The interdependence of certain consequences of the axiom of choice

by

A. Lévy (Jerusalem)

1. Introduction. Mostowski discusses in [7] the interdependence of the following statements.

$C^*(a)$ *For every function F on a* (¹) *such that, for each $\beta < a$, $F(\beta)$ is a non-void set there exists a function G on a such that $G(\beta) \in F(\beta)$ for $\beta < a$*

for a's which are cardinals. The main part of Mostowski's paper is devoted to Tarski's axiom of dependent choices:

D *If R is a binary relation and A a non-void set such that $(\forall x \in A)(\exists y \in A)xRy$, then there exists a sequence x such that $x_n R x_{n+1}$ for every $n \in \omega$.*

Mostowski shows in [7] that $D \to C^*(\omega_1)$ is unprovable without the axiom of choice (²) in a suitable system of set theory and also that $(\forall\beta)\big(\beta \textit{ is a cardinal} \vee \beta < a \to C^*(\beta)\big) \to C^*(a)$ is unprovable for any regular cardinal a in the same system (³). The aim of the present paper is to generalize and strengthen the results of Mostowski, using methods similar to his.

In order to single out the information contained in $C^*(a)$ in addition to that contained in $(\forall\beta)\big(\beta \textit{ is a cardinal} \wedge \beta < a \to C^*(\beta)\big)$ we shall consider the statement

(¹) Lower case Greek letters denote ordinals. An ordinal a is equal to the set of all ordinals smaller than a. ω_λ and $\aleph_\lambda$ will denote the λ-th infinite cardinal (where the counting starts with 0). $|x|$ will denote the cardinality of x.

(²) Whenever we shall say *provable* or *unprovable* we shall mean provable in the Zermelo-Fraenkel set theory without using the axiom of choice, or unprovable in the system $\mathfrak{S}$ of set theory which will be specified in § 3, respectively.

(³) The exact meaning of this and similar statements, in which mathematical and metamathematical variables are confused, will be discussed later.

Actually, Mostowski proved a somewhat stronger result, i.e., that some weaker form of $C^*(a)$ (in which also $|F(\beta)| \leqslant 2$ is required) does not follow from $(\forall\beta)$ $(\beta$ *is a cardinal* $\wedge\ \beta < a \to C^*(\beta))$. However, the question of what effects do restrictions on the cardinalities of the $F(\beta)$'s have will not be discussed in the present paper.

$C(\alpha)$ *Let F be a function on α such that, for each $\beta < \alpha$, $F(\beta)$ is a non-void set. If for every $\gamma < \alpha$ there exists a function H on γ such that $H(\beta) \in F(\beta)$ for $\beta < \gamma$, then there exists a function G on α such that $G(\beta) \in F(\beta)$ for $\beta < \alpha$.*

It will be shown that the following statements are provable:

(i) $C^*(\alpha) \leftrightarrow (\forall \beta \leqslant \alpha) C(\beta)$.

(ii) $C(0)$, $C(\alpha+1)$.

(iii) $C(\alpha) \leftrightarrow C(\beta)$ if α and β are confinal with the same regular ordinal.

It will also be shown that no $C(\beta)$ follows from other $C(\alpha)$'s unless it does so by (ii) or (iii).

We shall now generalize Tarski's axiom of dependent choices, which was originally formulated only for denumerably many choices to arbitrarily many choices. Given a binary relation R we shall say that a sequence λ of length γ is an *R-admissible* sequence if and only if $h \upharpoonright \lambda R h(\lambda)$ for every $\lambda < \gamma$ [4]. We shall consider the statement

$D^*(\alpha)$ *If R is a binary relation and A a set such that* [5] $(\forall \gamma < \alpha)(\forall f \in A^\gamma)(\exists x \in A) f R x$, *then there is an R-admissible sequence in A^α.*

It can be easily shown that $D^*(\omega) \leftrightarrow D$ is provable, hence D is a special case of $D^*(\alpha)$. We shall now carry out here the same transition which was carried out for $C^*(\alpha)$, i.e., we shall consider the statement

$D(\alpha)$ *If R is a binary relation and A a set such that $(\forall \gamma < \alpha)(\forall f \in A^\gamma)(\exists x \in A) f R x$ and such that if $\gamma < \alpha$ and f is an R-admissible sequence in A^γ then for every $\gamma \leqslant \delta < \alpha$ there is an R-admissible sequence g in A^δ which extends f (i.e., $f = g \upharpoonright \gamma$) then there is an R-admissible sequence G in A^α.*

It will be shown that the following statements are provable:

(i) $D^*(\alpha) \leftrightarrow (\forall \beta \leqslant \alpha) D(\beta)$.

(ii) If α is regular, then $D^*(\alpha) \leftrightarrow D(\alpha)$.

(iii) $D(0)$, $D(\alpha+1)$.

(iv) $D(\alpha) \leftrightarrow D(\beta)$ if α and β are confinal with the same regular ordinal.

It will also be shown that no $D(\beta)$ follows from other $D(\alpha)$'s unless it does so by (i)-(iv).

(4) By a sequence of length γ we mean a function on γ. $h \upharpoonright \lambda$ is the restriction of h to $\lambda = \{\langle \mu, h(\mu) \rangle | \mu < \lambda\}$.

(5) Unless otherwise mentioned x^y is the set of all functions on y into x.

Finally, we shall consider the statement [6]

$H(\omega_\alpha)$ *If A is a set such that $|A| > \omega_\beta$ for every $\beta < \alpha$ (and, for $\alpha = 0$, $|A| > n$ for every $n \in \omega$), then $|A| \geqslant \omega_\alpha$ (i.e., there are no sets A for which $\aleph(A) = \omega_\alpha$ except the obvious ones).*

It will be shown that the following statements are provable:

(i) If $\beta \geqslant \alpha$, then $H(\omega_{\beta+1}) \to H(\omega_\alpha)$.

(ii) If $\beta > \alpha$ and α, β are limit numbers confinal with the same regular ordinal, then $H(\omega_\beta) \to H(\omega_\alpha)$.

To prove (ii) we use some form of the generalized continuum hypothesis.

It will also be shown that $H(\omega_\beta)$ does not follow from other $H(\omega_\alpha)$'s unless it does so by (i) and (ii).

The interdependence of the $C(\alpha)$'s, the $D(\alpha)$'s and the $H(\omega_\alpha)$'s will be studied. Some of the results are: $D(\alpha) \to C(\alpha)$ for every α, $D(\omega_\alpha) \to H(\omega_\alpha)$ for every α, $C(\omega_\alpha) \to H(\omega_\alpha)$ for every limit number α. On the negative side, it will be shown, among other things, that $(\forall \alpha) C(\alpha)$ does not imply $D(\beta)$, where β is a limit number not confinal with ω. It is unknown to the author whether $C(\omega) \to D(\omega)$, and more generally, whether for any regular ordinal ω_α where α is a limit number (i.e., a weakly inaccessible number) $(\forall \beta < \omega_\alpha) D(\beta) \wedge C(\omega_\alpha)$ implies $D(\omega_\alpha)$.

2. Positive results. The following Theorems 1-16 are Theorems of Zermelo-Fraenkel's set theory without the axiom of choice.

THEOREM 1. $C(0)$, $C(\alpha+1)$.

The proof is obvious.

We shall say that α and β are *cofinal* if and only if α and β are limit numbers confinal with the same regular ordinal [7].

THEOREM 2. *If α and β are cofinal, then* $C(\alpha) \leftrightarrow C(\beta)$.

Proof. Without loss of generality we can assume that α is regular (otherwise there is a regular ordinal γ with which α and β are confinal and then $C(\alpha) \leftrightarrow C(\gamma) \leftrightarrow C(\beta)$) and $\beta > \alpha$.

Assume $C(\beta)$. Let f be a function on α such that $f(\gamma)$ is a non-void set for $\gamma < \alpha$ and such that for every $\gamma < \alpha$ there is a function h on γ such that $h(\lambda) \in f(\lambda)$ for $\lambda < \gamma$. Let μ be a sequence of length α ascending

(6) We assume that formulas containing the notion $|x|$ are abbreviations of other formulas which do not contain this notion. $\aleph(x)$ is the least ordinal λ such that $|\lambda|$ non $\leqslant |x|$.

(7) For the definitions of the notions of a regular ordinal and confinality and for theorems concerning these notions which are proved without using the axiom of choice see [1]. Unlike [1] we shall use the term *regular ordinal* only for limit ordinals. Note that every regular ordinal is an infinite cardinal.

to β. Define F on β by $F(\mu_\gamma) = f(\gamma)$ for $\gamma < \alpha$ and $F(\lambda) = \{t\}$ otherwise, where t is a fixed element. Let $\delta < \beta$, then for some $\gamma < \alpha$ $\delta < \mu_\gamma$; let h be a function on γ as above; put $H(\mu_\lambda) = h(\lambda)$ for $\mu_\lambda < \delta$ (in which case $\lambda < \gamma$) and $H(\lambda) = t$ for $\lambda < \delta$ otherwise; then $H(\lambda) \in F(\lambda)$ for $\lambda < \delta$. By $C(\beta)$ there is a function G on β such that $G(\gamma) \in F(\gamma)$ for $\gamma < \beta$. Put $g(\lambda) = G(\mu_\lambda)$, hence $g(\lambda) = G(\mu_\lambda) \in F(\mu_\lambda) = f(\lambda)$ for $\lambda < \alpha$ and thus $C(\alpha)$ is proved.

Assume $C(\alpha)$. Let f be a function on β such that $f(\gamma) \neq 0$ for $\gamma < \beta$ and such that for every $\delta < \beta$ there is a function h on δ such that $h(\gamma) \in f(\gamma)$ for $\gamma < \delta$. Let μ be a sequence of length α ascending to β. Let $F(\lambda) = \{h \mid h \text{ is a function on } \mu_\lambda \text{ with } h(\gamma) \in f(\gamma) \text{ for } \gamma < \mu_\lambda\}$ for $\lambda < \alpha$. By our hypothesis concerning f, and since $\mu_\lambda < \beta$ for $\lambda < \alpha$, $F(\lambda) \neq 0$ for $\lambda < \alpha$. Given any $\sigma < \alpha$ let $h \in F(\sigma)$ and put $H(\lambda) = h \upharpoonright \mu_\lambda$ for $\lambda < \sigma$; obviously $H(\lambda) \in F(\lambda)$ for $\lambda < \sigma$. Thus, by $C(\alpha)$, there is a function G on α with $G(\lambda) \in F(\lambda)$ for $\lambda < \alpha$. Define g on β by $g(\gamma) = G(\lambda)(\gamma)$ for $\gamma < \beta$, where λ is the least ordinal λ for which $\gamma < \mu_\lambda$. Obviously $g(\gamma) \in f(\gamma)$ for $\gamma < \beta$, which proves $C(\beta)$.

THEOREM 3. $C^*(\alpha) \leftrightarrow (\nabla\beta \leqslant \alpha)\, C(\beta)$.

Proof. (i) $\beta < \alpha \to \big(C^*(\alpha) \to C^*(\beta)\big)$. Let f be a function on β such that $f(\gamma)$ is a non-void set for $\gamma < \beta$. Let t be a fixed element. Put

$$F(\gamma) = \begin{cases} f(\gamma) & \text{for} \quad \gamma < \beta\,, \\ \{t\} & \text{for} \quad \beta \leqslant \gamma < \alpha\,. \end{cases}$$

By $C^*(\alpha)$ there is a function G on α with $G(\gamma) \in F(\gamma)$ for $\gamma < \alpha$. $G \upharpoonright \beta$ is the function required by $C^*(\beta)$.

(ii) $C^*(\alpha) \to C(\alpha)$ (obvious).

(iii) $C^*(\alpha) \to (\nabla\beta \leqslant \alpha)\, C(\beta)$ (by (i) and (ii)).

(iv) $(\nabla\beta \leqslant \alpha)\, C(\beta) \to C^*(\alpha)$. We shall prove it by induction on α. Assume $(\nabla\beta \leqslant \alpha)\, C(\beta)$. By the induction hypothesis and $(\nabla\beta \leqslant \alpha)\, C(\beta)$ we have $(\nabla\beta < \alpha)\, C^*(\beta)$. If $\alpha = 0$, $C^*(\alpha)$ is obvious. If $\alpha = \delta + 1$ for some δ, then $C^*(\alpha)$ follows immediately from $C^*(\delta)$. If α is a limit number let f be a function on α such that $f(\gamma)$ is a non-void set for $\gamma < \alpha$. The condition of $C(\alpha)$ is fulfilled since $(\nabla\beta < \alpha)\, C^*(\beta)$, hence, by $C(\alpha)$, there is a function g on α such that $g(\gamma) \in f(\gamma)$ for $\gamma < \alpha$; thus $C^*(\alpha)$ is proved.

THEOREM 4. $D(0)$, $D(\alpha + 1)$.

Proof. $D(0)$ is obvious. To prove $D(\alpha+1)$ extend an R-admissible sequence in A^α to an R-admissible sequence in $A^{\alpha+1}$.

THEOREM 5. *If α and β are cofinal, then* $D(\alpha) \leftrightarrow D(\beta)$.

Proof. Without loss of generality we assume that α is regular and $\beta > \alpha$.

Assume $D(\beta)$. Let A be a set and R a binary relation such that the conditions of $D(\alpha)$ hold, then $A \neq 0$ since $\alpha > 0$. Let μ be a sequence of length α ascending to β. Define the binary relation S as follows ([8]):

FSx iff $\mathfrak{D}(F) = \gamma < \beta$, $x \in A$ and

(i) $\gamma = \mu_\delta$ for some $\delta < \alpha$ and fRx where f is the function on δ defined by $f(\lambda) = F(\mu_\lambda)$, for $\lambda < \delta$,

or

(ii) $\gamma \neq \mu_\delta$ for every $\delta < \alpha$ and x is an arbitrary member of A.

Since, by our assumption, A and R satisfy the conditions of $D(\alpha)$ and since $A \neq 0$, one can easily verify that A and S satisfy the conditions of $D(\beta)$. Thus, by $D(\beta)$, there is an S-admissible sequence G in A^β. Put $g(\sigma) = G(\mu_\sigma)$ for $\sigma < \alpha$. As is easily seen g is an R-admissible sequence in A^α, which proves $D(\alpha)$.

Assume $D(\alpha)$. Let A be a set and R a binary relation which satisfy the conditions of $D(\beta)$. Let $B = \bigcup_{\lambda < \alpha} A^{\mu_\lambda}$. We define the binary relation S as follows:

FSh iff $\mathfrak{D}(F) = \gamma < \alpha$ and

(i) $F(\lambda)$ is, for every $\lambda < \gamma$, an R-admissible sequence in A^{μ_λ}, $F(\xi) \subseteq F(\lambda)$ for $\xi < \lambda < \gamma$, h is an R-admissible sequence in A^{μ_γ} and $F(\lambda) \subseteq h$ for every $\lambda < \gamma$,

or

(ii) F does not satisfy the requirements above and h is any member of B.

Since A and R satisfy the conditions of $D(\beta)$, B and S can be shown to satisfy the conditions of $D(\alpha)$. Therefore, by $D(\alpha)$, there is an S-admissible sequence G in B^α. One can prove, by induction on $\lambda < \alpha$, that $G \upharpoonright \lambda$ satisfies the requirements on F in (i) and hence that $G(\lambda) \in A^{\mu_\lambda}$. We define $g \in A^\beta$ by $g(\gamma) = G(\lambda)(\gamma)$ where λ is the least ordinal λ such that $\gamma < \mu_\lambda$. Since $G \upharpoonright \lambda$ satisfies for $\lambda < \alpha$ the requirements on F in (i) one can prove that $g \upharpoonright \mu_\lambda = G(\lambda)$ for $\lambda < \alpha$ and that g is R-admissible, which proves $D(\beta)$.

LEMMA 6. *$D^*(\alpha) \to$ If A and R are as in $D^*(\alpha)$, $\gamma < \alpha$ and f is an R-admissible sequence in A^γ, then there is an R-admissible sequence g in A^α extending f.*

Proof. Define the binary relation S by hSx iff hRx *and if* $h = f \upharpoonright \sigma$ *for some* $\sigma < \gamma$ *then* $x = f(\sigma)$. As immediately seen, for every $\sigma < \alpha$ if $h \in A^\sigma$ there is an $x \in A$ such that hSx. By $D^*(\alpha)$, there is an S-admissible sequence g in A^α. g is, obviously, also R-admissible. To see that g extends f one proves $g(\sigma) = f(\sigma)$ by induction on $\sigma < \gamma$.

([8]) $\mathfrak{D}(R)$ is the domain of the relation R, $\mathfrak{D}(R) = \{x \mid (\exists y)(\langle xy \rangle \in R)\}$. $\mathfrak{R}(R)$ is the range of R.

THEOREM 7. $D^*(\alpha)\leftrightarrow(\forall\beta\leqslant\alpha)D(\alpha)$.

Proof. (i) $\beta<\alpha\to\big(D^*(\alpha)\to D^*(\beta)\big)$. If $\beta=0$, then $D^*(\beta)$ is obvious. If $\beta>0$, let A and R be as in $D^*(\beta)$. We define the binary relation S as follows:

fSa iff $\mathfrak{D}(f)<\beta$ and fRa or $\mathfrak{D}(f)\geqslant\beta$ and $a\in A$.

Since $\beta>0$, $A\neq0$ and the requirements of $D^*(\alpha)$ are fulfilled. By $D^*(\alpha)$ there is an S-admissible sequence g in A^α. $g\upharpoonright\beta$ is an R-admissible sequence in A^β.

(ii) $D^*(\alpha)\to D(\alpha)$ (obvious).

(iii) $D^*(\alpha)\to(\forall\beta\leqslant\alpha)D(\beta)$ (by (i) and (ii)).

(iv) $(\forall\beta\leqslant\alpha)D(\beta)\to D^*(\alpha)$. We shall prove it by induction on α. Assume $(\forall\beta\leqslant\alpha)D(\beta)$. By the induction hypothesis and $(\forall\beta\leqslant\alpha)D(\beta)$ we have $(\forall\beta<\alpha)D^*(\beta)$. If $\alpha=0$, $D^*(\alpha)$ is obvious. If $\alpha=\delta+1$, for some δ, then $D^*(\alpha)$ follows immediately from $D^*(\delta)$. If α is a limit number let A and R be as in $D^*(\alpha)$. The additional condition of $D(\alpha)$ with respect to A and R is fulfilled because of $(\forall\beta<\alpha)D^*(\beta)$ and Lemma 6, hence the consequence of $D(\alpha)$ holds, which proves $D^*(\alpha)$.

THEOREM 8. *If α is regular, then $D(\alpha)\to D^*(\alpha)$.*

Proof. Let α be regular and assume $D(\alpha)$. Let A be a set and R a binary relation such that for every $\gamma<\alpha$ and for every $f\in A^\gamma$ there is an $x\in A$ such that fRx. We have to prove that there exists an R-admissible sequence in A^α.

Let $B=(\alpha\times A)\times(\alpha\times\alpha)$. We define the binary relation S as follows:

gSy iff $g\in B^\delta$ for some $\delta<\alpha$ and

(i) *$\mathfrak{D}(\mathfrak{R}(g))$ is a function f, $\mathfrak{D}(f)=\sigma$ for some $\sigma<\alpha$, σ is a limit number or 0 or σ is not a limit number and for some $z\in A$ and $\beta<\alpha$ $\langle\langle\sigma-1,z\rangle\langle\beta\beta\rangle\rangle\in\mathfrak{R}(g)$, and $y=\langle\langle\sigma x\rangle\langle\xi0\rangle\rangle$, where $x\in A$, fRx and $\xi<\alpha$,*

or

(ii) *$\mathfrak{D}(\mathfrak{R}(g))$ is a function f, $\mathfrak{D}(f)=\sigma<\alpha$, σ is not a limit number, for no $z\in A$ and $\beta<\alpha$ is $\langle\langle\sigma-1,z\rangle\langle\beta\beta\rangle\rangle\in\mathfrak{R}(g)$, and $y=\langle\langle\sigma-1,x\rangle\langle\zeta\eta\rangle\rangle$, where for some ϑ $\langle\langle\sigma-1,x\rangle\langle\zeta\vartheta\rangle\rangle\in\mathfrak{R}(g)$ and $\langle\langle\sigma-1,x\rangle\langle\zeta\xi\rangle\rangle\in\mathfrak{R}(g)$, for every $\xi<\eta$, but $\langle\langle\sigma-1,x\rangle\langle\zeta\eta\rangle\rangle\notin\mathfrak{R}(g)$,*

or

(iii) *$\mathfrak{D}(\mathfrak{R}(g))$ is not a function, or it is a function f and $\mathfrak{D}(f)\notin\alpha$ and $y\in B$.*

We shall now see that B and S satisfy the conditions of $D(\alpha)$. Let $g\in B^\delta$, $\delta<\alpha$. Obviously, g must satisfy the requirements of one of the cases (i), (ii) or (iii) above. In cases (i) and (iii) there is obviously a $y\in B$ such that gSy. In case (ii), since $f=\mathfrak{D}(\mathfrak{R}(g))$ and $\mathfrak{D}(f)=\sigma$ there are $\zeta,\vartheta<\alpha$ such that $\langle\langle\sigma-1,f(\sigma-1)\rangle\langle\zeta\vartheta\rangle\rangle\in\mathfrak{R}(g)$. Let η be the least ordinal $<\alpha$ for which $\langle\langle\sigma-1,f(\sigma-1)\rangle\langle\zeta\eta\rangle\rangle\notin\mathfrak{R}(g)$; then $gS\langle\langle\sigma-1,f(\sigma-1)\rangle\langle\zeta\eta\rangle\rangle$.

There is always such an η since $\langle\langle\sigma-1,f(\sigma-1)\rangle\langle\zeta\zeta\rangle\rangle\notin\mathfrak{R}(g)$ (note that $f(\sigma-1)\in A$ since $g\in B^\delta$ and $f=\mathfrak{D}(\mathfrak{R}(g))$).

Before we shall continue to show that B and S have the properties required by $D(\alpha)$ we shall show first that if g is an S-admissible sequence of length $\delta\leqslant\alpha$, then $\mathfrak{D}(\mathfrak{R}(g))$ is an R-admissible sequence of length $\sigma\leqslant\alpha$, and if $\delta<\alpha$, then also $\sigma<\alpha$. This will be shown by transfinite induction on δ. If δ is a limit number or 0, then $g=\bigcup_{\tau<\delta}g\upharpoonright\tau$, $f=\mathfrak{D}(\mathfrak{R}(g))$ $=\bigcup_{\tau<\delta}\mathfrak{D}(\mathfrak{R}(g\upharpoonright\tau))$. The right-hand side is, by the induction hypothesis, the union of an ascending sequence of R-admissible sequences, therefore f is an R-admissible sequence. Also by the induction hypothesis the lengths of the sequences are ordinals $<\alpha$, therefore the length of the union f is an ordinal $\leqslant\alpha$, and it is $<\alpha$ if $\delta<\alpha$ (because α is regular and hence α is not the limit of a sequence of length $\delta<\alpha$ of ordinals less than α). If δ is not a limit number, then $\delta<\alpha$ and, by the induction hypothesis, $\mathfrak{D}(\mathfrak{R}(g\upharpoonright(\delta-1)))$ is an R-admissible sequence f' of length $\sigma'<\alpha$ and thus $g\upharpoonright(\delta-1)$ satisfies the requirements for g in (i) or (ii) above. If $g\upharpoonright(\delta-1)$ satisfies (i) then, since $g\upharpoonright(\delta-1)Sg(\delta-1)$, $g(\delta-1)=\langle\langle\sigma'x\rangle\langle\zeta0\rangle\rangle$, where $x\in A$, $f'Rx$ and $\zeta<\alpha$. $g=g\upharpoonright(\delta-1)\cup\{\langle\delta-1\,g(\delta-1)\rangle\}$, $f=\mathfrak{D}(\mathfrak{R}(g))$ $=\mathfrak{D}(\mathfrak{R}(g\upharpoonright(\delta-1)))\cup\{\langle\sigma'x\rangle\}=f'\cup\{\langle\sigma'x\rangle\}$. Since $\mathfrak{D}(f')=\sigma'$, f is a sequence of length $\sigma'+1$; f is R-admissible since f' is R-admissible and $f'Rx$. If $g\upharpoonright(\delta-1)$ satisfies (ii) then $g(\delta-1)=\langle\langle\sigma'-1\,x\rangle\langle\zeta\eta\rangle\rangle$, where $\langle\langle\sigma'-1\,x\rangle\langle\zeta\vartheta\rangle\rangle\in\mathfrak{R}(g)$ for some $\vartheta<\alpha$, hence $\langle\sigma'-1\,x\rangle\in f'$, $g=g\upharpoonright(\delta-1)\cup$ $\cup\{\langle\delta g(\delta)\rangle\}$, $f=\mathfrak{D}(\mathfrak{R}(g))=\mathfrak{D}(\mathfrak{R}(g\upharpoonright(\delta-1)))\cup\{\langle\sigma'-1\,x\rangle\}=f'\cup\{\langle\sigma'-1\,x\rangle\}$ $=f'$ and then what we claim follows immediately from the induction hypothesis.

Now, let g be as above, $\delta<\alpha$, and $\delta\leqslant\varrho<\alpha$. We shall show that there exists an S-admissible sequence h of length ϱ extending g. From what was shown above it follows that g satisfies (i) or (ii) above. If g satisfies (i) let $\varrho=\delta+\tau$, $\tau<\alpha$ and we define $h(\lambda)=g(\lambda)$ for $\lambda<\delta$, $h(\delta+\lambda)=\langle\langle\sigma x\rangle\langle\tau\lambda\rangle\rangle$, where $x\in A$ and fRx, for $\lambda<\tau$. It is easy to see that h is S-admissible. If g satisfies (ii) then, as we saw, $gS\langle\langle\sigma-1\,x\rangle\langle\zeta\eta\rangle\rangle$ for some $x\in A$ and $\zeta,\eta<\alpha$. $\eta\leqslant\zeta$ since in case (ii) there is no $z\in A$ and $\beta<\alpha$ for which $\langle\langle\sigma-1\,z\rangle\langle\beta\beta\rangle\rangle\in\mathfrak{R}(g)$ and $\langle\langle\sigma-1\,x\rangle\langle\zeta\xi\rangle\rangle\in\mathfrak{R}(g)$ for all $\xi<\eta$. Let $\zeta=\eta+\tau$. We shall define a function h' on $\delta+\tau+1$ as follows: $h'(\lambda)=g(\lambda)$ for $\lambda<\delta$, $h'(\delta+\lambda)=\langle\langle\sigma-1\,x\rangle\langle\zeta\,\eta+\lambda\rangle\rangle$ for $\lambda\leqslant\tau$. As is easily seen h' is an S-admissible sequence of length $\delta+\tau+1$. If $\varrho\leqslant\delta+\tau+1$, then put $h=h'\upharpoonright\varrho$ and h is an S-admissible sequence of length ϱ extending g. If $\delta+\tau+1<\varrho$ then, since $h'(\delta+\tau)=\langle\langle\sigma-1\,x\rangle\langle\zeta\zeta\rangle\rangle$, h' satisfies (i) and then, by what we saw above, h' can be extended to an S-admissible sequence h of length ϱ, which is also an extension of g.

By $D(\alpha)$ there is an S-admissible sequence $G\in B^\alpha$. As we saw above $F=\mathfrak{D}(\mathfrak{R}(G))$ is an R-admissible sequence of length $\sigma\leqslant\alpha$. If we prove

$\sigma = \alpha$ we are done. For $\lambda < \alpha$ let $G(\lambda) = \langle\langle G_1(\lambda)G_2(\lambda)\rangle\langle G_3(\lambda)G_4(\lambda)\rangle\rangle$, then $\Re(G_1) = \sigma$. It is easy to see that G_1 is a non-decreasing function on α. For $\mu < \sigma$ let λ_μ be the least member of $\{\lambda < \alpha \mid G_1(\lambda) = \mu\}$; then since G is S-admissible, we have, for $\gamma \leqslant G_3(\lambda_\mu)$, $G_1(\lambda_\mu+\gamma) = G_1(\lambda_\mu) = \mu$, $G_2(\lambda_\mu+\gamma) = G_2(\lambda_\mu)$, $G_3(\lambda_\mu+\gamma) = G_3(\lambda_\mu)$, $G_4(\lambda_\mu+\gamma) = \gamma$, and $G_1\big(\lambda_\mu+G_3(\lambda_\mu)+1\big) = G_1(\lambda_\mu)+1 = \mu+1$. Thus G_1 gets the value $\mu<\sigma$ exactly $G_3(\lambda_\mu)+1$ consecutive times. Therefore $\alpha = \sum_{\mu<\sigma\leqslant\alpha}\big(G_3(\lambda_\mu)+1\big)$. For $\mu<\sigma$ we have $G_3(\lambda_\mu)+1<\alpha$ since $G(\lambda_\mu) \in B = (\alpha\times A)\times(\alpha\times\alpha)$ and α is a limit number. Therefore $\alpha = \sum_{\mu<\sigma\leqslant\alpha}\big(G_3(\lambda_\mu)+1\big)$ implies $\sigma=\alpha$ since α is regular.

THEOREM 9. $D(\alpha)\rightarrow C(\alpha)$.

Proof. Assume $D(\alpha)$. Let F be a function on α as assumed in $C(\alpha)$. Put $A = \bigcup_{\beta<\alpha} F(\beta)$; *$fRx$ iff f is a sequence of members of A of length $\gamma<\alpha$ and $x \in F(\gamma)$*. R and A are easily seen to satisfy the requirements of $D(\alpha)$, hence there is an R-admissible sequence G in A^α. By the definition of R we have $G(\beta)\in F(\beta)$ for $\beta<\alpha$, thus proving $C(\alpha)$.

LEMMA 10 (⁹). *$\sim H(\omega_{\alpha+1})\rightarrow$ there exists a set A such that $|A|$ non $\geqslant \omega_{\alpha+1}$ and whenever $A = C\cup D$ if $|C|\leqslant\omega_\alpha$, then $|D|\geqslant\omega_\alpha$.*

Proof. By $\sim H(\omega_{\alpha+1})$ there is a set B with $|B|$ non $\geqslant\omega_{\alpha+1}$, $|B|>\omega_\alpha$. Put $A = \omega_\alpha\times B$.

(i) $|A|$ non $\geqslant\omega_{\alpha+1}$ because if $|A|\geqslant\omega_{\alpha+1}$ then, for some subset C of A, $|C| = \omega_{\alpha+1}$. Let $D = \{x \mid (\omega_\alpha\times\{x\})\cap C\neq 0\}$; then $C\subseteq\omega_\alpha\times D$, hence $\omega_{\alpha+1}\leqslant\omega_\alpha|D|$. But D is a projection of the well ordered set C, hence D can be well ordered and therefore, by $\omega_{\alpha+1}\leqslant\omega_\alpha|D|$, we have $|D|\geqslant\omega_{\alpha+1}$, hence $|B|\geqslant\omega_{\alpha+1}$, a contradiction.

(ii) Let $A = C\cup D$ and $|C|\leqslant\omega_\alpha$. Let $E = \{x\mid(\omega_\alpha\times\{x\})\cap C\neq 0\}$. E can be well ordered since it is a projection of the well ordered set C. Since B cannot be well ordered (because $|B|>\omega_\alpha$, $|B|$ non $\geqslant\omega_{\alpha+1}$), E is a proper subset of B and there is a $y\in B-E$. $(\omega_\alpha\times\{y\})\cap C = 0$ hence $\omega_\alpha\times\{y\}\subseteq D$, $|D|\geqslant\omega_\alpha$.

THEOREM 11. $D(\omega_\alpha)\rightarrow H(\omega_\alpha)$.

Proof. Assume $\sim H(\omega_\alpha)$. Let A be an infinite set with $|A|\geqslant\omega_\beta$, for $\beta<\alpha$, $|A|$ non $\geqslant\omega_\alpha$. If α is not a limit number we can assume, by Lemma 10, that whenever $A = C\cup D$ and $|C|<\omega_\alpha$ then $|D|\geqslant\omega_\beta$ for each $\beta<\alpha$. This is always true if α is a limit number (since then, if $|C|<\omega_\alpha$, $|C| = \omega_\gamma$ for some $\gamma<\alpha$ and, $|A|\geqslant\omega_\beta$, for $\gamma+1\leqslant\beta<\alpha$, hence A has a subset E with $|E| = \omega_\beta$; $E = (E\cap C)\cup(E\cap D)$ and since $|E\cap C|\leqslant|C| = \omega_\gamma<\omega_\beta$, we have $|E\cap D| = \omega_\beta$, hence $|D|\geqslant\omega_\beta$) or if $\alpha = 0$. Define the binary relation R as follows: *fRx iff f is a one-one mapping of an ordinal into A and $x\in A-\Re(f)$ or f is not a one-one mapping of an ordinal into A and $x\in A$.* If f is an R-admissible sequence of members of A of length $\delta<\omega_\alpha$, then f is one-one and $|\Re(f)| = |\delta|<\omega_\alpha$. Let $\delta\leqslant\sigma<\omega_\alpha$, $|\sigma|<\omega_\alpha$. $A = \Re(f)\cup\big(A-\Re(f)\big)$; since $|\Re(f)|<\omega_\alpha$, $|A-\Re(f)|\geqslant|\sigma|$. Therefore there is a one-one function $g\in A^\sigma$ which extends f and which is, hence, R-admissible. If f is not one-one, then fRx for every $x\in A\neq 0$. By $D(\omega_\alpha)$ there exists an R-admissible sequence F in A^{ω_α}. By the definition of R, F is one-one and hence $|A|\geqslant|\Re(F)| = \omega_\alpha$, contradicting our choice of A.

(⁹) $\sim$ is the negation sign.

THEOREM 12 (¹⁰). $(\nabla\alpha)D(\alpha)\leftrightarrow(\nabla\alpha)H(\omega_\alpha)\leftrightarrow$ *the axiom of choice.*

Proof. (i) $(\nabla\alpha)D(\alpha)\rightarrow(\nabla\alpha)H(\omega_\alpha)$. This follows from Theorem 11.

(ii) $(\nabla\alpha)H(\omega_\alpha)\rightarrow$ *the axiom of choice*. By the theorem of Hartogs [3] for every set x there is an aleph ω_λ such that ω_λ non $\leqslant|x|$, whereas $\omega_\mu\leqslant|x|$ for every $\mu<\lambda$. By $H(\omega_\lambda)$ there is a $\mu<\lambda$, namely $=\lambda-1$, for which $|x| = \omega_\mu$. Hence x can be well ordered. Thus we proved the well ordering theorem which implies the axiom of choice.

(iii) *The axiom of choice* $\rightarrow(\nabla\alpha)D(\alpha)$. Let A and R be as in $D(\alpha)$. By the axiom of choice we can assume that A is well ordered. Define G on α as follows: For $\gamma<\alpha$, $G(\gamma)$ = *the first member of A for which $(G\upharpoonright\gamma)Ra$*; there is always such an $a\in A$ by the assumptions on R and A. This function G is obviously R-admissible.

THEOREM 13. *If α is a limit number or 0, then $C(\omega_\alpha)\rightarrow H(\omega_\alpha)$.*

Proof. Let A be an infinite set with $|A|\geqslant\omega_\beta$ for $\beta<\alpha$. We define F on ω_α as follows: $F(\gamma) = \{f\in A^\gamma\mid f$ *is one-one*$\}$ for $\gamma<\omega_\alpha$. Let $\delta<\omega_\alpha$, $|\delta|<\omega_\alpha$, hence there is a $g\in A^\delta$ which is one-one. $g\upharpoonright\gamma\in F(\gamma)$ for $\gamma<\delta$. By $C(\omega_\alpha)$ there is a function G on ω_α with $G(\gamma)\in F(\gamma)$ for $\gamma<\omega_\alpha$. Put $h(\gamma,\delta) = G(\gamma)(\delta)$ for $\delta<\gamma<\omega_\alpha$. $\Re(h)\supseteq\Re\big(G(\omega_\beta)\big)$ for $\beta<\alpha$, hence $|\Re(h)|\geqslant\omega_\beta$ for $\beta<\alpha$. Also $\Re(h)\supseteq\Re\big(G(n)\big)$ for $n<\omega$, hence $|\Re(h)|\geqslant n$ for $n<\omega$. Since $\Re(h)$ can be well ordered (because $\mathfrak{D}(h)$ can be well ordered) $|\Re(h)|\geqslant\omega_\alpha$, hence $|A|\geqslant\omega_\alpha$.

THEOREM 14. $\beta\leqslant\alpha\rightarrow\big(H(\omega_{\alpha+1})\rightarrow H(\omega_\beta)\big)$.

Proof. Assume $H(\omega_{\alpha+1})$ and let $\beta\leqslant\alpha$. Let $|A|>\omega_\gamma$ for $\gamma<\beta$. Let $B = \{0\}\times A\cup\{1\}\times\omega_\alpha$. Obviously $|B|\geqslant\omega_\alpha$. If $B = \omega_\alpha$, then $|A|\leqslant\omega_\alpha$ and, since $|A|>\omega_\gamma$ for $\gamma<\beta$, $|A|\geqslant\omega_\beta$. If $|B|>\omega_\alpha$ then, by $H(\omega_{\alpha+1})$, $|B|\geqslant\omega_{\alpha+1}$, i.e., B has a subset C with $|C| = \omega_{\alpha+1}$. Since $|C\cap(\{1\}\times\omega_\alpha)|\leqslant\omega_\alpha$, $|C\cap(\{0\}\times A)| = \omega_{\alpha+1}$, hence $|A|\geqslant\omega_{\alpha+1}>\omega_\beta$, $|A|\geqslant\omega_\beta$. Thus we proved $H(\omega_\beta)$.

(¹⁰) The theorems $D^*(\omega_\alpha)\rightarrow H(\omega_\alpha)$ as well as $(\nabla\alpha)D^*(\alpha)\leftrightarrow(\nabla\alpha)H(\omega_\alpha)\leftrightarrow$ *the axiom of choice* were known also, independently, to Mycielski (in an unpublished paper).

THEOREM 15. *Let σ be a limit number or 0 and let $\lambda > \sigma$ be such that ω_λ and ω_σ are cofinal; let* [11] $\aleph_\lambda \text{ non} \leqslant 2^{\aleph_\tau}$ *for every* $\tau < \sigma$; *then* [12] $H(\omega_\lambda) \to H(\omega_\sigma)$.

Proof. Assume $H(\omega_\lambda)$ and let A be a set such that $|A| > \aleph_\eta$ for every $\eta < \sigma$. We shall prove that $|A| \geqslant \aleph_\sigma$ by contradiction. Assume $|A| \text{ non} \geqslant \aleph_\sigma$. Let γ be the regular ordinal with which ω_λ and ω_σ are cofinal, and let μ and ν be sequences of length γ ascending to ω_λ and ω_σ, respectively. Let $B = \{\langle f\zeta\rangle \mid (\exists\beta < \gamma)$ (f *is a one-one function in* A^{ν_β} *and* $\zeta < \mu_\beta)\}$.

Since $|A| \geqslant |\xi|$ for $\xi < \omega_\sigma$, we have $|B| \geqslant |\mu_\beta|$ for $\beta < \gamma$ and hence, since $\lim\limits_{\beta<\gamma} \mu_\beta = \omega_\lambda$, we have $|B| \geqslant \omega_\eta$ for $\eta < \lambda$, and since λ is a limit number (because ω_λ is singular), $|B| > \omega_\eta$ for $\eta < \lambda$. By $H(\omega_\lambda)$ we have $|B| \geqslant \omega_\lambda$. Let C be a subset of B with $|C| = \omega_\lambda$. Consider the set $D = \bigcup\limits_{f \in \mathfrak{D}(C)} \mathfrak{R}(f)$. Since C is well ordered and all the $\mathfrak{R}(f)$'s are well ordered uniformly (since the f's are functions on ordinals), D is well ordered. Since $D \subseteq A$ and $\aleph_\sigma \text{ non} \leqslant |A|$, we have $|D| < \aleph_\sigma$; let δ be the least ordinal δ with $|D| < |\nu_\delta|$. $C \subseteq \{\langle f\zeta\rangle \mid (\exists\beta < \delta)$ (f *is a one-one function in* D^{ν_β} *and* $\zeta < \mu_\beta)\} \subseteq \bigcup\limits_{\beta<\delta} \{f \mid f$ *is a one-one function in* $D^{\nu_\beta}\} \times \mu_\beta$.

If we put, for $\beta < \delta$, $E_\beta = C \cap (\{f \mid f$ *is a one-one function in* $D^{\nu_\beta}\} \times \mu_\beta)$ we get $C = \bigcup\limits_{\beta<\delta} E_\beta$. E_β is, for $\beta < \delta$, well ordered by the well ordering of C, hence $\mathfrak{D}(E_\beta)$ is also well ordered. Since $\mathfrak{D}(E_\beta) \subseteq D^{\nu_\beta}$, $|\mathfrak{D}(E_\beta)| \leqslant |D|^{|\nu_\beta|} \leqslant \max(|D|, |\nu_\beta|)^{\max(|D|,|\nu_\beta|)}$. If $\max(|D|, |\nu_\beta|)$ is infinite, then

$$\max(|D|, |\nu_\beta|)^{\max(|D|,|\nu_\beta|)} = 2^{\max(|D|,|\nu_\beta|)}$$

and since $\max(|D|, |\nu_\beta|) < \omega_\sigma$, we have, by one of the assumptions of the theorem $\aleph_\lambda \text{ non} \leqslant 2^{\max(|D|,|\nu_\beta|)}$. If $\max(|D|, |\nu_\beta|)$ is finite then obviously, $\aleph_\lambda \text{ non} \leqslant 2^{\max(|D|,|\nu_\beta|)}$. Thus, in any case $\aleph_\lambda \text{ non} \leqslant |\mathfrak{D}(E_\beta)|$. Since $\mathfrak{D}(E_\beta)$ can be well ordered, we have $|\mathfrak{D}(E_\beta)| < \aleph_\lambda$ for $\beta < \delta$. Since $E_\beta \subseteq \mathfrak{D}(E_\beta) \times \mu_\beta$ and $\mu_\beta < \omega_\lambda$, we have $|E_\beta| < \aleph_\lambda$; and since γ is the regular ordinal with which ω_λ is confinal and $\delta < \gamma$, $|C| = |\bigcup\limits_{\beta<\delta} E_\beta| < \aleph_\lambda$, which is a contradiction.

THEOREM 16. $C(\omega_\alpha) \leftrightarrow (\forall\beta)(\exists\gamma \geqslant \beta)$ (ω_γ *and* ω_α *are confinal* $\wedge$ $H(\omega_\gamma)$).

Proof. Assume $C(\omega_\alpha)$, then by Theorems 2 and 13 we have even $(\forall\gamma > \alpha)$ (ω_γ *and* ω_α *are confinal* $\to H(\omega_\gamma)$). Now, on the other hand, assume $(\forall\beta)(\exists\gamma \geqslant \beta)$ (ω_γ *and* ω_α *are cofinal* $\wedge$ $H(\omega_\gamma)$). Let f be a function on ω_α as required by $C(\omega_\alpha)$. We have to prove the existence of a function g on ω_α such that $g(\gamma) \in f(\gamma)$ for $\gamma < \omega_\alpha$. Define F on ω_α as follows: $F(\gamma)$

[11] By $2^{\aleph_\tau}$, $|x|^{|y|}$, etc., we mean cardinal exponentiation.

[12] Note that for $\sigma = 0$ the last hypothesis is trivially true.

$= \{h \mid \mathfrak{D}(h) = \gamma \wedge (\forall\delta < \gamma)\big(h(\delta) \in f(\delta)\big)\}$ for $\gamma < \omega_\alpha$. By the assumption on f we have $F(\gamma) \neq 0$ for $\gamma < \omega_\alpha$. Let [6]

$$\omega_\beta = \max\Big(\sup_{\gamma<\omega_\alpha} \aleph\big(F(\gamma)\big),\ \omega_{\alpha+1}\Big).$$

Let $\gamma \geqslant \beta$ be such that ω_γ is confinal with ω_α and $H(\omega_\gamma)$ holds. Let σ be the regular ordinal with which ω_α and ω_γ are confinal. Let μ and ν be sequences of length σ ascending to ω_γ and ω_α, respectively. Let

$$B = \bigcup_{\lambda<\sigma} \mu_\lambda \times F(\nu_\lambda).$$

Obviously $|B| \geqslant |\mu_\lambda|$ for $\lambda < \sigma$, hence $|B| \geqslant \omega_\eta$ for $\eta < \gamma$. γ is a limit ordinal (because it is singular, being cofinal with $\omega_\alpha < \omega_\beta \leqslant \omega_\gamma$ and hence, by $H(\omega_\gamma)$, $|B| \geqslant \omega_\gamma$. Let $C \subseteq B$, $|C| = \omega_\gamma$. $C = \bigcup\limits_{\lambda<\sigma} C \cap \big(\mu_\lambda \times F(\nu_\lambda)\big)$. $\mu_\lambda < \omega_\gamma$, also $\aleph\big(F(\nu_\lambda)\big) \leqslant \omega_\gamma$, hence $\big|C \cap \big(\mu_\lambda \times F(\nu_\lambda)\big)\big| < \omega_\gamma$. Since ω_γ, is not the union of less than $|\sigma|$ sets of cardinality less than ω_γ, we have that $C \cap \big(\mu_\lambda \times F(\nu_\lambda)\big) \neq 0$ for arbitrarily large λ's less than σ. Let τ be a sequence of length σ ascending to σ such that $C \cap \big(\mu_{\tau_\lambda} \times F(\nu_{\tau_\lambda})\big) \neq 0$ for $\lambda < \sigma$. Define G on σ by $G(\lambda) =$ *the second component of the first member of* $C \cap \big(\mu_{\tau_\lambda} \times F(\nu_{\tau_\lambda})\big)$ (*in a fixed well ordering of* C), for $\lambda < \sigma$. Define g on ω_α as follows: $g(\delta) = G(\lambda)(\delta)$ for $\delta < \omega_\alpha$, where λ is the least ordinal for which $\delta < \nu_{\tau_\lambda}$. Thus we have $g(\delta) = G(\lambda)(\delta) \in f(\delta)$ for $\delta < \omega_\alpha$, since $G(\lambda) \in F(\nu_{\tau_\lambda})$.

3. Negative results. Now we shall show that certain cases of the statements considered above ($C(a)$, $D(a)$ and $H(\omega_a)$) do not follow from appropriate conjunctions of other cases of these statements. This will be done by methods developed by Fraenkel and Mostowski.

Even the independence of the axiom of choice itself is still an open problem for systems of set theory which do not admit urelements or non-founded sets. Thus we can hope, for the time being, to prove the above mentioned independence results only for a set theory which admits either urelements or non-founded sets. We shall choose a set theory which admits only urelements; however, the same results hold for a set theory which admits only non-founded sets (see, e.g., Mendelson [5]). To prove those independence results one can use a set theory with or without classes. For the sake of convenience we choose the system $\mathfrak{S}$ of set theory given in Mostowski [6]. This is a set theory of the Bernays-Gödel type, i.e., with classes, which permits the existence of urelements, but not of non-founded sets, and which does not have the axiom of choice among its axioms.

If $\mathfrak{S}$ is consistent, or if the system A, B, C, of Gödel [2] is consistent, then, by the construction of Gödel in [2] the system A, B, C, D, E, H

of [2] is consistent, where H is the generalized continuum hypothesis — $(\forall\alpha)(2^{\aleph_\alpha}=\aleph_{\alpha+1})$. This consistency will be assumed throughout this paragraph. This assumption cannot be avoided since if $\mathfrak{S}$ is not consistent no statement can be independent of any other statement in $\mathfrak{S}$. Let $\mathfrak{S}^*$ be the set theory obtained from $\mathfrak{S}$ by adding two symbols for constants L and $\precsim$ and axioms D, E, H and "*L is a class of urelements, L is not a set and $\precsim$ well-orders L*". Within the system A, B, C, D, E, H we define the classes $U_0=\{\alpha-\{0\}|\ \alpha\geqslant 2\}$, $U=\{y|\ (\exists t)\big(y\in t\wedge(\forall x\in t)\big(x\neq 0\wedge(x\subseteq t\vee x\in U_0)\big)\big)\}$. We interpret $\mathfrak{S}^*$ in the system A, B, C, D, E, H by translating "*class*" as "*subclass of U*", "*element*" as "*member of U*", 0 as $2-\{0\}$, L as $U_0-\{2-\{0\}\}$ and $\precsim$ as $\{\langle\alpha-\{0\},\beta-\{0\}\rangle|3\leqslant\alpha<\beta\}$. It is easy to see that the axioms of $\mathfrak{S}^*$ go over to theorems of A, B, C, D, E, H in this interpretation, and hence $\mathfrak{S}^*$ is consistent.

In § 4 we shall give an interpretation $\mathfrak{J}_\beta$ of $\mathfrak{S}$ in $\mathfrak{S}^*$ which depends on a formal parameter β, i.e., every sentence of $\mathfrak{S}$ goes over to a formula of $\mathfrak{S}^*$ with the free variable β. In $\mathfrak{J}_\beta$ every axiom of $\mathfrak{S}$ goes over to a formula $\varphi(\beta)$ such that $(\forall\beta)\varphi(\beta)$ is a theorem of $\mathfrak{S}^*$. It will be shown in § 4 that the following statements (a)-(g) go over by $\mathfrak{J}_\beta$ to theorems of $\mathfrak{S}^*$. In the following we write ω_α for the regular ordinal with which ω_β is cofinal. Note that in the following (a)-(g) β and α are fixed, whereas γ is a variable.

(a) *If ω_γ is not cofinal with ω_β, then $C(\omega_\gamma)$. If β is not a limit number or* 0, *then also $C(\omega_\beta)$.*

(b) *If β is a limit number or* 0, *then $\sim C(\omega_\beta)$.*

(c) *If $\gamma<\alpha$, then $D(\omega_\gamma)$.*

(d) *If ω_γ is regular and $\gamma\geqslant\alpha$ then $\sim D(\omega_\gamma)$.*

(e) *If $\gamma<\beta$, then $H(\omega_\gamma)$.*

(f) *If γ is a limit number or* 0 *and ω_γ is not cofinal with ω_β, or if β is not a limit number or* 0 *and γ is any limit number or* 0, *then $H(\omega_\gamma)$.*

(g) *If $\gamma\geqslant\beta$ and either γ is not a limit number or* 0, *or β is a limit number or* 0 *and ω_γ is cofinal with ω_β, then $\sim H(\omega_\gamma)$.*

By Theorems 1, 2, 4 and 5, (a)-(g) give all possible information concerning the $C(\gamma)$'s, the $D(\gamma)$'s and the $H(\omega_\gamma)$'s.

Let us see now what conclusions can be reached from $\mathfrak{J}_\beta$, using (a)-(g), concerning the independence of the various statements. We shall now arrive at some independence results informally and only later we shall discuss the exact formal meaning of what will be said now. By Theorems 1, 2, 4, and 5 we shall deal with the $C(\gamma)$'s and $D(\gamma)$'s only for regular ordinals γ.

Independence of $C(\omega_\alpha)$, where ω_α is regular. $(\forall\gamma)\big(\omega_\gamma$ *is regular* $\wedge\ \gamma\neq\alpha\to C(\omega_\gamma)\big)\wedge(\forall\gamma)\big(\gamma<\alpha\to D(\omega_\gamma)\big)\wedge(\forall\gamma)\big(\gamma<\beta\vee\gamma$ *is a limit number or* 0 *and ω_γ is not confinal with* $\omega_\alpha\to H(\omega_\gamma)\big)$ does not imply $C(\omega_\alpha)$ in $\mathfrak{S}$, for any β.

In (a)-(g) ω_β is confinal with ω_α, but since for every ordinal β there is an ordinal $\beta'>\beta$ such that $\omega_{\beta'}$ is cofinal with ω_α, e.g. $\beta'=\beta+\omega_\alpha$, we can drop the assumption concerning the confinality of ω_β.

It is easy to check that this result concerning the independence of $C(\omega_\alpha)$ cannot be improved since the addition of anything new to the conjunction will cause it to imply $C(\omega_\alpha)$ by the theorems of § 2. Note that one cannot expect $H(\omega_\gamma)$ to hold for arbitrarily large γ's such that ω_γ is cofinal with ω_α, by Theorem 16, and hence, by Theorem 14, the same is true also for $H(\omega_\gamma)$ with non-limit number γ.

Independence of $H(\omega_{\alpha+1})$. $(\forall\gamma)C(\omega_\gamma)\wedge(\forall\gamma\leqslant\alpha)D(\omega_\gamma)\wedge(\forall\gamma)\big(\gamma\leqslant\alpha\vee\gamma$ *is a limit number or* $0\to H(\omega_\gamma)\big)$ does not imply $H(\omega_{\alpha+1})$ in $\mathfrak{S}$.

This is obtained by substituting $\alpha+1$ for α and β in (a)-(g). It is easy to see that this result cannot be improved.

Independence of $H(\omega_\beta)$, where β is a limit number. ω_α *is the regular ordinal with which ω_β is confinal* $\wedge\ (\forall\gamma)\big(\omega_\gamma$ *is regular* $\wedge\ \gamma\neq\alpha\to\to C(\omega_\gamma)\big)\wedge(\forall\gamma<\alpha)D(\omega_\gamma)\wedge(\forall\gamma)\big(\gamma<\beta\vee\gamma$ *is a limit number or* 0 *and ω_γ is not confinal with* $\omega_\alpha\to H(\omega_\gamma)\big)$ does not imply $H(\omega_\beta)$ in $\mathfrak{S}$.

Also this result cannot be improved for the theory obtained from $\mathfrak{S}$ by adding to its axioms the generalized continuum hypothesis, by Theorem 15. Hence, an improvement of this result for the theory $\mathfrak{S}$ itself would give, as a by-product, the independence of the generalized continuum hypothesis in $\mathfrak{S}$, and hence in the system A, B, C, D of [2].

Independence of $D(\omega_{\alpha+1})$. (a) $(\forall\gamma)C(\omega_\gamma)\wedge(\forall\gamma\leqslant\alpha)D(\omega_\gamma)\wedge(\forall\gamma)\big(\gamma\leqslant\alpha\vee\gamma$ *is a limit number or* $0\to H(\omega_\gamma)\big)$ does not imply $D(\omega_{\alpha+1})$ in $\mathfrak{S}$.

(b) $(\forall\gamma)\big(\omega_\gamma$ *is regular* $\wedge\ \gamma\neq\alpha+1\to C(\omega_\gamma)\big)\wedge(\forall\gamma\leqslant\alpha)D(\omega_\gamma)\wedge(\forall\gamma)\big(\gamma<\beta\vee\gamma$ *is a limit ordinal or* 0 *and ω_γ is not confinal with* $\omega_{\alpha+1}\to H(\omega_\gamma)\big)$ does not imply $D(\omega_{\alpha+1})$ in $\mathfrak{S}$, where β is any ordinal.

The best independence result for $D(\omega_{\alpha+1})$ one can expect in the light of § 2 is stronger than (a) and (b). It is: $(\forall\gamma)C(\omega_\gamma)\wedge(\forall\gamma\leqslant\alpha)D(\omega_\gamma)\wedge\wedge(\forall\gamma)\big(\gamma<\beta\vee\gamma$ *is a limit number or* $0\to H(\omega_\gamma)\big)$ does not imply $D(\omega_{\alpha+1})$ in $\mathfrak{S}$, where β is any ordinal. Whether this is true or not is unknown to the author. In the special case of $D(\omega_1)$, (a) can be slightly improved, as will be mentioned later.

Independence of $D(\omega_\alpha)$, where α is weakly inacessible or 0. (a) $(\forall\gamma)C(\omega_\gamma)\wedge(\forall\gamma\leqslant\beta)D(\omega_\gamma)\wedge(\forall\gamma)\big(\gamma\leqslant\beta\vee\gamma$ *is a limit number or*

$0 \to H(\omega_\gamma)\big)$ does not imply $D(\omega_\alpha)$ in $\mathfrak{S}$, where β is an ordinal such that $\beta+1<\alpha$. (This is obtained by substituting $\beta+1$ for α and β in (a)-(g).)

(b) $(\forall\gamma)\big(\omega_\gamma$ *is regular* $\wedge\ \gamma\neq\alpha\to C(\omega_\gamma)\big)\wedge(\forall\gamma<\alpha)D(\omega_\gamma)\wedge(\forall\gamma)\big(\gamma<\beta\ \vee$ $\vee\ \gamma$ *is a limit number or* 0 *such that* ω_α *is not confinal with* $\omega_\gamma\to H(\omega_\gamma)\big)$ does not imply $D(\omega_\alpha)$ in $\mathfrak{S}$, where β is any ordinal.

Here, again, one can expect a stronger result, i.e. that $(\forall\gamma)C(\omega_\gamma)$ $\wedge\ (\forall\gamma<\alpha)D(\omega_\gamma)\wedge(\forall\gamma)\big(\gamma<\beta\vee\gamma$ *is a limit number or* $0\to H(\omega_\gamma)\big)$ does not imply $D(\omega_\alpha)$ in $\mathfrak{S}$, where β is any ordinal. It is unknown to the author whether this is true, even in the case $\alpha=0$, where it is unknown whether $C(\omega)$ implies $D(\omega)$ in $\mathfrak{S}$.

To study the formal meaning of the independence results we shall deal only with one simple independence result since the same procedure can be applied to all other independence results. Only occasional remarks will be made concerning the other independence results.

What does it mean that, for every ordinal α, $(\forall\gamma\leqslant\alpha)D(\omega_\gamma)$ does not imply $D(\omega_{\alpha+1})$ in $\mathfrak{S}$? Of course, this has no immediate meaning at all since we cannot quantify with a formal variable α the metamathematical statement "$(\forall\gamma\leqslant\alpha)D(\omega_\gamma)$ does not imply $D(\omega_{\alpha+1})$ in $\mathfrak{S}$". However, we can give the quantified statement two different formal meanings.

The first meaning is as follows. Let $\boldsymbol{\alpha}$ be a term in $\mathfrak{S}$ which can be proved in $\mathfrak{S}^*$ to be an ordinal, and let $\boldsymbol{\alpha}$ be absolute with respect to the interpretations $\mathfrak{J}_\beta$, i.e., if we denote with $\boldsymbol{\alpha}^{(\beta)}$ the translation of the term $\boldsymbol{\alpha}$ in $\mathfrak{J}_\beta$, then $(\forall\beta)(\boldsymbol{\alpha}=\boldsymbol{\alpha}^{(\beta)})$ is provable in $\mathfrak{S}^*$. The independence result is that $(\forall\gamma\leqslant\boldsymbol{\alpha})D(\omega_\gamma)$ does not imply $D(\omega_{\boldsymbol{\alpha}+1})$ in $\mathfrak{S}$. The same meaning can be given to all other independence results. Whenever β occurs we have just another term $\boldsymbol{\beta}$ which meets the same requirements like $\boldsymbol{\alpha}$, and if $\beta<\alpha$ is mentioned we require also $\mathfrak{S}^*\vdash\boldsymbol{\beta}<\boldsymbol{\alpha}$. To see that the independence results are correct under this meaning all one has to do is to substitute $\boldsymbol{\alpha}$ for α throughout § 4 and notice that all proofs remain correct.

The requirement of absoluteness (with respect to $\mathfrak{J}_\beta$) of the term $\boldsymbol{\alpha}$ is quite natural. Every decent term like $0, 1, \omega_0, \omega_1, \omega_\omega$, etc. is absolute. If $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ are absolute so are $\boldsymbol{\tau}+\boldsymbol{\sigma}$, $\boldsymbol{\tau}\cdot\boldsymbol{\sigma}$, $\omega_{\boldsymbol{\tau}}$, $\aleph_{\boldsymbol{\tau}}^{\aleph_{\boldsymbol{\sigma}}}$, etc. Indeed, any term in whose definition all the quantifiers are restricted to the part of the universe which is founded on the void set (i.e., restricted to the class U given by $x\in U\leftrightarrow(\exists t)\big(x\in t\wedge(\forall y\in t)(y\subseteq t)\big)$) is absolute. To see why the requirement of absoluteness is necessary consider the case where $\boldsymbol{\alpha}$ is the term "*the least ordinal* α *such that* $D(\omega_{\alpha+1})$ *if there is such an ordinal, and* 1 *otherwise*". We shall see that $(\forall\gamma\leqslant\boldsymbol{\alpha})D(\omega_\gamma)$ does imply $D(\omega_{\boldsymbol{\alpha}+1})$. If there is an α such that $D(\omega_{\alpha+1})$, then $D(\omega_{\boldsymbol{\alpha}+1})$, hence $(\forall\gamma\leqslant\boldsymbol{\alpha})D(\omega_\gamma)\to$ $\to D(\omega_{\boldsymbol{\alpha}+1})$; if there is no such α, then $\boldsymbol{\alpha}=1$ and since $\sim D(\omega_1)$, we have $(\forall\gamma\leqslant\boldsymbol{\alpha})D(\omega_\gamma)\to D(\omega_{\boldsymbol{\alpha}+1})$.

The second meaning is more natural that the first, but requires semantical notions rather than the simple syntactical notions used for the first meaning. Now, the above independence result will be understood as follows. For every model $\mathcal{M}$ of $\mathfrak{S}$ and for every ordinal α of $\mathcal{M}$ there is a model $\mathcal{M}'$ of $\mathfrak{S}$ with the same ordinal numbers such that, for every $\gamma<\alpha$, γ satisfies the formula $D(\omega_\lambda)$ in $\mathcal{M}'$ whereas $\alpha+1$ does not satisfy $D(\omega_\lambda)$ in $\mathcal{M}'$. The other independence results are given the same meaning. To see that the independence results are correct under this meaning notice that the interpretation of $\mathfrak{S}^*$ in $\mathfrak{S}$ (via A, B, C, D, E, H) given in the beginning of this paragraph actually gives us a construction of a model $\mathcal{M}''$ of $\mathfrak{S}^*$ for every given model $\mathcal{M}$ of $\mathfrak{S}$ (or of A, B, C of [2]) such that the ordinals with the relation $<$ in $\mathcal{M}''$ are isomorphic to the ordinals with the relation $<$ in $\mathcal{M}$. Thus we can replace $\mathcal{M}''$ by an isomorphic model $\mathcal{M}'''$ of $\mathfrak{S}^*$ in which the ordinals and the relation $<$ are the same as the ordinals and the relation $<$ of $\mathcal{M}$. The interpretation $\mathfrak{J}_\beta$ produces, starting with $\mathcal{M}'''$, a model $\mathcal{M}'$ of $\mathfrak{S}$ which has the same ordinals with the relation $<$ as $\mathcal{M}$ and in which (a)-(g) hold for the given β.

In § 4 we shall give also another interpretation $\mathfrak{J}$ of $\mathfrak{S}$ in $\mathfrak{S}^*$ in which the following sentences go over the theorems of $\mathfrak{S}^*$:

(a) $(\forall\gamma)C(\omega_\gamma)$.

(b) $D(\omega)$.

(c) *If* ω_γ *is regular and* $\gamma\neq 0$, *then* $\sim D(\omega_\gamma)$.

(d) *If* γ *is* $0, 1$, *or a limit number, then* $H(\omega_\gamma)$.

(e) *If* $\gamma\neq 0, 1$ *and is not a limit number, then* $\sim H(\omega_\gamma)$.

Independence of $D(\omega_1)$. (a) $(\forall\gamma)C(\omega_\gamma)\wedge D(\omega)\wedge(\forall\gamma)\big(\gamma=0\vee\gamma$ $=1\vee\gamma$ *is a limit number* $\to H(\omega_\gamma)\big)$ does not imply $D(\omega_1)$ in $\mathfrak{S}$. (Part (b) is just part (b) of the theorem on the independence of $D(\omega_{\alpha+1})$, where 0 is substituted for α.)

Part (a) is a slight improvement of what we get from part (a) of the theorem on the independence of $D(\omega_{\alpha+1})$ by substituting 0 for α, since here $H(\omega_1)$ is added.

4. The interpretations. We proceed now within $\mathfrak{S}^*$. Let K be a non-void set of urelements. We define: $K_0=K$, $K_\eta=K\cup\bigcup_{\mu<\eta}P(K_\eta)$, where $P(x)$ is the power set of x. We say that x is a K-element iff there exists an ordinal η such that $x\in K_\eta$. Let Φ be the group of all permutations of K. For $\varphi\in\Phi$ and a K-element x we define $\varphi(x)$ as follows:

(1) $\varphi(x)$ *is, for* $x\in K$, *already defined*; $\varphi(x)=\{\varphi(y)\,|\,y\in x\}$ *for* $x\notin K$.

One can easily verify that

(2) $\varphi\psi(x)=\varphi\big(\psi(x)\big)$; $1(x)=x$, *where* 1 *is the identity on* K.

Let Ψ be a subgroup of Φ. Let $Q \subseteq P(K)$ have the following properties:

(i) $\bigcup Q \subseteq K$,

(ii) $a, b \in Q \rightarrow a \cup b \in Q$,

(iii) $a \in Q \wedge \varphi \in \Psi \rightarrow \varphi(a) \in Q$.

Let the variables a, b, c range over Q. A permutation $\varphi \in \Psi$ is said to be b-*identical* iff $\varphi(x) = x$ for every $x \in b$. The group of all b-identical members of Ψ will be denoted with Ψ^b. An element x is said to be b-*symmetric* iff x is a K-element and $\varphi(x) = x$ for every $\varphi \in \Psi^b$. We define

(3) *x is an M-element iff x is a member of some set t such that t is transitive (i.e., $u \in v \in t \rightarrow u \in t$) and such that for every member y of t there is a $b \in Q$ such that y is b-symmetric.*

(4) *X is an M-class iff every member of X is an M-element and* $(\exists b \in Q)(\forall \varphi \in \Psi^b)(\forall y \in X)(\varphi(y) \in X)$.

By an M-*function* (M-*relation*, M-*set*, etc.) we shall mean an M-class which is a function (a relation, a set, etc.). The following are easy to prove (see [6]):

(5) *If x is an M-element and $y \in x$, then y is an M-element.*

(6) *If a set x is an M-element it is also an M-class.*

(7) *If an M-class X is a set it is also an M-element* (to show this one proves that the transitive closure of X, i.e. the intersection of all transitive classes which contain X, satisfies the requirement for t in (3)).

(8) *If all the members of x are M-elements and x is b-symmetric for some $b \in Q$, then x is an M-element.*

(9) *Every ordinal is a 0-symmetric M-element* (the proof is by induction — 0 can be assumed to be a member of Q without loss of generality).

(10) *Every finite set of M-elements is an M-element, and hence every ordered n-tuple of M-elements is an M-element.*

(11) *Let F be a function mapping a class of ordinals onto X, and let $b \in Q$. F is a b-symmetric M-function iff each member y of X is a b-symmetric M-element.*

(12) *If x is a b-symmetric M-element and $\psi \in \Psi$, then $\psi(x)$ is a $\psi(b)$-symmetric M-element.*

As shown, essentially, by Mostowski in [6] if we interpret $\mathfrak{S}$ in $\mathfrak{S}^*$ by replacing the primitive notions "*class*", "*element*", "ϵ" and "0" by "M-*class*", "M-*element*", "ϵ" and "0", respectively, all the axioms (and the theorems) of $\mathfrak{S}$, as well as the sentences $(\forall a)(P(a)$ *can be well-ordered*$)$ and $(\forall a)(2^{\aleph_a} = \aleph_{a+1})$ go over to theorems of $\mathfrak{S}^*$. It follows easily from

the properties of the interpretation that the formulas $C(\lambda)$, $D(\lambda)$ and $H(\omega_\lambda)$ go over, respectively, to the formulas

$\mathfrak{C}(\lambda)$ *Let F be an M-function on λ such that $F(\gamma)$ is, for each $\gamma < \lambda$, a non-void set. If for every $\delta < \lambda$ there exists an M-function H on δ such that $H(\gamma) \in F(\gamma)$ for $\gamma < \delta$ then there exists an M-function G on λ such that $G(\gamma) \in F(\gamma)$ for $\gamma < \lambda$.*

$\mathfrak{D}(\lambda)$ *If R is a binary M-relation and A an M-set such that $(\forall \gamma < \lambda)$ $(\forall f \in A^\gamma)(f$ is M-function$\rightarrow (\exists x \in A) fRx)$ and such that if f is an R-admissible M-sequence in A^γ, where $\gamma < \lambda$, and $\gamma \leqslant \delta < \lambda$, then f can be extended to an R-admissible M-sequence g in A^δ, then there is an R-admissible M-sequence in A^λ.*

$\mathfrak{H}(\omega_\lambda)$ *If A is an M-set such that for every $\gamma < \lambda$ there is a one-one M-function F mapping ω_γ into A, but there is no such F mapping ω_γ onto A (and, for $\lambda = 0$, $|A| \geqslant n$ for every $n \in \omega$)* [13]*, then there is a one-one M-function G mapping ω_λ into A.*

Finally we remark that, as easily seen, the ordering of the ordinals, the function ω_λ and the notions of confinality and regularity are absolute with respect to this interpretation.

The interpretation $\mathfrak{J}_\beta$. We take for K the subset of L which consists of the first ω_β members of L. For Ψ we take the group of all permutations of K. We take $Q = \{a | a \subseteq K \wedge |a| < \omega_\beta\}$. We recall that the regular ordinal with which ω_β is confinal is denoted with ω_a.

(a) *If ω_γ is not confinal with ω_β, then $\mathfrak{C}(\omega_\gamma)$. If β is not a limit number or* 0, *then also* $\mathfrak{C}(\omega_\beta)$.

Let F be an M-function on ω_γ as assumed in $\mathfrak{C}(\omega_\gamma)$. Since F is an M-function, it is a b-symmetric M-element for some $b \in Q$ and hence, by (11), each $F(\zeta)$ is a b-symmetric M-element, for $\zeta < \omega_\gamma$. By the assumption of $\mathfrak{C}(\omega_\gamma)$ for each $\zeta < \omega_\gamma$ there is an M-function f on ζ such that $f(\delta) \in F(\delta)$ for $\delta < \zeta$. By the axiom of choice there is a function f on ω_γ such that f_ζ is, for $\zeta < \omega_\gamma$, an M-function on ζ such that $f_\zeta(\delta) \in F(\delta)$ for $\delta < \zeta$. Since f_ζ is an M-function, it is a-symmetric for some $a \in Q$; by the axiom of choice there is a function a on ω_γ such that, for $\zeta < \omega_\gamma$, $a_\zeta \in Q$ and f_ζ is a_ζ-symmetric. Thus $|a_\zeta|$ is a sequence of length ω_γ of cardinals $< \omega_\beta$. Let ω_σ be the regular ordinal with which ω_γ is confinal. By a general theorem about sequences of ordinals we get that if $\sigma \neq a$ or if β is not a limit number then there is a cardinal $\omega_\xi < \omega_\beta$ and a sequence μ of length ω_σ ascending to ω_γ such that $|a_{\mu_\lambda}| \leqslant \omega_\xi$ for $\lambda < \omega_\sigma$. Let c be a subset of $K - b$ of cardinality ω_ξ; then, for a fixed $\lambda < \omega_\sigma$, let φ be

[13] The absoluteness of the notion of finiteness for such an interpretation is shown, essentially, in [4].

a permutation of K which is the identity on b and which maps a_{μ_λ} into $b \cup c$. There is always such a permutation φ since $|a_{\mu_\lambda}| \leqslant \omega_\xi = |c|$. By (12), $\varphi(f_{\mu_\lambda})$ is a $\varphi(a_{\mu_\lambda})$-symmetric M-element, and hence also a $b \cup c$-symmetric M-element. As follows easily from (2) and (9), $\varphi(f_{\mu_\lambda})$ is a function on μ_λ. $\langle \zeta, f_{\mu_\lambda}(\zeta)\rangle \in f_{\mu_\lambda}$, for $\zeta < \mu_\lambda$, hence $\langle \zeta, \varphi(f_{\mu_\lambda}(\zeta))\rangle = \langle \varphi(\zeta), \varphi(f_{\mu_\lambda}(\zeta))\rangle = \varphi(\langle \zeta, f_{\mu_\lambda}(\zeta)\rangle) \in \varphi(f_{\mu_\lambda})$, i.e., $\varphi(f_{\mu_\lambda})(\zeta) = \varphi(f_{\mu_\lambda}(\zeta))$. $F(\zeta)$ is b-symmetric, $\varphi \in \Psi^\beta$ and $f_{\mu_\lambda}(\zeta) \in F(\zeta)$, for $\zeta < \mu_\lambda$, hence $\varphi(f_{\mu_\lambda})(\zeta) = \varphi(f_{\mu_\lambda}(\zeta)) \in \varphi(F(\zeta)) = F(\zeta)$. Denote $\varphi(f_{\mu_\lambda})$ with g; then g is a $b \cup c$-symmetric M-function on μ_λ and $g(\zeta) \in F(\zeta)$ for $\zeta < \mu_\lambda$. By the axiom of choice there is a function g on ω_σ such that, for $\lambda < \omega_\sigma$, g_λ is a $b \cup c$-symmetric M-function and $g_\lambda(\zeta) \in F(\zeta)$ for $\zeta < \mu_\lambda$. We define the function G on ω_γ by $G(\zeta) = g_\lambda(\zeta)$, where λ is the least ordinal such that $\zeta < \mu_\lambda$. Since $G(\zeta) = g_\lambda(\zeta)$ and g_λ is a $b \cup c$-symmetric M-function, $G(\zeta)$ is a $b \cup c$-symmetric M-element. Thus, by (11), G is a $b \cup c$-symmetric M-function and $G(\zeta) \in F(\zeta)$ for $\zeta < \omega_\gamma$.

(b) *If β is a limit number or 0, then* $\sim\mathfrak{C}(\omega_\beta)$.

We define the function F on ω_β by $F(\gamma) = \{h \mid h \in K^\gamma \wedge h \text{ is one-one}\}$. Let $h \in F(\gamma)$; then h is obviously an $\Re(h)$-symmetric M-element, hence $F(\gamma)$ is a 0-symmetric M-element, hence F itself is a 0-symmetric M-function. $F(\gamma)$ is, for every $\gamma < \omega_\beta$, a non-void set, since $|K| = \omega_\beta$. Let $\delta < \omega_\beta$ and let $h \in F(\delta)$. Define the function g on δ by $g(\gamma) = h \upharpoonright \gamma$ for $\gamma < \delta$. Obviously $g(\gamma) \in F(\gamma)$ for $\gamma < \delta$. Thus all the hypotheses of $\mathfrak{C}(\omega_\beta)$ are fulfilled. Assume that the conclusion of $\mathfrak{C}(\omega_\beta)$ holds also, i.e., that there is an M-function G on ω_β such that $G(\gamma) \in F(\gamma)$ for $\gamma < \omega_\beta$. Since G is an M-function, G is b-symmetric for some $b \in Q$. Since β is a limit number or 0 and $|b| < \omega_\beta$, there is a $\xi < \omega_\beta$ such that $|b| < |\xi| < \omega_\beta$. Since G is b-symmetric, $G(\xi)$ is also b-symmetric, hence also $G(\xi)(\gamma)$ is b-symmetric for $\gamma < \xi$. But $G(\xi)(\gamma) \in K$, hence it is b-symmetric only if $G(\xi)(\gamma) \in b$. Thus $\Re(G(\xi)) \subseteq b$, but this is impossible since $G(\xi)$ is one-one, being a member of $F(\xi)$, and hence $|\Re(G(\xi))| = |\xi| > |b|$, a contradiction.

(c) $\gamma < \alpha \rightarrow \mathfrak{D}(\omega_\gamma)$.

Let R and A be as assumed in $\mathfrak{D}(\omega_\gamma)$. Let $\zeta < \omega_\alpha$ and let $f \in A^\zeta$. Since A is an M-set, $f(\delta)$ is, for $\delta < \zeta$, a b-symmetric M-element for some $b \in Q$. By the axiom of choice there is a function b on ζ such that $b_\delta \in Q$ for $\delta < \zeta$ and $f(\delta)$ is a b_δ-symmetric M-element. $|\bigcup_{\delta<\zeta} b_\delta| < \omega_\beta$ since ω_β is confinal with ω_α, $\zeta < \omega_\alpha$ and $|b_\delta| < \omega_\beta$ for $\delta < \zeta$. Hence $f(\delta)$ is, for $\delta < \zeta$, a $\bigcup_{\delta<\zeta} b_\delta$-symmetric M-element and by (11) f itself is a $\bigcup_{\delta<\zeta} b_\delta$-symmetric M-element. By the assumption of $\mathfrak{D}(\omega_\gamma)$ we have now that for every $\zeta < \omega_\gamma$ and every $f \in A^\zeta$ (which is necessarily an M-function) there is an $x \in A$ such that fRx. By $D^*(\omega_\gamma)$, which follows from the axiom of

choice, there is an R-admissible function F in A^{ω_γ}. F is an M-function by what was shown above, since $\omega_\gamma < \omega_\alpha$.

(e) $\gamma < \beta \rightarrow \mathfrak{K}(\omega_\gamma)$.

Let A be an M-set as required in $\mathfrak{K}(\omega_\gamma)$. A is b-symmetric for some $b \in Q$. If every member of x is also b-symmetric, then let F be a one-one function mapping $|A|$ onto A. By (11), F is also a b-symmetric M-function. By the hypothesis of $\mathfrak{K}(\omega_\gamma)$ there is no one-one M-function mapping ω_δ, with $\delta < \gamma$, onto A, hence $|A| \geqslant \omega_\gamma$. $F \upharpoonright \omega_\gamma$ satisfies the conclusion of $\mathfrak{K}(\omega_\gamma)$. We shall now consider the other case, namely where there exists a member y of A which is not b-symmetric.

We shall see now that if z is an M-element which is $b \cup d$-symmetric and $b \cup d'$-symmetric, where $b, d, d' \in Q$ and b, d, d' are pairwise disjoint, then z is b-symmetric.

Let $\varphi \in \Psi^b$. We shall now show that φ is a product of members of $\Psi^{b\cup d} \cup \Psi^{b \cup d'}$. Let e be a subset of $K - (b \cup d \cup d' \cup \varphi^{-1}(d) \cup \varphi^{-1}(d'))$ of cardinality $|d|$ (this is always possible since $|b \cup d \cup d' \cup \varphi^{-1}(d) \cup \varphi^{-1}(d')| < \omega_\beta$ and hence $|K - (b \cup d \cup d' \cup \varphi^{-1}(d) \cup \varphi^{-1}(d'))| = \omega_\beta$). Let ψ be a one-one function mapping d on e. We define:

$$\varphi_1(t) = \begin{cases} \psi(t) & \text{for} \quad t \in d, \\ \psi^{-1}(t) & \text{for} \quad t \in e, \\ t & \text{for} \quad t \in K - (d \cup e). \end{cases}$$

φ_1 is well-defined because $d \cap e = 0$. φ_1 is obviously a permutation of K, and since $(d \cup e) \cap (b \cup d') = 0$, we have $\varphi_1 \in \Psi^{b\cup d'}$. We define:

$$\varphi_2(t) = \begin{cases} \varphi(t) & \text{for} \quad t \in K - (d \cup e \cup \varphi^{-1}(d)), \\ \varphi(\psi^{-1}(t)) & \text{for} \quad t \in e - \{u \mid \varphi(\psi^{-1}(u)) \in d\}, \\ \varphi(\psi(\varphi(t))) & \text{for} \quad t \in \varphi^{-1}(d) - d, \\ \varphi(\psi(\varphi(\psi^{-1}(t)))) & \text{for} \quad t \in \{u \mid \varphi(\psi^{-1}(u)) \in d\} \ (\subseteq e), \\ t & \text{for} \quad t \in d. \end{cases}$$

φ_2 is well-defined because $e \cap (d \cup \varphi^{-1}(d)) = 0$. φ_2 is easily seen to be a permutation of K. $b \cap (d \cup e \cup \varphi^{-1}(d)) = 0$ since $b \cap d = 0$ by hypothesis, $b \cap e = 0$ by definition of e and $b \cap \varphi^{-1}(d) = 0$ since $b \cap d = 0$ and φ^{-1} is the identity on b, by $\varphi \in \Psi^b$. Thus $b \subseteq K - (d \cup e \cup \varphi^{-1}(d))$ and hence, for $t \in b$, $\varphi_2(t) = \varphi(t) = t$. Thus $\varphi_2 \in \Psi^{b \cup d}$. We define:

$$\varphi_3(t) = \begin{cases} \varphi(\psi(t)) & \text{for} \quad t \in d, \\ \psi^{-1}(\varphi^{-1}(t)) & \text{for} \quad t \in \varphi(e), \\ t & \text{for} \quad t \in K - (d \cup \varphi(e)). \end{cases}$$

φ_3 is well-defined since $d \cap \varphi(e) = 0$ because $\varphi^{-1}(d) \cap e = 0$. φ_3 is easily seen to be a permutation of K. $(b \cup d') \cap (d \cup \varphi(e)) = 0$ since $d' \cap \varphi(e) = 0$, because $\varphi^{-1}(d') \cap e = 0$, and $b \cap \varphi(e) = 0$, because $b \cap e = 0$ and $\varphi \in \Psi^b$. Thus we have $\varphi_3 \in \Psi^{b \cup d'}$. Now, $\varphi = \varphi_3 \varphi_2 \varphi_1$ as can be easily checked. Thus φ is a product of members of $\Psi^{b \cup d} \cup \Psi^{b \cup d'}$, hence φ maps z on itself since the members of $\Psi^{b \cup d} \cup \Psi^{b \cup d'}$ map z on itself. Therefore, z is b-symmetric.

Returning now to the member y of A which is not b-symmetric, y is c-symmetric for some $c \in Q$. $|c-b| < \omega_\beta$ hence $|c-b| \cdot \omega_\gamma < \omega_\beta$. Therefore, there is a sequence d of length ω_γ such that, for $\delta \neq \delta'$ and $\delta, \delta' < \omega_\gamma$, $d_\delta \subseteq K-b$, $|d_\delta| = |c-b|$ and $d_\delta \cap d_{\delta'} = 0$. By the axiom of choice there is a sequence φ of length ω_γ such that $\varphi_\delta \in \Psi^b$ for $\delta < \omega_\gamma$ and $\varphi_\delta(c-b) = d_\delta$. Since A is b-symmetric and $y \in A$, also $\varphi_\delta(y) \in A$. Since y is $b \cup (c-b)$-symmetric, $\varphi_\delta(y)$ is $b \cup d_\delta$-symmetric, by (12). Assume $\varphi_\delta(y) = \varphi_{\delta'}(y)$ for $\delta \neq \delta'$; then $\varphi_\delta(y)$ is $b \cup d_\delta$-symmetric and also $b \cup d_{\delta'}$-symmetric, hence, but what was shown above, $\varphi_\delta(y)$ is also b-symmetric. Since $\varphi_\delta \in \Psi^b$, we have, by (12), that y is also b-symmetric, which is a contradiction. Thus $|\{\varphi_\delta(y) | \delta < \omega_\gamma\}| = \omega_\gamma$. Each $\varphi_\delta(y)$ is $b \cup \bigcup_{\delta<\omega_\gamma} d_\delta$-symmetric, where $b \cup \bigcup_{\delta<\omega_\gamma} d_\delta \in Q$, since $|b \cup \bigcup_{\delta<\omega_\gamma} d_\delta| = |b| + |b-c| \cdot \omega_\gamma$. Therefore, by (11), $F = \{\langle \delta, \varphi_\delta(y) \rangle | \delta < \omega_\gamma\}$ is a one-one M-function mapping ω_γ into A.

(f) *If γ is a limit number or 0 and ω_γ is not confinal with ω_β, or if β is not a limit number or 0 and γ is any limit number or 0, then* $\mathcal{H}(\omega_\gamma)$.

This follows from (a) by (the translation of) Theorem 13.

(g) *If $\gamma \geqslant \beta$ and either γ is not a limit number or 0, or β is a limit number or 0 and ω_γ is confinal with ω_β, then* $\sim\mathcal{H}(\omega_\gamma)$.

By (the translations of) Theorems 14 and 15 and the generalized continuum hypothesis, which is an axiom of $\mathfrak{S}^*$, it is enough to prove $\sim\mathcal{H}(\omega_\beta)$. Consider the 0-symmetric M-set K. For any given $\zeta < \omega_\beta$, let a be a subset of K of cardinality $|\zeta|$, thus $a \in Q$. Let F be a one-one mapping of ζ into a, then $F(\delta) \in a$ for every $\delta < \zeta$, hence $F(\delta)$ is an a-symmetric M-element, thus F is an a-symmetric M-function. There is no one-one function mapping ζ onto K because $|K| = \omega_\beta > \zeta$. On the other hand, there is no one-one function mapping ω_β into K, because if G is such a function then it is b-symmetric for some $b \in Q$, hence $G(\delta)$ is b-symmetric for every $\delta < \omega_\beta$, hence $G(\delta) \in b$, $\mathfrak{R}(G) \subseteq b$, i.e., $|\mathfrak{R}(G)| \leqslant |b| < \omega_\beta$, a contradiction.

(d) *If ω_γ is regular and $\gamma \geqslant a$, then* $\sim\mathfrak{D}(\omega_\gamma)$.

This follows immediately from (g) by (the translations of) Theorems 8, 7, 5 and 11.

The interpretation $\mathfrak{J}$. We take for K the subset of L consisting of the first $\aleph_1$ $(= 2^{\aleph_0})$ members of L. Let S be an order relation on K similar to the natural order of the real numbers. For Ψ we take the group of all order preserving permutations of K. We take $Q = \{(-\infty, x) | x \in K\}$ $((-\infty, x) = \{y | y \in K \wedge ySx\})$. We shall write Ψ^x, x-*symmetric*, etc., for $\Psi^{(-\infty,x)}$, $(-\infty, x)$-symmetric, etc.

(a) $(\forall\gamma)\mathfrak{C}(\omega_\gamma)$.

Let F be an M-function on ω_γ such that $F(\delta)$ is a non-void set for each $\delta < \omega_\gamma$. Since F is an M-function F is x-symmetric for some $x \in K$. Let y be a fixed member of K with xSy. We shall see that $F(\delta)$ has, for every $\delta < \omega_\gamma$, a y-symmetric member. Let $u \in F(\delta)$; then u is a z-symmetric M-element for some $z \in K$. Without loss of generality we can assume xSz (since if wSw' and z is w-symmetric u is also w'-symmetric). There is an order-preserving premutation φ of K which is the identity on $(-\infty, x)$ and which maps z on y. By (12), $\varphi(u)$ is a $\varphi((-\infty, z))$-symmetric M-element; but $\varphi((-\infty, z)) = (-\infty, \varphi(z)) = (-\infty, y)$, hence $\varphi(u)$ is a y-symmetric M-element. $u \in F(\delta)$, $F(\delta)$ is x-symmetric (because F is) and $\varphi \in \Psi^x$, hence $\varphi(u) \in F(\delta)$. Thus $\varphi(u)$ is the required y-symmetric member of $F(\delta)$. By the axiom of choice there is a function G on ω_γ such that $G(\delta)$ is a y-symmetric member of $F(\delta)$ for every $\delta < \omega_\gamma$. By (11), G is a y-symmetric M-function.

(b) $\mathfrak{D}(\omega)$.

Let A and R be as assumed in $\mathfrak{D}(\omega)$. Since A and R are M-sets, they are u-symmetric for some $u \in K$. Let f be a v-symmetric sequence of members of A of length $<\omega$ and let $w \in K$ be such that vSw, uSw. We shall show that there is a w-symmetric member y of A such that fRy. By the assumption of $\mathfrak{D}(\omega)$, and since f is an M-sequence by (10), there is a member x of A such that fRx. Since $x \in A$, x is an M-element and hence x is r-symmetric for some $r \in K$. Without loss of generality we may assume vSr and uSr. Let φ be an order preserving permutation of K which is the identity on $(-\infty, u)$ and $(-\infty, v)$ and such that $\varphi(r) = w$. $\varphi(x)$ is a w-symmetric M-element by (12). A is u-symmetric, $\varphi \in \Psi^u$ and $x \in A$, hence $\varphi(x) \in A$. By fRx we have $\varphi(f)\varphi(R)\varphi(x)$, but f is v-symmetric and R is u-symmetric, hence $\varphi(f) = f$, $\varphi(R) = R$ and we have $fR\varphi(x)$. Thus $\varphi(x)$ is the required y. Let v be an ascending bounded sequence of length ω of members of K such that uSv_0. Put $u = v_{-1}$. We define the relation T as follows. fTy iff for some $n \in \omega$, $f \in A^n$, fRy and y is v_n-symmetric, or, if there is no such y, y is any element. By $D^*(\omega)$ there is a T-admissible function $g \in A^\omega$. We shall prove by induction that $g(n)$ is v_n-symmetric and that $(g \upharpoonright n)Rg(n)$ for every $n \in \omega$. By the induction hypothesis, $g(m)$ is, for every $m < n$, v_m-symmetric and since the sequence v is ascending, $g(m)$ is v_{n-1}-symmetric, where $v_{n-1} = u$

or uSv_{n-1}. By (11), $g \upharpoonright n$ is v_{n-1}-symmetric and hence, by what was shown above, there is a $y \in A$ such that $(g \upharpoonright n)Ry$ and y is v_n-symmetric. Thus, by $(g \upharpoonright n)Tg(n)$, we get $(g \upharpoonright n)Rg(n)$ and $g(n)$ is v_n-symmetric. Let w be an upper bound of v, then each $g(n)$ is w-symmetric and, by (11), g is a w-symmetric R-admissible M-sequence.

(d) *If γ is $0, 1$ or a limit number, then $\mathcal{H}(\omega_\gamma)$.*

If $\gamma = 0$ or if γ is a limit number, then $\mathcal{H}(\omega_\gamma)$ follows from (a) by (the translation of) Theorem 13. If $\gamma = 1$, let A be an infinite M-set satisfying the assumption of $\mathcal{H}(\omega_1)$. Since A is an M-set, A is u-symmetric for some $u \in K$. Let $r \in K$ be such that uSr. If every $x \in A$ is r-symmetric, then, by (11), there is an r-symmetric M-function f mapping $|A|$ onto A. By our assumptions on A, $\mathfrak{D}(f)$ cannot be ω_0, hence $|A| \geqslant \omega_1$, and $f \upharpoonright \omega_1$ is the M-function required by the conclusion of $\mathcal{H}(\omega_1)$. If, on the other hand, there is a $y \in A$ such that y is not r-symmetric, then y is s-symmetric for some $s \in K$, rSs. Let v be the greatest lower bound of the set of all t's for which y is t-symmetric. Since y is not r-symmetric, we have rSv or $r = v$, hence uSv. Let $w \in K$, vSw. Let $v' \in K$, $uSv'Sw$, and let φ be an order preserving premutation of K which is the identity on $(-\infty, u)$ and such that $\varphi(v) = v'$. It follows easily from (12) that v' is the greatest lower bound of the set of all t's such that $\varphi(y)$ is t-symmetric. Since A is u-symmetric and $\varphi \in \Psi^u$, we have $\varphi(y) \in A$. Thus for every v' between u and w there is at least one member of A such that the greatest lower bound of the set of all t's for which it is t-symmetric is v'. Since there are $2^{\aleph_0} = \aleph_1$, such v''s A has at least $\aleph_1$ w-symmetric members. Let f be a one-one function mapping ω_1 into the set of w-symmetric members of A. By (11), f is an M-function which satisfies the conclusion of $\mathcal{H}(\omega_1)$.

(e) *If $\gamma \neq 0, 1$ and γ is not a limit number, then $\sim\mathcal{H}(\omega_\gamma)$.*

By (the translation of) Theorem 14 it is enough to show $\sim\mathcal{H}(\omega_2)$. Consider the set K. Let f be a one-one function mapping ω_1 into $(-\infty, u)$ where $u \in K$. f is, by (11), a u-symmetric M-function. There is no M-function g mapping ω_1 onto K, because, if g is an M-function, g is u-symmetric for some $u \in K$, hence, by (11), $g(\lambda)$ is u-symmetric for every $\lambda < \omega_1$. Therefore, $g(\lambda)Su$ or $g(\lambda) = u$, for $\lambda < \omega_1$, i.e. $\mathfrak{R}(g) \subseteq (-\infty, u] \subset K$. Thus the assumptions of $\mathcal{H}(\omega_2)$ are satisfied by K. On the other hand, there is no one-one M-function mapping ω_2 into K since $|K| = \aleph_1 < \aleph_2$.

(c) *If ω_γ is regular and $\gamma \neq 0$, then $\sim\mathfrak{D}(\omega_\gamma)$.*

By (the translations of) Theorems 8 and 7 it is enough to show $\sim\mathfrak{D}(\omega_1)$. Let A be the set K and let R be the binary relation defined by: fRx iff f is a bounded above sequence of members of K of length $< \omega_1$ and x is a strict upper bound of f (i.e., $u \in \mathfrak{R}(f) \rightarrow uSx$). A and R are easily seen to be M-sets (note that if f is a bounded above sequence of members of K and y is an upper bounded of f, then f is a y-symmetric M-element). Let $\zeta < \omega_1$ and let f be an M-sequence $f \in K^\zeta$. Since f is an M-sequence it is y-symmetric for some $y \in K$. $f(\delta)$ is y-symmetric, for $\delta < \zeta$, by (11), hence $f(\delta) \in (-\infty, y]$, $\mathfrak{R}(f) \subseteq (-\infty, y]$. Let $x \in K$, ySx, then x is a strict upper bound of f and fRx. Furthermore, if $\zeta \leqslant \lambda < \omega_1$, let $\zeta + \eta = \lambda$, let x be a strictly ascending sequence in $A^{\eta+1}$ with ySx_0 (by the well-known properties of the natural order of the real numbers there is always such a sequence x as long as $\eta + 1 < \omega_1$). We define a sequence $g \in A^\lambda$ by $g(\delta) = f(\delta)$ for $\delta < \zeta$, $g(\zeta + \delta) = x_\delta$ for $\delta < \eta$. It is easily seen that $f \subseteq g$ and that if f is R-admissible, then g is R-admissible too. g is strictly bounded by x_η, hence $\mathfrak{R}(g) \subseteq (-\infty, x_\eta)$ and, by (11), g is an x_η-symmetric M-sequence. Thus all the assumptions of $\mathfrak{D}(\omega_1)$ hold. The conclusion of $\mathfrak{D}(\omega_1)$ cannot hold since if F is an R-admissible sequence in K^{ω_1} F is obviously strictly increasing, but this is impossible since there is no strictly ascending sequence of real numbers of length ω_1.

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HEBREW UNIVERSITY, JERUSALEM, ISRAEL

Reçu par la Rédaction le 14. 1. 1963