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Additions to some results of Erdös and Tarski

by

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Erdös and Tarski in [3], Theorem 4.3, establish a connection between a representation problem for certain Boolean algebras and a problem about ramification systems (the exact problem depends on a given cardinal number). In this note we obtain a result (Theorem 2.1) which yields the converse of the implication proved in [3]. Actually we show that the ramification problem is equivalent to a compactness problem involving some special topological spaces. The definition of these spaces is given in Section 1, where the compactness problem is related to a prime ideal problem studied by Keisler and Tarski in [4]. The proof of equivalence of the representation problem of [3] and the prime ideal problem may be found in [4], Theorem 4.16.

In Section 3 the compactness problem is reformulated in simpler set-theoretical terms which make no reference to topological spaces.

§ 1. α-products of topological spaces. Throughout this note α, β, and γ will denote infinite cardinal numbers. Cardinals are considered as special kinds of ordinal numbers (initial numbers), and each ordinal coincides with the set of all smaller ordinals. The ξ-th infinite cardinal is denoted by ω_ξ. If $\alpha = \omega_\xi$, then $\alpha^+ = \omega_{\xi+1}$. The cardinal number of a set A is denoted by $|A|$. The set of all subsets of a set A is denoted by $S(A)$, and further

$$S_\alpha(A) = \{B \in S(A)\colon\ |B| < \alpha\}.$$

A topological space X is *α-complete* if the intersection of a family of power smaller than α of open sets is again open. Every space is of course ω_0-complete. Note that if α is a singular cardinal, then a space X is α-complete if and only if X is α^+-complete. The notion of an α-complete space is a natural generalization of the ordinary notion of a topological space, and many of the usual topological concepts may be appropriately modified for this class (see, e.g., [8]). We shall be concerned with two of these concepts, namely *compactness* and the formation of the *product topology*.

A topological space X is α-*compact* if each open cover of X can be reduced to one of power less than α, or, equivalently, if the intersection of a family of closed subsets of X is non-empty whenever the intersection of every subfamily of power less than α is non-empty. The ordinary notion of compactness is obviously the same as ω_0-compactness. If a space is α-compact, and $\alpha \leqslant \beta$, it is also β-compact. For α singular, the properties of α-compactness and α^+-compactness do *not* coincide. For example, the set ω_ω with the discrete topology is $\omega_{\omega+1}$-compact but not ω_ω-compact.

The α-*product topology* on the Cartesian product $\boldsymbol{P}_{t\in T} X_t$ of spaces X_t is the smallest α-complete topology which includes as open sets the usual cylinder sets $\{f \in \boldsymbol{P}_{t\in T} X_t\colon f_{t_0} \in U\}$, where $t_0 \in T$ and U is open in X_{t_0}. In other words, to obtain the α-topology on $\boldsymbol{P}_{t\in T} X_t$, close the class of cylinder sets under the formation of intersections of families of power less than α; this gives a base for the α-topology, and open sets of the new topology are arbitrary unions of these intersections. The ω_0-product topology is the same as the ordinary product topology. If α is regular, the base for the α-topology is just the class of all intersections of families of power less than α of cylinder sets. For α singular, the α-product topology and the α^+-topology are identical.

The main cases of products dealt with in this note are the products of discrete 2-point spaces. Let T be any index set. The product of T copies of $2 = \{0, 1\}$ is denoted by 2^T. The α-topology on 2^T is especially easy to describe (for α regular). Every open subset of 2^T in the α-topology is a union of sets obtained by the following method: select a subset $T' \in S_\alpha(T)$ and choose a function $g \in 2^{T'}$; the basic open set of 2^T corresponding to g is the set $\{f \in 2^T\colon f_t = g_t \text{ for all } t \in T'\}$. For α regular, 2^T is discrete if and only if $|T| < \alpha$, while for α singular 2^T is discrete if and only if $|T| \leqslant \alpha$. Of course, each 2^T with the α-topology is a totally disconnected Hausdorff space, which is ω_0-compact only when $\alpha = \omega_0$ or $|T| < \omega_0$. The spaces 2^α were used in [7] and [8], where it is noted that in the α-topology the appropriate generalization of the Baire Category Theorem holds for all regular α. In [7], p. 259, last paragraph, it is stated that these spaces are always α-compact. That statement is *false* (take $\alpha = \omega_1$, for example). We shall see from the results below how much α-compactness can reasonably be expected.

Inasmuch as compactness is not automatically obtained, it is reasonable to study the following relation between cardinals: $T(\alpha, \beta)$ holds if and only if the space 2^β with the α-topology is α-compact. The letter "T" has been chosen to denote this relation to remind us that topological spaces are involved, and also that Tychonoff's Theorem tells us that $T(\omega, \beta)$ holds for all β. We formulate next a series of seven lemmas which lead to the two main results of this section (1.8 and 1.9).

LEMMA 1.1. *If $\gamma \leqslant \beta$, then $T(\alpha, \beta)$ implies $T(\alpha, \gamma)$.*

Proof. Obviously 2^γ is a continuous image of 2^β in the sense of the α-topology. Hence, the α-compactness of 2^β at once implies the α-compactness of 2^γ.

LEMMA 1.2. *If $|2^\beta| < \alpha$, then $T(\alpha, \beta)$ holds.*

Proof. Every α-complete space of power less than α is α-compact.

LEMMA 1.3. *If $\beta < \alpha \leqslant |2^\beta|$, then $T(\alpha, \beta)$ fails to hold.*

Proof. Under the hypothesis, 2^β is discrete in the α-topology. No discrete space of power greater than or equal to α is α-compact.

LEMMA 1.4. *If α is singular, then $T(\alpha, \alpha)$ fails to hold.*

Proof. If α is singular, then 2^α is discrete in the α-topology ($= \alpha^+$-topology). The conclusion follows by the argument of 1.3.

LEMMA 1.5. *If $\beta \geqslant \alpha$, then $T(\alpha, \beta)$ implies that α is strongly inaccessible.*

Proof. By 1.1 and 1.4, $\beta \geqslant \alpha$ and $T(\alpha, \beta)$ imply that α is regular (i.e. not singular). By 1.1 and 1.3, the same hypothesis implies that $|2^\gamma| < \alpha$ whenever $\gamma < \alpha$. Hence, α is strongly inaccessible.

From 1.5 we see that 2^{ω_1} is not ω_1-compact in the ω_1-topology. Indeed, if $\alpha > \omega_0$ is less than the first uncountable strongly inaccessible cardinal, then 2^α fails to be α-compact. If α is the first uncountable inaccessible, then it can be shown that 2^α is still not α-compact. This will follow from 1.8 below and the results in [4].

The relation T is closely related to another relation $\widetilde{R}$, which we now define. An ideal I in a field of sets A is called α-*complete* if $\bigcup F \in I$ whenever $F \in S_\alpha(I)$. Since A is an ideal of A itself, this concept applies to A, and we refer to α-*complete fields of sets*. A subset B of A α-*generates* A if A is the least α-complete field of sets including B and containing $\bigcup A$ as an element. We say that A is (α, β)-*generated* if some subset of A of power at most β does α-generate A. By definition, $\widetilde{R}(\alpha, \beta)$ holds if and only if every α-complete proper ideal can be extended to an α-complete prime ideal in every α-complete field of sets that is (α, β)-generated. In [4] Keisler and Tarski studied the negation of the relation $\widetilde{R}$ (which they call R); we recall the following facts established in [4], Section 4.

LEMMA 1.6. (i) *If $|2^\beta| < \alpha$, then $\widetilde{R}(\alpha, \beta)$ holds.*

(ii) *If $\beta < \alpha < |2^\beta|$, then $\widetilde{R}(\alpha, \beta)$ fails to hold.*

(iii) *If $\alpha = |2^\beta|$ and α is regular, then $\widetilde{R}(\alpha, \beta)$ fails to hold.*

(iv) *If $\alpha = |2^\beta|$ and α is singular, then $\widetilde{R}(\alpha, \beta)$ holds.*

(v) *If $\alpha \leqslant \beta$ and α is singular, then $\widetilde{R}(\alpha, \beta)$ fails to hold.*

The Generalized Continuum Hypothesis implies the negation of the hypothesis of 1.6 (ii) and (iv). These cases were only included because we do not have any need to assume the Continuum Hypothesis. The main result of this section is that, aside from the very exceptional situation of 1.6 (iv) where they behave in opposite ways, the relations T and $\widetilde{R}$ prove to be equivalent. Before giving this result we require one further lemma which is essentially known: it is a generalization of Tychonoff's Theorem to the α-topology. In [2], [5], and [6] generalizations of Tychonoff's theorem were discussed, but the ordinary product topology was used. It seems to us more natural (and it gives stronger results) to use the α-topology. For this reason, and for the sake of completeness, we shall outline here a proof of the lemma; our proof is a simple modification of the well-known Bourbaki proof. In stating the lemma we use the notation $C_\alpha(Y)$, where $Y = P_{t\in T} X_t$, to denote the α-complete field of subsets of Y generated by the cylinder sets $\{f \in Y\colon f_{t_0} \in U\}$, where $t_0 \in T$ and U is open in X_{t_0}.

LEMMA 1.7. *Let α be regular, and suppose that the spaces X_t are α-compact for each $t \in T$. Let $Y = P_{t\in T} X_t$ be given the α-product topology. Then Y is α-compact, if every α-complete proper ideal in $C_\alpha(Y)$ can be extended to an α-complete prime ideal.*

Proof. Suppose that B is a family of basic open sets of Y which has the property that no subfamily in $S_\alpha(B)$ covers Y. We wish to prove that $\bigcup B \neq Y$ (1). Now $B \subseteq C_\alpha(Y)$, and the α-complete ideal of $C_\alpha(Y)$ generated by B is obviously proper in view of our assumption on B. Let $P \supseteq B$ be an α-complete prime ideal of $C_\alpha(Y)$. For each $t \in T$, let

$$P_t = \big\{U \in Op(X_t)\colon \{f \in Y\colon f_t \in U\} \in P\big\},$$

where $Op(X_t)$ denotes the class of open subsets of the space X_t. Since P is α-complete and $Y \notin P$, it is clear that no subfamily in $S_\alpha(P_t)$ covers X_t. Let the function f be an element of the non-empty product $P_{t\in T}(X_t \sim \bigcup P_t)$ (2). If $f \in \bigcup B$, then $f \in A \in B$ where A is a basic open subset of Y. In other words, A would be a less-than-α-termed intersection of cylinder sets. Now $A \in P$ and P is α-complete; whence, at least one of the cylinder sets including A would belong to P. But by construction, f belongs to no such cylinder set; therefore, $f \in Y \sim \bigcup B$, which completes the proof.

THEOREM 1.8. *The conditions $\widetilde{R}(\alpha, \beta)$ and $T(\alpha, \beta)$ are equivalent, unless α is singular and $\alpha = |2^\beta|$.*

(1) The fact that this is sufficient in order to prove the theorem requires the axiom of choice (for families of power less than α). However, for $\alpha = \omega_0$ choice is not needed. (Cf. footnote 2).

(2) The axiom of choice is used at this stage. If we assumed that each X_t were Hausdorff, then it would follow that each $X_t \sim \bigcup P_t$ consists of a single point. Thus a choice would be unnecessary. Hence in cojnunction with footnote 1 we recover the known fact that the prime ideal theorem alone implies Tychonoff's theorem for Hausdorff spaces when $\alpha = \omega_0$.

Proof. That $\widetilde{R}(\alpha, \beta)$ implies $T(\alpha, \beta)$ is an immediate consequence of 1.7 for α regular. (Hint: take $T = \beta$ and $X_t = 2$ for all $t \in T$, and note that $C_\alpha(Y)$ is (α, β)-generated.) For α singular the implication is immediate from 1.2, 1.6 and the hypothesis. For the other implication, we may assume that $\beta \geqslant \alpha$ by virtue of 1.3 and 1.6. Then according to 1.5, α is strongly inaccessible. Suppose that A is an α-complete field of sets which is α-generated by a subset B of power at most β, and that I is an α-complete proper ideal of A. Let B' be the set of complements of elements of B in the field A. Since α is strongly inaccessible, every element of A can be obtained as a less-than-α-termed union of less-than-α-termed intersections of sets in $B \cup B'$. (Hint: use the general distributive law to show that this collection of unions of intersections is an α-complete field.) Let us use the notation $a^1 = a$ and $a^0 = \bigcup A \sim a$, for $a \in A$. From the foregoing remarks it readily follows that the problem of finding an α-complete prime ideal of A including I is equivalent to finding a function $f \in 2^\beta$ such that the set $I \cup \{b^{f(b)}\colon b \in B\}$ generates a *proper* α-complete ideal of A (the ideal so generated will always be prime or trivial).

This last problem can be solved by using our α-compactness assumption for 2^β. Let $B = \{b_\xi\colon \xi < \beta\}$. For each subset $M \in S_\alpha(\beta)$, let

$$F_M = \{f \in 2^\beta\colon I \cup \{b_\xi^{f(\xi)}\colon \xi \in M\} \textit{ generates a proper } \alpha\textit{-complete ideal of } A\}.$$

If F_M were empty, it would mean that

$$\bigcup A \sim \bigcup_{\xi\in M} b_\xi^{f(\xi)} \in I \quad \textit{for all} \quad f \in 2^M.$$

Now $|M| < \alpha$, so $|2^M| < \alpha$, and I is α-complete. Hence,

$$\bigcup_{f\in 2^M} \big(\bigcup A \sim \bigcup_{\xi\in M} b_\xi^{f(\xi)}\big) \in I.$$

Using De Morgan's law and the distributive law we would at once have $\bigcup A \in I$, which is impossible. Thus F_M is never empty for $M \in S_\alpha(\beta)$. Notice also that if $M \in S_\alpha(\beta)^\gamma$, where $\gamma < \alpha$, and if $N = \bigcup_{\xi<\gamma} M_\xi$, then

$$F_N \subseteq \bigcap_{\xi<\gamma} F_{M_\xi}.$$

Hence, no less-than-α-termed intersection of the sets F_M, $M \in S_\alpha(\beta)$, is ever empty. Finally, if we note the simple fact that F_M is a closed subset of 2^β (because F_M *is* an at most $|2^M|$ union of basic open-*closed* subsets of 2^β), it follows from compactness that $\bigcap\{F_M\colon M \in S_\alpha(\beta)\}$ is non-empty. Any function in this intersection will give the desired result.

Perhaps the most natural property of cardinal numbers which would correspond to Tychonoff's Theorem is this: we write $C(\alpha)$ to mean that every product of α-compact spaces is α-compact in the α-product topology. The connection between C and T is easy to establish using the above results.

THEOREM 1.9. *$C(\alpha)$ holds if and only if $T(\alpha, \beta)$ holds for all β.*

Proof. The necessity of the second condition is obvious. If $T(\alpha, \beta)$ holds for all β, then by 1.5, α is strongly inaccessible and hence regular. In view of 1.8, $\widetilde{R}(\alpha, \beta)$ must hold for all β. Hence, $C(\alpha)$ follows by 1.7.

§ 2. Ramification systems. A relational system $\langle A, \leqslant\rangle$ is called a *ramification system* if the following conditions hold:

(i) *A is non-empty*;

(ii) *$\leqslant$ partially orders A*;

(iii) *for each $x \in A$, the set $\{y \in A\colon y \leqslant x\}$ is well-ordered by $\leqslant$.*

The *order* of an element x of a ramification system $\langle A, \leqslant\rangle$ is the ordinal type of the well-ordered set $\{y \in A\colon y \leqslant x,\ y \neq x\}$. The *order* of the system $\langle A, \leqslant\rangle$ is the least upper bound of the orders of its elements. By an *α-ramification system* we understand a system of order α where for each $\xi < \alpha$, the set of all elements of order ξ has power less than α.

We say that $\widetilde{Q}(\alpha)$ holds if every α-ramification system has a subset ordered by the partial ordering in type α. The negation of the property $\widetilde{Q}$ was studied in [3]. Our purpose here is to relate $\widetilde{Q}$ to T.

THEOREM 2.1. *If α is strongly inaccessible, then $\widetilde{Q}(\alpha)$ and $T(\alpha, \alpha)$ are equivalent.*

Proof. In [3], Theorem 3.4, it is shown that $\widetilde{Q}(\alpha)$ implies a property of α that is stronger than $\widetilde{R}(\alpha, \alpha)$ (cf. [4], Theorem 4.31). Hence, by 1.8, $\widetilde{Q}(\alpha)$ implies $T(\alpha, \alpha)$, if α is inaccessible. Assume now that $T(\alpha, \alpha)$ holds, and let $\langle A, \leqslant\rangle$ be an α-ramification system. Since α is strongly inaccessible, it follows that $|A| = \alpha$. We may assume $A = \alpha$. If $f \in 2^\alpha$, let $f^{-1}(1) = \{\xi < \alpha\colon f(\xi) = 1\}$. For each $\xi < \alpha$, let A_ξ be the elements of $\langle A, \leqslant\rangle$ of order less than ξ. Let

$$C_\xi = \{f \in 2^\alpha\colon f^{-1}(1) \cap A_\xi \text{ is ordered by } \leqslant \text{ in type } \xi\},$$

for $\xi < \alpha$. Clearly since $A_{\xi+1} \sim A_\xi \neq 0$, each C_ξ is a non-empty closed subset of 2^α in the α-topology. It is also clear that $C_\xi \subseteq C_\eta$, whenever $\eta < \xi < \alpha$. Since $T(\alpha, \alpha)$ holds, there must be an $f \in \bigcap\{C_\xi\colon \xi < \alpha\}$, and $f^{-1}(1)$ is the subset of A ordered in type α (3).

Thus for α strongly inaccessible the properties $\widetilde{R}(\alpha, \alpha)$, $T(\alpha, \alpha)$, and $\widetilde{Q}(\alpha)$ are equivalent. By 1.8, $\widetilde{R}(\alpha, \alpha)$ and $T(\alpha, \alpha)$ are equivalent for all α. We know that $T(\alpha, \alpha)$ fails if α is not inaccessible. Specker in [9] has shown that $\widetilde{Q}(\alpha)$ fails if $\alpha = \beta^+$, where β is inaccessible, or assuming the Generalized Continuum Hypothesis, if $\alpha = \gamma^+$, where γ is regular. Also, it is obvious that $\widetilde{Q}(\alpha)$ fails to hold if α is singular. Whence, $\widetilde{Q}(\alpha)$ is equivalent to $T(\alpha, \alpha)$ in these cases also. The remaining case where $\alpha = \gamma^+$, γ singular, seems to be still an open problem.

(3) Alfred Tarski has pointed out to us that the proof of Theorem 4.2 of [3] can easily be modified to show that $\widetilde{R}(\alpha, \alpha)$ implies $\widetilde{Q}(\alpha)$ for α strongly inaccessible; by virtue of our Theorem 1.8 this gives an alternate proof of 2.1.

§ 3. The separation principles. Even though the topological terminology used in the first two sections is suggestive and convenient in many instances, the combinatorial simplicity of the statement of the α-compactness of 2^β is rather hidden by the product space formulation. To place the ideas in sharper focus, we shall first express the condition $T(\alpha, \beta)$ in terms of the family $S(\beta)$ of all subsets of β (Theorem 3.1) and then simplify the resulting statement (Theorem 3.2).

To start out, recall the usual one-one correspondence between $S(\beta)$ and 2^β obtained from the notion of a characteristic function of a subset of a given set. Under this correspondence an open set in the base for the α-topology on 2^β is matched with a subset of $S(\beta)$ of the form

$$[x, \beta\sim y] = \{X \subseteq \beta\colon x \subseteq X \subseteq \beta\sim y\},$$

where $|x \cup y| < \alpha$. That is to say, the open sets in the induced α-complete topology on $S(\beta)$ are simply unions of intervals of this form. Notice further that the interval $[x, \beta\sim y]$ is completely determined by the ordered pair $(x, y) \in S_\alpha(\beta) \times S_\alpha(\beta)$. Hence, a covering of $S(\beta)$ by basic open sets is determined by a subset $R \subseteq S_\alpha(\beta) \times S_\alpha(\beta)$, that is, by a relation. Since the α-compactness of $S(\beta)$ can be formulated in terms of coverings by basic open sets, we see that $T(\alpha, \beta)$ can be expressed as a property of relations over $S_\alpha(\beta)$.

To find the property of relations that corresponds to $T(\alpha, \beta)$, we must define what it means for a relation $R \subseteq S_\alpha(\beta) \times S_\alpha(\beta)$ to determine a cover of $S(\beta)$. For some purposes, however, the authors prefer to place the emphasis on the non-covers. A relation which determines a non-cover will be called a *separable* relation. To be precise, we say that a relation $R \subseteq S_\alpha(\beta) \times S_\alpha(\beta)$ is *separable over* β, in symbols: $\mathrm{Sep}_\beta(R)$, if there is an $X \subseteq \beta$, such that $R \cap \big(S(X) \times S(\beta\sim X)\big) = 0$. That condition means exactly that X belongs to no interval $[x, \beta\sim y]$ for $(x, y) \in R$; hence, there is one point of the space 2^β that is not in the union of the open sets determined by R. We may also use the notion $\mathrm{Sep}_b(R)$, where β is replaced by an arbitrary set b, since the fact that β is an ordinal is clearly irrelevant in the definition.

The foregoing discussion leads at once to what we call the *First Separation Principle* for a pair of cardinals α, β. It is condition $(*)$ of the next theorem.

THEOREM 3.1. *$T(\alpha, \beta)$ is equivalent to the condition:*

(*) *whenever $R \subseteq S_\alpha(\beta) \times S_\alpha(\beta)$ and $\mathrm{Sep}_\beta(r)$ holds for all $r \in S_\alpha(R)$, then $\mathrm{Sep}_\beta(R)$ holds.*

Proof. Obvious.

Using a suggestion of Tarski (4), condition 3.1 (i) can be further simplified. It turns out that it is not necessary to use arbitrary relations $R \subseteq S_\alpha(\beta) \times S_\alpha(\beta)$; rather those relations R, where $(x, y) \in R$ implies $x = 0$ or $y = 0$, prove to be sufficient. Now each such R is determined by two classes:

$$A = \{x \in S_\alpha(\beta)\colon (x, 0) \in R\}, \quad \text{and} \quad B = \{y \in S_\alpha(\beta)\colon (0, y) \in R\}.$$

The condition $\mathrm{Sep}_\beta(R)$ is then equivalent to a new condition $A)(_\beta B$ (read: A and B are *separable* over β), which in general means that for some $X \subseteq \beta$, we have $A \cap S(X) = 0 = S(\beta \sim X) \cup B$. Thus $A)(_\beta B$ and $\mathrm{Sep}_\beta(A \times \{0\} \cup \{0\} \times B)$ are equivalent for all $A, B \subseteq S_\alpha(\beta)$. We shall also use $A)(_b B$ for arbitrary sets b. In this way we arrive at the *Second Separation Principle* given in condition (**) of the next theorem.

THEOREM 3.2. *$T(\alpha, \beta)$ is equivalent to the condition:*

(**) *whenever $A, B \subseteq S_\alpha(\beta)$ and $a)(_\beta b$ holds for all $a \in S_\alpha(A)$, $b \in S_\alpha(B)$, then $A)(_\beta B$ holds.*

Proof. The implication from left to right follows at once from 3.1 and the above discussion. Suppose then that the right hand side holds. Clearly the right-hand side will continue to hold with β replaced by $\beta \times 2$, because β is infinite and $\beta = |\beta \times 2|$. Let then C be a cover of 2^β by basic open sets in the α-topology. Notice that since functions are sets of ordered pairs, $2^\beta \subseteq S(\beta \times 2)$. Notice also that there is a set $K \subseteq S_\alpha(\beta \times 2)$ such that for the given cover:

$$C = \{[x, \beta \times 2] \cap 2^\beta\colon x \in K\}.$$

Let

$$A = K \cup \big\{\{(\xi, 0), (\xi, 1)\}\colon \xi < \beta\big\}, \qquad B = \big\{\{(\xi, 0), (\xi, 1)\}\colon \xi < \beta\big\}.$$

Suppose that $A)(_{\beta \times 2} B$ holds. Let $X \subseteq \beta \times 2$ be chosen so that

$$A \cap S(X) = 0 = S\big((\beta \times 2) \sim X\big) \cap B.$$

If $\xi < \beta$, then $\{(\xi, 0), (\xi, 1)\} \nsubseteq X$ by the first equation, and $X \cap \{(\xi, 0), (\xi, 1)\} \neq 0$ by the second equation. Hence, for each $\xi < \beta$, there is a unique $\zeta < 2$, such that $(\xi, \zeta) \in X$; in other words, $X \in 2^\beta$. Now C covers 2^β; whence, $x \subseteq X$ for some $x \in K$. This contradicts the equation $A \cap S(X) = 0$. Therefore, A and B are not separable over $\beta \times 2$. From our assumption it follows that there are $a \in S_\alpha(A)$ and $b \in S_\alpha(B)$ which are also not separable. Let $K' = K \cap a$. Let

$$C' = \{[x, \beta \times 2] \cap 2^\beta\colon x \in K'\}.$$

It is trivial to verify that $C' \subseteq C$ covers 2^β and that $|C'| < \alpha$. Thus $T(\alpha, \beta)$ is proved.

(4) The idea of the First Separation Principle was formulated by Scott in 1956 and communicated to Tarski in a version suitable for an axiomatic set theory with both sets and classes and with the cardinal α equal to the cardinal number of the universe. At that time Tarski suggested the simpler Second Separation Principle and Tarski and Scott independently verified its equivalence with the first. The details of the axiomatic version will be published elsewhere.

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