Ordering relations admitting automorphisms

by

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In this paper we find (1) a necessary and sufficient condition that an ordering relation have a (non-trivial) automorphism group, (2) a necessary and sufficient condition that an ordering relation have a non-Abelian automorphism group, and (3) a lower bound to the cardinality of a non-Abelian automorphism group of an ordering relation (1), (2).

For notation and definitions, the reader is first referred to [1], the content of which is closely related to that of this paper; we shall freely use terms defined in [1]. Further background material can be found in [5] and [7]. If A is any set, we write $\varkappa(A)$ for the cardinality of A. We use the symbols $\subseteq$ and $\subset$ for inclusion and proper inclusion respectively. We write $A\times B$ for the Cartesian product of A and B. Let S be any simply ordering relation; we shall not distinguish between the group $G(S)$ and the set of automorphisms of S. We write e for the identity of the group $G(S)$. We will use the letters m and n for finite order types. If a is an order type, then a^* denotes the inverse of a. For any relation S, $\tau(S)$ is the order type of S. The order type of the natural numbers in their usual order will be written as ω; hence ω^* (or $\omega^*+\omega$) is the order type of the negative integers (or the integers) in their usual order. We shall denote by I the ordering relation of the integers. The relations R and S are *strictly disjoint* if $F(R)\cap F(S)=\emptyset$. If R, S, and T are initial, middle and final segments of U, we write $R\mathfrak{I}U$, $S\mathcal{M}U$ and $T\mathcal{F}U$; similarly, for types ϱ, σ, τ and v we write $\varrho\mathfrak{I}v$, $\sigma\mathcal{M}v$ and $\tau\mathcal{F}v$. If $x, y\in F(R)$, then $[x, y)$ is the set of $z\in F(R)$ such that xRz, zRy, and $z\neq y$. For any relation R and any set A, we put $R\langle A\rangle = R\cap(A\times A)$.

Let S be an ordering relation with $x, y\in F(S)$. Let $x\sim y$ if either (i) xSy and there exist $f\in G(S)$ and $x_1, y_1\in F(S)$ such that x_1Sx, ySy_1, and $f(x_1)=y_1$, or (ii) ySx and there exist $f\in G(S)$ and $x_1, y_1\in F(S)$ such that y_1Sy, xSx_1 and $f(y_1)=x_1$. Clearly, $\sim$ is an equivalence relation over

(1) The results of this paper were stated without proof in [3] and [6].

(2) Theorems (1) and (2) answer problems (a) and (b) proposed by Goffman in [4]; problem (c) is solved in the Abelian case in [1]; the general case, which appears to be very difficult, is open.

$F(S)$; the equivalence class $x/\sim$ will be denoted by C_x. We note that $S\langle C_x\rangle$ is a segment of S. Now let $f \in G(S)$ and $x \in F(S)$; if $xSf(x)$, we put

$$C_{f,x} = \{y \mid f^i(x)Sy \text{ and } ySf^{i+1}(x) \text{ for some integer } i\};$$

if $f(x)Sx$, we put

$$C_{f,x} = \{y \mid f^{i+1}(x)Sy \text{ and } ySf^i(x) \text{ for some integer } i\}.$$

We observe that $S\langle C_{f,x}\rangle$ is a segment of S and that

$$C_{f,x} \subseteq C_x.$$

LEMMA. *Let S be an ordering relation, let $f \in G(S)$ and let $x \in F(S)$. Then either*

(i) $\tau(S\langle C_{f,x}\rangle) = 1$,

or

(ii) *For some type $\gamma \neq 0$, $\tau(S\langle C_{f,x}\rangle) = \gamma\cdot(\omega^*+\omega)$.*

Proof. If $f(x) = x$, then (i). If $f(x) \neq x$, we may assume $xSf(x)$. We put

$$\gamma_i = \tau\Big(S\big\langle[f^i(x), f^{i+1}(x))\big\rangle\Big) \quad \text{for every integer } i.$$

Now $\gamma_i \neq 0$ for each i and

$$\tau(S\langle C_{f,x}\rangle) = \sum_{i,I}\gamma_i.$$

Since f is an automorphism, it is clear that $\gamma_i = \gamma_{i+1}$ always; hence we have (ii).

THEOREM 1. *Let S be an ordering relation and let $\alpha = \tau(S)$. Then the following conditions are equivalent:*

(i) *$G(S)$ is non-trivial.*

(ii) *There exists a type $\gamma \neq 0$ such that $\gamma\cdot(\omega^*+\omega)\mathcal{M}\alpha$.*

Proof. Suppose (i) holds. Choose $f \in G(S)$ with $f \neq e$. Then (ii) follows from the Lemma. If (ii) holds, then S is representable in the form

$$S = T + \sum_{i,I} U_i + V,$$

where $\tau(U_i) = \gamma$ for each integer i and the relations occurring on the right side of the identity are pairwise strictly disjoint. For each i, let f_i be an automorphism of U_i onto U_{i+1}. Now we can define an $f \in G(S)$ with $f \neq e$ as follows:

$$f(x) = x \quad \text{for} \quad x \in F(T) \cup F(V),$$
$$f(x) = f_i(x) \quad \text{elsewhere.}$$

THEOREM 2. *Let S be an ordering relation and let $\alpha = \tau(S)$. Then the following conditions are equivalent:*

(i) *$G(S)$ is non-Abelian.*

(ii) *There exist types β and δ and a type $\gamma \neq 0$ such that*

$$\big(\beta + \gamma\cdot(\omega^*+\omega) + \delta\big)\cdot(\omega^*+\omega)\mathcal{M}\alpha.$$

Proof. Suppose (i) holds. By Theorem 3 of [2], the group $G(S)$ cannot be simply ordered. Now using Theorem 2 of [2], we easily find that there must exist an $f \in G(S)$ and $x, y \in F(S)$ such that $y \in C_x$ and

$$f(y)Sy, \quad xSf(x) \quad \text{and} \quad x \neq f(x).$$

Since $zSf(z)$ and $z \neq f(z)$ for every $z \in C_{f,x}$, we have

$$C_{f,x} \subset C_x.$$

Hence either (a) tSy for every $t \in C_{f,x}$, or (b) ySt for every $t \in C_{f,x}$; for definiteness assume (a). There must be a $g \in G(S)$ and $x_1, y_1 \in C_x$ such that x_1Sx, ySy_1, and $g(x_1) = y_1$; obviously $g \neq e$. We have obtained

(1) There exist $x \in F(S)$ and $f, g \in G(S)$ such that $x \neq f(x)$, $x \neq g(x)$, and $C_{f,x} \neq C_{g,x}$.

We can assume that $xSf(x)$ and $xSg(x)$ in (1). Put

(2) $$\varphi = \tau\Big(S\big\langle[x, f(x)]\big\rangle\Big) \text{ and } \theta = \tau\Big(S\big\langle(x, g(x))\big\rangle\Big).$$

Using (1), (2), the Lemma, and the fact that $S\langle C_{f,x}\rangle$ is a segment of S, we find that at least one of the following cases holds:

(3) There is an $\varepsilon \neq 0$ such that $\theta\cdot\omega = \varphi\cdot\omega + \varepsilon$ and $\varphi\cdot\omega^* = \theta\cdot\omega^*$.

(4) There is an $\varepsilon \neq 0$ such that $\theta\cdot\omega^* = \varepsilon + \varphi\cdot\omega^*$ and $\varphi\cdot\omega = \theta\cdot\omega$.

(5) There exist non-zero types ε and ζ such that $\theta\cdot\omega = \varphi\cdot\omega + \varepsilon$ and $\varphi\cdot\omega^* = \zeta + \theta\cdot\omega^*$.

(6) There exist non-zero types ζ and ε such that $\theta\cdot\omega = \varphi\cdot\omega + \zeta$ and $\theta\cdot\omega^* = \varepsilon + \varphi\cdot\omega^*$.

(7) A case obtained from (3)-(6) by interchanging φ and θ everywhere.

Suppose (3) holds. Applying the elementary Lemma 3.5 of [5] to the first identity of (3), we obtain at least one of the two statements below:

(8) There is a finite type $n \neq 0$ such that $\varphi\cdot\omega = \theta\cdot n$ and $\varepsilon = \theta\cdot\omega$.

(9) There is a finite type n and non-zero types $\theta^{(1)}$ and $\theta^{(2)}$ such that $\varphi\cdot\omega = \theta\cdot n + \theta^{(1)}$, $\varepsilon = \theta^{(2)} + \theta\cdot\omega$ and $\theta^{(1)} + \theta^{(2)} = \theta$.

If (9) holds, add $\theta^{(2)}$ on the right to the first identity of (9); we now see that (3) implies

(10) There exist types $n \neq 0$ and π such that $\varphi\cdot\omega + \pi = \theta\cdot n$.

From the second identity of (3) we get

$$\varphi\cdot\omega^* = \theta\cdot\omega^* + \theta\cdot n.$$

Now applying Lemma 3.5 of [5] to the identity above, we obtain either

(11) There is a type $m \neq 0$ such that $\theta \cdot \omega^* = \varphi \cdot \omega^*$ and $\varphi \cdot m = \theta \cdot n$,

or

(12) There exist a finite type m as well as non-zero types $\varphi^{(1)}$ and $\varphi^{(2)}$ such that $\varphi^{(1)} + \varphi^{(2)} = \varphi$, $\varphi \cdot \omega^* + \varphi^{(1)} = \theta \cdot \omega^*$, and $\varphi^{(2)} + \varphi \cdot m = \theta \cdot n$.

Suppose (11) is satisfied; put

(13) $$\varphi' = \varphi \cdot m \text{ and } \theta' = \theta \cdot n.$$

If (12) holds, we add $\varphi^{(1)}$ on the left to the third identity of (12), obtaining

(14) $$\varphi^{(1)} + \varphi^{(2)} + \varphi \cdot m = \varphi \cdot (m+1) = \varphi_1^{(1)} + \theta \cdot n.$$

In this case we put

(15) $$\varphi' = \varphi \cdot (m+1) \text{ and } \theta' = \theta \cdot n.$$

From (15) and (10), we have $\varphi' \mathfrak{J} \theta'$; from (15) and (14) we have $\theta' \mathcal{F} \varphi'$. Now using Corollary 1.28 of [7], we get

(16) $$\varphi' = \theta'.$$

Note that if φ' and θ' are defined by (13) rather than by (15), then the statement (16) is a part of (11). Now using (10), (16), and (13) or (15), we find that

(17) $$\varphi \cdot \omega + \pi = \varphi' \cdot \omega + \pi = \theta' = \varphi',$$

and hence

(18) $$\varphi' + \varphi' = \varphi'.$$

From (18) and Theorem 1.37 of [7] we get

(19) There is a type ι such that $\varphi' = \varphi' \cdot \omega + \iota + \varphi' \cdot \omega^*$.

Using (2), (13) or (15), (18) and (19), we see that

(20) $$\begin{aligned} \tau(S\langle C_{j,x}\rangle) &= \varphi \cdot (\omega^* + \omega) = \varphi' \cdot (\omega^* + \omega) \\ &= \big((\varphi' \cdot \omega + \iota + \varphi' \cdot \omega^*) + (\varphi' \cdot \omega + \iota + \varphi' \cdot \omega^*)\big) \cdot (\omega^* + \omega) \\ &= \big((\varphi' \cdot \omega + \iota) + \varphi' \cdot (\omega^* + \omega) + (\iota + \varphi' \cdot \omega^*)\big) \cdot (\omega^* + \omega). \end{aligned}$$

Now set

$$\varphi' \cdot \omega + \iota = \beta, \quad \varphi' = \gamma, \quad \iota + \varphi' \cdot \omega^* = \delta$$

in (20). Then we have

$$\tau(S\langle C_{j,x}\rangle) = \big(\beta + \gamma \cdot (\omega^* + \omega) + \delta\big) \cdot (\omega^* + \omega).$$

Since $\varphi \neq 0$, $\gamma = \varphi' \neq 0$, and we obtain (ii).

The case (4) is symmetric by inverses with (3). Now assume that (5) holds. Since (10) was derived from the first identity of (3), we again have (10). Using the fact that the first and second identities of (5) are symmetric by inverses, we see that the second identity yields

(21) $$\mu + \theta \cdot \omega^* = \varphi \cdot m \text{ for some } \mu \text{ and some finite } m \neq 0.$$

As in case (3), put

$$\varphi' = \varphi \cdot m \quad \text{and} \quad \theta' = \theta \cdot n.$$

Then by (10) and (21), we have

$$\varphi' \mathfrak{J} \theta' \quad \text{and} \quad \theta' \mathcal{F} \varphi';$$

using Corollary 1.28 of [7] again, we obtain

$$\theta' = \varphi'.$$

Now the argument leading from (17) to (ii) is applicable here.

Suppose that (6) holds. Then we have (10) and also

$$\nu + \varphi \cdot \omega^* = \theta \cdot m \quad \text{for some } \nu \text{ and some } m \neq 0.$$

Moreover,

$$\theta \cdot m + \theta \cdot n = \nu + \varphi \cdot \omega^* + \varphi \cdot \omega + \pi = \nu + \varphi \cdot (\omega^* + \omega) + \pi = \theta \cdot (m+n);$$

and

$$\theta \cdot (\omega^* + \omega) = \theta \cdot (m+n) \cdot (\omega^* + \omega) = \big(\nu + \varphi \cdot (\omega^* + \omega) + \pi\big) \cdot (\omega^* + \omega);$$

hence (ii) holds.

Now let (ii) hold. Then S is representable in the form

(22) $$S = T_1 + \sum_{j,I} \Big(U^{(j)} + \sum_{i,I} R_i^{(j)} + V^{(j)}\Big) + T_2,$$

where

$$\tau(R_i^{(j)}) = \gamma \text{ for all integers } i, j,$$
$$\tau(U^{(j)}) = \beta \text{ and } \tau(V^{(j)}) = \delta \text{ for all integers } j,$$

and any two relations on the right of the identity sign in (22) are pairwise strictly disjoint. For all integers i, j, i', j' we choose functions $f_U^{(j,j')}$, $f_V^{(j,j')}$, $f_{i,i'}^{(j,j')}$ as follows:

(23) $$\begin{aligned} &f_U^{(j,j')} \text{ maps } U^{(j)} \text{ isomorphically onto } U^{(j')}, \\ &f_V^{(j,j')} \text{ maps } V^{(j)} \text{ isomorphically onto } V^{(j')}, \\ &f_{i,i'}^{(j,j')} \text{ maps } R_i^{(j)} \text{ isomorphically onto } R_{i'}^{(j')}. \end{aligned}$$

Then there exists an automorphism $f \in G(S)$ defined as follows:

$$\begin{aligned} f(x) &= x \quad \text{for} \quad x \in F(T_1) \cup F(T_2), \\ f(x) &= f_U^{(j,j+1)}(x) \text{ for all } j \text{ and all } x \in F(U^{(j)}), \\ f(x) &= f_V^{(j,j+1)}(x) \text{ for all } j \text{ and all } x \in F(V^{(j)}), \\ f(x) &= f_{i,i+1}^{(j,j+1)}(x) \text{ for all } i, j \text{ and all } x \in F(R_i^{(j)}). \end{aligned}$$

There also exists a $g \in G(S)$ defined as follows:

$$g(x) = f_{i,i+2}^{(1,2)}(x) \text{ for all } i \text{ and all } x \in F(R_i^{(1)}),$$
$$g(x) = f(x) \text{ elsewhere.}$$

Now suppose $x \in F(R_0^{(0)})$. Then $f(x) \in F(R_1^{(1)})$ and $gf(x) \in F(R_3^{(2)})$; on the other hand, $g(x) \in F(R_1^{(1)})$ and $fg(x) \in F(R_2^{(2)})$. Hence (i) holds.

COROLLARY 3. *If S is an ordering relation and if $G(S)$ is non-Abelian, then*

$$\varkappa(G(S)) \geqslant 2^{\aleph_0}.$$

Proof. By Theorem 2, S is representable in the form (22) above. For each function h on the set of integers to $\{0, 1\}$ we use (22) above to define an automorphism f_h of S:

$$f_h(x) = x \text{ for } x \in F(T_1) \cup F(T_2),$$
$$f_h(x) = f_U^{(j,j+1)}(x) \text{ for all } j \text{ and all } x \in F(U^{(j)}),$$
$$f_h(x) = f_V^{(j,j+1)}(x) \text{ for all } j \text{ and all } x \in F(V^{(j)}),$$
$$f_h(x) = f_{i,i+h(j)}^{(j,j+1)}(x) \text{ for all } i, j \text{ and all } x \in F(R_i^{(j)}).$$

Clearly, $f_h \neq f_{h'}$ for $h \neq h'$.

We note in conclusion that every non-Abelian $G(S)$ has as a subgroup the automorphism group of a relation of type $(\omega^* + \omega)^2$.

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On the imbedding of a regular ring in a regular ring with identity

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1. Introduction. Throughout this note we shall suppose that R is an associative ring, regular in the sense of von Neumann (this means that for every $a \in R$, $axa = a$ for some $x \in R$). We shall prove the following theorem:

THEOREM 1. *A regular ring is isomorphic to a two-sided ideal of a regular ring with identity.*

A special case of this theorem has been established previously by Kohls [3]. Also, Johnson [2] has shown that, for a certain class of rings which includes all regular rings, each of the rings is isomorphic to a subring of a regular ring with identity.

In this note we shall not require familiarity with the theory of regular rings.

Our procedure is to imbed the regular ring R in the ring of all pairs (a, ϱ) with $a \in R$ and ϱ from a suitable commutative regular ring M with identity such that R is an algebra over M. If every non-zero element of R has the same additive order, necessarily a prime or infinity, then we can choose M as the prime field of the corresponding characteristic (and begin our proof with section 4). In the most general case we shall need the following result:

THEOREM 2. *There exists a commutative regular ring M with identity such that every regular ring R is an algebra over M.*

First we shall construct this ring M and then turn each regular ring into an algebra over M. Finally, we verify the main result, i.e. Theorem 1.

2. Construction of M. Let $p_1, \ldots, p_i, \ldots$ be the set of rational primes in some order, and K_{p_i} the prime field of characteristic p_i. Define K as the complete direct sum of the K_{p_i}:

$$K = \sum_{p_i}^{*} K_{p_i},$$