(4.8) After this paper was written, we became aware of the paper *Uniform boundedness for groups* by Irving Glicksberg [Canadian J. Math. 14 (1962), pp. 269-276]. In this paper, Glicksberg has proved that G_1 has at least one continuous character that is G_2-discontinuous.

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Reçu par la Rédaction le 20. 7. 1962

On compactifications allowing extensions of mappings

by

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In this paper we shall deal with compactifications of a space X which allow extensions of some mappings of X into itself. By a topological space we always mean a completely regular topological space, by a mapping we mean a continuous function. We shall say that a compact space Y is a *compactification* of a topological space X if Y contains a dense subspace X' homeomorphic to X. Compactifications of the space X will be denoted by aX, a_iX, etc. The letter a will also denote the homeomorphism of X onto X'. Hence we have $a\colon X\to aX$ and a is a homeomorphism of X onto $a(X)$. We can define a partial order $\succcurlyeq$ in the class of all compactifications of X. Namely, we put $a_1X \succcurlyeq a_2X$ if there exists a mapping $f\colon a_1X\to a_2X$ such that $fa_1 = a_2$. In that case we shall write $f\colon a_1X\succcurlyeq a_2X$. The Čech-Stone compactification βX is the maximal element in the partially ordered class. By the *weight* of a space X we shall mean the smallest cardinality of bases of X. The weight of X will be denoted by $w(X)$. Let us notice the well-known result that $a_1X\succcurlyeq a_2X$ implies $w(a_1X)\geqslant w(a_2X)$.

Let $\Phi = \{\varphi_s\}_{s\in S}$ be a family of mappings of X into itself, i.e. $\varphi_s\colon X\to X$ for every $s\in S$. A compactification aX of the space X will be called a *Φ-compactification* if, for every $s\in S$, there exists a mapping $\tilde{\varphi}_s\colon aX\to aX$ such that $\tilde{\varphi}_s|X = \varphi$ (more exactly $\tilde{\varphi}_s a = a\varphi_s$). The notion of Φ-compactification was introduced in [7]. The paper contains some theorems on the existence of Φ-compactifications for metric spaces. Other results are in [3], [4], [12] and [19]. Of course, βX is a Φ-compactification for every family Φ. However, it is known that the weight of βX is much greater than the weight of X and one can set the following problem: *Determine the minimal weight of Φ-compactifications for a given space X and a family Φ.* The paper contains some results concerning this subject.

The paper is divided into two parts. In the first part we consider Φ-compactifications of X preserving the dimension of X. The second part is devoted to investigations of Φ-compactifications of X, where X is a peripherically compact space.

1. Φ-compactifications preserving dimension. The following lemma plays an important rôle in the proofs of all the theorems of the paper:

LEMMA 1. *Let X be a topological space and let $\Phi = \{\varphi_s\}_{s\in S}$ be a family of mappings of X into itself. Let us consider an enumerable family $a_1X, a_2X, \ldots$ of compactifications of X and a family of mappings $\pi_1^2, \pi_2^3, \ldots$ where $\pi_i^{i+1}\colon a_{i+1}X \succ a_iX$. If for every $s \in S$ there exists extensions $\tilde{\varphi}_s^{i+1}\colon a_{i+1}X \to a_iX$, $i = 1, 2, \ldots$, of mappings φ_s, then the limit aX of the inverse system $\{a_iX, \pi_j^i\}$, where $\pi_j^i = \pi_j^{j+1}\pi_{j+1}^{j+2}\ldots\pi_{i-1}^i$ for $i > j$, is a Φ-compactification of the space X.*

Proof. Mappings π_i^{i+1} and φ_s^{i+1} satisfy the following conditions:

(1) $$\pi_i^{i+1}a_{i+1} = a_i\,.$$

(2) $$\tilde{\varphi}_s^{i+1}a_{i+1} = a_i\varphi_s\,,$$

Let $Y = \varprojlim\{a_iX, \pi_j^i\}$. Then Y is a compact space. The mapping $a\colon X \to \mathop{P}\limits_{i=1}^{\infty} a_iX$ defined by $a(x) = \{a_i(x)\}$ is a homeomorphism into $\mathop{P}\limits_{i=1}^{\infty} a_iX$. Of course, $a\colon X \to Y \subset \mathop{P}\limits_{i=1}^{\infty} a_iX$, and we shall show that $\overline{a(X)} = Y$. The family of all subsets $\hat{W}_j = Y \cap (\mathop{P}\limits_{i<j}^{\infty} a_iX \times W_j \times \mathop{P}\limits_{i>j}^{\infty} a_iX)$, where W_j is an open subset of a_jX, is a base of Y. Since $a_j(X)$ is dense in a_jX, there exists a point $x \in X$ such that $a_j(x) \in W_j$. But $a(x) \in \hat{W}_j$ and therefore $\overline{a(X)} = Y$. Thus Y is a compactification of the space X. According to our convention, we shall denote it by aX.

For every $s \in S$, let us consider the following diagram:

$$\begin{tikzcd}
a_2X & a_3X \arrow[l, "\pi_2^3"'] & \ldots \arrow[l] & a_iX \arrow[l] & a_{i+1}X \arrow[l, "\pi_i^{i+1}"'] & \ldots \arrow[l] \\
a_1X \arrow[u, leftarrow, "\tilde{\varphi}_s^2"'] & a_2X \arrow[l, "\pi_1^2"'] \arrow[u, leftarrow, "\tilde{\varphi}_s^3"] & \ldots \arrow[l] & a_{i-1}X \arrow[l] \arrow[u, leftarrow, "\tilde{\varphi}_s^i"'] & a_iX \arrow[l, "\pi_{i-1}^i"'] \arrow[u, leftarrow, "\tilde{\varphi}_s^{i+1}"] & \ldots \arrow[l]
\end{tikzcd}$$

We shall show that every square of the above diagram is commutative, i.e.

(3) $$\tilde{\varphi}_s^i\pi_i^{i+1} = \pi_{i-1}^i\tilde{\varphi}_s^{i+1} \quad \text{for} \quad i = 2, 3, \ldots$$

In order to prove this it suffices to show that equality (3) holds on a dense subset $a_{i+1}(X) \subset a_{i+1}X$, i.e., it suffices to show that

(4) $$\tilde{\varphi}_s^i\pi_i^{i+1}a_{i+1} = \pi_{i-1}^i\tilde{\varphi}_s^{i+1}a_{i+1} \quad \text{for} \quad i = 2, 3, \ldots$$

However, it follows from (1) and (2) that

$$\tilde{\varphi}_s^i\pi_i^{i+1}a_{i+1} = \tilde{\varphi}_s^i a_i = a_{i-1}\varphi_s = \pi_{i-1}^i a_i\varphi_s = \pi_{i-1}^i\tilde{\varphi}_s^{i+1}a_{i+1}\,,$$

whence (4) and (3) are proved.

The family $\{\tilde{\varphi}_s^i\}_{i=2}^{\infty}$ is a map of the inverse system $\{a_iX, \pi_j^i\}_{i\geq 2}$ into $\{a_iX, \pi_j^i\}_{i\geq 1}$. It determines a mapping $\tilde{\varphi}_s\colon aX \to aX$, because aX is a limit of both systems. We shall show that $\tilde{\varphi}_s$ is an extension of φ_s. We have to prove

(5) $$\tilde{\varphi}_s a = a\varphi_s\,.$$

Let $p_i\colon \mathop{P}\limits_{i=1}^{\infty} a_iX \to a_iX$ be the projection into the i^{th} axis. In order to prove (5) it suffices to show that

(6) $$p_i\tilde{\varphi}_s a = p_i a\varphi_s \quad \text{for} \quad i = 2, 3, \ldots$$

It follows from (2) and the equality $p_i a = a_i$ that

$$p_i\tilde{\varphi}_s a = \tilde{\varphi}_s^{i+1}p_{i+1}a = \tilde{\varphi}_s^{i+1}a_{i+1} = a_i\varphi_s = p_i a\varphi_s\,,$$

and we have proved (6).

Hence, for every $s \in S$, the mapping $\tilde{\varphi}_s\colon aX \to aX$ is an extension of φ_s. Thus aX is a π-compactification of X and the proof of the lemma is complete.

In the sequel we shall also use two well-known lemmas.

LEMMA 2. *Let aX be a compactification of X and let $\{f_s\}_{s\in S}$ be a family of mappings, $f_s\colon X \to Y_s$, where Y_s is a compact space, for every $s \in S$. Then there exists a minimal compactification a_SX of X such that $a_SX \succ aX$ and, for every $s \in S$, there is an extension $\tilde{f}_s\colon a_SX \to Y_s$ of f_s, i.e., there exists a compactification a_SX such that if, for a compactification $a'X$ of X, $a'X \succ aX$ and if there is an extension $f_s'\colon a'X \to Y_s$ of f_s for every $s \in S$, then $a'X \succ a_SX$.*

Moreover, if $w(aX) \leqslant \tau$, $w(Y_s) \leqslant \tau$ for every $s \in S$ and $\overline{\overline{S}} \leqslant \tau$, then $w(a_SX) \leqslant \tau$.

Proof. Let $P = aX \times \mathop{P}\limits_{s\in S} Y_s$ and consider the mapping $a_S\colon X \to P$ determined by $a_S(x) = \{a(x), \{f_s(x)\}\}$. Then a_S is a homeomorphism of the space X into P. We shall show that we can take the subspace $a_S(X) \subset P$ as a_SX. Indeed, denoting by p a projection of P onto aX and by p_s the projection of P onto Y_s, we have $pa_S = a$ and $p_s a_S = f_s$. Hence $p\colon a_SX \succ aX$ and the mappings $\{f_s\}_{s\in S}$ can be extended.

Now, let us suppose that for a compactification $a'X$ there exist a function $g\colon a'X \succ aX$ and extensions $f_s'\colon a'X \to Y_s$ of mappings f_s for all $s \in S$. Then the mapping $F\colon a'X \to a_SX$ defined by $F(x) = \{g(x), \{f_s'(x)\}\}$ satisfies condition $Fa' = a_S$. Hence $a'X \succ a_SX$. The last part of the lemma follows from the given construction of a_SX.

COROLLARY. *Let $\mathfrak{R} = \{a_SX\}_{s\in S}$ be a family of compactifications of a space X. Then there exists a compactification a_SX that is the least upper bound of $\mathfrak{R}$ with respect to the partial order $\succ$.*

Indeed, it suffices to replace Y_s, f_s and aX occurring in Lemma 2 by a_sX, a_s: $X \to a_sX$ and an arbitrary compactification from the family $\mathcal{R}$, respectively.

Remark. Using the construction from the proof of Lemma 2 one can construct the compactification βX. In fact, it suffices to take in the Corollary for $\mathcal{R}$ the family of all compactifications of the space X. The consideration of the family does not lead to a contradiction, since for every compactification aX of X we have $\overline{\overline{aX}} \leqslant 2^{2^{\overline{\overline{X}}}}$.

LEMMA 3. *Let $\{a_sX\}_{s\in S}$ be a family of compactifications that are greater than a fixed compactification aX of X. Then the family $\{a_sX\}_{s\in S}$ has the greater lower bound.*

If a mapping φ: $X \to Y$, where Y is an arbitrary compact space, can be extended over all compactifications belonging to the family $\{a_sX\}_{s\in S}$, then φ can be extended over a_SX.

Proof. Let us consider the family $\mathcal{R}$ of all compactifications smaller than a_sX for every $s \in S$. The family $\mathcal{R}$ is not void since $aX \in \mathcal{R}$. It follows from Lemma 2 that $\mathcal{R}$ has the least upper bound a_SX. It is easy to check that a_SX is the greatest lower bound of the family $\{a_sX\}_{s\in S}$.

In order to prove the second part of the lemma, let us consider the mapping a_1: $X \to aX \times Y$ determined by $a_1(x) = (a(x), \varphi(x))$. The compactification $a_1X = \overline{a_1(X)} \subset aX \times Y$ is smaller than a_sX for every $s \in S$. Therefore we have $a_SX \succ a_1X$ and, for some mapping f: $a_SX \to a_1X$, $fa_S = a_1$. Since $\tilde{\varphi}$: $a_1X \to Y$, where $\tilde{\varphi}(x, y) = y$, is an extension of φ over a_1X, $\tilde{\varphi}f$ is the extension of φ over a_SX. Thus the proof of Lemma 3 is complete.

Let us denote the greatest lower bound of $\{a_sX\}_{s\in S}$ either by $\bigwedge_{s\in S} a_sX$ or by $a_1X \wedge a_2X \wedge \ldots \wedge a_kX$ in the case where S is finite.

It is known that if $a_0X \succ a_1X$ and a map φ: $X \to Y$, where Y is a compact space, can be extended over a_1X, then φ can also be extended over a_0X. An analogous fact does not hold when φ: $X \to X$. In this case it may happen that there exists an extension φ_1: $a_1X \to a_1X$, though there is no extension of φ over a_0X.

The following theorem is proved in [13]

THEOREM S. *Let X be a normal space. For every compactification a_1X such that $w(a_1X) = w(X)$ there is a compactification aX of X satisfying the following conditions:*

1) $aX \succ a_1X$;
2) $\dim aX = \dim X$ (¹);
3) $w(aX) = w(X)$.

(¹) By $\dim X$ we mean a *Lebesgue covering dimension*, i.e., $\dim X \leqslant n$ if and only if every finite open covering of X has a finite open refinement of order at most $n+1$.

Remark. In paper [13] it is not stated that aX satisfies condition 1). However, it follows from the construction given there that the condition can also be satisfied.

In [9] and, independently, in [19] and [4] the following generalizations of theorem S are proved:

THEOREM M. *For any space X and any pair of compactifications a_0X, a_1X of X such that $a_0X \succ a_1X$ there exists a compactification aX satisfying the following conditions:*

1) $a_0X \succ aX \succ a_1X$;
2) $\dim aX = \dim a_0X$;
3) $w(aX) = w(a_1X)$.

THEOREM VE. *For every normal space X, any family $\Phi = \{\varphi_s\}_{s\in S}$ of mappings of X into itself such that $\overline{\overline{S}} \leqslant w(X)$ and for any compactification a_1X such that $w(a_1X) = w(X)$ there exists a compactification aX of X satisfying the following conditions:*

1) $aX \succ a_1X$;
2) $\dim aX = \dim X$;
3) $w(aX) = w(X)$;
4) *aX is a Φ-compactification.*

Remark. As in [13], it is not stated explicitly in [19] and [4] that aX satisfies condition 1), but it follows from the construction given in [4] that the condition can also be satisfied.

We shall prove a theorem (Theorem 1) which generalizes Theorem M and Theorem VE. We shall use Theorem M in the proof of Theorem 1. In order to make the paper self-contained we shall include a proof of Theorem M. The proof we are going to give here is quite different from the original proof from [9], and seems to be somewhat simpler. In fact, we get the proof by a little modification of the proof of Theorem S given in [13].

First, we shall fix the terminology. By a *cover* we always mean a cover by open subsets. By a *uniformity* for X we mean a uniformity in the sense of Tukey [18], i.e. a family $\mathcal{U}$ of covers of X that satisfy the following conditions (²):

(i) For $\mathfrak{A}, \mathfrak{B} \in \mathcal{U}$ there exists $\mathfrak{C} \in \mathcal{U}$ such that $\mathfrak{C} > \mathfrak{A} \wedge \mathfrak{B}$;
(ii) For $\mathfrak{A} \in \mathcal{U}$ there exists $\mathfrak{B} \in \mathcal{U}$ such that $\mathfrak{B} * > \mathfrak{A}$;
(iii) If $\mathfrak{A} \in \mathcal{U}$ and $\mathfrak{A} > \mathfrak{B}$ then $\mathfrak{B} \in \mathcal{U}$;
(iv) For any $x \in X$ and any neighbourhood V of x there exist $\mathfrak{A} \in \mathcal{U}$ and $U \in \mathfrak{A}$ such that $x \in U \subset V$.

(²) Let $\mathfrak{A}$ and $\mathfrak{B}$ be covers of X. By $\mathfrak{A} \wedge \mathfrak{B}$ we denote the cover composed of all sets of the form $U \cap V$, where $U \in \mathfrak{A}$ and $V \in \mathfrak{B}$. We say that the cover $\mathfrak{B}$ is a *refinement* of $\mathfrak{A}$ if for every $V \in \mathfrak{B}$ there exists $U \in \mathfrak{A}$ such that $V \subset U$: we then write $\mathfrak{B} > \mathfrak{A}$. We say that the cover $\mathfrak{B}$ is a *star-refinement* of $\mathfrak{A}$ if for every $V \in \mathfrak{B}$ the star $\mathrm{St}(V, \mathfrak{B})$ of V in the cover $\mathfrak{B}$, i.e., the union of all elements of $\mathfrak{B}$ which are not disjoint from V, is contained in an element $U \in \mathfrak{A}$: we then write $\mathfrak{B} * > \mathfrak{A}$.

By a *base for uniformity* we mean any family of covers of X that satisfy conditions (i), (ii) and (iv).

Proof of Theorem M. First, let us suppose that the dimension of α_0X is finite and equal to n. A cover $\{U_s\}_{s\in S}$ of X is said to be *extendable* over α_0X if there exists a cover $\{\widetilde{U}_s\}_{s\in S}$ of α_0X such that $\alpha_0(U_s) = \alpha_0(X)\cap\widetilde{U}_s$ for every $s\in S$. Let us denote by $\mathcal{W}$ the family of all finite covers of X that are extendable over α_0X, and by $\mathcal{W}_0$ the subset of $\mathcal{W}$ composed of all covers of order at most $n+1$. Since $\dim\alpha_0X = n$, there exists a cover $\mathfrak{R}_0\in\mathcal{W}_0$ which has no refinement belonging to $\mathcal{W}_0$ and of order $\leqslant n$.

In order to prove Theorem M it suffices to show that there exists a base $\mathcal{V}$ for uniformity in X which satisfies the following conditions:

(1) $\mathcal{V}\subset\mathcal{W}_0$;
(2) $\overline{\overline{\mathcal{V}}} = w(\alpha_1X)$;
(3) $\mathfrak{R}_0\in\mathcal{V}$;
(4) For every cover $\{U_s\}_{s\in S}$ of α_1X there exists in $\mathcal{V}$ a refinement of $\{\alpha_1^{-1}(U_s)\}_{s\in S}$.

In fact, it follows from some fundamental theorems on uniformities that a completion of a uniform space $(X,\mathcal{U})$, where $\mathcal{U}$ is a uniformity generated by $\mathcal{V}$, is a compactification of X that satisfies the conditions of the theorem.

First, we are going to construct in X a base for the uniformity $\mathcal{V}_0\subset\mathcal{W}$ satisfying the following conditions:

(5) $\mathcal{V}_0$ is partially ordered by a relation $\geqslant$ that is stronger than $>$, i.e., such that $\mathfrak{B}\geqslant\mathfrak{A}$ implies $\mathfrak{B}>\mathfrak{A}$;
(6) For every $\mathfrak{B}_0\in\mathcal{V}_0$, the family of covers $\mathfrak{B}\in\mathcal{V}_0$ such that $\mathfrak{B}_0\geqslant\mathfrak{B}$ is finite;
(7) $\{X\}\in\mathcal{V}_0$, $\overline{\overline{\mathcal{V}}}_0 = w(\alpha_1X)$;
(8) For any cover $\{U_s\}_{s\in S}$ of α_1X one can find a refinement of $\{\alpha_1^{-1}(U_s)\}_{s\in S}$ belonging to $\mathcal{V}_0$.

The space α_1X can be treated as a subspace of the Tichonov cube $I^\tau = \mathop{\mathbf{P}}_{t\in T} I_t$, where $T=\tau=w(\alpha_1X)$, and the space X as the subspace of α_1X. Let $\mathcal{V}_I = \{\mathfrak{J}_n\}_{n=1}$ be a family of covers of the interval I, where $\mathfrak{J}_n = \{(k/2^n,\ k+2/2^n)\}_{k=0}^{2^n-2}$. One can easily verify that $\mathfrak{J}_{n+1}*>\mathfrak{J}_n$, and hence $\mathcal{V}_I$ is a base for the uniformity in I. Let us denote by $\mathcal{V}_1$ a base for the product uniformity in I^τ composed of all covers of the form $\mathfrak{A}(t_1,t_2,\dots,t_k;\ n_1,n_2,\dots,n_k) = \mathop{\mathbf{P}}_{t\in T} V_t$, where $V_t = I_t$ for $t\neq t_i$ and $V_{t_i}\in\mathfrak{J}_{n_i}$ for $i=1,2,\dots,k$.

For every $\mathfrak{A} = \{U_s\}_{s\in S}\in\mathcal{V}_1$ let $\mathfrak{A}|X$ denote the cover $\{U_s\cap X\}_{s\in S}$ of X. Let $\mathcal{V}_0 = \{\mathfrak{A}|X\}_{\mathfrak{A}\in\mathcal{V}_1}$ and put $\mathfrak{A}|X\geqslant\mathfrak{B}|X$ if and only if $\mathfrak{A}>\mathfrak{B}$ and $\mathfrak{A}\neq\mathfrak{B}$. It is easy to check that $\mathcal{V}_0$ is contained in $\mathcal{W}$ and satisfies conditions (5)-(8).

Let $\mathbf{T}$ denote the set of all triples $(\mathcal{V}',\mathcal{V},\varphi)$, where $\mathcal{V}'\subset\mathcal{V}_0$, $\mathcal{V}\subset\mathcal{W}_0$ and φ is a one-one map of $\mathcal{V}'$ onto $\mathcal{V}$ that satisfies the following conditions:

(9) $\mathfrak{R}_0\in\mathcal{V}$;
(10) $\varphi(\mathfrak{A})>\mathfrak{A}$ for every $\mathfrak{A}\in\mathcal{V}'$;
(11) If $\mathfrak{A},\mathfrak{B}\in\mathcal{V}'$ and $\mathfrak{B}\geqslant\mathfrak{A}$ then $\varphi(\mathfrak{B})*>\varphi(\mathfrak{A})$.

Family $\mathbf{T}$ is not empty. Indeed, the triple $\big(\{\{X\}\},\{\mathfrak{R}_0\},\varphi\big)$, where $\varphi(\{X\}) = \mathfrak{R}_0$, belongs to $\mathbf{T}$.

Let us define a partial order in $\mathbf{T}$ agreeing that $(\mathcal{V}_1',\mathcal{V}_1,\varphi_1)>(\mathcal{V}_2',\mathcal{V}_2,\varphi_2)$ if and only if $\mathcal{V}_1'\supset\mathcal{V}_2'$ and $\varphi_2(\mathfrak{A}) = \varphi_1(\mathfrak{A})$ for every $\mathfrak{A}\in\mathcal{V}_2'$. One can easily see that for any subset $\mathbf{T}_0\subset\mathbf{T}$ ordered by $>$ there exists in $\mathbf{T}$ an element greater than any element from $\mathbf{T}_0$. Hence, by the Kuratowski—Zorn Lemma, there exists a maximal element in $\mathbf{T}$, say $(\mathcal{V}',\mathcal{V},\varphi)$. We shall show that the element in question satisfies also condition

(12) For every $\mathfrak{A}\in\mathcal{V}_0$ there is $\mathfrak{B}\in\mathcal{V}'$ such that $\mathfrak{B}\geqslant\mathfrak{A}$.

Let us suppose that condition (12) is not satisfied. Then there exists $\mathfrak{A}_0\in\mathcal{V}_0$ such that the relation $\mathfrak{B}\geqslant\mathfrak{A}_0$ does not hold for any element $\mathfrak{B}\in\mathcal{V}'$. Let us take any $\mathfrak{B}_0'\in\mathcal{V}_0$ such that $\mathfrak{B}_0'\geqslant\mathfrak{A}_0$ and let $\mathfrak{B}_1,\mathfrak{B}_2,\dots,\mathfrak{B}_k$ be all elements of $\mathcal{V}'$ which are smaller (with respect to $\geqslant$) than $\mathfrak{B}_0'$. Let $\widetilde{\mathfrak{B}}_0,\widetilde{\mathfrak{B}}_1,\dots,\widetilde{\mathfrak{B}}_k$ be extensions of covers $\mathfrak{B}_0',\varphi(\mathfrak{B}_1),\dots,\varphi(\mathfrak{B}_k)$ over α_0X, respectively, and let us consider the finite cover $\widetilde{\mathfrak{B}}_0\wedge\widetilde{\mathfrak{B}}_1\wedge\dots\wedge\widetilde{\mathfrak{B}}_k$ of α_0X. The cover has a finite star refinement $\widetilde{\mathfrak{B}}$ because the space α_0X is normal. Finally, let $\widetilde{\mathfrak{B}}_{k+1} = \{U_s\}_{s\in S}$ be a finite cover of order $\leqslant n+1$ which is a refinement of $\widetilde{\mathfrak{B}}$. The cover $\mathfrak{B} = \{\alpha_0^{-1}(U_s)\}_{s\in S}$ belongs to $\mathcal{W}_0$. Moreover, it is easy to check that the triple $(\mathcal{V}_1',\mathcal{V}_1,\varphi_1)$ where $\mathcal{V}_1' = \mathcal{V}'\cup\{\mathfrak{B}_0'\}$, $\mathcal{V}_1 = \mathcal{V}\cup\{\mathfrak{B}\}$, $\varphi_1(\mathfrak{B}_0') = \mathfrak{B}$ and $\varphi_1(\mathfrak{A}) = \varphi(\mathfrak{A})$ for $\mathfrak{A}\in\mathcal{V}'$ is greater than the maximal triple $(\mathcal{V}',\mathcal{V},\varphi)$. Thus condition (12) is satisfied.

It follows from conditions (9)-(12) that the family $\mathcal{V}$, where $(\mathcal{V}',\mathcal{V},\varphi)$ is any maximal element of $\mathbf{T}$ satisfies conditions (i), (ii), (iv) and conditions (1)-(4).

In the case where $\dim\alpha_0X = \infty$ the proof is a little more complicated. In fact, instead of $\mathcal{W}_0$ one should consider $\mathcal{W}$ and instead of $\mathfrak{R}_0$ the family of covers $\{\mathfrak{R}_i\}_{i=1}^\infty$, where $\mathfrak{R}_i$ is a cover of X that has no refinement extendable over α_0X and of order at most i.

THEOREM 1. *Let X be a space and $\Phi = \{\varphi_s\}_{s\in S}$ a family of mappings of X into itself, and let α_0X be a Φ-compactification of X. Then for every compactification α_1X such that f: $\alpha_0X\succ\alpha_1X$ and $\overline{\overline{S}}\leqslant w(\alpha_1X)$ there exists a compactification αX of X such that:*

1) $\alpha_0X\succ\alpha X\succ\alpha_1X$;
2) $\dim\alpha X = \dim\alpha_0X$;
3) $w(\alpha X) = w(\alpha_1X)$;
4) *αX is a Φ-compactification.*

Proof. Let us consider the mappings $\alpha_1\varphi_s$: $X\to\alpha_1X$, for $s\in S$. It follows from Lemma 2 that there exists a compactification $\alpha_2'X$ greater than α_1X such that, for every $s\in S$, there is an extension $\widetilde{\varphi}_s^2$: $\alpha_2'X\to\alpha_1X$ of $\alpha_1\varphi_s$ and $w(\alpha_2'X) = w(\alpha_1X)$.

Let us put $\alpha_2X = \alpha_0X\wedge\alpha_2'X$; we then have

(7) $$\alpha_0X\succ\alpha_2X\succ\alpha_1X\,.$$

The mappings $\alpha_1\varphi_s$ can be extended to the mappings $f\widetilde{\varphi}_s$: $\alpha_0X\to\alpha_1X$, where $\widetilde{\varphi}_s$: $\alpha_0X\to\alpha_0X$ is an extension of φ. It follows from Lemma 3 that one can find extensions of the mappings $\alpha_1\varphi_s$, say

(8) $$\widetilde{\varphi}_s^2\colon\ \alpha_2X\to\alpha_1X\quad\text{for}\quad s\in S\,;$$

moreover $w(\alpha_2'X)\geqslant w(\alpha_2X)\geqslant w(\alpha_1X)$ and hence $w(\alpha_2X) = w(\alpha_1X)$. By Theorem M, there exists a compactification α_3X such that

(a_1) $\alpha_0X\succ\alpha_3X\succ\alpha_2X\succ\alpha_1X$;
(b_1) $\dim\alpha_3X = \dim\alpha_0X$;
(c_1) $w(\alpha_3X) = w(\alpha_2X) = w(\alpha_1X)$.

By (a_1) and (8) we infer that

(d_1) *for every $s\in S$, there exists an extension*

$$\widetilde{\varphi}_s^3\colon\ \alpha_3X\to\alpha_1X\quad\text{of}\quad\varphi_s\colon\ X\to X\,.$$

Now let us suppose that for every $i < k$ we have already defined all compactifications $\alpha_{2i+1}X$ such that

(a_i) $\alpha_0 X \succ \alpha_{2i+1} X \succ \alpha_{2i-1} X$;
(b_i) $\dim \alpha_{2i+1} X = \dim \alpha_0 X$;
(c_i) $w(\alpha_{2i+1}X) = w(\alpha_1 X)$;
(d_i) *for every* $s \in S$, *there exists an extension*

$$\tilde{\varphi}_s^{2i+1}\colon \alpha_{2i+1}X \to \alpha_{2i-1}X \quad \text{of} \quad \varphi_s\colon X \to X .$$

Next we shall define the compactification $\alpha_{2k+1}X$ in such a way that conditions (a_k)-(d_k) are satisfied.

Let us consider the mappings $\alpha_{2k-1}\varphi_s\colon X \to \alpha_{2k-1}X$. There exists a compactification $\alpha'_{2k}X$ greater than $\alpha_{2k-1}X$ such that, for any $s \in S$, one can find an extension $\tilde{\varphi}_s^{2k}\colon \alpha'_{2k}X \to \alpha_{2k-1}X$ of $\alpha_{2k-1}\varphi_s$ and $w(\alpha'_{2k}X) = w(\alpha_{2k-1}X) = w(\alpha_1 X)$.

Put $\alpha_{2k}X = \alpha_0 X \wedge \alpha'_{2k}X$, we then have

$$\alpha_0 X \succ \alpha_{2k} X \succ \alpha_{2k-1} X . \tag{9}$$

The mappings $\alpha_{2k-1}\varphi_s$ can be extended to the mappings $g\tilde{\varphi}_s\colon \alpha_0 X \to \alpha_{2k-1}X$, where $g\colon \alpha_0 X \succ \alpha_{2k-1}X$. By Lemma 3 there exist extensions of $\alpha_{2k-1}\varphi_s$

$$\tilde{\varphi}_s^{2k}\colon \alpha_{2k}X \to \alpha_{2k-1}X \quad \text{for} \quad s \in S \tag{10}$$

and $w(\alpha_{2k}X) = w(\alpha_1 X)$.

Now, the existence of a compactification $\alpha_{2k+1}X$ which satisfies conditions (a_k)-(d_k) follows from Theorem M and (10).

The family of compactifications $\{\alpha_{2i+1}X\}$ satisfies the assumptions of Lemma 1, whence the limit αX of the corresponding inverse system is a Φ-compactification. Moreover, from the construction we infer that αX is the least upper bound of the family $\{\alpha_{2i+1}X\}$. Therefore, conditions 1) and 3) are satisfied since $w(\mathop{P}\limits_{i=1}^{\infty} \alpha_{2i+1}X) = w(\alpha_1 X)$. Finally, it follows from a well-known theorem on inverse systems (which follows for example from Lemma 3.7 [2] p. 217) that condition 2) is satisfied.

This completes the proof of Theorem 1.

Let us notice that the assumption that $\alpha_0 X$ is a Φ-compactification cannot be omitted. Indeed, let X be the real line, let $\alpha_1 X$ be the minimal (one-point) compactification of X and let $\alpha_0 X$ be any compactification different from βX. Then there exists a function $\varphi\colon X \to I \subset X$, where I denotes the interval $[0,1]$, which cannot be extended over $\alpha_0 X$. It is easy to see that for $\Phi = \{\varphi\}$ there is no compactification satisfying conditions 1) and 4) of our theorem.

If $\alpha_0 X = \beta X$ then Theorem M becomes Theorem S and Theorem 1 becomes Theorem VE. The proof of Theorem VE obtained in this way is much simpler than that given in [4].

Finally, we want to pay attention to some consequences of Theorem M. Let us consider a metric compact space X. Let D be a countable dense subset of $X \times I$. Then D does not contain any isolated points since $X \times I$ is dense in itself. Put $\alpha_0 D = \beta D$, $\alpha_1 D = X \times I$. Then it follows from Theorem M that there exists a compactification αD such that $w(\alpha D) = \aleph_0$ and $\dim \alpha D = \dim \beta D = \dim D = 0$. But αD does not contain any isolated points and hence is homeomorphic to the Cantor set (see [8], p. 58). X is the continuous image of $X \times I$; hence

COROLLARY. *Each metric compact space is the continuous image of the Cantor set.*

If X does not contain any isolated points, then one can take X instead of $X \times I$ and we have the following

THEOREM 2. *Let X be a compact metric space with no isolated points. Then there exists a mapping $f\colon C \to X$ of the Cantor set onto X. Moreover, for a dense subset D of C the mapping $f|D\colon D \to f(D) \subset X$ is a homeomorphism.*

If we consider, in addition, a countable family $\Phi = \{\varphi_i\}_{i=1}^{\infty}$ of mappings of X into itself, then the following theorem follows from Theorem 1:

THEOREM 3. *Let X be a compact metric space with no isolated points. Then, for every countable family $\Phi = \{\varphi_i\}_{i=1}^{\infty}$ of mappings of X into itself there exist a mapping $f\colon C \to X$ of the Cantor set C onto X that is a homeomorphism on a dense subset of C and mappings $\tilde{\varphi}_i\colon C \to C$ such that $f\tilde{\varphi}_i = \varphi_i f$ for $i = 1, 2, \ldots$*

Proof. The last equalities of the theorem are valid on a dense subset of the Cantor set and hence on the whole Cantor set.

Finally, let us notice that if $f\colon C \to X$ is an open mapping, then for every $\varphi\colon X \to X$ there exists a mapping $\tilde{\varphi}\colon C \to C$ such that $f\tilde{\varphi} = \varphi f$. In fact, the multivalued mapping $f^{-1}\varphi f$ is lower semi-continuous and the existence of a mapping $\tilde{\varphi}\colon C \to C$ such that $\tilde{\varphi}(x) \in f^{-1}\varphi f(x)$ follows from Theorem 2 of [10].

If we take $X = I = [0,1]$ and D equal to the set of rational numbers, then, by Theorem 2.1 of [12], we infer that for no $f\colon C \to X$, such that for some dense $D' \subset C$, $f|D'\colon D' \to D$ is a homeomorphism, there exist mappings $\tilde{\varphi}\colon C \to C$ for any $\varphi\colon X \to X$.

2. Φ-compactifications of peripherically compact spaces.

A space X is called *peripherically compact* if there exists a basis $\mathfrak{B}$ of open subsets of X such that $\mathrm{Fr}(U)$ is compact for any $U \in \mathfrak{B}$.

A basis $\mathfrak{B}$ of open subsets of X will be called a π-*basis*, if it satisfies the following conditions:

1) *if* $U_1, U_2 \in \mathfrak{B}$ *then* $U_1 \cap U_2 \in \mathfrak{B}$ *and* $U_1 \cup U_2 \in \mathfrak{B}$;
2) *if* $U \in \mathfrak{B}$ *then* $X \setminus \overline{U} \in \mathfrak{B}$;
3) $\mathrm{Fr}(U)$ *is compact for every* $U \in \mathfrak{B}$.

It is easy to prove the following

LEMMA 4. *Let X be a peripherically compact space and $\mathfrak{A}$ a family of subsets of X with compact boundaries. If $\overline{\overline{\mathfrak{A}}} \leqslant \tau = w(X)$ and a π-basis $\mathfrak{B}$ of X is given such that $\mathfrak{A} \subset \mathfrak{B}$, then there exists a π-basis $\mathfrak{B}_0$ satisfying $\mathfrak{A} \subset \mathfrak{B}_0 \subset \mathfrak{B}$ and $\overline{\overline{\mathfrak{B}}}_0 \leqslant \tau$.*

It is proved in [14] that every π-basis of X determines a compactification $\nu_{\mathfrak{B}} X$, namely the compactification which corresponds to the proximity $\delta_{\mathfrak{B}}$ defined by the following condition (3):

(1) $A \bar{\delta}_{\mathfrak{B}} B$ *if and only if there exists* $U \in \mathfrak{B}$ *such that* $\bar{A} \subset U$ *and* $\bar{B} \subset X \setminus \bar{U}$.

It is also shown in [14] that the basis $\tilde{\mathfrak{B}} = \{\tilde{U}\}$ of $\nu_{\mathfrak{B}} X$, where $\tilde{U} = \nu_{\mathfrak{B}} X \setminus \overline{X \setminus U}$ (closure in $\nu_{\mathfrak{B}} X$), satisfies

$$\mathrm{Fr}_X U = \mathrm{Fr}_{\nu_{\mathfrak{B}} X} \tilde{U} \quad \text{for every} \quad U \in \mathfrak{B}. \tag{2}$$

In the terminology used in [15] it means that $\nu_{\mathfrak{B}} X$ is a perfect compactification with respect to all $U \in \mathfrak{B}$. It is clear that (2) implies $\mathrm{ind}\big(\nu_{\mathfrak{B}} X \setminus \nu_{\mathfrak{B}}(X)\big) \leqslant 0$ (4). In [1] the notion of a compactification with a zero-dimensionally placed set of added points was introduced; here such compactifications will be called *π-compactifications*. A compactification νX is called *π-compactification* if and only if there exists a basis $\mathfrak{B}$ of open subsets of X such that $\mathrm{Fr}\, U \cap \big(\nu X \setminus \nu(X)\big) = 0$ for every $U \in \mathfrak{B}$. It follows from the results mentioned above that $\nu_{\mathfrak{B}} X$ is a π-compactification. Of course, if a space X has a π-compactification then, X is peripherically compact.

There exists a maximal π-compactification in the family of all π-compactifications of a peripherically compact space X. The compactification corresponds to the π-basis $\mathfrak{B}(X)$ composed of all open subsets with compact boundary. The compactification will be denoted by μX. For metrizable spaces, μX coincides with the compactification X^* of X constructed in [5] "*durch Endpunkte*" (see also [6] and [11]).

Let X, Y be two peripherically compact spaces and let $\mathfrak{B}$ and $\mathfrak{D}$ be π-bases of X and Y respectively. Consider the compactifications $\nu_{\mathfrak{B}} X$ and $\nu_{\mathfrak{D}} Y$ determined by the bases. A mapping $g\colon X \to Y$ is said to be a *π-mapping with respect to $\mathfrak{B}$ and $\mathfrak{D}$*, if, for every closed set $A \subset Y$ and for every $V \in \mathfrak{D}$ such that $A \subset V$, there exists $U \in \mathfrak{B}$ satisfying the following condition:

$$g^{-1}(A) \subset U \subset g^{-1}(V). \tag{3}$$

(3) For the definition of the proximity space and its properties (which shall be used in the sequel) see [17].

(4) by $\mathrm{ind} X$ we mean the inductive Menger-Urysohn dimension, i.e. $\mathrm{ind} X \leqslant 0$ if and only if X has a basis $\mathfrak{B}$ such that $\mathrm{Fr}(U) = 0$ for any $U \in \mathfrak{B}$.

If $g\colon X \to Y$ is a π-mapping with respect to the bases $\mathfrak{B}(X)$ and $\mathfrak{B}(Y)$ composed of all open sets with compact boundaries, then g is called briefly a *π-mapping*.

In the sequel, we shall mainly consider mappings of a peripherically compact space X into itself. If $g\colon X \to X$ is a π-mapping with respect to bases $\mathfrak{B}$ and $\mathfrak{B}$, then we shall say that g is a *π-mapping with respect to* $\mathfrak{B}$.

LEMMA 5. *Let X be a peripherically compact space and let $\mathfrak{B}$ be a π-basis of X. If $A = \bar{A} \subset V \in \mathfrak{B}$ then $A \bar{\delta}_{\mathfrak{B}} (X \setminus V)$.*

Proof. For every $x \in \mathrm{Fr} V$ we have $A \bar{\delta}_{\mathfrak{B}} \{x\}$ since $A = \bar{A} \subset V$. Let $V_x \in \mathfrak{B}$ be a neighbourhood of $x \in \mathrm{Fr} V$ such that $A \bar{\delta}_{\mathfrak{B}} V_x$. Since $\mathrm{Fr} V$ is compact, one can choose a finite subset $\{x_1, \ldots, x_k\} \subset \mathrm{Fr} V$ such that $V_{x_1}, \ldots, V_{x_k}$ is a covering of $\mathrm{Fr} V$. Let us put

$$W = (X \setminus \bar{V}) \cup V_{x_1} \cup \ldots \cup V_{x_k}.$$

Then we have

$$A \subset X \setminus \bar{W}, \quad \mathrm{Fr} V \subset W, \quad W \in \mathfrak{B}.$$

Hence $A \bar{\delta}_{\mathfrak{B}} \mathrm{Fr} V$ and now the lemma follows from Lemma 1 of [15].

LEMMA 6. *Let X, Y be two peripherically compact spaces and let $\mathfrak{B}$ and $\mathfrak{D}$ be π-bases of X and Y, respectively. Let us consider compactifications $\nu_{\mathfrak{B}} X$ and $\nu_{\mathfrak{D}} Y$ and a mapping $g\colon X \to Y$. Then g is a π-mapping with respect to $\mathfrak{B}$ and $\mathfrak{D}$ if and only if g can be extended to a map $G\colon \nu_{\mathfrak{B}} X \to \nu_{\mathfrak{D}} Y$.*

Proof. 1) Let $g\colon X \to Y$ be a π-mapping with respect to $\mathfrak{B}$ and $\mathfrak{D}$. It suffices to show that g is a δ-mapping of the proximity spaces $(X, \delta_{\mathfrak{B}})$ into $(Y, \delta_{\mathfrak{D}})$, i.e. that two remote sets A and B contained in Y have remote counter-images in X. Let $A \bar{\delta}_{\mathfrak{D}} B$; then we have $\bar{A} \bar{\delta}_{\mathfrak{D}} \bar{B}$ and hence we can suppose that A is closed. There exists $V \in \mathfrak{D}$ such that $A \subset V$, $B \subset Y \setminus \bar{V}$. It follows from our assumption that for some $U \in \mathfrak{B}$ condition (3) is satisfied. Therefore

$$\bar{U} \subset \overline{g^{-1}(V)} \subset g^{-1}(\bar{V})$$

and hence

$$g^{-1}(B) \subset g^{-1}(Y \setminus \bar{V}) = X \setminus g^{-1}(\bar{V}) \subset X \setminus \bar{U}. \tag{4}$$

But we infer from (3), (4) and (1) that $g^{-1}(A) \bar{\delta}_{\mathfrak{B}} g^{-1}(B)$.

2) Let A be a closed subset and let $V \in \mathfrak{D}$ be such a subset of Y that $A \subset V$. It follows from Lemma 5 that

$$A \bar{\delta}_{\mathfrak{D}} (X \setminus V). \tag{5}$$

Let us suppose that $G\colon \nu_{\mathfrak{B}} X \to \nu_{\mathfrak{D}} Y$ is an extension of g. Then the sets $g^{-1}(A)$ and $g^{-1}(Y \setminus V)$ are remote. Hence we infer from (1) that there exists $U \in \mathfrak{B}$ such that

$$g^{-1}(A) \subset U, \quad g^{-1}(Y \setminus V) \subset X \setminus \bar{U}.$$

It follows from the second equality that

$$U \subset \overline{U} = X \setminus (X \setminus \overline{U}) \subset X \setminus g^{-1}(Y \setminus V) = g^{-1}(V),$$

and hence the condition is also sufficient.

One can obtain from Lemma 6 the following particular case of Theorem 1 from [16].

COROLLARY. *Every perfect mapping $f\colon X \to Y$ (i.e. a closed mapping such that for any $y \in Y$ the set $f^{-1}(y)$ is compact) is a π-mapping, and hence there exists an extension $G\colon \mu X \to \mu Y$ of g.*

Proof. Since f is a perfect map, $f^{-1}(C)$ is compact for every compact set $C \subset Y$. Let $A = \overline{A} \subset Y$ and $V \in \mathfrak{B}(Y)$ be such that $A \subset V$. Let us consider the set $U = g^{-1}(V)$. We then have

$$\operatorname{Fr} U = \overline{g^{-1}(V)} \setminus g^{-1}(V) \subset g^{-1}(\overline{V}) \setminus g^{-1}(V) = g^{-1}(\operatorname{Fr} V).$$

The last set is compact and hence $\operatorname{Fr} U$ is also compact as a closed subset of a compact set. Therefore we have obtained

$$U \in \mathfrak{B}(X) \quad \text{and} \quad g^{-1}(A) \subset U \subset g^{-1}(V).$$

Thus g is a π-mapping.

LEMMA 7. *A mapping $g\colon X \to Y$ is a π-mapping with respect to $\mathfrak{B}$ and $\mathfrak{D}$ if and only if for any pair V_1, V_2 of elements of $\mathfrak{D}$ such that $\overline{V}_1 \subset V_2$ there exists $U \in \mathfrak{B}$ such that*

$$g^{-1}(\overline{V}_1) \subset U \subset g^{-1}(V_2).$$

Proof. The necessity of the condition is obvious. Let A be a closed subset of Y and let V_2 be an element of $\mathfrak{D}$ such that $A \subset V_2$. It follows from Lemma 5 that there exists $V_1 \in \mathfrak{D}$ such that

$$A \subset V_1 \quad \text{and} \quad X \setminus V_2 \subset X \setminus \overline{V}_1, \quad \text{i.e.} \quad \overline{V}_1 \subset V_2.$$

If U is a subset satisfying the condition of the lemma, then

$$g^{-1}(A) \subset U \subset g^{-1}(V_2).$$

Thus g is a π-mapping with respect to $\mathfrak{B}$ and $\mathfrak{D}$.

LEMMA 8. *Let X be a peripherically compact space. Let us consider an inverse system $\{\alpha_\sigma X, \varrho^\sigma_{\sigma'}\}_{\sigma \in \Sigma}$ of compactifications of the space X, where $\varrho^\sigma_{\sigma'}\colon \alpha_\sigma X \succ \alpha_{\sigma'} X$. Let $\alpha X = \lim\{\alpha_\sigma X, \varrho^\sigma_{\sigma'}\}_{\sigma \in \Sigma}$ be the limit of the system. If $\alpha_\sigma X$ is a π-compactification of X, for every $\sigma \in \Sigma$, then αX is also a π-compactification of X.*

Proof. Let us take for every $\sigma \in \Sigma$ a basis $\{U^\sigma_\lambda\}_{\lambda \in L_\sigma}$ of $\alpha_\sigma X$ such that $\operatorname{Fr} U^\sigma_\lambda \cap (\alpha_\sigma X \setminus \alpha_\sigma(X)) = 0$ for $\lambda \in L_\sigma$. Consider the base in $\underset{\sigma \in \Sigma}{P} \alpha_\sigma X$

composed of all subsets of the form $\underset{\sigma \in \Sigma}{P} U^\sigma_{\lambda_\sigma}$, where $\lambda_\sigma \in L_\sigma$ and $U^\sigma_{\lambda_\sigma} = \alpha_\sigma X$ for all but a finite number of indices $\sigma \in \Sigma$. We shall show that

$$\operatorname{Fr}\big(\underset{\sigma \in \Sigma}{P} U^\sigma_{\lambda_\sigma}\big) \cap \big(\alpha X \setminus \alpha(X)\big) = 0. \tag{6}$$

Of course, it suffices to prove (6) for $\underset{\sigma \in \Sigma}{P} U^\sigma_{\lambda_\sigma}$, where $U^\sigma_{\lambda_\sigma} \neq \alpha_\sigma X$ for exactly one index $\sigma \in \Sigma$, say σ_0.

Therefore we have

$$\operatorname{Fr}\big(\underset{\sigma \in \Sigma}{P} U^\sigma_{\lambda_\sigma}\big) = \underset{\sigma_0 \neq \sigma \in \Sigma}{P} U^\sigma_{\lambda_\sigma} \times \operatorname{Fr} U^{\sigma_0}_{\lambda_{\sigma_0}},$$

and hence if $\boldsymbol{x} = \{x_\sigma\} \in \operatorname{Fr}\big(\underset{\sigma \in \Sigma}{P} U^\sigma_{\lambda_\sigma}\big) \cap \alpha X$ then $x_{\sigma_0} \in \operatorname{Fr} U^{\sigma_0}_{\lambda_{\sigma_0}} \cap \alpha_{\sigma_0} X \subset \alpha_{\sigma_0}(X)$. Since

$$\varrho^\sigma_{\sigma'}\big(\alpha_\sigma(X)\big) = \alpha_{\sigma'}(X) \quad \text{and} \quad \varrho^\sigma_{\sigma'}\big(\alpha_\sigma X \setminus \alpha_\sigma(X)\big) = \alpha_{\sigma'} X \setminus \alpha_{\sigma'}(X),$$

we have $x_\sigma \in \alpha_\sigma(X)$ for every $\sigma \in \Sigma$. Thus $\boldsymbol{x} \in \alpha(X)$ and (6) is valid. This completes the proof of the lemma.

THEOREM 4. *Let X be a peripherically compact space of weight τ, let $\mathfrak{B}$ be a π-base of X and let Φ', Φ'' be two families of mappings defined on X and satisfying the following conditions:*

(a) *for any $\varphi \in \Phi'$, $\varphi\colon X \to X$ and φ is a π-mapping with respect to $\mathfrak{B}$;*
(b) *for any $\varphi \in \Phi''$, $\varphi\colon X \to Y_\varphi$, where Y_φ is a compact space of weight $\leqslant \tau$ and φ is a π-mapping with respect to $\mathfrak{B}$ and a π-basis $\mathfrak{B}_\varphi$ of Y_φ;*
(c) *$\overline{\overline{\Phi' \cup \Phi''}} \leqslant \tau$.*

There exists a compactification νX of X satisfying the following conditions:

(1) *$\nu_{\mathfrak{B}} X \succ \nu X$;*
(2) *νX is a π-compactification;*
(3) *$w(\nu X) \leqslant \tau$;*
(4) *νX is a Φ'-compactification;*
(5) *for every $\varphi \in \Phi''$, there exists an extension $\tilde{\varphi}\colon \nu X \to Y_\varphi$ of φ.*

Proof. We may assume that for every $\varphi \in \Phi''$ the π-base $\mathfrak{B}_\varphi$ has the cardinality $\leqslant \tau$. Consider any $\varphi \in \Phi''$ and the set of all pairs (V_1, V_2) such that $\overline{V}_1 \subset V_2 \subset Y_\varphi$ and $V_i \in \mathfrak{B}_\varphi$, for $i = 1, 2$. The cardinality of the set is evidently not greater than τ. It follows from (b) and Lemma 7 that for any pair (V_1, V_2) from our set there exists $U \in \mathfrak{B}$ such that

$$\varphi^{-1}(\overline{V}_1) \subset U \subset \varphi^{-1}(V_2).$$

Let us denote by $\mathfrak{A}$ the subset of $\mathfrak{B}$ obtained in this way for all φ in Φ''. Of course we have $\overline{\overline{\mathfrak{A}}} \leqslant \tau$. Let $\mathfrak{B}_1$ be a π-basis satisfying the conditions $\mathfrak{A}\subset\mathfrak{B}_1\subset\mathfrak{B}$ and $\overline{\overline{\mathfrak{B}}}_1\leqslant\tau$. The existence of such a base follows from Lemma 4. It is easy to see that, for $i=1$, $\alpha_1 X=\nu_{\mathfrak{B}}X$ satisfies the following conditions:

(1_i) *$\alpha_i X$ is a π-compactification determined by the base $\mathfrak{B}_i$;*

(2_i) *$\overline{\overline{\mathfrak{B}}}_i\leqslant\tau$, $\mathfrak{A}\subset\mathfrak{B}_1\subset\mathfrak{B}$, if $i>1$ then $\mathfrak{B}_{i-1}\subset\mathfrak{B}_i\subset\mathfrak{B}$;*

(3_i) *if $i>1$, then every $\varphi\in\Phi'$ is a π-mapping with respect to $\mathfrak{B}_i$ and $\mathfrak{B}_{i-1}$.*

Let us suppose that, for $i<k$, some bases $\mathfrak{B}_i$ and compactifications $\alpha_i X$ satisfying conditions (1_i)-(3_i) are already defined. Let us consider the base $\mathfrak{B}_{k-1}$ and the set of all pairs (V_1, V_2) of elements of $\mathfrak{B}_{k-1}$ which satisfy condition $\overline{V}_1\subset V_2$. By ($2_{k-1}$) the cardinality of the set is not greater than τ. It follows from (a) and Lemma 7 that for any pair (V_1, V_2) from our set and any $\varphi\in\Phi'$ there exists $U\in\mathfrak{B}$ such that

$$\varphi^{-1}(\overline{V}_1)\subset U\subset\varphi^{-1}(V_2)\,.$$

Let us denote by $\mathfrak{A}'_k$ the subset of $\mathfrak{B}$ obtained in this way for all φ in Φ'' and letus put $\mathfrak{A}_k=\mathfrak{A}'_k\cup\mathfrak{B}_{k-1}$; of course we have $\overline{\overline{\mathfrak{A}}}_k\leqslant\tau$. Let $\mathfrak{B}_k$ denote a base satisfying the conditions of Lemma 4, where $\mathfrak{A}$ is replaced by $\mathfrak{A}_k$, and let $\alpha_k X=\nu_{\mathfrak{B}_k}X$. Conditions ($1_k$)-($3_k$) are evidently satisfied. Now let us consider the inverse limit $\{\alpha_i X, \varrho^i_j\}$ of compactifications obtained in this way, where ϱ^i_j for $i>j$ is the natural map of the greater compactification $\alpha_i X$ onto the smaller compactification $\alpha_j X$. Let αX be the limit of the system. We shall show that αX satisfies conditions (1)-(5) of the theorem. Since by (2_i), $\nu_{\mathfrak{B}}X\succ\alpha_i X$ for every i, condition (1) is satisfied. By Lemma 8 condition (2) is also satisfied. Condition (3) is satisfied because for every i we infer from (2_i) that $w(\alpha_i X)\leqslant\tau$. Condition (4) follows from Lemmas 6 and 1. Finally, (5) follows from the definition of $\mathfrak{A}$, the inclusion $\mathfrak{A}\subset\mathfrak{B}_1\subset\mathfrak{B}_2\subset\ldots$ and Lemmas 7 and 6.

In an analogous way one can prove the following

THEOREM 5. *Let X be a peripherically compact space of weight τ, let $\mathfrak{B}$ be a π-base of X and let Φ', Φ'' be two families of mappings defined on X and satisfying conditions* (a) *and* (c) *of Theorem* 4 *and the following condition:*

(b') *for any $\varphi\in\Phi''$, $\varphi\colon X\to Y_\varphi$, where Y_φ is a compact space of weight $\leqslant\tau$, and φ can be extended to a map $\tilde{\varphi}\colon \nu_\varphi X\to Y_\varphi$ of a π-compactification $\nu_\varphi X$ of X into Y_φ.*

Then there exists a compactification αX which satisfies conditions (2)-(5) *of Theorem* 4.

Remark. In the first part of the paper we did not have to consider the family Φ'' because we constructed there a compactification greater than a compactification given in advance. In particular we could take any compactification over which every mapping from Φ'' can be extended.

In the present situation that is not possible. In fact, if μX is different from βX, then there exists a function $f\colon X\to I$ that cannot be extended over μX, and hence it cannot be extended over any π-compactification.

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Reçu par la Rédaction le 4. 8. 1962