha \in {}^{\beta}\alpha$.

We shall show that

$$\alpha = \bigcup\nolimits_{\zeta<\beta} \big(\psi_\alpha(\zeta)+1\big) , \tag{7}$$

and hence that $cf(\alpha) \leqslant \beta$ and $\alpha \in \boldsymbol{SN}$. Since we have assumed $\alpha \notin \boldsymbol{SN}$, this contradiction will complete our proof.

For every $\zeta < \alpha$, the function $g_{4,\zeta}$ defined in (6) belongs to A. By (6) we have

$$\tau_{A,I}(g_{4,\zeta}) < \beta ,$$

and for brevity we write

$$\eta_\zeta = \tau_{A,I}(g_{4,\zeta}) .$$

It is clear that, whenever $\zeta < f(\xi) < \alpha$, we have

$$\zeta \leqslant \psi_{f(\xi)}\big(g_{4,\zeta}(\xi)\big) ,$$

but

$$\{\xi < \alpha \colon g_{4,\zeta}(\xi) = \eta_\zeta\} \notin I$$

so that

$$\{\xi < \alpha \colon \zeta \leqslant \psi_{f(\xi)}(\eta_\zeta) = g_{3,\eta_\zeta}(\xi)\} \notin I$$

and therefore

$$\zeta \leqslant \psi_\alpha(\eta_\zeta) .$$

Consequently,

$$\alpha \leqslant \bigcup\nolimits_{\eta<\beta} \psi_\alpha(\eta) ,$$

and hence, since $\psi_\alpha \in {}^{\beta}\alpha$, the equation (7) follows. This completes our proof.

From this point on the development becomes very sketchy, for the reasons explained in the introduction. We here wish to reiterate our advice to the reader: skip the remaining portion of § 3 at first reading, making only a mental note of the main results. Notice particularly the last result of this section, Theorem 3.36, which improves Theorem 1.32 by replacing $\boldsymbol{C}_1$ by $\boldsymbol{C}_0$, and thus shows that $\boldsymbol{C}_0$ is already very large.

LEMMA 3.16. *The following condition is necessary and sufficient for* $\alpha \in \boldsymbol{C}_0$: *there is a set* A *of power* α *such that* 3.11 (i), (ii), (vi), *and* (vii) *hold but* $\boldsymbol{P}(\alpha, A, I)$ *fails for all* I.

3.16 follows easily from Theorem 4.14, which we shall prove in § 4.

DEFINITION 3.17. *Let* $\eta < \omega$ *and let* F *be an* η*-ary function on* α, *that is,* $F \in {}^{({}^{\eta}\alpha)}\alpha$. *We define the function* F^* *on* ${}^{\eta}({}^{\alpha}\alpha)$ *into* ${}^{\alpha}\alpha$ *by the condition*

$$F^*(f_1, \ldots, f_\eta)(\xi) = F\big(f_1(\xi), \ldots, f_\eta(\xi)\big)$$

for all $f_1, \ldots, f_\eta \in {}^{\alpha}\alpha$ *and* $\xi < \alpha$. *We shall say that the subset* A *of* ${}^{\alpha}\alpha$ *is* F*-closed if, for all* $f_1, \ldots, f_\eta \in A$, *we have* $F^*(f_1, \ldots, f_\eta) \in A$. *Let* $\boldsymbol{F}$ *be any set of finitary functions on* α, *that is,* $\boldsymbol{F} \subseteq \bigcup\{{}^{({}^{\eta}\alpha)}\alpha \colon \eta \in \omega\}$. *A subset* A *of* ${}^{\alpha}\alpha$ *is said to be* $\boldsymbol{F}$*-closed if it is* F*-closed for every* $F \in \boldsymbol{F}$. *By the minimal* $\boldsymbol{F}$*-closed set we mean the intersection of all sets* A *which are* $\boldsymbol{F}$*-closed, are included in* ${}^{\alpha}\alpha$, *and contain the identity function on* α *and all the constant functions on* α *into* α.

The notation introduced in 3.17 could have been used in the proof of Theorem 3.15. In fact, in the proof of 3.15, we have $g_1 = \varphi^*(f)$ and $g_2 = cf^*(f)$; moreover, if for all $\xi < \alpha$ we write

$$\hat{\psi}_\zeta(\xi) = \begin{cases} \psi_\xi(\zeta), & \text{if} \quad cf(\xi) > \zeta , \\ \xi, & \text{if} \quad cf(\xi) \leqslant \zeta \end{cases}$$

and $\hat{\theta}_\zeta(\xi) = \theta(\xi, \zeta)$, then we have $g_{3,\zeta} = \hat{\psi}^*_\zeta(f)$ and $g_{4,\zeta} = \hat{\theta}^*_\zeta(f)$. If $\boldsymbol{F}$ consists of the set of functions φ, cf (restricted to α), $\hat{\psi}_\zeta$ for all $\zeta < \alpha$, and $\hat{\theta}_\zeta$ for all $\zeta < \alpha$, then $\boldsymbol{F} \subseteq {}^{\alpha}\alpha$, $|\boldsymbol{F}| = \alpha$, and A is just the minimal $\boldsymbol{F}$-closed set. We did not use the notation of 3.17 in the proof of 3.15 because it would not simplify the statement of the proof very much. However, 3.17 will be seen to be necessary for the introduction of the notion of a strongly normal class in 3.23.

Clearly ${}^{a}\alpha$ is $\boldsymbol{F}$-closed for any set $\boldsymbol{F}$ of finitary functions on α.

LEMMA 3.18. *Suppose that $\boldsymbol{F}$ is a set of finitary functions on α and A is the minimal $\boldsymbol{F}$-closed set. Then A has the following properties:*

(i) *A is $\boldsymbol{F}$-closed;*

(ii) *A contains the identity function on a and all the constant functions on a into α;*

(iii) $\alpha \leqslant |A| \leqslant \alpha \cup |\boldsymbol{F}|$.

LEMMA 3.19. *Assume that $\boldsymbol{P}(\alpha, A, I)$ holds, and furthermore that $\eta < \omega$, $F \in {}^{({}^{\eta}\alpha)}\alpha$, and A is F-closed. Then whenever $f_1, \ldots, f_\eta, g_1, \ldots, g_\eta \in A$ and*

$$\tau_{A,I}(f_1) = \tau_{A,I}(g_1), \quad \ldots, \quad \tau_{A,I}(f_\eta) = \tau_{A,I}(g_\eta),$$

we have

$$\tau_{A,I}(F^*(f_1, \ldots, f_\eta)) = \tau_{A,I}(F^*(g_1, \ldots, g_\eta)).$$

DEFINITION 3.20. *Assume the hypotheses of 3.19. We denote by $F_{A,I}$ the function on ${}^{\eta}(\alpha_{A,I})$ into $\alpha_{A,I}$ determined by the condition*

$$F_{A,I}(\tau_{A,I}(f_1), \ldots, \tau_{A,I}(f_\eta)) = \tau_{A,I}(F^*(f_1, \ldots, f_\eta))$$

for all $f_1, \ldots, f_\eta \in A$.

Notice that, under the hypotheses of 3.19, $F_{A,I}$ properly includes F. In the proofs of the theorems which follow the fact that many properties possessed by F can be carried over to $F_{A,I}$ plays an important role. However, the notation $F_{A,I}$ will never appear in the statements of those theorems.

The notation $F_{A,I}$ could have been used in the proof of 3.15; in fact, we have $\beta = cf_{A,I}(\alpha)$, $\psi_\alpha(\zeta) = (\hat{\varphi}_\zeta)_{A,I}(\alpha)$, and $\eta_\zeta = (\hat{\varphi}_\zeta)_{A,I}(\alpha)$ for each $\zeta < \alpha$. (Cf. the remark following 3.17.) The proof of 3.15 actually depends on the fact that enough properties of F are carried over to $F_{A,I}$ to insure that ψ_α is a function on β into α such that α is the union of the range of ψ_α and thus that α is confinal with β.

DEFINITION 3.21. *α is said to be $\boldsymbol{F}$-representable by f if $\boldsymbol{F}$ is a set of finitary functions on α and there exist I, A such that $\boldsymbol{P}(\alpha, I, A)$ holds, A is $\boldsymbol{F}$-closed, $f \in A$, and $\tau_{I,A}(f) = \alpha$.*

Notice that if α is representable by f in the sense of § 1, then α is $\boldsymbol{F}$-representable by f for every set $\boldsymbol{F}$ of finitary functions on α. Also, ω is never $\boldsymbol{F}$-representable, in view of 3.11 (i).

LEMMA 3.22. *Let $\alpha \in \boldsymbol{C}$. Then a necessary and sufficient condition for $\alpha \in \boldsymbol{C}_0$ is that there exists a set $\boldsymbol{F}$ of finitary functions on α such that $|\boldsymbol{F}| \leqslant \alpha$ and α is not $\boldsymbol{F}$-representable by any function $f \in {}^{\alpha}\alpha$.*

DEFINITION 3.23. *$\boldsymbol{X}$ is said to be a strongly normal class if $\boldsymbol{X} \subseteq \boldsymbol{C}_0$ and, for each cardinal α, there is a set $\boldsymbol{F}$ of finitary functions on α such that $|\boldsymbol{F}| \leqslant \alpha$ and α is not $\boldsymbol{F}$-representable by any function $f \in {}^{\alpha}(\boldsymbol{X} \cap \alpha)$.*

Notice that any strongly normal class is normal in the sense of § 1.

THEOREM 3.24. *If $\boldsymbol{X}$ is a strongly normal class and $\boldsymbol{Y} \subseteq \boldsymbol{X}$, then $\boldsymbol{Y}$ is strongly normal.*

THEOREM 3.25. *A sufficient condition for $\boldsymbol{X}$ to be strongly normal is that the union of any non-empty subset of $\boldsymbol{X}$ belongs to $\boldsymbol{C}_0$.*

In particular it follows that any finite subclass of $\boldsymbol{C}_0$ is strongly normal and furthermore, if $\boldsymbol{X} \subseteq [\alpha, \beta] \subseteq \boldsymbol{C}_0$, then $\boldsymbol{X}$ is strongly normal.

THEOREM 3.26. *The following three conditions are equivalent:*

(i) $\boldsymbol{C}_0 = \boldsymbol{C}$;

(ii) *$\boldsymbol{C}_0$ is strongly normal;*

(iii) *the class $\{\alpha \in \boldsymbol{C}\colon [\omega^+, \alpha] \subseteq \boldsymbol{C}_0\}$ is strongly normal.*

THEOREM 3.27. *Suppose that, for each ordinal μ, $\boldsymbol{X}_\mu$ is strongly normal. Then $\{\beta\colon \beta \in \bigcup_{\mu<\beta} \boldsymbol{X}_\mu\}$ is strongly normal.*

3.27 is the analogue of the first induction principle, Theorem 1.19.

COROLLARY 3.28. *Suppose that, for each $\mu < \nu$, $\boldsymbol{X}_\mu$ is strongly normal. Then $(\bigcup_{\mu<\nu} \boldsymbol{X}_\mu) \sim \nu$ is strongly normal.*

COROLLARY 3.29. (i) *If $\boldsymbol{X}$, $\boldsymbol{Y}$ are strongly normal, then $\boldsymbol{X} \cup \boldsymbol{Y}$ is strongly normal.*

(ii) *If $[\omega^+, \alpha] \subseteq \boldsymbol{C}_0$ and, for each $\mu < \alpha$, $\boldsymbol{X}_\mu$ is strongly normal, then $\bigcup_{\mu<\alpha} \boldsymbol{X}_\mu$ is strongly normal.*

COROLLARY 3.30. *If $\boldsymbol{C}_0 \neq \boldsymbol{C}$, then there is no maximal strongly normal class.*

DEFINITION 3.31. *$\boldsymbol{F}$ is said to preserve strong normality if $\boldsymbol{F}(\boldsymbol{X})$ is strongly normal whenever $\boldsymbol{X}$ is strongly normal.*

THEOREM 3.32. *If $\boldsymbol{F}$ preserves strong normality, then $\boldsymbol{F}^{(\infty)}$ preserves strong normality.*

THEOREM 3.33. *The operation $\boldsymbol{M}$ preserves strong normality.*

Theorems 3.32 and 3.33 are the analogues of the second induction principle 1.27 and the third induction principle 1.31, respectively.

COROLLARY 3.34. *The operations $\boldsymbol{M}^{(\infty)}$, $(\boldsymbol{M}^{(\infty)})^{(\infty)}$, etc. preserve strong normality.*

THEOREM 3.35. *The class $\boldsymbol{AC}$ is strongly normal.*

3.35 is an improvement of Theorem 1.33.

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THEOREM 3.36.

(i) $\boldsymbol{M}(\boldsymbol{AC}) \subseteq \boldsymbol{C}_0$;

(ii) $\boldsymbol{M}^{(\infty)}(\boldsymbol{AC}) \subseteq \boldsymbol{C}_0$;

(iii) $\boldsymbol{M}^{(\infty)}(\boldsymbol{M}^{(\infty)}(\boldsymbol{AC})) \subseteq \boldsymbol{C}_0$;

(iv) $(\boldsymbol{M}^{(\infty)})^{(\infty)}(\boldsymbol{AC}) \subseteq \boldsymbol{C}_0$.

Theorem 3.36 should be compared with the analogous results 1.34 and 3.5.

As in the case of 1.34 and 3.5, the process indicated by 3.36 can be continued indefinitely. In fact, if $\boldsymbol{X}$ is strongly normal and $\boldsymbol{AC} \subseteq \boldsymbol{X}$, then $\boldsymbol{M}(\boldsymbol{X})$ is a subclass of $\boldsymbol{C}_0$ which is strictly larger than $\boldsymbol{X}$. We thus see that the class $\boldsymbol{C}_0$ already extends very far into the hyperinaccessible cardinals, although it is included in $\boldsymbol{C}_1$.

§ 4. Characteristic properties of cardinals in the class C_0.

This section performs for the class $\boldsymbol{C}_0$ exactly the same task which has been performed in § 2 for the class $\boldsymbol{C}_1$. In fact, we shall state here a number of conditions, formulated in terms of set theory and related branches of mathematics, which will prove to be necessary and sufficient for a cardinal α to belong to $\boldsymbol{C}_0$. Actually, we shall discuss a more general problem involving two cardinals α and β; we shall investigate the conditions under which there is an α-complete field of sets B with at most β generators in which some α-complete proper ideal I cannot be extended to an α-complete prime ideal in B. To obtain a convenient notation for the relevant results, we formulate the following

DEFINITION 4.1. *We shall say that α is in the relation $\boldsymbol{R}$ to β, in symbols $\alpha\boldsymbol{R}\beta$, if there exists an α-complete field of sets, with at most β generators, in which some α-complete proper ideal cannot be extended to an α-complete prime ideal.*

From definitions 3.1 and 4.1 we obtain at once

COROLLARY 4.2. $\alpha \in \boldsymbol{C}_0$ *iff* $\alpha\boldsymbol{R}\alpha$.

We shall begin with some elementary properties of the relation $\boldsymbol{R}$.

THEOREM 4.3. *There is no β such that $\omega\boldsymbol{R}\beta$.*

Proof. By 0.2 and 4.1.

THEOREM 4.4. *If $\alpha\boldsymbol{R}\beta$ and $\beta<\gamma$, then $\alpha\boldsymbol{R}\gamma$.*

Proof. By 4.1.

THEOREM 4.5. *Suppose that α is singular. Then $\alpha\boldsymbol{R}\beta$ iff $\alpha^{+}\boldsymbol{R}\beta$.*

Proof. By 0.4 and 4.1.

LEMMA 4.6. *If $2^\beta<\alpha$, then $\alpha\boldsymbol{R}\beta$ does not hold.*

Proof. Suppose that B is an α-complete field of sets which is α-generated by a set B_0 of power $\leqslant\beta$. Let I be any α-complete proper ideal in B. For each subset C of B_0 let

$$x_C = \bigcap (C \cup \{\bigcup B_0\}) \sim \bigcup (B_0 \sim C)\,.$$

It is easily seen that $\bigcup\{x_C\colon\ C\subseteq B_0\} = \bigcup B_0$. Moreover it follows from the α-completeness of B that $x_C \in B$ whenever $C\subseteq B_0$. In fact, for each subset C of B_0, x_C is either the empty set or an atom of B. For some subset C_0 of B_0 we must have $x_{C_0}\notin I$, for otherwise, by the α-completeness of I, we would have $\bigcup\{x_C\colon\ C\subseteq B_0\} = \bigcup B_0 \in I$, contradicting the assumption that I is proper. Now let $J=\{y\in B\colon\ x_{C_0}\sim y\neq 0\}$. Since $0\in I$, we have $x_{C_0}\neq 0$, and hence x_{C_0} is an atom of B. Therefore J is an α-complete prime ideal which includes I, and it follows that $\alpha\boldsymbol{R}\beta$ does not hold.

LEMMA 4.7. *If α is regular and $\beta<\alpha\leqslant 2^\beta$, then $\alpha\boldsymbol{R}\beta$.*

Proof. We have already established this fact in the proof of 3.7.

The results established so far provide us with a full answer to the question: for which cardinals α and β does $\alpha\boldsymbol{R}\beta$ hold in case $\beta<\alpha$? In fact, the results 4.3-4.7 imply directly

THEOREM 4.8. *In case $\beta<\alpha$, we have $\alpha\boldsymbol{R}\beta$ iff either α is regular and $\alpha\leqslant 2^\beta$, or α is singular and $\alpha<2^\beta$.*

As an immediate consequence of 4.8, we obtain

COROLLARY 4.9. *Assume the generalized continuum hypothesis. Then, in case $\beta\leqslant\alpha$, we have $\alpha\boldsymbol{R}\beta$ iff $\alpha=\beta^+$.*

COROLLARY 4.10. *The following two conditions are equivalent:*

(i) *α is not a strong limit number;*

(ii) *$\alpha\boldsymbol{R}\beta$ holds for some $\beta<\alpha$.*

Proof. If α is regular, the result follows at once from 4.8. Suppose that α is singular. Then, by 4.8, there exists $\beta<\alpha$ such that $\alpha\boldsymbol{R}\beta$ iff there exists γ such that $\gamma<\alpha<2^\gamma$. But if $\beta<\alpha\leqslant 2^\beta$ and we set $\gamma=\beta\cup cf(\alpha)$, then we have $\gamma<\alpha<2^\gamma$. This completes the proof.

For a given cardinal α, the smallest cardinal β for which Theorem 4.8 does not give any answer to the question whether $\alpha\boldsymbol{R}\beta$ holds is obviously $\beta=\alpha$. Moreover, the whole problem is completely solved for those cardinals α for which, in the particular case $\beta=\alpha$, $\alpha\boldsymbol{R}\beta$ actually holds, i.e. which belong to $\boldsymbol{C}_0$; for, as an immediate consequence of Theorem 4.4, we have

THEOREM 4.11. *If $\alpha\in\boldsymbol{C}_0$ and $\alpha\leqslant\beta$, then $\alpha\boldsymbol{R}\beta$.*

Thus, for example, in view of Theorem 3.36, the problem of whether $\alpha\boldsymbol{R}\beta$ holds is settled whenever α belongs to any of the classes $\boldsymbol{AC}$, $\boldsymbol{M}(\boldsymbol{AC})$, $\boldsymbol{M}^{(\infty)}(\boldsymbol{AC})$, $\boldsymbol{M}^{(\infty)}(\boldsymbol{M}^{(\infty)}(\boldsymbol{AC}))$, $(\boldsymbol{M}^{(\infty)})^{(\infty)}(\boldsymbol{AC})$, etc.

In the next three theorems we shall consider conditions which are closely related to the one formulated in Definition 4.1.

THEOREM 4.12. *The following condition is necessary and sufficient for* $\alpha \boldsymbol{R} \beta$:

(i) *there exists an* α*-complete field of sets, with at most* β *generators, whose power is at least* β *and in which some* α*-complete proper ideal cannot be extended to an* α*-complete prime ideal.*

Proof. Condition (i) is obviously sufficient for $\alpha \boldsymbol{R} \beta$.

Suppose $\alpha \boldsymbol{R} \beta$. Then there exists an α-complete field B of sets which is α-generated by a set B_0 of power $\leqslant \beta$ and which has an α-complete proper ideal I that cannot be extended to an α-complete prime ideal. Let X be a set of power β which is disjoint from $\bigcup B$. Let B' be the α-complete field of subsets of $X \cup (\bigcup B)$ which is α-generated by the set $\{\bigcup B\} \cup B_0 \cup \{\{x\}: x \in X\}$ of power β. Let $I' = \{y: y \in B', y \cap \bigcup B \in I\}$. Then I' is an α-complete proper ideal in B'. If J' were an α-complete prime ideal in B' which includes I', then $J' \cap B$ would be an α-complete prime ideal in B which includes I. Therefore I' cannot be extended to an α-complete prime ideal in B'. This proves the necessity of (i).

THEOREM 4.13. *Suppose that either* α *is regular and* $\alpha \leqslant \beta$, *or* α *is singular and* $\alpha < \beta$. *Then the following condition is necessary and sufficient for* $\alpha \boldsymbol{R} \beta$:

(i) *there exists an* α*-complete field of sets with* β *generators in which some* α*-complete proper ideal cannot be extended to an* α*-complete prime ideal.*

Proof. Condition (i) is obviously sufficient.

Suppose that α is regular and $\alpha \leqslant \beta$. Let C be the α-complete field of subsets of β which is α-generated by the set $\{\{\xi\}: \xi < \beta\}$. Then we have

$$C = \{x: x \subseteq \beta, \text{ and either } |x| < \alpha \text{ or } |\beta \sim x| < \alpha\}.$$

Suppose that C is α-generated by X. We shall prove that $|X| \geqslant \beta$. We may assume without loss of generality that every element of X has power $< \alpha$. If $|X| < \beta$, then $|\bigcup X| < \beta$, and thus X could not possibly generate C. Thus $|X| \geqslant \beta$, and C is an α-complete field of sets with β generators. In case $\alpha \boldsymbol{R} \beta$, the field B' constructed in the proof of 4.12 is therefore an α-complete field of sets with β generators, and 4.13 (i) follows.

Suppose α is singular and $\alpha < \beta$. Then α^+ is regular and $\alpha^+ \leqslant \beta$. If $\alpha \boldsymbol{R} \beta$, then by 4.5 we have $\alpha^+ \boldsymbol{R} \beta$. Condition (i) thus holds for α^+ and β and, by 0.4, it also holds for α and β. Our proof is complete.

THEOREM 4.14. *If* $\beta = \beta^{\underline{\alpha}}$, *then the following three conditions are equivalent:*

(i) $\alpha \boldsymbol{R} \beta$;

(ii) *there is an* α*-complete field of sets* B *of power* β *in which some* α*-complete proper ideal* I *cannot be extended to an* α*-complete prime ideal;*

(iii) *there is an* α*-complete field of subsets of* β *which includes* $S_\omega(\beta)$, *is of power* β, *and in which some* α*-complete proper ideal cannot be extended to an* α*-complete prime ideal.*

Proof. The implication (iii) $\Rightarrow$ (i) is obvious.

The implication (i) $\Rightarrow$ (ii) follows from 4.12 and the fact that, since $\beta = \beta^{\underline{\alpha}}$, any α-complete field of sets which is α-generated by a set of power β has power β.

Assume (ii), and choose a function $f \in {}^B(\bigcup B)$ such that, for each $x \in B \sim \{0\}$, $f(x) \in x$.

Let

$$Y = \{f(x): x \in B \sim \{0\}\}.$$

Clearly, $Y \subseteq \bigcup B$ and $|Y| = \beta$. For each $x \in B$, let

$$x' = x \cap Y, \quad B' = \{x': x \in B\}, \quad \text{and} \quad I' = \{x': x \in I\}.$$

It is then easily seen that B' is an α-complete field of subsets of Y, and in fact that the mapping $x \to x'$ is an isomorphism on the set algebra $\mathfrak{B}$ onto the set algebra $\mathfrak{B}'$. Therefore I' is an α-complete proper ideal in B' which cannot be extended to an α-complete prime ideal.

We now consider the field B'' of subsets of Y which is α-generated by the set $B' \cup S_\alpha(Y)$. Since $\beta = \beta^{\underline{\alpha}}$ and $|B'| = |Y| = \beta$, B'' is of power β. Let

$$I'' = \{y \in B'': y \subseteq x \text{ for some } x \in I'\}.$$

Then I'' is an α-complete proper ideal in B''. Moreover, I'' cannot be extended to an α-complete prime ideal in B''; for, if J'' were an α-complete prime ideal in B'' including I', then the set $J' = J'' \cap B'$ would be an α-complete prime ideal in B' including I'. We have shown that (iii) holds when β (in its first two occurrences) is replaced by a set Y of power β, and it follows that (iii) itself holds.

Conditions 4.14 (ii) and 4.14 (iii) are each sufficient for $\alpha \boldsymbol{R} \beta$ even if $\beta \neq \beta^{\underline{\alpha}}$. If $\alpha \leqslant \beta$ and β is inaccessible, then we have $\beta = \beta^{\underline{\alpha}}$, so 4.14 (i), (ii), and (iii) are all equivalent. Moreover, if we assume the generalized continuum hypothesis, then $\beta = \beta^{\underline{\alpha}}$ holds, and thus 4.14 (i), (ii), and (iii) are equivalent, whenever $\alpha \leqslant \beta$ and β is regular.

The next phase of our discussion involves Boolean algebras.

DEFINITION 4.15. *We shall denote by $W_{\alpha,\beta}$ the class of all weakly α-representable Boolean algebras with at most β generators.*

THEOREM 4.16. *The following six conditions are equivalent:*

(i) $\alpha R \beta$;

(ii) *there exists $\mathfrak{B} \in W_{\alpha,\beta}$ which has power >1 and has no α-complete prime ideals;*

(iii) *there exists $\mathfrak{B} \in W_{\alpha,\beta}$ which is not α-generated by its atoms and in which every α-complete prime ideal is principal;*

(iv) *there exists $\mathfrak{B} \in W_{\alpha,\beta}$ in which some α-complete proper ideal cannot be extended to an α-complete prime ideal;*

(v) *there exists $\mathfrak{B} \in W_{\alpha,\beta}$ in which some α-complete proper ideal is not the intersection of all α-complete prime ideals including it;*

(vi) *there exists $\mathfrak{B} \in W_{\alpha,\beta}$ which is not α-representable.*

Proof. We shall successively establish the implications (i)$\Rightarrow$(ii), (ii)$\Rightarrow$(iii), (iii)$\Rightarrow$(iv), (iv)$\Rightarrow$(v), (v)$\Rightarrow$(vi), and (vi)$\Rightarrow$(i).

Assume (i); then there is a set algebra $\mathfrak{B}$ which is α-complete and α-generated by a set B_0 of power $\leqslant\beta$ and which has an α-complete proper ideal I that is not included in any α-complete prime ideal. The quotient algebra $\mathfrak{B}/I$ is weakly α-representable. Moreover, $\mathfrak{B}/I$ is α-generated by the set $\{x/I\colon x \in B_0\}$ of power $\leqslant\beta$, where $x/I = \{y \in B\colon (x-y)+(y-x) \in I\}$. Since I is a proper ideal in $\mathfrak{B}$, $\mathfrak{B}/I$ has power >1. If J were an α-complete prime ideal in $\mathfrak{B}/I$, then $\{x \in B\colon x/I \in J\}$ would be an α-complete prime ideal in $\mathfrak{B}$ which includes I. Therefore $\mathfrak{B}/I$ does not have any α-complete prime ideals. Hence (ii) holds.

Assuming (ii), we see at once that $\mathfrak{B}$ has no atoms. Since $|B|>1$, $\mathfrak{B}$ is not α-generated by its atoms. Hence (iii) holds.

Now assume (iii). Let I be the α-complete ideal which is α-generated by the atoms of $\mathfrak{B}$. Then I is proper, and any prime ideal which includes I is non-principal. Hence I is not included in any α-complete prime ideal, and thus I is equal to the intersection of all α-complete prime ideals which include I. This verifies (iv).

It is obvious that (iv) implies (v).

Now assume (v), let B_0 be a set of power $\leqslant\beta$ which α-generates $\mathfrak{B}$, and let I be an α-complete proper ideal in $\mathfrak{B}$ which is not equal to the intersection I' of all α-complete prime ideals including it. (We assume $I'=B$ in case no α-complete ideal includes I.) Clearly $I \subseteq I'$. By 0.7, the quotient algebra $\mathfrak{B}/I$ is weakly α-representable. Moreover, $\mathfrak{B}/I$ is α-generated by the set $\{x/I\colon x \in B_0\}$ of power $\leqslant\beta$, so $\mathfrak{B}/I \in W_{\alpha,\beta}$. Choose an element $x_0 \in I' \sim I$, and let

$$J = \{y/I\colon y \in B \text{ and } y \leqslant \overline{x}_0\};$$

that is, J is the principal ideal in $\mathfrak{B}/I$ whose sum is $\overline{x}_0/I$. If J could be extended to an α-complete prime ideal K in $\mathfrak{B}/I$, then the set

$$\{x \in B\colon x/I \in K\}$$

would be an α-complete prime ideal in $\mathfrak{B}$ including $I \cup \{\overline{x}_0\}$. But this would contradict our hypothesis that every α-complete prime ideal in $\mathfrak{B}$ which includes I also contains x_0. Therefore J cannot be extended to an α-complete prime ideal in $\mathfrak{B}/I$. J is clearly a proper ideal because $x_0 \notin I$, and hence

$$\sum J = x_0/I \neq \sum(\mathfrak{B}/I).$$

By 0.5, $\mathfrak{B}/I$ is not α-representable. This verifies (vi).

Assuming (vi), we consider a set B_0 of power $\leqslant\beta$ which α-generates $\mathfrak{B}$. Since $\mathfrak{B} \in W_{\alpha,\beta}$, $\mathfrak{B}$ is isomorphic to $\mathfrak{C}'/I'$ for some α-complete set algebra $\mathfrak{C}'$ and some α-complete ideal I' in $\mathfrak{C}'$. By 0.5, there is a proper principal ideal J in $\mathfrak{B}$ which is not included in any α-complete prime ideal. Let f be an isomorphism on $\mathfrak{C}'/I'$ onto $\mathfrak{B}$. Choose a function $g \in {}^{B_0}C'$ such that, for each element $y \in B_0$, $f(g(y))/I' = y$; the range of g is clearly of power $\leqslant\beta$. Let $\mathfrak{C}$ be the α-complete subalgebra of $\mathfrak{C}'$ which is α-generated by the range of g. Then

$$\mathfrak{C}'/I' = \{x/I'\colon x \in C\}.$$

The set

$$K = \{x\colon x \in C \text{ and } f(x/I') \in J\}$$

is a proper α-complete ideal in $\mathfrak{C}$, and $I' \cap C \subset K$. If L were an α-complete prime ideal in $\mathfrak{C}$ which included K, then the set

$$\{f(x/I')\colon x \in L\}$$

would be an α-complete prime ideal in $\mathfrak{B}$ which included J, and this is impossible. Therefore K cannot be extended to an α-complete prime ideal in $\mathfrak{C}$, and (i) follows.

DEFINITION 4.17. *We shall denote by $D_{\alpha,\beta}$ the class of all α-distributive Boolean algebras with at most β generators.*

THEOREM 4.18. *The following three conditions are equivalent:*

(i) $\alpha R \beta$;

(ii) *there exists $\mathfrak{B} \in D_{\alpha,\beta}$ in which some α-complete proper ideal cannot be extended to an α-complete prime ideal;*

(iii) *there exists $\mathfrak{B} \in D_{\alpha,\beta}$ in which some α-complete proper ideal is not the intersection of all α-complete prime ideals including it.*

Proof. It is obvious, by 4.1 and 0.6, that (i) implies (ii), and we see also at once that (ii) implies (iii).

Assume that (iii) holds. If α is regular, then $D_{\alpha,\beta} \subseteq W_{\alpha,\beta}$ by 0.8, and therefore 4.16 (v) holds; hence, by 4.16, we have $\alpha R \beta$, i.e., (i) holds.

If α is singular and $\alpha \leqslant \beta$, then $\alpha \in \boldsymbol{C}_0$ by 3.7, and thus by 4.11 we have $\alpha \boldsymbol{R} \beta$. If α is singular and $\beta < \alpha < 2^\beta$, then $\alpha \boldsymbol{R} \beta$ follows by 4.8.

It remains to consider the case in which $\alpha \in \boldsymbol{SN}$ and $2^\beta \leqslant \alpha$. In this case $\mathfrak{B}$ is β^+-distributive, and it follows by a familiar argument that $\mathfrak{B}$ is an atomistic Boolean algebra with at most 2^β atoms (cf. [4], proof of Theorem 4.5, or [36], p. 82). Let I be a proper α-complete ideal in $\mathfrak{B}$ and let $x \in B \sim I$. By 0.4, I is $(2^\beta)^+$-complete, and hence there is an atom $a \leqslant x$ which does not belong to I. The principal ideal J of $\mathfrak{B}$ generated by $\bar{a}$ is then an α-complete prime ideal such that $I \subseteq J$ and $x \notin J$. But we have thus contradicted (iii), and with this contradiction our proof is complete.

By comparing Theorems 4.16 and 4.18 we see that 4.16 remains valid if $\boldsymbol{W}_{\alpha,\beta}$ is replaced by $\boldsymbol{D}_{\alpha,\beta}$ in conditions (iv) and (v). The problem naturally arises whether 4.16 in its entirety remains valid when $\boldsymbol{W}_{\alpha,\beta}$ is replaced everywhere by $\boldsymbol{D}_{\alpha,\beta}$. Let us denote by (ii*)-(vi*) the statements which are obtained form 4.16 (ii)-(vi), respectively, by replacing $\boldsymbol{W}_{\alpha,\beta}$ by $\boldsymbol{D}_{\alpha,\beta}$. In case α is inaccessible, we have $\boldsymbol{W}_{\alpha,\beta} = \boldsymbol{D}_{\alpha,\beta}$ by 0.10, and hence all the conditions (ii*)-(vi*) are equivalent to $\alpha \boldsymbol{R} \beta$. We shall show later that, as a consequence of Theorem 4.31, (iii*) holds whenever α is a regular accessible cardinal and $\alpha \leqslant \beta$. On the other hand, it can be seen from the proof of Theorem 4.5 in [4] that, if α is a singular strong limit cardinal and $\alpha = \beta$, then (ii*) and (vi*) fail, even though $\alpha \boldsymbol{R} \beta$ holds. Aside from the special cases we have just mentioned, the problem of which pairs α, β of cardinals satisfy the conditions (ii*), (iii*), and (vi*) is, as far as we know, still open.

DEFINITION 4.19. *Let $\mathfrak{B} = \langle B, +, \cdot, ^- \rangle$ be a Boolean algebra and I a principal ideal in $\mathfrak{B}$. By the Boolean algebra induced in $\mathfrak{B}$ by I, denoted by $\mathfrak{B}_I$, we mean the algebra $\langle I, +', \cdot', ^{-'} \rangle$ such that, for all $x, y \in I$, we have*

$$x +' y = x + y, \quad x \cdot' y = x \cdot y, \quad \text{and} \quad \bar{x}' = \bar{x} \cdot \sum I.$$

THEOREM 4.20. *The following four conditions are equivalent:*

(i) $\alpha \boldsymbol{R} \beta$;

(ii) *there exists $\mathfrak{B} \in \boldsymbol{W}_{\alpha,\beta}$ which has a principal ideal I such that $|I| > 1$ and $\mathfrak{B}_I$ is not isomorphic to any α-subalgebra of $\mathfrak{B}$;*

(iii) *there exists $\mathfrak{B} \in \boldsymbol{W}_{\alpha,\beta}$ which has a proper principal ideal J such that $\mathfrak{B}/J$ is not isomorphic to any α-subalgebra of $\mathfrak{B}$;*

(iv) *there exists $\mathfrak{B} \in \boldsymbol{W}_{\alpha,\beta}$ which has a direct factor of power >1 that is not isomorphic to any α-subalgebra of $\mathfrak{B}$.*

Proof. The equivalence of (ii), (iii), and (iv) follows from two well known facts about Boolean algebras: if I and J are principal ideals in $\mathfrak{B}$ such that $\sum J = \overline{\sum I}$, then $\mathfrak{B}/J$ is isomorphic to $\mathfrak{B}_I$; $\mathfrak{C}$ is a direct factor of $\mathfrak{B}$ iff there is a principal ideal I in $\mathfrak{B}$ such that $\mathfrak{B}_I$ is isomorphic to $\mathfrak{C}$.

We wish to show that (i) is equivalent to (ii). In view of 4.16, it suffices to prove that 4.16 (ii) implies 4.20 (iv), and that 4.20 (iii) implies 4.16 (vi).

Assume 4.16 (ii), and let $\mathfrak{C}$ be the direct product

$$\mathfrak{C} = \mathfrak{B} \times \mathfrak{S}(\{0\}).$$

It is easily seen that $\mathfrak{C} \in \boldsymbol{W}_{\alpha,\beta}$. By 4.16 (ii) we have $|B| > 1$. Moreover, there is no isomorphism on the direct factor $\mathfrak{B}$ of $\mathfrak{C}$ onto an α-subalgebra of $\mathfrak{C}$, for, if f were such an isomorphism, then the set

$$f^{-1}(B \times \{0\})$$

would be an α-complete prime ideal in $\mathfrak{B}$, contradicting 4.16 (ii). This verifies 4.20 (iv).

Now let us assume 4.20 (iii). We shall show that $\mathfrak{B}$ is not α-representable. Indeed, suppose that $\mathfrak{B}$ were α-representable. Then, by 0.5, J could be extended to an α-complete prime ideal K in $\mathfrak{B}$. But then we could define an isomorphism g on $\mathfrak{B}/J$ onto an α-subalgebra of $\mathfrak{B}$ by putting, for each $x \in B$,

$$g(x/J) = \begin{cases} x \cdot \overline{\sum J} & \text{if} \quad x \in K, \\ x + \sum J & \text{if} \quad x \notin K. \end{cases}$$

This contradiction completes our proof.

We shall now generalize some portions of Theorem 4.16 by considering δ-saturated ideals in place of prime ideals.

LEMMA 4.21. *Suppose that $\delta \leqslant \omega$, $\mathfrak{B}$ is an α-complete Boolean algebra, and I is an α-complete and δ-saturated proper ideal in $\mathfrak{B}$. Then there is an α-complete prime ideal J in $\mathfrak{B}$ which includes I; if, moreover, I is non-principal, then J can also be assumed to be non-principal.*

Proof. Since I is proper, there must be an element $x_0 \in B \sim I$ with the property that, if $y \in B$ and $y \leqslant x_0$, then either $y \in I$ or $x_0 - y \in I$; otherwise we would have a countably infinite disjointed subset of $B \sim I$. Set $J = \{y\colon y \in B \text{ and } x_0 \cdot y \in I\}$. Obviously $I \subseteq J$ and $x_0 \notin J$. If $x \in B$ and $x \leqslant y \in J$, then $x_0 \cdot x \leqslant x_0 \cdot y \in I$, so $x \in J$. If $y \in B \sim J$, then $x_0 \cdot y \notin I$, so $x_0 - y \in I$ and thus $\bar{y} \in J$. Finally, if $Y \subseteq J$ and $|Y| < \alpha$, we put $X = \{x_0 \cdot y\colon y \in Y\}$ and find that $X \subseteq I$ and $|X| < \alpha$. By α-completeness and 0.1, $\sum X = x_0 \cdot \sum Y$, and since $\sum X \in I$ we have $\sum Y \in J$. Therefore J is an α-complete prime ideal in $\mathfrak{B}$.

If I is non-principal, we can first extend I to a proper α-complete ideal I' containing all atoms of $\mathfrak{B}$ (cf. the remarks following 0.2), and then embed I' in an α-complete prime ideal J by the method indicated above. Since the prime ideal J thus obtained contains all atoms, it cannot be principal.

THEOREM 4.22. *Suppose that $2 \leqslant \delta \leqslant \omega$. Then Theorems* 4.16 *and* 4.18 *remain valid if the phrase "α-complete prime ideal(s)" is replaced in them everywhere by "α-complete and δ-saturated proper ideal(s)".*

Proof. This is an easy consequence of 4.16, 4.18, and 4.21, in view of the fact that for $\delta \geqslant 2$ every prime ideal is a δ-saturated proper ideal.

Notice that Theorems 4.16 and 4.18 are special cases of Theorem 4.22 obtained by letting $\delta = 2$.

LEMMA 4.23. *Suppose that $2^{\underset{\smile}{\delta}} < \alpha$, $\mathfrak{B}$ is an α-distributive Boolean algebra, and I is an α-complete and δ-saturated proper ideal in $\mathfrak{B}$. Then there is an α-complete prime ideal J in $\mathfrak{B}$ which includes I; if, moreover, I is non-principal, then J can also be assumed to be non-principal.*

Proof. We shall show that there is an element $x_0 \in B \sim I$ with the property that, if $y \in B$ and $y \leqslant x_0$, then either $y \in I$ or $x_0 - y \in I$. Suppose there is no such element x_0. Then by the axiom of choice we can correlate with each element $x \in B$ an element $F(x) \in B$ such that $F(x) \leqslant x$ and, if $x \notin I$, then $F(x)$, $x - F(x) \notin I$. Assuming $\zeta \leqslant \xi \leqslant \delta$ and $\varphi \in {}^{\xi}2$, let $\varphi^{(\zeta)}$ denote the restriction of φ to ζ, i.e., the function $\varphi \cap (\zeta \times 2)$. By transfinite recursion on ordinals $\xi \leqslant \delta$ we correlate with each sequence $\varphi \in {}^{\xi}2$ a certain set x_φ by stipulating as follows:

$$x_\varphi = \sum B \quad \text{for} \quad \varphi \in {}^{0}2$$

(${}^{0}2$ has only one member, the empty sequence);

if $\varphi \in {}^{(\xi+1)}2$, then $x_\varphi = F(x_{\varphi(\xi)})$ or $x_\varphi = x_{\varphi(\xi)} - F(x_{\varphi(\xi)})$, according as $\varphi_\xi = 0$ or $\varphi_\xi = 1$;

if ξ is a limit ordinal then $x_\varphi = \prod_{\zeta<\xi} x_{\varphi(\xi)}$.

It follows by induction on ξ that if $\zeta \leqslant \xi \leqslant \delta$ and $\varphi \in {}^{\xi}2$, then $x_\varphi \leqslant x_{\varphi(\zeta)} \in B$. We wish to show that for each ordinal $\xi \leqslant \delta$ the following formula holds:

$$(1) \qquad \sum B = \sum\nolimits_{\varphi \in {}^{\xi}2} x_\varphi .$$

It is clear that (1) holds when $\xi = 0$.

Since ${}^{(\xi+1)}2 = \{\varphi \cup \{\langle \xi, 0\rangle\} \colon \varphi \in {}^{\xi}2\} \cup \{\varphi \cup \{\langle \xi, 1\rangle\} \colon \varphi \in {}^{\xi}2\}$, we have

$$\sum\nolimits_{\varphi \in {}^{\xi}2} x_\varphi = \sum\nolimits_{\varphi \in {}^{\xi}2} (x_{\varphi \cup \{\langle \xi, 0\rangle\}} + x_{\varphi \cup \{\langle \xi, 1\rangle\}}) = \sum\nolimits_{\psi \in {}^{(\xi+1)}2} x_\psi .$$

Thus if (1) holds for ξ, it also holds for $\xi+1$. Finally suppose that $0 < \xi \leqslant \delta$, ξ is a limit ordinal, and (1) holds for every ordinal $\zeta < \xi$. Then

$$\sum B = \prod\nolimits_{\zeta<\xi} \sum\nolimits_{\varphi \in {}^{\zeta}2} x_\varphi .$$

By the α-distributivity of $\mathfrak{B}$ and the fact that $2^{\underset{\smile}{\delta}} < \alpha$, we have

$$(2) \qquad \sum B = \sum\nolimits_{\sigma \in K} \prod\nolimits_{\zeta<\xi} x_{\sigma(\zeta)} , \quad \text{where} \quad K = \mathsf{P}_{\zeta<\xi}\, {}^{\zeta}2 .$$

Among the summands of (2) appear all products of the form $\prod_{\zeta<\xi} x_{\varphi(\zeta)}$ where $\varphi \in {}^{\xi}2$; these terms arise when $\sigma_\zeta = \varphi^{(\zeta)}$ for each $\zeta < \xi$. We shall show, however, that all other summands of (2) are equal to zero, and therefore may be omitted from the sum. To this end, suppose that $\eta \leqslant \zeta < \xi$, $\sigma \in K$, $\varphi = \sigma_\zeta$, $\psi = \sigma_\eta$, and $\varphi^{(\eta)} \neq \psi$. Let ϱ be the smallest ordinal for which $\psi_\varrho \neq \varphi_\varrho$; accordingly $\psi^{(\varrho)} = \varphi^{(\varrho)}$. Furthermore $\varphi_\varrho = 0$ and $\psi_\varrho = 1$, or *vice versa*. In the first case, since $\varrho+1 \leqslant \eta \leqslant \zeta$,

$$x_\varphi \cdot x_\psi \leqslant x_{\varphi(\varrho+1)} \cdot x_{\psi(\varrho+1)} = \big(x_{\varphi(\varrho)} \cdot F(x_{\varphi(\varrho)})\big) \cdot \big(x_{\varphi(\varrho)} - F(x_{\varphi(\varrho)})\big) = \prod B ,$$

and the second case is similar. Therefore

$$\prod\nolimits_{\zeta<\xi} x_{\sigma_\zeta} = 0 .$$

The formula (2) thus reduces to

$$\sum B = \sum\nolimits_{\varphi \in {}^{\xi}2} \prod\nolimits_{\zeta<\xi} x_{\varphi(\zeta)} ,$$

and hence ξ satisfies (1). Consequently (1) holds for every $\xi \leqslant \delta$.

Now put

$$X = I \cap \{x_\varphi \colon \varphi \in \textstyle\bigcup_{\xi<\delta} {}^{\xi}2\} .$$

Then $|X| \leqslant 2^{\underset{\smile}{\delta}} < \alpha$, and it follows that $\sum X \in I$. For each $\varphi \in {}^{\delta}2$ there exists $\xi < \delta$ such that $x_{\varphi(\xi)} \in I$, for otherwise

$$\{x_{\varphi(\xi)} - x_{\varphi(\xi+1)} \colon \xi < \delta\}$$

would be a disjointed subset of $B \sim I$ of power δ, and I is δ-saturated. Therefore for each $\varphi \in {}^{\delta}2$ there is an element $y \in X$ such that $x_\varphi \leqslant y$. It follows that

$$\sum B = \sum\nolimits_{\varphi \in {}^{\delta}2} x_\varphi = \sum X ,$$

and hence $\sum B \in I$. But this is impossible because I is a proper ideal. We therefore conclude that there is an element $x_0 \in B \sim I$ with the desired property that, if $y \in B$ and $y \leqslant x_0$, then either $y \in I$ or $x_0 - y \in I$. To complete the proof, we now argue exactly as in the proof of Lemma 4.21.

It may be noticed that, in case $\delta \leqslant \omega$, 4.21 is stronger than 4.23, because in 4.21 $\mathfrak{B}$ is only assumed to be α-complete, while in 4.23, $\mathfrak{B}$ is assumed to be α-distributive.

Lemma 4.23 (as well as 4.21) is essentially known from the literature. With some restrictions concerning either the cardinals or the Boolean algebras involved, the result was stated and established first in [43] as Theorem 4.12 and later in [38] as Theorem 3.1. However, just because of those restrictions, it seemed to us desirable to outline here a proof of the lemma in its whole generality.

THEOREM 4.24. *Suppose that $\delta \geqslant 2$ and $2^{\delta} < \alpha$. Then the following condition is necessary and sufficient for $\alpha \boldsymbol{R} \beta$:*

(i) *there exists an α-complete field of sets, with at most β generators, which has an α-complete proper ideal that is not included in any α-complete and δ-saturated proper ideal.*

Proof. This is an easy consequence of 0.6, 4.23, and 4.1.

THEOREM 4.25. *Suppose that $\delta \geqslant 2$ and $2^{\delta} < \alpha$. Then Theorem 4.18 remains valid if the phrase "α-complete prime ideal(s)" is replaced in it everywhere by the phrase "α-complete and δ-saturated proper ideal(s)".*

Proof. By 4.18 and 4.23.

The next two theorems, 4.27 and 4.29, are proved in [28] and will be stated below without proof. They essentially provide characterizations of the relation $\boldsymbol{R}$.

Although the first of these results is formulated in topological terms and the second in terms of abstract set theory, the two results are closely related to each other.

DEFINITION 4.26. *A topological space T is said to be α-complete if the intersection of any set of fewer than α open sets in T is itself open in T. T is said to be α-compact if every open covering of T includes an open covering of power $< \alpha$. By the α-product space of the sequence T_{ζ}, $\zeta < \xi$, of topological spaces we mean the least α-complete topology T on the cartesian product set such that, for all $\zeta < \xi$, the projection function on T onto T_{ζ} is continuous.*

THEOREM 4.27. *The following condition is necessary and sufficient for $\alpha \boldsymbol{R} \beta$, unless $\alpha \in \boldsymbol{SN}$ and $\alpha = 2^{\beta}$:*

(i) *the α-product space of a β-termed sequence of two-point discrete topological spaces is not α-compact.*

Notice that, if the generalized continuum hypothesis holds, then 2^{β} is never singular, so that condition 4.27 (i) is always equivalent to $\alpha \boldsymbol{R} \beta$. In case $\alpha = \omega$ the notions of α-product and α-compactness coincide with the ordinary notions of product and compactness, and it is well known that 4.27 (i) fails.

DEFINITION 4.28. *A binary relation R is said to be separable over a set c if $R \subseteq S(c) \times S(c)$ and there exists a set $x \subseteq c$ such that*

$$R \cap \big(S(x) \times S(c \sim x)\big) = 0 .$$

Two sets of sets A and B are said to be separable over a set c if $A, B \subseteq S(c)$ and there exists a set $x \subseteq c$ such that

$$A \cap S(x) = 0 = B \cap S(c \sim x) .$$

THEOREM 4.29. *Each of the following two conditions is equivalent to the condition 4.27 (i) and moreover, unless $\alpha \in \boldsymbol{SN}$ and $\alpha = 2^{\beta}$, it is also equivalent to $\alpha \boldsymbol{R} \beta$:*

(i) *there is a relation $R \subseteq S_{\alpha}(\beta) \times S_{\alpha}(\beta)$ such that every relation $r \in S_{\alpha}(R)$ is separable over β, but R is not separable over β;*

(ii) *there are two sets $A, B \subseteq S_{\alpha}(\beta)$ such that any two sets $a \in S_{\alpha}(A)$ and $b \in S_{\alpha}(B)$ are separable over β, but A and B are not separable over β.*

The negations of the conditions 4.29 (i) and 4.29 (ii) are called respectively the *first* and the *second separation principle*. The two conditions, and especially 4.29 (ii), are distinguished by their simple formulations involving only elementary notions of general set theory.

From each of the theorems 4.12, 4.13, 4.14, 4.16, 4.18, 4.20, 4.22, 4.24, 4.25, 4.27, and 4.29 we obtain necessary and sufficient conditions for $\alpha \in \boldsymbol{C}_0$ by everywhere replacing β by α. We shall state below some further necessary and sufficient conditions for $\alpha \in \boldsymbol{C}_0$ which have not yet been generalized to conditions for $\alpha \boldsymbol{R} \beta$.

DEFINITION 4.30. *A ramification system is an ordered pair $\langle X, \leqslant \rangle$ such that the set X is partially ordered by $\leqslant$ and, for each $x \in X$, the set $P(x) = \{y\colon\, y \in X,\ y \leqslant x, \text{ and } y \neq x\}$ is well ordered by $\leqslant$; the type of the well ordering of $P(x)$ is called the order of x, and the least ordinal greater than the orders of all elements of X is called the order of the ramification system.*

THEOREM 4.31. *The following four conditions are equivalent:*

(i) $\alpha \in \boldsymbol{C}_0$;

(ii) *either $\alpha \in \boldsymbol{AC}$ or there is a ramification system of order α such that, for every $\xi < \alpha$, the set of elements of order ξ has power $< \alpha$, and every well-ordered subset has power $< \alpha$;*

(iii) *there is an α-complete field B of subsets of α which includes $S_{\omega}(\alpha)$, has at most α generators, and in which every α-complete prime ideal is principal;*

(iv) *there is an α-complete field of sets with at most α generators, whose power is at least α, and in which every α-complete prime ideal is principal.*

Proof. For the proof that (i) implies (ii), we refer to [4], Theorem 4.3. The implication (ii) $\Rightarrow$ (i) was first established in [28]. However, we shall present here a different argument. We shall first outline a proof that (ii) implies (iii) by a method which is in part a modification of the proof of a weaker result in [4], Theorem 4.2; we shall complete the argument by showing that (iii) implies (iv), and (iv) implies (i).

Suppose that (ii) holds. Assume first that α is singular. Then the field $S(\alpha)$ is α-generated by the set $S_{\alpha}(\alpha)$ of power α, and by 1.2 every α-complete prime ideal in $S(\alpha)$ is principal. Hence (iii) holds.

If α is a regular accessible cardinal, then there exists $\beta < \alpha$ such that $\alpha \leqslant 2^\beta$. For all $\xi, \eta < \beta$ let

$$y(\xi, \eta) = \{\varphi: \varphi \in {}^\beta\beta \text{ and } \varphi(\xi) = \eta\}.$$

We choose a subset $z \subseteq {}^\beta\beta$ such that $|z| = \alpha$, and for all $\xi, \eta < \beta$ we let

$$x(\xi, \eta) = z \cap y(\xi, \eta).$$

Let B be the α-complete field of subsets of z which is α-generated by the set

$$\{x(\xi, \eta): \xi, \eta < \beta\}$$

of power β. Clearly we have

$$S_\omega(z) \subseteq B.$$

Moreover, B is a β^+-complete field of sets such that conditions 3.6 (i) and 3.6 (ii) are satisfied when $\gamma = \beta$. Therefore, by 3.6, every β^+-complete prime ideal, and hence also every α-complete prime ideal, in B is principal. We have shown that (iii) holds whenever $\alpha \in \boldsymbol{AC}$.

Now assume that α is inaccessible. Then there is a ramification system $\langle\alpha, \leqslant\rangle$ such that for every $\xi < \alpha$ the set of elements of α of order ξ has power $< \alpha$, and every subset of α well ordered by $\leqslant$ has power $< \alpha$. Let B be the α-complete field of subsets of α which is α-generated by the set

$$S_\omega(\alpha) \cup \{F(\xi): \xi < \alpha\}, \quad \text{where} \quad F(\xi) = \{\zeta < \alpha: \xi \leqslant \zeta \text{ and } \xi \neq \zeta\}.$$

We may then argue exactly as in the proof of Theorem 4.2 of [4], but with B in place of $S(\alpha)$, to show that any α-complete prime ideal in B is principal, and thus verify (iii).

It is obvious that (iii) implies (iv).

It remains to prove that (iv) implies (i), and this is very easy. Suppose that B is an α-complete field of sets which is α-generated by a set of power α and in which every α-complete prime ideal is principal. If α is accessible, then $\alpha \in \boldsymbol{C}_0$ by Theorem 3.7. Assume that α is inaccessible. Then $\bigcup B$ cannot be the union of fewer than α atoms of B, for otherwise B would have power $< \alpha$. Let I be the set of all elements of B which are unions of fewer than α atoms of B. Then I is a proper α-complete ideal in B. Any prime ideal which includes I is non-principal, for it must contain each atom of B. Therefore I is not included in any α-complete prime ideal. Thus $\alpha \in C_0$ and (i) holds. The proof is complete.

As indicated in the proof above, the implications 4.31 (i) $\Rightarrow$ 4.31 (ii) and 4.31 (ii) $\Rightarrow$ 4.31 (i) were established in [4] and [28], respectively. The fact that 4.31 (iii) and 4.31 (iv) are necessary and sufficient conditions for $\alpha \in \boldsymbol{C}_0$ was first arrived at through metamathematical considerations, and independently of 4.31 (ii).

In [4] a cardinal is said to have the property $\boldsymbol{Q}$ if it satisfies the condition 4.31 (ii) with the phrase "either α is accessible or" deleted. The property $\boldsymbol{Q}$ is discussed there for accessible as well as for inaccessible cardinals; we refer to [4], pp. 76-77, for an account of known results (of Specker and others) as well as open problems concerning the question of which accessible cardinals have the property $\boldsymbol{Q}$.

In the discussion following Theorem 4.18 it was mentioned that, if α is a regular accessible cardinal and $\alpha \leqslant \beta$, then condition 4.16 (iii*) holds. We can now give a proof of that fact. It is obviously sufficient to establish 4.16 (iii*) in case $\alpha = \beta$. Since $\alpha \in \boldsymbol{AC}$, we have $\alpha \in \boldsymbol{C}_0$ by 3.7. Hence 4.31 (iii) holds. Since $\mathfrak{B}$ is α-representable and has at most α generators, we conclude by 0.6 that $B \in \boldsymbol{D}_{\alpha,\alpha}$. The set $S_\alpha(\alpha)$ is a non-principal α-complete ideal in $\mathfrak{B}$ because α is regular. Thus $S_\alpha(\alpha)$ cannot be prime in $\mathfrak{B}$, and consequently $\mathfrak{B}$ cannot be α-generated by its atoms. Therefore 4.16 (iii*) follows from 4.31 (iii).

In connection with 4.31 (iii) we mention another relation between two cardinals whose formulation is similar to that of $\alpha\boldsymbol{R}\beta$.

We shall say that $\alpha\boldsymbol{R}'\beta$ holds if there is an α-complete field of sets, with at most β generators, whose power is at least β and in which every α-complete prime ideal is principal.

It follows from 4.31 that $\alpha\boldsymbol{R}\alpha$ is equivalent to $\alpha\boldsymbol{R}'\alpha$. In case $\alpha > \beta$ we always have $\alpha\boldsymbol{R}'\beta$. On the other hand, if $\alpha \leqslant \beta$, then it is easily seen that $\alpha\boldsymbol{R}'\beta$ implies $\alpha\boldsymbol{R}\beta$. Theorem 4.4 states that $\alpha\boldsymbol{R}\beta$ and $\beta \leqslant \gamma$ imply $\alpha\boldsymbol{R}\gamma$; the relation $\boldsymbol{R}'$, however, exhibits to a certain extent the opposite behavior, and in fact one can show without difficulty that $\alpha\boldsymbol{R}'\beta$ and $\beta \geqslant 2^\gamma$ imply $\alpha\boldsymbol{R}'\gamma$. It follows from Theorem 2.1 that, for any cardinals α and γ, we have $\alpha\boldsymbol{R}'2^\gamma$ iff $[\alpha, \gamma] \subseteq \boldsymbol{C}_1$.

THEOREM 4.32. *If $\alpha = \alpha^{\underline{\alpha}}$, then the following condition is necessary and sufficient for $\alpha \in \boldsymbol{C}_0$:*

(i) *for some α-termed sequence of discrete topological spaces, each with $< \alpha$ points, there exists a set of α points in the product space which has no accumulation point.*

Proof. Assume $\alpha \in \boldsymbol{C}_0$; then, by 4.31, condition 4.31 (iii) holds for some field B. Since $\alpha = \alpha^{\underline{\alpha}}$, B has power α. Consider the set

$$A = \Big\{g: g \in \bigcup_{\delta<\alpha} {}^\alpha\delta \text{ and, for all } \eta < \alpha,\ g^{-1}(\{\eta\}) \in B\Big\}.$$

Using the equation $\alpha = \alpha^{\underline{\alpha}}$, it is easily seen that $|A| = \alpha$, and hence we may choose an α-termed sequence f whose range is A. For each $\zeta < \alpha$, let T_ζ be the discrete topological space whose set of points is the range of the function f_ζ; thus each of the spaces T_ζ has fewer than α points.

Let T be the product space of the sequence T_ζ, $\zeta < \alpha$, and for every $\eta < \alpha$ let t_η be the point of T defined by

$$t_\eta(\zeta) = f_\zeta(\eta) \quad \text{for all} \quad \zeta < \alpha .$$

We may now argue exactly in the proof of 2.30 (except that we replace ξ and β everywhere by α, and replace $S(\beta)$ by B), to show that the set

$$U = \{t_\eta \colon \eta < \alpha\}$$

of α points of T has no accumulation point in T.

For the converse, suppose that $\alpha \notin \boldsymbol{C}_0$, let T be the product space of an arbitrary α-termed sequence T_ζ, $\zeta < \alpha$, of discrete topological spaces, each with $< \alpha$ points, and let U be any set of α points of T. We wish to show that U has an accumulation point in T.

For each $\zeta < \alpha$ and each point i in T_ζ let

$$x(\zeta, i) = \{t \in U \colon t(\zeta) = i\} .$$

Let B be the α-complete field of subsets of U which is α-generated by the set

$$\{x(\zeta, i) \colon \zeta < \alpha,\ i \in T_\zeta\}$$

of power $\leqslant \alpha$. Since $|U| = \alpha$ and α is regular (because $\alpha = \alpha^{\underline{\alpha}}$, or alternatively because $\alpha \notin C_0$), the set $I = B \cap S_\alpha(U)$ is a proper α-complete ideal in B. Hence, by the hypothesis $\alpha \in \boldsymbol{C}_0$, I can be extended to an α-complete prime ideal J in B. For each $\zeta < \alpha$ we have

$$\textstyle\bigcup_{i \in T_\zeta} x(\zeta, i) = U ,$$

and, because each T_ζ has fewer than α points, we may choose for each $\zeta < \alpha$ a point $u(\zeta)$ of T_ζ such that

$$x\big(\zeta, u(\zeta)\big) \in B \sim I .$$

It follows that the point u of T determined by the points $u(\zeta)$, $\zeta < \alpha$, has the property that, for every $y \in S_\omega(\alpha)$,

$$\textstyle\bigcap_{\zeta \in y} x\big(\zeta, u(\zeta)\big) \in B \sim I .$$

Since each member of $B \sim I$ is an infinite subset of U, we conclude that u is an accumulation point of U, and our proof is complete.

The hypothesis $\alpha = \alpha^{\underline{\alpha}}$ of the foregoing theorem holds whenever $\alpha \in \boldsymbol{AC}$, and under the generalized continuum hypothesis it holds whenever $\alpha \notin \boldsymbol{SN}$. Both the formulation and the proof of 4.32 exhibit a close similarity to 2.30.

DEFINITION 4.33. *By a graph we mean any set of unordered pairs $\{x, y\}$ with $x \neq y$. By a complete graph on a set X we mean the set of all two-element subsets of X.*

THEOREM 4.34. *The following three conditions are equivalent:*

(i) $\alpha \in \boldsymbol{C}_0$;

(ii) *there is a relation $\preceq$ which simply orders α in such a way that every subset of α which is well ordered by $\preceq$ or by the converse relation $\succeq$ has power $< \alpha$;*

(iii) *the complete graph G on α can be divided into two disjoint graphs whose union is G and neither of which includes the complete graph on any set of power α.*

In the paper [4] the property of a cardinal which consists in satisfying (ii), or (iii), is denoted by $\boldsymbol{P}_1$, or $\boldsymbol{P}_2$, respectively. In that paper it is shown that every accessible cardinal satisfies (ii) (Theorem 1.1), that (ii) implies (iii) (Theorem 1.2), and that (iii) implies 4.31 (ii) (Theorem 4.1). Combining this with Theorem 4.31, or with the results of [28], the implication (iii)$\Rightarrow$(i) follows. The third implication, (i)$\Rightarrow$(ii), was recently announced for the case $\alpha \notin \boldsymbol{AC}$ in [10], and the whole Theorem 4.34 was thus established.

In the next theorem (which is the last theorem in this section) we shall concern ourselves with the hypothesis $\boldsymbol{C} = \boldsymbol{C}_0$. We wish to do for this hypothesis what has been done in Theorems 2.50-2.52 for $\boldsymbol{C} = \boldsymbol{C}_1$, namely to sum up its main implications in general set theory and related domains.

THEOREM 4.35. *The hypothesis $\boldsymbol{C} = \boldsymbol{C}_0$ implies (and is implied by) each of the following statements:*

(i) *For every $\alpha > \omega$ there is an α-complete field of subsets of α which includes $S_\omega(\alpha)$, is α-generated by a set of power α, and in which every α-complete prime ideal (more generally, every α-complete and δ-saturated ideal with $2^{\underline{\delta}} < \alpha$) is principal.*

(ii) *If $\omega < \alpha \leqslant \beta$, then there is an α-complete field of sets which is α-generated by a set of power β and in which some α-complete proper ideal cannot be extended to any α-complete prime ideal (more generally, to any α-complete and δ-saturated proper ideal with $2^{\underline{\delta}} < \alpha$).*

(iii) *For every $\alpha > \omega$ there is a weakly α-representable Boolean algebra, with at least two elements and at most α generators, which has no α-complete prime ideals (more generally, no α-complete and δ-saturated proper ideals with $\delta \leqslant \omega$) and which therefore is not strongly α-representable.*

(iv) *For every $\alpha > \omega$ there is a weakly α-representable Boolean algebra $\mathfrak{B}$, with at most α generators, which has a proper principal ideal I such that $\mathfrak{B}/I$ is not isomorphic to any α-subalgebra of $\mathfrak{B}$.*

(v) *Every non-denumerable set a can be simply ordered by some relation $\preceq$ in such a way that every subset of a which is well ordered by $\preceq$ or by the converse relation $\succeq$ is of power smaller than a.*

(vi) *The complete graph G on any non-denumerable set a can be divided into two disjoint graphs whose union is G and neither of which includes the complete graph on any set of the same power as a.*

(vii) *For every inaccessible $\alpha > \omega$ there is a ramification system of order α such that, for every $\xi < \alpha$, the set of elements of order ξ has power $< \alpha$ and every well-ordered subset has power $< \alpha$.*

(viii) *If $\omega < \alpha \leqslant 2^\beta$, then there are sets $A, B \subseteq S_\alpha(\beta)$ such that any two sets $a \in S_\alpha(A)$ and $b \in S_\alpha(B)$ are separable over β, but A and B are not separable over β.*

(ix) *If $\omega < \alpha \leqslant 2^\beta$, then the α-product space of a β-termed sequence of discrete topological spaces, each with two points, is not α-compact.*

(x) *If $\omega < \alpha = \alpha^{\underline{\alpha}}$, then there is an α-termed sequence of discrete topological spaces, each with less than α points, such that not every set of α points in the product space of this sequence has an accumulation point.*

This theorem is essentially a simple corollary from other results of this section. In deriving 4.35 (viii), (ix) we make use, of course, of 4.27 and 4.29; however, in order to include the case $\alpha = 2^\beta \in \boldsymbol{SN}$, we have to apply in addition a result from [28].

From 4.14 (iii) we see that the conclusion of 4.35 (ii) can be strenghened when restricted to those cardinals β for which $\beta = \beta^{\underline{\omega}}$.

Assuming the generalized continuum hypothesis, various portions of Theorem 4.35 can be improved and simplified. Thus we can replace $2^{\underline{\delta}} < \alpha$ by $\delta < \alpha$ in (i), (ii). Moreover, applying certain results from the literature (cf. [4], pp. 76-77), we can extend (vii) from inaccessible cardinals $\alpha > \omega$ to all cardinals $\alpha > \omega$ which are not of the form $\alpha = \beta^+$ where β is singular. Finally, (x) extends to all regular cardinals $\alpha > \omega$.

It follows from 1.2 (i) and 3.3 that the hypothesis $\boldsymbol{C} = \boldsymbol{C}_0$ implies $\boldsymbol{C} = \boldsymbol{C}_1$ (while the problem is open whether the implication in the opposite direction also holds). Hence $\boldsymbol{C} = \boldsymbol{C}_0$ implies all the statements equivalent to $\boldsymbol{C} = \boldsymbol{C}_1$, thus in particular the statements 2.50 (i)-(xii) and also, under additional assumptions, 2.51 (i)-(iii) and 2.52 (i), (ii). As further consequences of both $\boldsymbol{C} = \boldsymbol{C}_0$ and $\boldsymbol{C} = \boldsymbol{C}_1$ we may mention the statements 5.12 (i)-(v) which will be formulated in the following section.

§ 5. The class $\boldsymbol{C}_2$. In this section we shall study a class of cardinals which arises naturally in connection with the relation $\boldsymbol{R}$, and which is at least as large as, and may prove to be even larger than, the class $\boldsymbol{C}_1$.

DEFINITION 5.1. *We shall denote by $\boldsymbol{C}_2$ the class of all cardinals α such that $\alpha \boldsymbol{R} \beta$ for some β (i.e. such that there exists an α-complete field of sets in which some α-complete proper ideal cannot be extended to an α-complete prime ideal).*

THEOREM 5.2. $\boldsymbol{C}_0 \subseteq \boldsymbol{C}_2$.

Proof. By 4.2, 5.1.

THEOREM 5.3. $\boldsymbol{C}_2 \subseteq \boldsymbol{C}$.

Proof. By 4.3.

In particular, it follows from 5.3 that $\omega \notin \boldsymbol{C}_2$.

LEMMA 5.4. *If $\alpha \in \boldsymbol{C}_1$, then $\alpha \boldsymbol{R} 2^\alpha$.*

Proof. Suppose that $\alpha \in \boldsymbol{C}_1$. If α is singular, then $\alpha \in \boldsymbol{C}_0$ by 3.7, so $\alpha \boldsymbol{R} \alpha$ by 4.2, and $\alpha \boldsymbol{R} 2^\alpha$ by 4.4. Suppose that α is regular. Then the set $I = S_\alpha(\alpha)$ is an α-complete proper ideal in $S(\alpha)$. Since any prime ideal which includes I is non-principal and $\alpha \in \boldsymbol{C}_1$, I is not included in any α-complete prime ideal in $S(\alpha)$. Moreover, $S(\alpha)$ is an α-complete field of sets with at most 2^α generators. Hence $\alpha \boldsymbol{R} 2^\alpha$.

THEOREM 5.5. $\boldsymbol{C}_1 \subseteq \boldsymbol{C}_2$.

Proof. By 5.1, 5.4.

Theorem 5.5 above is also proved in [4] as Theorem 3.2.

THEOREM 5.6. *A necessary and sufficient condition for $\alpha \in \boldsymbol{C}_2$ is:*

(i) *there is a complete field of sets in which some α-complete proper ideal cannot be extended to an α-complete prime ideal.*

Proof. It is obvious that (i) implies $\alpha \in \boldsymbol{C}_2$.

Assuming $\alpha \in \boldsymbol{C}_2$, let B be an α-complete field of subsets of some cardinal β such that some α-complete proper ideal I in B cannot be extended to an α-complete prime ideal. Let

$$J = \{x \in S(\beta) \colon \text{for some } y \in I,\ x \subseteq y\}\,.$$

Then J is an α-complete proper ideal in the complete atomistic field of sets $S(\beta)$. If J could be extended to an α-complete prime ideal J' in $S(\beta)$, then the set $J' \cap B$ would be an α-complete prime ideal in B which would include I. It follows that no such J' exists, and hence (i) holds.

The results of § 4 yield at once various characteristic properties of the class $\boldsymbol{C}_2$.

DEFINITION 5.7. *We shall denote by $\boldsymbol{W}_\alpha$ the class of all weakly α-representable Boolean algebras.*

THEOREM 5.8. *Each of the conditions* 4.16 (ii)-(vi) *and* 4.20 (ii)-(iv) *becomes a necessary and sufficient condition for $\alpha \in \boldsymbol{C}_2$ when we replace "$\boldsymbol{W}_{\alpha,\beta}$" by "$\boldsymbol{W}_\alpha$".*

Proof. By 5.1, 4.16, 4.20.

DEFINITION 5.9. *We shall denote by $\boldsymbol{D}_\alpha$ the class of all α-distributive Boolean algebras.*

THEOREM 5.10. *Each of the conditions* 4.16 (ii)-(vi) *and* 4.20 (ii)-(iv) *becomes a necessary and sufficient condition for $\alpha \in \boldsymbol{C}_2$ if we replace "$\boldsymbol{W}_{\alpha,\beta}$" by "$\boldsymbol{D}_\alpha$".*

Proof. In case $\alpha \notin \boldsymbol{AC}$ the result follows from 0.10, which states that $\boldsymbol{W}_\alpha = \boldsymbol{D}_\alpha$, and from 5.8.

Suppose that $\alpha \in \boldsymbol{AC}$. Then $\alpha \in \boldsymbol{C}_2$ by 5.2 and 3.7. Let (ii')-(vi') and (ii'')-(iv'') denote the conditions which result from 4.16 (ii)-(vi) and 4.20 (ii)-(iv), respectively, by replacing "$\boldsymbol{W}_{\alpha,\beta}$" by "$\boldsymbol{D}_\alpha$". We must show that each of the conditions (ii')-(vi'), (ii'')-(iv'') holds. Since $\boldsymbol{D}_{\alpha^+} \subseteq \boldsymbol{D}_\alpha$ and in view of 0.5, we may assume that α is regular.

It follows at once from 4.18 that (iv') and (v') hold.

Let $\beta = (2^\alpha)^+$. By 1.2, we have $[\alpha, \beta] \subseteq \boldsymbol{C}_1$. Then by 2.1 it follows that every α-complete prime ideal in $S(\beta)$ is principal. The set $S_\beta(\beta)$ is a β-complete proper ideal in $S(\beta)$. Let $\mathfrak{B} = S(\beta)/S_\beta(\beta)$; then $\mathfrak{B} \in \boldsymbol{W}_\beta$. By 0.9 we have $\boldsymbol{W}_\beta \subseteq \boldsymbol{D}_{\alpha^+}$. Clearly $\boldsymbol{D}_{\alpha^+} \subseteq \boldsymbol{D}_\alpha$; hence $\mathfrak{B} \in \boldsymbol{D}_\alpha$. $\mathfrak{B}$ has no α-complete prime ideals, for, if J were an α-complete prime ideal in $\mathfrak{B}$, then $\{x\colon x \in S(\beta),\ x/S_\beta(\beta) \in J\}$ would be an α-complete non-principal prime ideal in $S(\beta)$, which is impossible. Since $\{0\}$ is a principal ideal in B, it follows by 0.5 that (vi') holds.

Since $|B| > 1$, (ii') holds; (iii') follows from the fact that $S(\beta)$ is not α-generated by its atoms.

The direct product $\mathfrak{C}$ of $\mathfrak{B}$ and $\mathfrak{S}(\{0\})$ obviously belongs to $\boldsymbol{D}_\alpha$. Suppose that f is an isomorphism on $\mathfrak{B}$ onto an α-subalgebra of $\mathfrak{C}$. Then the set $\{x\colon x \in B,\ f(x) \leqslant \langle \beta, 0\rangle\}$ is an α-complete prime ideal in $\mathfrak{B}$, which is impossible. Therefore there is no such isomorphism f, and (iv'') holds. The conditions (ii'') and (iii'') follow at once from (iv'').

THEOREM 5.11. *The following three conditions are equivalent:*

(i) $\alpha \in \boldsymbol{C}_2$;

(ii) *there is a sequence of α-compact topological spaces whose α-product space is not α-compact;*

(iii) *there is a sequence of α-compact Hausdorff spaces whose α-product space is not α-compact.*

The above theorem is established in [28], Theorem 1.9. It is obvious that (iii) $\Rightarrow$ (ii), and the implication (i) $\Rightarrow$ (iii) is immediate from 4.27. Note that, in case $\alpha = \omega$, we have $\alpha \notin \boldsymbol{C}_2$, and hence (ii) fails; the result that (ii) fails for $\alpha = \omega$ is just the classical theorem of Tychonoff on topological products (see [20]).

We mention at this point two natural conditions on a cardinal α which are necessary for $\alpha \in \boldsymbol{C}_1$ and sufficient for $\alpha \in \boldsymbol{C}_2$.

We shall say that $\alpha \in \boldsymbol{C}_1^*$ if either $\alpha \in \boldsymbol{SN}$ or there is an α-complete proper ideal in the field $S(\alpha)$ which cannot be extended to an α-complete prime ideal. (Note the similarity between the condition $\alpha \in \boldsymbol{C}_1^*$ and 5.6 (i).) Moreover, we shall say that $\alpha \in \boldsymbol{C}_1^{**}$ if there is a sequence of α-compact spaces whose product space is not α-compact.

It is easily seen from the definitions involved that $\boldsymbol{C}_1 \subseteq \boldsymbol{C}_1^* \subseteq \boldsymbol{C}_2$. It follows from 2.30 that $\boldsymbol{C}_1 \subseteq \boldsymbol{C}_1^{**}$, and from 5.11 that $\boldsymbol{C}_1^{**} \subseteq \boldsymbol{C}_2$ (13). We know of no further connections between the classes $\boldsymbol{C}_1, \boldsymbol{C}_1^*, \boldsymbol{C}_1^{**}$, and $\boldsymbol{C}_2$.

In the last theorem of this paper we sum up the main implications of the hypothesis $\boldsymbol{C} = \boldsymbol{C}_2$.

THEOREM 5.12. *The hypothesis $\boldsymbol{C} = \boldsymbol{C}_2$ implies (and is implied by) each of the following statements:*

(i) *For every α there is a β such that, for every $\gamma \geqslant \beta$, some α-complete proper ideal in $S(\gamma)$ cannot be extended to any α-complete prime ideal (more generally, to any α-complete and δ-saturated proper ideal with $2^\delta < \alpha$).*

(ii) *For every $\alpha > \omega$ there is an α-distributive and weakly α-representable Boolean algebra, with at least two elements, which has no α-complete prime ideals (more generally, no α-complete and δ-saturated proper ideals with $2^\delta < \alpha$) and which therefore is not strongly α-representable.*

(iii) *For every $\alpha > \omega$ there is an α-distributive and weakly α-representable Boolean algebra $\mathfrak{B}$ which has a proper principal ideal I such that $\mathfrak{B}/I$ is not isomorphic to any α-subalgebra of $\mathfrak{B}$.*

(iv) *For every $\alpha > \omega$ there is a β with the following property: for every $\gamma \geqslant \beta$ there are two sets $A, B \subseteq S_\alpha(\gamma)$ such that any two sets $a \in S_\alpha^*(A)$ and $b \in S_\alpha(B)$ are separable over γ, but A and B are not separable over γ.*

(v) *For every $\alpha > \omega$ there is a β such that for every $\gamma \geqslant \beta$ the α-product space of a γ-termed sequence of two-point discrete topological spaces is not α-compact.*

The derivation of this theorem from other results of this and the preceding section presents no difficulty.

In the main results of §§ 2, 4, and 5 we have established various necessary and sufficient conditions for a cardinal α to belong to the class $\boldsymbol{C}_1$, $\boldsymbol{C}_0$, or $\boldsymbol{C}_2$, respectively. In the special case $\alpha = \omega$ one can show that most of these results can in fact be established without the axiom of choice. Various observations in this direction are known from the literature. For instance, the results of [24], [39], [44], and [45] yield the

(13) In [23] the term "Tychonoff number" was used to refer to those infinite cardinals which do not belong to $\boldsymbol{C}_1^{**}$; it is also stated there that $\boldsymbol{C}_1' \sim \{\omega\} \subseteq \boldsymbol{C}_1^{**}$ and $\boldsymbol{C}_1^{**} \subseteq \boldsymbol{C}_2$, and some further references are given.

information that, in the special case $a = \omega$, the following theorems can be proved without using the axiom of choice: Theorems 5.6, 5.8, and 5.10 (which provide characteristic properties of the class C_2), Theorem 5.5 (i.e., $\omega \in C_1 \Rightarrow \omega \in C_2$), and Theorem 0.8 (i.e., $D_\omega \subseteq W_\omega$) [14]. Moreover, in [24] and [30] it is established that for $a = \omega$ the equivalence of 5.11 (iii) with $a \in C_2$ can be proved without the axiom of choice. In [49] and [50] it is stated that certain metamathematical conditions involving a are equivalent to the formula $a \in C_2$; as had been announced in [12], the proof of these equivalences does not require the axiom of choice in case $a = \omega$.

On the basis of the above examples, and also to some degree on the basis of our intuitions concerning the character of the axiom of choice, there seems to be good reason to believe that there is a comprehensive naturally defined class of statements $s(a)$ which can be shown to hold for all infinite cardinals a if and only if they can be proved without the axiom of chôice for $a = \omega$. We are not able to provide any more precise information about this class of statements. The following may serve as an indication of the difficulties involved: as has been shown in [19], the statement that 5.11 (ii) fails for $a = \omega$ is equivalent to the axiom of choice, while it is stated in [44] that (in a suitable set theory) the formula $\omega \notin C_2$ does not imply the axiom of choice; nevertheless Theorem 5.11 shows that $a \in C_2$ and 5.11 (ii) are equivalent for all a.

In conclusion we should like to restate the fundamental connections between the classes C_0, C_1, C_2, and C, and to formulate the related problems which remain open. In Theorems 3.3, 5.5, and 5.2, we have established the following inclusions:

$$C_0 \subseteq C_1, \quad C_1 \subseteq C_2, \quad C_2 \subseteq C.$$

It is not known whether any of these inclusions can be replaced by an identity. Furthermore, consider the six identities which may hold between two of the four classes discussed:

$$C_0 = C_1, \quad C_0 = C_2, \quad C_0 = C, \quad C_1 = C_2, \quad C_1 = C, \quad C_2 = C.$$

It is known that the first identity implies the third one; this is a consequence of 1.14 and 3.4. From this and the inclusions stated above we easily conclude that the first three identities are equivalent and each of them implies the last three identities; moreover, the fifth identity implies the fourth and the sixth (and is actually equivalent to their conjunction). We do not know whether any further implications hold between these six identities.

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[14] It follows trivially from the definitions involved that D_ω is the class of all Boolean algebras.

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Incompactness in languages with infinitely long expressions

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1. First order predicate logic has the well-known compactness property: If every finite subset of a set Γ of sentences has a model, then the set Γ has a model. This property is not, in general, shared by languages with a greater power of expression. In this paper we consider, corresponding to any given infinite cardinal α, the logic L_α which differs from first order predicate logic in that disjunctions and conjunctions of sequences of formulas of type less than α and quantifications over fewer than α variables are allowed. For many cardinals α we shall show that L_α is incompact; that is, we shall exhibit a set Γ of sentences of L_α which has no model but is such that every subset of Γ of power less than α has a model (1).

For the most part, the set theoretical part of our discussion can be carried out on the basis of Zermelo-Fraenkel set theory. Without some modification, however, some of the set theoretical statements we make would not have their proper meaning in Z-F. For example, we will have occasion to speak of the class of all accessible cardinals. This is not, of course, a set in the sense of Z-F at all. On the other hand, it would do little good to use the Bernays type of set theory (where we have sets and arbitrary classes of sets) since we also have occasion to consider functions defined on classes, etc. Although it could easily be avoided, at one point in Section 3, we use "ordinals" to index the set of all ordinals in the sequence determined by an arbitrary well-order-

(1) Languages with infinitely long expressions were introduced by Tarski in [16]. The author's results on incompactness of L_α for inaccessible α were first stated in the abstract [3]. This paper represents an essential portion of the author's doctoral dissertation presented to the Graduate Division of the University of California, Berkeley, California. Grateful acknowledgement is made to Professor Tarski for his encouragement and assisstance at many stages of this work. It should be pointed out that the Löwenheim-Skolem theorems given in [3] are incorrectly formulated. The correct formulations and references to the earlier work of Carol Karp are given in [5].