A 3-dimensional absolute retract which does not contain any disk

by

R. H. Bing (Madison) and K. Borsuk (Warszawa) *

Some examples of 2-dimensional AR-spaces which do not contain any disks were discovered long ago (see [4], [5], [6]). However, the question as to whether or not there exists a 3-dimensional AR-space which does not contain any disk remained open. The aim of the present note is to give an affirmative answer to this question.

1. Loops and their linking. By a map $f\colon X\to Y$ of a space X into a space Y we mean a single-valued continuous function from X to Y. In particular, if $X=C$ is a simple closed curve, then a map

$$f\colon C\to Y$$

is called a *loop* in the space Y. Two loops $f_0, f_1\colon C\to Y$ are said to be *homotopic in* Y if there exists a map φ of the Cartesian product $C\times I$ of C by the unit interval $I\colon 0\leqslant t\leqslant 1$ into Y such that

$$f_0(x)=\varphi(x,0) \text{ and } f_1(x)=\varphi(x,1) \quad \text{for every} \quad x\in C.$$

In particular, the loop f_0 is said to be *homotopic to zero in* Y (symbolically: $f_0\simeq 0$ in Y) provided f_0 is homotopic in Y to a loop f_1 mapping C onto a single point of Y. If C lies in a plane H, then the homotopy $f_0\simeq 0$ in Y is equivalent to the extendability of the map f_0 to a map f of the disk $D\subset H$ bounded by C into Y.

Let us consider two loops in the Euclidean 3-space E^3:

$$f_1\colon C_1\to E^3, \quad f_2\colon C_2\to E^3,$$

and let us suppose that they are disjoint, i.e. $f_1(C_1)\cap f_2(C_2)=0$. If orientations of C_1, C_2 and E^3 are given, then an integer $\lambda(f_1,f_2)$ called the *linking coefficient* of f_1 and f_2 is defined (see, for instance, Lefschetz [11], p. 124). It describes the algebraic number of times each of loops f_1 and f_2 twists around the other. For our aims it suffices to consider only the absolute value of this coefficient

$$\bar{\lambda}(f_1,f_2)=|\lambda(f_1,f_2)|,$$

* The first author is supported by NSF grants NSF-G21514 and NSF-G11665.

which does not depend on the choice of orientations of C_1, C_2 and E^3. We shall apply the following properties of the coefficients $\bar{\lambda}$:

(1) $\bar{\lambda}(f_1, f_2) = \bar{\lambda}(f_2, f_1)$.

(2) *If h is a homeomorphism of E^3 into itself, then*

$$\bar{\lambda}(f_1, f_2) = \bar{\lambda}(hf_1, hf_2).$$

(3) *If $0 \leqslant \varepsilon < \varrho(f_1(x_1), f_2(x_2))$ for every $x_1 \in C_1$, $x_2 \in C_2$, then for every two maps g_1: $f_1(C_1) \to E^3$, g_2: $f_2(C_2) \to E^3$ such that $\varrho(g_\nu(y), y) \leqslant \frac{1}{2}\varepsilon$ for every $y \in f_\nu(C_\nu)$ and $\nu = 1, 2$ one has $\bar{\lambda}(f_1, f_2) = \bar{\lambda}(g_1 f_1, g_2 f_2)$.*

If f_1: $C_1 \to E^3$ and f_1': $C_1' \to E^3$ are homeomorphisms of simple closed curves C_1, C_1' onto the same simple closed curve $C \subset E^3$, then for every loop f_2: $C_2 \to E^3$ with $f_2(C_2) \subset E^3 - C$, the coefficients $\bar{\lambda}(f_1, f_2)$ and $\bar{\lambda}(f_1', f_2)$ are equal. Let us denote the common value of them by $\bar{\lambda}(C, f_2)$ or by $\bar{\lambda}(f_2, C)$. The vanishing of the number $\bar{\lambda}(C, f_2)$ implies that each of C, $f_2(C_2)$ bounds in the complement of the other.

The corresponding homotopy linking is not symmetric, for of two polygonal simple closed curves in E^3, it may be that the first is homotopic to zero in the complement of the second but the second is not homotopic to zero in the complement of the first. Since the first homology group is obtained by abelianizing the first homotopy group (fundamental group), it follows that if C is a simple closed curve in E^3, then a loop f lying in $E^3 - C$ is not homotopic to zero in $E^3 - C$ if $\bar{\lambda}(C, f) > 0$. However, if C is an unknotted simple closed curve as would lie in a plane, then the first homotopy group of $E^3 - C$ is infinite cyclic and hence already abelian so f is homotopic to zero in $E^3 - C$ if and only if $\bar{\lambda}(C, f) = 0$. In this paper we make use of the following

(4) *If C is a simple closed curve in a plane in E^3, then a loop f lying in $E^3 - C$ is homotopic to zero in $E^3 - C$ if and only if $\bar{\lambda}(C, f) = 0$.*

If both disjoint loops f_1: $C_1 \to E^3$ and f_2: $C_2 \to E^3$ are homeomorphisms of C_1 onto C_1' and of C_2 onto C_2', then the coefficient $\bar{\lambda}(f_1, f_2)$ depends only on C_1' and C_2'. We shall denote it by $\bar{\lambda}(C_1', C_2')$. If $\bar{\lambda}(C_1', C_2') \neq 0$, then the simple closed curves C_1', C_2' are said to be *homologically linked*. It can be shown that:

(5) *Given:*

1° A disk D_1 with boundary C_1 and a disk $D_2 \subset D_1$ with boundary C_2,

2° a simple closed curve $C \subset E^3$,

3° a map f: $D_1 \to E^3$ such that $f((D_1 - D_2) \cup C_2) \subset E^3 - C$,

then the loops $f_1 = f/C_1$, $f_2 = f/C_2$ satisfy the equation $\bar{\lambda}(f_1, C) = \bar{\lambda}(f_2, C)$.

(6) *Given:*

1° A simple closed curve $C \subset E^3$ decomposed into the sum of two simple arcs J_1, J_2 with common ends a and b,

2° a plane $H \subset E^3$ decomposing E^3 between a and b,

3° a simple closed curve $C' \subset H$ decomposing H between the sets $J_1 \cap H$ and $J_2 \cap H$,

then $\bar{\lambda}(J_1 \cup J_2, C') \neq 0$.

2. Anchor ring and its core. Now let us suppose that C_0 is a circle with radius r_0 lying in a plane $H \subset E^3$. Let D_0 be the disk bounded (in H) by C_0 and let r be a positive number less than r_0. The set

$$A_0 = \{x | x \in E^3,\ \varrho(x, C_0) \leqslant r\}$$

is an anchor ring with the *core* (center line) C_0. The numbers r and r_0 will be said to be the *inner* and the *outer radius* of A_0 and the number $\vartheta = r/r_0$ is called the *profile* of A_0. Evidently two anchor rings are similar if and only if they have the same profile. Let B_0 denote the boundary of A_0. A circle $C_1 \subset E^3 - A_0$ will be said to be *transversal* to the anchor ring A_0 provided $\bar{\lambda}(C_0, C_1) = 1$. We observe:

(7) *Given in E^3 an anchor ring A_0 with core C_0 and a circle C_1 transversal to A_0. Then there exists a neighborhood U of the boundary B_0 of A_0 such that every loop f lying in U is homotopic to zero in the set U if and only if $\bar{\lambda}(f, C_0) = \bar{\lambda}(f, C_1) = 0$.*

Evidently, if the number $\varepsilon > 0$ is sufficiently small then the set

$$U = \{x | x \in E^3,\ \varrho(x, B_0) < \varepsilon\}$$

satisfies condition (7).

3. Chain which substitutes for an anchor ring. Let A_0 be an anchor ring with outer radius r_0, inner radius r, core C_0, and boundary B_0. Let n be a natural number greater than 3 and let $a_1, a_2, \ldots, a_n$ (where the indices are understood mod n) be vertices of a regular n-polygon inscribed on C_0 in their natural order. Evidently

$$\varrho(a_0, a_1) = \varrho(a_i, a_{i+1}) = 2r_0 \sin\frac{\pi}{n}$$

is small if n is large. Set

$$s_0 = \frac{1}{3}\varrho(a_0, a_1) = \frac{2}{3}r_0 \sin\frac{\pi}{n}.$$

Let a_k' denote the center of the segment $a_k a_{k+1}$, D_k denote the geometrical disk with center a_k, radius s_0, and lying in the plane of the circle C_0. Let D_k' denote the geometrical disk with center a_k', radius s_0, and lying in the plane perpendicular to the segment joining a_k' with the center of C_0. Let C_k denote the boundary of D_k and C_k' the boundary of D_k'. One sees easily that:

1° $C_k \cap C_j' = 0$ for every k and j.

2° $C_k \cap C_j = C_k' \cap C_k' = 0$ if $k \neq j \pmod n$.

3° $\bar{\lambda}(C_k, C_j) = \bar{\lambda}(C'_k, C'_j) = 0$ if $k \neq j \pmod{n}$.

4° $\bar{\lambda}(C_k, C'_j) = 1$ if $0 \leqslant k-j \leqslant 1 \pmod{n}$, and $\bar{\lambda}(C_k, C'_j) = 0$ in all other cases.

5° If x, y are points belonging to two different circles of the collection $C_1, C'_1, C_2, C'_2, \ldots, C_n, C'_n$, then $\varrho(x, y) > \frac{1}{2} s_0$.

Now let us suppose that $\vartheta = r/r_0 \leqslant \frac{1}{4}$, that ε is a positive number less than $2r$, and that the natural number n is so large that $\varrho(a_0, a_1) < \varepsilon$. Denote by A_k the anchor ring with core C_k and inner radius $s = \vartheta s_0$, and by A'_k the anchor ring with core C'_k and with the same inner radius s. It follows that A_k and A'_k are similar to A_0 and their outer radius is s_0. Their diameters are equal to $2(s+s_0) = 2(1+\vartheta)s_0 < 2(1+\frac{1}{4})\frac{1}{3}\varepsilon < \varepsilon$. It follows from 5°, that all anchor rings $A_1, A'_1, A_2, A'_2, \ldots, A_n, A'_n$ are disjoint. Moreover it is clear that for every point $x \in \bigcup_{k=1}^{n} (A_k \cup A'_k)$

$$\varphi(x, C_0) < s_0 + \vartheta s_0 \leqslant \frac{5}{4} s_0 < \frac{1}{2}\varrho(a_0, a_1) < \frac{\varepsilon}{2} < r.$$

We conclude that all anchor rings A_i, A'_i are contained in the interior of the anchor ring A_0 and also in the ε-neighborhood of its core C_0.

The set

$$(8) \qquad M^1 = \bigcup_{k=1}^{n} (A_k \cup A'_k) \subset A_0$$

will be said to be *chain substituting for the anchor ring* A_0, and the anchor rings A_k, A'_k, $k = 1, 2, \ldots, n$, will be said to be *links* of this chain. Thus the chain M^1 has $2n$ links $A_1, A'_1, A_2, A'_2, \ldots, A_n, A'_n$; we shall denote these links (in the same order) by $L_1^1, L_2^1, \ldots, L_{2n}^1$, where the lower indices are understood mod $2n$. It follows by our construction that the cores of two links L_i^1 and L_j^1 are linked if and only if $|i-j| = 1 \pmod{2n}$. Thus we have:

(9) *For every anchor ring A_0 with core C_0 and profile $\vartheta \leqslant \frac{1}{4}$ and for every $\varepsilon > 0$ there exists an integer $n > 3$ such that in the interior of A there exists a chain M^1 with $2n$ links $L_1^1, L_2^1, \ldots, L_{2n}^1$, these links being congruent anchor rings with profiles ϑ, diameters $\leqslant \varepsilon$, and each lying in the ε-neighborhood of C_0. If C_i^1 denotes the core of the link L_i^1, then $\bar{\lambda}(C_i^1, C_j^1) = 1$ if $|i-j| = 1 \pmod{2n}$ and $\bar{\lambda}(C_i^1, C_j^1) = 0$ if $|i-j| > 1 \pmod{2n}$.*

The set

$$N^1 = \bigcup_{i=1}^{2n} C_i^1 = \bigcup_{k=1}^{n} (C_k \cup C'_k)$$

will be called the *core of the chain* M^1.

It is known (see, for instance, [8]) that a loop f lying in $E^3 - A_0$ is homotopic to zero in $E^3 - M^1$ if and only if it is homotopic to zero in $E^3 - A_0$, that is if $\bar{\lambda}(f, C_0) = 0$. This fact may be also formulated as follows: Let D be a disk with the boundary C.

(10) *A loop $f\colon C \to E^3 - A_0$ can be extended to a map $f'\colon D \to E^3 - N^1$ if and only if $\bar{\lambda}(f, C_0) = 0$.*

Now let us prove (with the same notation):

(11) *If $f\colon C \to E^3 - A_0$ and $\bar{\lambda}(f, C_0) > 0$, then for each extension $f'\colon D \to E^3$ of f and for every $\varepsilon > 0$ there exist a link L_i^1 and two disjoint subdisks D_1, D_2 of D with boundaries C'_1 and C'_2 such that each partial map $f_\nu = f'/C'_\nu$, $\nu = 1, 2$, is a lop in $A_0 - M^1$ lying in the ε-neighborhood of link L_i^1 and homologically linked with the core of this link.*

Proof. By the construction of M^1, we infer that the core C_{i+1}^1 of the link L_{i+1}^1 is transversal to the anchor ring L_i^1. It follows by (7) that there exists an open neighborhood U_i of the boundary B_i^1 of L_i^1 contained in A_0 and such that a loop g lying in U_i is homotopic to zero in U_i if and only if $\bar{\lambda}(g, C_\nu^1) = 0$ for $\nu = i, i+1$. Since the sets $B_1^1, B_2^1, \ldots, B_{2n}^1$ are disjoint, we may assume that the neighborhoods $U_1, U_2, \ldots, U_{2n}$ are mutually disjoint and also disjoint from the core N^1 of M^1, and that each U_i lies in the ε-neighborhood of L_i^1. Now let us write:

$$F_i = f'^{-1}(L_i^1), \quad G_i = f'^{-1}(U_i) \quad \text{for every} \quad i = 1, 2, \ldots, 2n.$$

The sets $G_1, G_2, \ldots, G_{2n}$ are open and mutually disjoint while F_i is a compact subset of G_i. It follows that G_i contains an open subset G_i^0 such that

$$F_i \subset G_i^0 \subset G_i \quad \text{for} \quad i = 1, 2, \ldots, 2n$$

and the boundary Z_i of G_i^0 is the union of a finite number of disjoint simple closed curves $C_{i,1}^0, C_{i,2}^0, \ldots, C_{i,m_i}^0$. Each of the partial functions

$$g_{i,j} = f'/C_{i,j}^0, \quad i = 1, 2, \ldots, 2n, \ j = 1, 2, \ldots, m_i,$$

is a loop in U_i. If $\bar{\lambda}(g_{i,j}, C_\nu^1) = 0$ for $\nu = i, i+1$, then $g_{i,j}$ is homotopic to zero in U_i by (7). Consider the component E of $D - \bigcup C_{i,j}^0$ that contains C. Each boundary component of $\bar{E}$ other than C is a $C_{i,j}^0$. If for each such $C_{i,j}^0$, $\bar{\lambda}(g_{i,j}, C_\nu^1) = 0$ for $\nu = i, i+1$, we find that the map $f'/C = f$ can be extended to map of D into $E^3 - N^1$ — namely, the extension is f' on E and takes each component of $D - E$ into an appropriate U_i. Since the existence of such an extension is contrary to (10) we conclude that for at least one $C_{i,j}^0$ and $\nu = i$ or $i+1$ it is $\bar{\lambda}(g_{i,j}, C'_\nu) > 0$.

Thus we have shown that there exists a loop $g_{i,j}$ such that for some index ν we have $\bar{\lambda}(g_{i,j}, C_\nu^1) > 0$. Now, fixing the index ν, we can find a maximal C_{i_0,j_0}^0 in the sense that there are indices i_0, j_0 such that $\bar{\lambda}(g_{i_0,j_0}, C_\nu^1)$ is positive but for every other pair of indices i, j satisfying the condition $\bar{\lambda}(g_{i,j}, C_\nu^1) > 0$, the disk $D_{i,j}$, bounded in D by the curve $C_{i,j}^0$, does not contain C_{i_0,j_0}^0 on its interior.

Now let us consider the domain $V \subset D$ bounded by the disjoint simple closed curves C and $C^0_{i_0,j_0}$. Consider the various $C^0_{i,j}$'s in V. For some such $C^0_{i,j}$, $\bar\lambda(g_{i,j}, C^1_\nu) > 0$ or else we could obtain a map $g\colon \bar V \to E^3 - C^1_\nu$ such that $g/C = f'$, $g/C^0_{i_0,j_0} = f'/C^0_{i_0,j_0}$. But since $\bar\lambda(g/C, C^1_\nu) = 0$, we arrive at the contradiction from (5) that $\bar\lambda(g_{i_0,j_0}, C^1_\nu) = 0$. Consequently there is a $C^0_{i,j}$ in V such that $\bar\lambda(g_{i,j}, C^1_\nu) > 0$. It follows from the maximality of $C^0_{i_0,j_0}$ that this $C^0_{i,j}$ bounds a disk in V. Consequently the proof of (11) is complete.

As an easy consequence of (11) we get:

(11′) *If C is the boundary of a disk D and if the loop $f\colon C \to E^3 - A_0$ satisfies the condition $\bar\lambda(f, C_0) > 0$, then for every extension $f'\colon D \to E^3$ of f there exists a disk $D' \subset D$ with boundary C' such that the loop f'/C' is homologically linked with the core of one of the links of the chain M^1.*

4. Wreath substituting for an anchor ring. Suppose we are given an anchor ring A_0 with outer radius 1, profile $\vartheta \leqslant \frac14$, core C_0, and chain M^1 substituting for A_0 having $2n$ links $L^1_1, L^1_2, \ldots, L^1_{2n}$ similar to A_0, with outer radii equal to $\frac23 \sin(\pi/n)$. Let us construct a decreasing sequence of compacta

$$M^1 \supset M^2 \supset \ldots \supset M^k \supset \ldots$$

in the following manner: Suppose that for a natural number k we have a set M^k such that

(12_k) *M^k is the union of $(2n)^k$ disjoint anchor rings L^k_i $(i = 1, 2, \ldots, (2n)^k)$ with the outer radii equal to $(\frac23 \sin\pi/n)^k$ and with profile ϑ.*

Let φ_i denote the similarity mapping A_0 onto L^k_i. Then φ_i maps the links L^1_j $(j = 1, 2, \ldots, 2n)$ of the chain M^1 onto disjoint anchor rings similar to A_0 and contained in the interior of L^k_i. The inner radii of these anchor rings are equal to $(\frac23 \sin\pi/n)^{n+k}$. If we set

$$M^{k+1} = \bigcup_{i=1}^{(2n)^k} \varphi_i(M^1),$$

then we get a set M^{k+1} satisfying the condition (12_{k+1}). The anchor rings L^k_i $(i = 1, 2, \ldots, (2n)^k)$ are components of M^k and their diameters converge to zero when $k \to \infty$. It follows that the set

$$M = \bigcap_{k=1}^{\infty} M^k$$

is a 0-dimensional compactum, actually a set of Antoine [3]. Evidently each of the sets

$$M_i = M \cap L^1_i, \quad i = 1, 2, \ldots, 2n,$$

is similar to M, and consequently M_i is a set of Antoine lying in the interior of the anchor ring L^1_i. It is known [13] that there exists in the interior of L^1_i a simple arc J_i containing M_i. The set

$$W = \bigcup_{i=1}^{2n} J_i$$

will be said to be a *wreath* substituting for the anchor ring A_0, and the arcs J_i (components of W) will be said to be *links of the wreath* W. We infer the following from (9):

(13) *For every anchor ring A_0 with core C_0 and profile $\vartheta \leqslant \frac14$ and for every $\varepsilon > 0$ there exists in the interior of A_0 a wreath W substituting for A_0, lying in the ε-neighborhood of C_0 and such that the diameters of its links are less than ε.*

Let us observe that (11′) implies that if a simple closed curve C is the boundary of a disk D and if the loop $f\colon C \to E^3 - A_0$ is homologically linked with C_0, then for every continuous extension $f'\colon D \to E^3$ of f and for every $k = 1, 2, \ldots$ there exists a disk $D^k_0 \subset D$ with the boundary C^0_k such that the core C' of one of the components of M^k is homologically linked with the loop f'/C^0_k. It follows that $f'(D) \cap M^k \neq 0$, for every $k = 1, 2, \ldots$, and consequently $f'(D) \cap M \neq 0$.

Moreover, by virtue of (11), there exist two disjoint disks $D_1, D_2 \subset D$ with boundaries C'_1, C'_2, and link L^1_i of the chain M^1 such that the core of L^1_i is homologically linked with each of the loops $f'_\nu = f'/C'_\nu$, $\nu = 1, 2$. Moreover, we can assume that the loops f'_1, f'_2 lie in $A_0 - M^1$ and in the ε-neighborhood of the link L^1_i. It follows that both sets $f'(D_1)$ and $f'(D_2)$ intersect the set M_i, and consequently also the arc $J_i \supset M_i$. Hence, we have the following:

(14) *Let C be the boundary of a disk D and let $f\colon C \to E^3 - A_0$ with $\bar\lambda(f, C_0) > 0$. Then for every continuous extension $f'\colon D \to E^3$ of f and for every $\varepsilon > 0$ there exist two disjoint disks $D^0_1, D^0_2 \subset D$ with boundaries C^0_1, C^0_2 and a link L^1_i of M^1 such that both loops f'/C^0_1, f'/C^0_2 lie in $A_0 - M^1$ in the ε-neighborhood of L^1_i and that $f'(D^0_1) \cap L^1_i \cap W \neq 0 \neq f'(D^0_2) \cap L^1_i \cap W$.*

In particular it follows (under the hypotheses of (14)) that

$$f'(D) \cap W \neq 0. \tag{15}$$

5. Broken anchor ring and its substituting wreath. By a *topological anchor ring* we understand the image $A = h(A_0)$ of A_0 by a homeomorphism h. Then the set $h(C_0)$ will be said to be the *core* of A and the sets $h(M^1)$ and $h(W)$ will be said to be the *chain* and the *wreath* substituting for the topological anchor ring A. The links of $h(M^1)$ and of $h(W)$ are defined as components of these sets.

It follows from (13):

(16) *For every topological anchor ring A with core C and for every $\varepsilon > 0$ there exists a chain and a wreath substituting for A, lying in the ε-neighborhood of C and having links with diameters less than ε.*

In particular, if $\hat{C}_0$ is a simple polygon (i.e. a polygonal simple closed curve) lying in a plane $H \subset E^3$ and η is a positive number sufficiently small, then the set

$$\hat{A}_0 = \{x | x \in E^3,\ \varrho(x, C_0) \leqslant \eta\}$$

is a topological anchor ring. More exactly, there exists a homeomorphism h mapping the space E^3 onto itself so that $h(A_0) = \hat{A}_0$ and $h(C_0) = \hat{C}_0$. The set $\hat{A}_0$ will be called a *broken anchor ring* with core $\hat{C}_0$ and *inner radius* η. The sets

$$\hat{M}^1 = h(M^1) \quad \text{and} \quad \hat{W} = h(W)$$

will be said to be the *chain* and the *wreath substituting for the broken anchor ring* $\hat{A}_0$.

The following is a consequence of (2) and (11):

(17) *If C is the boundary of a disk D and f: $C \to E^3 - \hat{A}_0$ is a loop with $\bar{l}(f, \hat{C}_0) > 0$ then, for every continuous extension f': $D \to E^3$ of f and for every $\varepsilon > 0$, there exist in the interior of D two disjoint disks D_1, D_2 with boundaries C_1', C_2' and a link of the chain $\hat{M}^1$ such that the loops $\bar{f}_\nu = f'|C_\nu'$, $\nu = 1, 2$ lie in $\hat{A}_0 - \hat{M}^1$ in the ε-neighborhood of the link of $\hat{M}^1$ and they are both homologically linked with the core of this link.*

6. Dense sequences of chords. Let Q be a unit ball in E^3 with boundary S. A segment $K \subset E^3$ will be said to be a *chord* of S provided both its ends belong to S. A sequence $\{K_\nu\}$ of chords of S will be said to be *dense* on S provided for each open subset $G \neq 0$ of S there exists an index ν such that both ends of the chord K_ν belong to G. Let us show:

(18) *There exists a sequence of the disjoint chords $\{K_\nu\}$ dense on S and such that the diameters of K_ν converge to* 0.

Proof. Let $\{a_n\}$ be a sequence of points of S dense in S and such that $a_n \neq a_m$ for $n \neq m$. We define the sequence $\{K_\nu\}$ by the induction:

1. Let a_1' be a point of $S - \{a_n\}$ such that $\varrho(a_1, a_1') \leqslant 1$. We set $K_1 = \overline{a_1 a_1'}$.

2. Suppose that for $\nu = 1, 2, \ldots, m$ we have $K_\nu = \overline{a_\nu a_\nu'}$, where $a_\nu' \in S - \{a_n\}$, $\varrho(a_\nu, a_\nu') \leqslant 1/\nu$ and $K_\nu \cap K_\mu = 0$ for $\mu, \nu = 1, 2, \ldots, m$. Then we see at once that there exists a point $a_{m+1}' \in S - \{a_n\}$ such that $\varrho(a_{m+1}, a_{m+1}') \leqslant 1/(m+1)$ and that the chord $K_{m+1} = \overline{a_{m+1} a_{m+1}'}$ is disjoint from $K_1 \cup K_2 \cup \ldots \cup K_m$.

Thus we get a sequence $\{K_\nu\}$ of disjoint chords with diameters convergent to zero. If x is a point of an open subset G of S, then there exists an $\varepsilon > 0$ such that all points $y \in S$ with $\varrho(x, y) < \varepsilon$ belong to G. Since the sequence $\{a_n\}$ is dense in G, there exists an index $n > 2/\varepsilon$ such that $\varrho(a_n, x) < \varepsilon/2$. Then $\varrho(a_n', x) \leqslant \varrho(a_n', a_n) + \varrho(a_n, x) < \varepsilon$ and consequently both ends of $K_n = \overline{a_n a_n'}$ belong to G.

7. A sequence of broken anchor rings dense in Q. Let $\{\hat{A}_\nu\}$ be a sequence of broken anchor rings lying in the interior of the unit ball Q. We say that $\{\hat{A}_\nu\}$ is *dense* in Q if for every simple closed curve $C \subset Q - S$ there exists an index ν such that $\hat{A}_\nu \subset Q - C$ and the core $\hat{C}_\nu$ of $\hat{A}_\nu$ is homologically linked with C.

Now let us show:

(19) *There exists a sequence $\{A_\nu'\}$ of broken anchor rings lying in $Q - S$ and dense in Q.*

Proof. We apply the following elementary facts about the topology of the plane:

1° If X and Y are disjoint compact subsets of E^2 which lie in a region $G \subset E^2$ but do not cut it, then there exists a simple polygon $\hat{C} \subset G - X - Y$ which separates G between X and Y. Moreover, we can assume that all vertices of $\hat{C}$ have rational coordinates.

Since the set of all simple polygons in E^2 with rational coordinates of vertices is countable, we infer:

2° For every plane region G there exists a countable family of simple polygons lying in G such that every two disjoint compact subsets of G which do not cut G, may be separated by a simple polygon belonging to this family.

Applying 1° and 2° let us show:

3° There exists in $Q - S$ a countable family $\mathfrak{C}$ of plane simple polygons such that for every simple closed curve $C \subset Q - S$, there exists a $\hat{C} \in \mathfrak{C}$ such that $\bar{L}(C, \hat{C}) > 0$.

In order to prove 3°, consider the family $\mathfrak{H}$ of all planes $H \subset E^3$ which intersect Q and pass through three non collinear points with rational coefficients. Manifestly $\mathfrak{H}$ is countable. By 2°, for every $H \in \mathfrak{H}$ there exists a countable family $\mathfrak{C}_H$ of simple polygons lying in the region $G_H = H \cap (Q - S)$ and such that every two disjoint compact subsets of G which do not cut G, can be separated by a simple polygon $\hat{C} \in \mathfrak{C}_H$. Evidently the family

$$\mathfrak{C} = \bigcup_{H \in \mathfrak{H}} \mathfrak{C}_H$$

is also countable.

Now let us consider a simple closed curve $C \subset Q - S$ and two distinct points $a, b \in C$. Then C is the union of two simple arcs J_1 and J_2 with common ends a and b. Consider a plane $H \in \mathfrak{H}$ cutting E^3 between a and b. Then there exists a simple polygon $\hat{C} \in \mathfrak{C}_H$ separating G between the sets $J_1 \cap H$ and $J_2 \cap H$. It follows by (6) that $\bar{\lambda}(C, \hat{C}) > 0$. Thus 3° is proved.

Now let us order the polygons of the family $\mathfrak{C}$ in a sequence $\{C_i'\}$ such that each polygon $\hat{C} \in \mathfrak{C}$ appears in $\{C_i'\}$ an infinite number of times. Denote by A_ν' a broken anchor ring with core C_ν' and with radius $\eta_\nu < 1/\nu$. Evidently η_ν can be chosen so small that $A_\nu' \subset Q - S$. The proof of proposition (19) will be complete, if we shall show that for every simple closed curve $C \subset Q - S$ there exists an index ν such that $A_\nu' \subset Q - C$ and $\bar{\lambda}(C, C_\nu') > 0$.

In order to prove it, let us observe that, by the construction of the sequence $\{C_i'\}$ there exists an index i_0 such that $\bar{\lambda}(C, C_{i_0}') > 0$. Let ε denote the distance between C and C_{i_0}'. Since C_{i_0}' appears in the sequence $\{C_i'\}$ an infinite number of times, there exists an index i such that

$$C_i' = C_{i_0}' \quad \text{and} \quad \frac{1}{i} < \varepsilon .$$

Then the polygon C_i' satisfies the condition $\bar{\lambda}(C_i', C) > 0$. Moreover, if $x \in A_i$ then $\varrho(x, C_i') < 1/i < \varepsilon$, and consequently $A_i \subset Q - C$. Thus (19) is proved.

Let us observe that (19) implies:

(20) *For every neighborhood U_x of each point $x \in Q$ there exists an index ν such that $C_\nu' \cap U_x \neq 0$.*

In fact, there exists in U_x a disk $D \subset Q - S$. By (19), there exists an index ν such that the simple polygon C_ν' is linked with the boundary C of D. It follows $C_\nu' \cap D \neq 0$ and consequently also $C_\nu' \cap U_x \neq 0$.

Now let us prove:

(21) *There exists a sequence $\{A_i\}$ of broken anchor rings in $Q - S - \bigcup_{\nu=1}^{\infty} K_\nu$ and dense in Q such that for each $i = 1, 2, \ldots$ the inner radius of A_i is less than $1/i$ and there exists in A_i a wreath W_i substituting for A_i and such that $W_i \cap W_j = 0$ for $i \neq j$ and the diameters of the links of W_i are less than $1/i$.*

Proof. Consider the sequence $\{A_\nu'\}$ of broken anchor rings of (19). If we tried to prove (21) by shrinking the inner radii of the broken anchor rings of $\{A_\nu'\}$ and putting wreaths in these shrunken broken anchor rings, we would have to exercise care to insure that the wreaths did not intersect each other or the segments in $\{K_\nu\}$. We modify the cores of the broken anchor rings of $\{A_\nu'\}$ before shrinking them.

Let C_1' be the core of A_1'. At most a countable number of planes parallel to the plane containing C_1' contains a segment of $\{K_\nu\}$. We adjust C_1' to obtain a polygonal simple closed curve C_1 in the interior of A_1' such that C_1 lies in a plane, C_1 misses $\bigcup_{\nu=1}^{\infty} K_\nu$, and there is a map of an annulus into the interior of A_1' that takes the two boundary components of the annulus homeomorphically onto C_1' and C_1. We can obtain C_1 by first translating C_1' to a nearby plane H not containing any element of $\{K_\nu\}$ and then adjusting the image of C_1' in H so that it misses the 0-dimensional set consisting of the intersection of H with the segments of $\{K_\nu\}$. It follows from (5) that if C is a simple closed curve in $Q - A_1'$, then $\bar{\lambda}(C, C_1') = \bar{\lambda}(C, C_1)$.

Let A_1 be a broken anchor ring with core C_1 and inner radius less than 1 and so small that $A_1 \subset A_1'$ and $A_1 \cap \bigcup_{\nu=1}^{\infty} K_\nu = 0$. It follows from (16) that there exists a wreath W_1 substituting for the anchor ring A_1 such that the diameters of its links are less than 1.

Suppose the broken anchor rings $A_1, A_2, \ldots, A_k$ have been chosen so that for $i = 1, 2, \ldots, k$, $A_i \subset A_i' - \bigcup_{\nu=1}^{\infty} K_\nu$, the inner radius of A_i is less than $1/i$, each simple closed curve in $Q - A_i'$ that homologically links the core of A_i' also homologically links the core of A_i. Suppose that $W_1, W_2, \ldots, W_k$ have been chosen so that they substitute for the anchor rings $A_1, A_2, \ldots, A_k$ respectively, are mutually disjoint, and the diameter of the links of W_i are less than $1/i$. Suppose the core C_{i+1}' of A_{i+1}' lies in a plane H. Note that at most a countable number of planes parallel to H contain a non degenerate continuum that lies in $\bigcup_{i=1}^{k} W_i$ or any of $\{K_\nu\}$. Let C_{i+1} be a polygonal simple closed curve in the interior of A_{i+1} such that C_{i+1} lies in a plane, C_{i+1} misses $\bigcup_{\nu=1}^{\infty} K_\nu$, C_{i+1} misses each $W_1, W_2, \ldots, W_k$ and there is a map of an annulus into the interior of A_{k+1}' which takes the two boundary components of the annulus into C_{i+1}' and C_{i+1}. Then A_{i+1} is a broken anchor ring with core C_{i+1}' and inner radius less than $1/i+1$ and so small that $A_{i+1} \subset A_{i+1}'$, A_{i+1} misses each W_1, $W_2, \ldots, W_i$, and A_{i+1} misses $\bigcup_{\nu=1}^{\infty} K_\nu$. Then W_{i+1} is a wreath in A_{i+1} substituting for A_{i+1} such that the diameters of the links of W_{i+1} have diameters less than $1/i+1$. If we continue defining A_j's and W_j's in this way, the sequences $\{A_\nu\}$, $\{W_\nu\}$ satisfy (21).

It follows from a modified version of (20) applied to the broken anchor rings $\{A_\nu\}$ of (21) that for every neighborhood U_x of each point $x \in Q$, there are infinitely many indices i such that $A_i \cap U_x \neq 0$. Since each point of A_i lies within the distance $1/i$ of W_i we conclude.

(22) *Each open subset of Q contains a link of an element of $\{W_\nu\}$.*

8. The construction of the space Q^*. Now let us consider in the ball Q a sequence of chords $\{K_\nu\}$ satisfying (18), and sequences of anchor rings $\{A_\nu\}$ and wreaths $\{W_\nu\}$ satisfying (21). The chords K_ν

and the links of the wreaths W_ν constitute a countable family $\mathfrak{A}$ of disjoint arcs with diameters converging to zero. It follows that the decomposition $\mathfrak{M}$ of Q into arcs of the family $\mathfrak{A}$ and the individual points of Q not on one of these arcs is upper semicontinuous. Let Q^* denote the hyperspace of this decomposition and let

$$\varphi: Q \to Q^*$$

be the continuous map whose inverse sets $\varphi^{-1}(x)$, $x \in Q^*$, are exactly the elements of the decomposition $\mathfrak{M}$. It follows by a theorem of Lelek [12] that Q^* is an AR-set. In Section 9 we prove that $\dim Q^* = 3$ and in Section 11 we prove that Q^* does not contain any disk.

9. Dimension Q^*. Let us denote by P the subset of Q consisting of all individual points of the decomposition $\mathfrak{M}$. Then φ maps the set P homeomorphically onto a subset P^* of Q^*, and it maps the set $N = Q - P$ onto the countable set $N^* = \varphi(N) = Q^* - P^*$.

Since the set $N \cap S$ is countable (it consists of the ends of chords K_ν), there exists a simple closed curve $C \subset S - N$. Then S is the union of two disks D_1 and D_2 having C as their common boundary. Since φ is a homeomorphism on C, $C^* = \varphi(C)$ is a simple closed curve and this curve is homotopic to zero in each of the sets $D_1^* = \varphi(D_1)$ and $D_2^* = \varphi(D_2)$. Moreover we have:

$$C^* \subset D_1^* \cap D_2^* \subset C^* \cup N^*,$$

where N^* is a countable set. It follows that the set $S^* = D_1^* \cup D_2^* = \varphi(S)$ contains a 2-dimensional true cycle which is not homologous to zero in S^*, but it is homologous to zero in Q^* (because Q^* is an AR). Consequently (see [1], also p. 151 of [10]) we obtain:

$$\dim Q^* \geqslant 3\,. \tag{23}$$

On the other hand, let us observe that (22) implies that the set $N \subset \bigcup_{\nu=1}^{\infty} W_\nu$ is dense in Q and consequently (see p. 44 of [10]) the dimension of the set $P = Q - N \subset Q$ is less than or equal to 2. It follows that the dimension of the set P^*, homeomorphic to P, is less than or equal to 2, and since $Q^* = P^* \cup N^*$, where N^* is countable, we conclude (see p. 32 of [10]) that $\dim Q^* \leqslant 3$. By virtue of (23) we get:

$$\dim Q^* = 3\,. \tag{24}$$

10. Lemma on extension of maps. We now prove the following:

LEMMA. *Suppose we are given a closed AR-set Y_0 lying in a compact metric ANR-space Y. For every neighborhood V of Y_0 in Y there exists a neighborhood U of Y_0 in Y such that every map f_0 of a closed subset X_0 of an arbitrary metric space X into U can be extended to a map $f: X \to V$.*

Proof. We can suppose that Y is a subset of the Hilbert cube Q^ω, consisting of all points $z = (z_1, z_2, \ldots)$ of the Hilbert space E^ω satisfying the condition $0 \leqslant z_i \leqslant 1/i$. Since Y is an ANR, there exists a neighborhood G of Y in E^ω and a retraction

$$r: G \to Y$$

Evidently, there exists for G and r a positive number ε such that:

(25) *If $z \in E^\omega$ and $\varrho(z, Y) \leqslant \varepsilon$ then $z \in G$,*

(26) *If $z \in E^\omega$ and $\varrho(z, Y_0) \leqslant \varepsilon$ then $r(z) \in V$.*

Since Y_0 is an AR, there exists a retraction $r_0: E^\omega \to Y_0$. Then there is an open neighborhood U of Y_0 such that:

(27) *If $y \in U$ then $\varrho(y, r_0(y)) \leqslant \varepsilon$.*

Consider now a map $f_0: X_0 \to U$. Then $r_0 f_0: X_0 \to Y_0$ and consequently $r_0 f_0$ can be extended to a map $\alpha: X \to Y_0$. Since the values of f_0 and of $r_0 f_0$ lie in the Hilbert cube Q^ω, the formula

$$\beta_0(x) = f_0(x) - r_0 f_0(x) \quad \text{for every} \quad x \in X_0$$

defines a continuous function with values belonging (by (27)) to the set Z_ε consisting of all points $z = (z_1, z_2, \ldots) \in E^\omega$ such that $|z_i| \leqslant 2/i$ and $\varrho(z, 0) \leqslant \varepsilon$. Evidently Z_ε is an AR-set and consequently β_0 can be extended to a map

$$\beta: X \to Z_\varepsilon\,.$$

Now let us observe that the values of the map α lie in $Y_0 \subset Y$ and that $\varphi(\alpha(x) + \beta(x), \alpha(x)) = \varrho(\beta(x), 0) \leqslant \varepsilon$ for every $x \in X$. It follows by (25) that

$$\alpha(x) + \beta(x) \in G \quad \text{for every} \quad x \in X\,,$$

and consequently the formula

$$f(x) = r(\alpha(x) + \beta(x))$$

defines a continuous function. Since $\alpha(x) \in Y_0$ and $\beta(x) \in Z_\varepsilon$, we infer by (26) that the values of f belong to V. Hence

$$f: X \to V\,.$$

Moreover, if $x \in X_0$, then $f(x) = r(\alpha(x) + \beta(x)) = r(r_0 f_0(x) + f_0(x) - r_0 f_0(x)) = r f_0(x) = f_0(x)$, because $f_0(x) \in V \subset Y$. Thus f is an extension of f_0 and the lemma is proved.

11. Q^* does not contain any disk. Let us suppose that D^* is a disk lying in Q^*. Since the set $N^* \subset Q^*$ is countable, the disk D^* contains a subdisk with the boundary lying in the set $P^* = Q^* - N^*$. Consequently we may assume, without loss of generality, that the boundary C^* of D^* lies in P^*. Then $C = \varphi^{-1}(C^*)$ is a simple closed curve in Q and φ maps C onto C^* homeomorphically.

Now we shall consider two cases. First, let us suppose that $D^* \subset S^*$. Then S is the union of two disks D_1 and D_2 with C as their common boundary. Since the set N^* is countable, we infer that the set $D^*-C^*-N^*$ is arcwise connected, and consequently the set $\varphi^{-1}(D^*-C^*-N^*)$ is contained in one of the regions D_1-C and D_2-C. We can assume that $\varphi^{-1}(D^*-C^*-N^*) \subset D_1-C$. We now show that $D^*=\varphi(D_1)$ by showing that the assumption that there is a point p in $D_1-\varphi^{-1}(D^*)$ leads to an absurdity. First, we note that if D^* is given a definite metric, then for each positive integer n there is a simple closed curve J_n in D^*-N of diameter less than $1/n$ such that $\varphi^{-1}(J_n)$ separates p from D_2-C in S. We obtain J_n by considering a triangulation of D^* of mesh less than $1/n$ such that each 1-simplex of the triangulation misses N^* and find that the boundary of one 2-simplex of the triangulation serves as J_n. If $J_{n_1}, J_{n_2}, \ldots$ is a converging subsequence of $J_1, J_2, \ldots$, the limit of $\varphi^{-1}(J_{n_1}), \varphi^{-1}(J_{n_2}), \ldots$ separates p from D_2-C in S. However, this is impossible since this limit contains at most two points. Hence, we conclude that $D^*=\varphi(D_1)$. Now let us consider a chord K_ν of $\{K_\nu\}$ of (18) with ends a and b belonging to D_1. The set $D_1 \cap N$ is countable. Consequently there exists a simple arc J with ends a and b and with interior lying in D_1-C-N. Then $\varphi(J)$ is a simple closed curve lying in D^*-C^* and since D_1-J is arcwise connected, we infer that $D^*-\varphi(J)$ is connected. This contradiction disposes of the first case.

We now consider the second case, where D^* is not contained in S^*. It follows that D^* contains a subdisk lying in Q^*-S^*. Consequently we can assume that the given disk D^* lies in Q^*-S^*, and its boundary C^* lies in P^*. Then $C=\varphi^{-1}(C^*)$ is a simple closed curve lying in $Q-S$ and, by our construction, there exists in $Q-S$ a broken anchor ring $A \in \{A_\nu\}$ of (21) such that $C \subset Q-A$ and that C is linked with the core of A. Let $W \subset N$ be the wreath substituting for A. If $D^* \subset Q^*-N^*$, then $\varphi^{-1}(D^*)$ would be a disk in $Q-S-N \subset Q-W$ which is impossible by (15). Consequently every disk $D^* \subset Q^*-S^*$ intersects the countable set N^*-S^*. Let us order the points of $D^* \cap N^*$ into a sequence $\{a_n\}$ where $a_n \neq a_k$ for $n \neq k$. We can assume that $a_1, a_2, \ldots, a_m$ are the points of N^*-S^* of the form $a_i=\varphi(J_i)$, where J_i is a link of the wreath W. Let M be the chain substituting for A and $\hat{A}_i$ $(i=1,2,\ldots,m)$ be the link of M containing J_i. Let us observe that $\hat{A}_i \cap \hat{A}_j=0$ for $i \neq j$.

Now let us consider a positive number ε so small that

(28) $\varrho(x,y)>2\varepsilon$ for every $x \in J_i$ and $y \in Q-\hat{A}_i$, for $i=1,2,\ldots,m$,

and let V_i denote the ε-neighborhood of J_i in Q. Then

(29) $\varrho(x,y)>\varepsilon$ for every $x \in V_i$ and $y \in Q-\hat{A}_i$, for $i=1,2,\ldots,m$.

By the lemma of Section 10, there exists a neighborhood $U_i \subset V_i$ of the arc J_i such that every map of a closed subset X_0 of a space X into U_i can be extended to a map of X into V_i. Now let us consider, for $i=1,2,\ldots,m$ a disk $D_i^* \subset D^*$ containing in its interior the point a_i and such that its boundary C_i^* lies in D^*-N^* and that $\varphi^{-1}(D^*) \subset U_i$. Since $U_i \subset V_i \subset \hat{A}_i$, we infer that

(30) $$D_i^* \cap D_j^*=0 \quad \text{for} \quad i,j=1,2,\ldots,m,\ i \neq j.$$

It follows by (28) that, for $i=1,2,\ldots,m$:

(31) *The distance of every point of $\varphi^{-1}(D_i^*)$ from the boundary B_i of $\hat{A}_i$ is greater than ε.*

Moreover, by the definition of U_i, we infer that, for $i=1,2,\ldots,m$:

(32) *There exists a map $\psi_i\colon D_i^* \to V_i$ such that $\psi_i(x)=\varphi^{-1}(x)$ for every $x \in C_i^*$.*

Now suppose that, for some $k \geqslant m$, we have already defined the disks $D_1^*, D_2^*, \ldots, D_k^* \subset D^*$ such that $a_\nu \in D_\nu^*$ and that the boundary C_ν^* of D_ν^* lies in D^*-N^* for $\nu=1,2,\ldots,k$. Suppose also that we have defined the neighborhoods $V_1, V_2, \ldots, V_k$ in Q of sets $\varphi^{-1}(a_1), \varphi^{-1}(a_2), \ldots, \varphi^{-1}(a_k)$ respectively and also the maps $\psi_1, \psi_2, \ldots, \psi_k$ satisfying (32). Consider the point a_{k+1}. If $a_{k+1} \in \bigcup_{i=1}^{k} D_i^*$, then we choose an index $j \leqslant k$ such that $a_{k+1} \in D_j^*$ and we set $D_{k+1}^*=D_j^*$, $V_{k+1}=V_j$ and $\psi_{k+1}=\psi_j$. If, however, $a_{k+1} \in D^*-\bigcup_{i=1}^{k} D_i^*$, then we set $V_{k+1}=Q-W$ and we define D_{k+1}^* as a disk lying in $D^*-\bigcup_{i=1}^{k} D_i^*$ such that its boundary C_{k+1}^* lies in D^*-N^* and that it is a neighborhood (in D^*) of the point a_{k+1} so small that the partial map φ^{-1}/C_{k+1}^* can be extended to a map $\psi_{k+1}\colon D_{k+1}^* \to V_{k+1}$, for which the diameter of the set $\psi_{k+1}(D_{k+1}^*)$ is less than twice the diameter of $\varphi^{-1}(a_{k+1})$. By the lemma of Section 10 this is always possible.

Thus we have a sequence of disks $\{D_n^*\}$ and a sequence of maps $\{\psi_n\}$ such that:

(33) *If $n \neq n'$ then either $D_n^* \cap D_{n'}^*=0$, or $D_n^*=D_{n'}^*$ and $\psi_n=\psi_{n'}$.*

(34) *ψ_n is an extension of φ^{-1}/C_n^*.*

(35) *For every $\varepsilon<0$ there exists only a finite number of disks D_n^* such that the diameters of $\psi_n(D_n^*)$ are $\geqslant \varepsilon$.*

(36) *If a_n does not belong to $D_1^* \cup D_2^* \cup \ldots \cup D_m^*$ then $\psi_n(D^*) \subset Q-W$.*

Now let us set

(37) $$g(x)=\begin{cases}\varphi^{-1}(x) & \text{for every} \quad x \in D^*-\bigcup_{i=1}^{\infty} D_i^*,\\ \psi_i(x) & \text{for every} \quad x \in D_i^*,\ i=1,2,\ldots\end{cases}$$

It follows from (33) that g is a function $g\colon D^* \to Q$ and from (34) and (35) that this function is continuous. Moreover, we infer from (36) that

(38) $$g^{-1}(J_i) \subset D_i^* \quad \text{for} \quad i=1,2,\ldots,m.$$

Applying (17), we infer that there exists an index $i_0 \leqslant m$ such that the disk D^* contains two disjoint disks D' and D'' with boundaries C' and C'' such that the loops $g|C'$ and $g|C''$ are linked with the core C_{i_0} of a link $\hat{A}_{i_0}$ of the chain M and that $g(C')$ and $g(C'')$ are subsets of the ε-neighborhood of the boundary B_{i_0} of the link $\hat{A}_{i_0}$.

It follows by (15) that if J_{i_0} is the link of the wreath W lying in A_{i_0} then

$$g(D') \cap J_{i_0} \neq 0 \neq g(D'') \cap J_{i_0}.$$

By virtue of (38), we conclude that

$$D' \cap D^*_{i_0} \neq 0 \neq D'' \cap D^*_{i_0}.$$

Since the disks D' and D'' are disjoint, we infer that the disk $D^*_{i_0}$ intersects the boundary of D', and consequently $D^*_{i_0} \cap C' \neq 0$, hence also $g(D^*_{i_0}) \cap g(C') \neq 0$. But this is impossible, by (28) and (29), because the points of $g(D^*_{i_0})$ belong to the ε-neighborhood of J_{i_0} and the points of $g(C') = \varphi^{-1}(C')$ belong to the ε-neighborhood of B_{i_0}.

Thus the supposition that Q^* contains a disk leads to a contradiction.

Remark. It follows by our construction that the hyperspace X of the decomposition of the set $Q-S$ whose non degenerate elements coincide with the links of the wreaths W_ν, $\nu = 1, 2, \ldots$, does not contain any disk. By a generalization of the theorem of Andrews and Curtis [2] obtained recently by D. S. Gillman and J. M. Martin [9] the Cartesian product of E^1 by the decomposition-space of E^n into points and arcs, with only a countable number of arcs, is homeomorphic with E^{n+1}. It follows that $E^1 \times X$ is topologically E^4 and thus we get a 3-dimensional Cartesian divisor of E^4 which does not contain any disk.

12. Problems. The following questions remain open:

1° Is it true that the space Q^* (constructed in Section 8) does not contain any 2-dimensional AR? More exactly, is it true that the first Betti number of every 2-dimensional subset of Q^* is infinite?

2° Does there exist an AR-space of an arbitrarily given dimension which does not contain any disk?

3° Does there exist a Cartesian divisor of a Euclidean cube such that its dimension is greater than or equal to 2 but it does not contain any disk?

References

[1] P. Alexandroff, *Dimensionstheorie*, Math. Annalen 106 (1932), pp. 161-238.

[2] J. J. Andrews and M. L. Curtis, *n-space modulo an arc*, Annals of Math. 75 (1962), pp. 1-7.

[3] L. Antoine, *Sur l'homéomorphie de figures et de leurs voisinages*, Journ. de Math. 86 (1921), pp. 221-325.

[4] K. Borsuk, *Sur les rétractes*, Fund. Math. 17 (1931), pp. 152-170.

[5] — *On an irreducible 2-dimensional absolute retract*, ibidem 37 (1950), pp. 137-160.

[6] — *On a family of 2-dimensional AR-sets*, ibidem 51 (1962), pp. 283-297.

[7] — *Theorie der Retrakte, und ihre Problems*, Izwiestija na matiematiczeskaja institut 4 (1959), pp. 45-54.

[8] R. P. Coelho, *On the groups of certain linkages*, Portugaliae Mathematica 6 (1947), pp. 57-65.

[9] D. S. Gillman and J. M. Martin, *Countable decompositions of E^3 into point and pointlike arcs*, Amer. Math. Soc. Notices 10 (1963), pp. 74-75.

[10] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton (1941), pp. 1-165.

[11] S. Lefschetz, *Algebraic topology*, Amer. Math. Soc. Coll. Publ. 27 (1942), pp. 1-389.

[12] A. Lelek, *Some mappings of ANR-sets*, Fund. Math. 52 (1963), pp. 225-233.

[13] G. T. Whyburn, *Concerning the proposition that every closed, compact, and totally disconnected set of points is a subset of an arc*, ibidem 18 (1932), pp. 47-60.

Reçu par la Rédaction le 16. 1. 1963