to show that R^* is a regular ring. Given $(a, \varrho) \in R^*$, determine $\sigma \in M$ and $b \in R$ as in Lemma 3. Then we have

$$(a, \varrho)(b, \sigma)(a, \varrho) = (a, \varrho),$$

and since $(b, \sigma) \in R^*$, R^* is in fact regular. This completes the proof of Theorem 1 (5).

References

[1] L. Fuchs, *Abelian groups*, Budapest 1958.

[2] R. E. Johnson, *The extended centralizer of a ring over a module*, Proc. Amer. Math. Soc. 2 (1951), pp. 891-895.

[3] C. W. Kohls, *On the embedding of a generalized regular ring in a ring with identity*, Michigan Math. Journ. 3 (1955-56), pp. 165-168.

(5) As an alternative to (i) we could identify (a, ϱ) and (a', ϱ') in R^* if and only if for all $x \in R$, $ax+\varrho x = a'x+\varrho' x$. Our theorem holds with this new R^* (in general different from the previous R^*). The new R^* can be identified with a subring of the ring of left endomorphisms of R and, if R has an identity, this new R^* will coincide with R.

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Skolem-type normal forms for first-order languages with a generalized quantifier *

by

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Let L be a first-order language with countably many non-logical constants and let α be an ordinal number. With L and α we associate a new language L_α which is obtained from L by adding to the symbols of L a new quantifier Q which is read "there are at least $\aleph_\alpha$...". Let V_α be the set of all sentences of L_α which are logically valid (where Q is counted among the logical symbols).

Mostowski raised the question (1) whether V_α is axiomatizable. For $\alpha = 0$ he showed that the answer is negative (provided L has a sufficient supply of non-logical constants). One can show (see [3] and [5]) that for the majority of ordinals Mostowski's proof cannot be adapted. In fact for $\alpha = 1$ the answer is positive as shown by Vaught (see [16]).

Another natural question to ask is the following: What is the relationship between the various V's? (Note that the formation rules for L_α are independent of the particular α.) Here we obtain as partial results:

(1) $V_1 \subseteq V_\alpha$ *for any ordinal* α *for which* $\aleph_\alpha$ *is regular*;

(2) $V_\alpha \subseteq V_0$ *for any ordinal* α.

The following negative results are known (assuming that L has a sufficient supply of non-logical constants):

(3) $V_0 \not\subseteq V_\alpha$ *for any ordinal* $\alpha > 0$;

(4) $V_\alpha \not\subseteq V_\beta$ *for any limit ordinal* α *and any successor ordinal* β;

(5) $V_\alpha \not\subseteq V_\beta$ *for any ordinals* α *and* β *for which* $\aleph_\alpha$ *is regular and* $\aleph_\alpha$ *is singular*;

* The paper is mainly based on § 2 of Part I of the author's Doctoral Dissertation [2]. The main results have been summarized in [6]. The author takes the opportunity to express his gratitude to his Thesis Advisor Professor Robert L. Vaught and to Professor William Craig for the stimulation and help received. It was in Professor William Craig's Seminar conducted at the University of California in Berkeley in 1961 that the author learned about the problems treated in this paper and obtained the first results in this direction.

(1) In [10]; see also [11].

(6) $V_\alpha \nsubseteq V_\beta$ *for any ordinal* α *and* β *such that* (i) $\mathfrak{m} < \aleph_\alpha$ *implies* $2^{\mathfrak{m}} < \aleph_\alpha$, *for all cardinals* $\mathfrak{m}$; *while* (ii) $\mathfrak{m} < \aleph_\beta$ *and* $\aleph_\beta \leqslant 2^{\mathfrak{m}}$, *for some cardinal* $\mathfrak{m}$.

This leaves the following conjectures:

(A) $V_\alpha = V_\beta$ whenever

(a) α and β are both successor ordinals;

or (b) $\aleph_\alpha$ and $\aleph_\beta$ are both singular;

or (c) α and β are both weakly but not strongly inaccessible ordinals;

or (d) α and β are both strongly inaccessible ordinals (greater than ω).

(B) $V_\alpha \subseteq V_\beta$ whenever

(a) α is a successor ordinal or $\aleph_\alpha$ is a singular;

and (b) β is a weakly inaccessible ordinal.

(1) and (2) are obtained from results of first-order model theory with the help of Skolem-type normal forms. By a Skolem-type normal form of a sentence we mean, roughly, a sentence which may contain additional predicates, in which the quantifiers occur only in certain special contexts, and which has the property that it is satisfiable if and only if the original sentence is satisfiable. The method may be used to establish other results, e.g. a compactness theorem for L_1 is obtained. For a further application, see the following note by Vaught (this volume, pp. 303-304).

1. Preliminaries. (2) Let L have as non-logical constants countably many predicates; the list of (individual) variables is $v_0, v_1, \ldots$; the (primitive) sentential connectives are negation and conjunction; the (primitive) quantifier is the existential quantifier; the identity predicate is the only logical predicate.

The symbols of L_α are the symbols of L together with the new quantifier. The syntactical notions like being a formula, being a subformula, being a sentence of L_α are defined in the usual way with the obvious modifications. In particular:

(1.1) For every formula φ of L_α and every variable v_n there is a unique formula $Qv_n\varphi$ of L_α. $Qv_n\varphi$ in turn determines v_n and φ, and can only obtained in this way.

(1.2) A variable occurs free in $Qv_n\varphi$ if, and only if, it occurs free in φ and is distinct from v_n.

The notion of an assignment $\boldsymbol{a}$ over a relational system $\mathfrak{A}$ (of appropriate similarity type) satisfying a formula of L_α in $\mathfrak{A}$, and of a sentence of L_α being true in $\mathfrak{A}$ is defined in the usual way with the additional stipulation:

(2) For the model-theoretic notions see [12], [13], [14], and [9].

(1.3) $\boldsymbol{a}$ *satisfies* $Qv_n\varphi$ *in* $\mathfrak{A}$ *iff there are at least* $\aleph_\alpha$ *elements* x *in* A *for which* $\boldsymbol{a}(n/x)$ *satisfies* φ *in* $\mathfrak{A}$, *where* $\boldsymbol{a}(n/x)$ *is the assignment over* $\mathfrak{A}$ *which differs from* $\boldsymbol{a}$ *at most at the n-th place and has the value* x *at this place.*

Let Γ be a set of sentences of L_α. The following two properties hold:

(1.4) *If* Γ *has a model of power* $\aleph_\beta$, *then* Γ *has a model of every power* $\aleph_\gamma$ *for which* $\alpha \leqslant \gamma \leqslant \beta$.

(1.5) *For every sentence* φ *of* L_α *there is a first-order sentence* φ' *of* L_α *with the property that for every relational system* $\mathfrak{A}$, $\mathfrak{A}$ *is a model of* $\varphi \wedge \neg Qv_0v_0 = v_0$ *if and only if* $\mathfrak{A}$ *is a model of* $\varphi' \wedge \neg Qv_0v_0 = v_0$. *Let* Γ' *be the set of all* φ' *for* φ *in* Γ. Γ *and* Γ' *have the same models of power less than* $\aleph_\alpha$. *Note that* φ' *and* Γ' *do not depend on the particular* α.

(1.4) is in essence to be found already in [10]. It is a downward Löwenheim-Skolem theorem and can be proved by adapting the proof of the corresponding first-order theorem given in [13], obtaining at the same time a stronger version of (1.4). (1.5) is a consequence of the observation that in a relational system of power less than $\aleph_\alpha$ the $\aleph_\alpha$-quantifier acts trivially, i.e. no assignment satisfies any formula $Qv_n\varphi$.

We shall make use of the following theorems of model theory about first-order theories T with countably many non-logical constants:

(1.6) *Assume that among the non-logical constants of* T *there is a unary predicate* U. *If* T *has an infinite model* $\mathfrak{A}$ *for which* $|U^{\mathfrak{A}}| < |A|$ (3), *then* T *has an infinite model* $\mathfrak{A}$ *for which* $|U^{\mathfrak{A}}| < |A| = \aleph_1$.

This is Vaught's Löwenheim-Skolem theorem form two cardinals. For a proof of (1.6) see [9]: This theorem is stated there for complete T only, but this special case easily implies the general one.

(1.7) *Assume* T *and* U *are as in* (1.6). *Let* W *be a new unary predicate and* Δ *the set of all sentences*

$$\bigwedge v_0 \ldots \bigwedge v_{n-1}\left[Wv_0 \wedge \ldots \wedge Wv_{n-1} \to [\varphi \leftrightarrow \varphi^{(W)}]\right]$$

where φ *is any formula of (the language of)* T *having no other free variables than* $v_0, \ldots, v_{n-1}$, *and* $\varphi^{(W)}$ *is obtained from* φ *by relativising quantifiers to* W. *If* $T \cup \Delta \cup \{\bigwedge v_0[Uv_0 \to Wv_0], \bigvee v_0 \neg Wv_0\}$ *is consistent, then* T *has a model* $\mathfrak{A}$ *for which* $|U^{\mathfrak{A}}| < |A| = \aleph_1$, *and conversely, if* T *has an infinite model* $\mathfrak{A}$ *for which* $|U^{\mathfrak{A}}| < |A|$, *then the above set of sentences is consistent.*

This is a lemma underlying the proof of (1.6), though it is not stated explicitely in [9].

(3) $U^{\mathfrak{A}}$ is the interpretation of U in $\mathfrak{A}$; $|X|$ is the cardinality of X.

(1.8) *Assume that among the non-logical constants of T there is a binary predicate R, and that for every model $\mathfrak{A}$ of T, $R^{\mathfrak{A}}$ is a linear ordering of A. Assume furthermore that $\aleph_\alpha$ is regular. If T has a model of power $\aleph_\alpha$ every proper initial R-segment of which is of smaller power, then T has a model of power $\aleph_1$ every proper R-segment of which is countable.*

This generalizes (1.6). The proof is similar to the proof of (1.6) and runs, roughly, like follows:

First observe that T has a model $\mathfrak{A}_0$ which has a proper elementary extension $\mathfrak{A}_1$ such that 'all old elements precede all new elements'. (This corresponds in the proof of (1:6) to a model of T with a proper elementary extension "having the same U".) In fact, if $\alpha = 0$ take for $\mathfrak{A}_0$ the model given by hypothesis and for $\mathfrak{A}_1$ any proper elementary extension of $\mathfrak{A}_0$. (Note that $R^{\mathfrak{A}_0}$ is of type ω.) If $\alpha > 0$ take for $\mathfrak{A}_1$ the model given by hypothesis. Let $\mathfrak{B}_0$ be a denumerable elementary subsystem of $\mathfrak{A}_0$, and let X_0 be a proper initial R-segment of $\mathfrak{A}_1$ which contains B_0. Let $\mathfrak{B}_1$ be an elementary subsystem of $\mathfrak{A}_1$ which contains X_0 and is of the same power as X_0. Repeating this process we obtain an elementary chain [4] $\mathfrak{B}_0, \mathfrak{B}_1, \ldots$ of elementary subsystems of $\mathfrak{A}_1$. Take for $\mathfrak{A}_0$ the union of this chain.

As in the proof of (1.6) we can even find $\mathfrak{A}_0$ and $\mathfrak{A}_1$ which are in addition denumerable, homogenous, and have the same kinds of elements, and thus are isomorphic by a Cantor-type argument. Hence $\mathfrak{A}_1$, like $\mathfrak{A}_0$, has a proper elementary extension $\mathfrak{A}_2$ which is isomorphic to $\mathfrak{A}_1$ and "every new elements follows every old element". Repeating this process we arrive at an elementary chain $\mathfrak{A}_0, \mathfrak{A}_1, \ldots$ of isomorphic models where "new elements are allways added at the end". The union of such a countable chain is again denumerable, homogeneous, and has the same kind of elements as $\mathfrak{A}_0$. The process can thus be continued into the transfinite and ends after ω_1 steps with the desired model.

(1.9) *Assume that T and R are as in* (1.8). *If T has a model $\mathfrak{A}$ for which $R^{\mathfrak{A}}$ is of type ω, then for every ordinal α, T has a model $\mathfrak{A}$ of power $\aleph_\alpha$ such that every proper initial R-segment of $\mathfrak{A}$ is of smaller power.*

This is a consequence of a theorem in McDowell and Specker [7]. Let $\mathfrak{A}$ be the model given by hypothesis. We may assume $A = \omega$ and $\alpha > 0$. Expand $\mathfrak{A}$ by the addition and the multiplication on ω, i.e. form $\mathfrak{B}_0 = (\mathfrak{A}, +, \cdot)$. By the theorem just mentioned every model which is elementarily equivalent to $\mathfrak{B}_0$ has a proper elementary extension of the same power where "new elements are added at the end only". We thus obtain an elementary chain $\mathfrak{B}_0, \mathfrak{B}_1, \ldots$ of proper elementary extensions, continuing the process into the transfinite by taking unions at the limit stages. The appropriate reduct of $\mathfrak{B}_{\omega_\alpha}$ is the desired model.

(4) I.e. a chain with respect to being an elementary subsystem.

(1.10) *Assume T and R are as in* (1.8). *Let θ be a* (*weakly*) *inaccessible ordinal* (*greater than* ω). *If $\mathfrak{A}$ is a model of T for which $R^{\mathfrak{A}}$ is of type θ, then $\mathfrak{A}$ has an elementary subsystem $\mathfrak{B}$ which has the property that* (i) *$R^{\mathfrak{B}}$ has a type which is an initial ordinal cofinal with ω; and* (ii) *B is an initial R-segment of $\mathfrak{A}$.*

This is a rudimentary form of a theorem in [8]. The proof is similar to the argument in the beginning of the proof of (1.8) where the case $\alpha > 0$ is considered. The only difference is that one has to choose $X_0, X_1, \ldots$ such that they are initial segments which correspond to initial ordinals and that they form a properly increasing chain.

2. The Skolem-type normal forms. (5)

The first Skolem-type normal form is designed for the study of L_α when α is a successor ordinal, say $\alpha = \beta + 1$. The idea behind the normal form is as follows: Let $\mathfrak{A}$ be a model of power $\aleph_\alpha$. Choose a subset U of A of power $\aleph_\beta$. Instead of saying "the set S has power at least $\aleph_\alpha$" we shall say "there is a biunique correspondence between a subset of S and A". Instead of saying "the set S has power less than $\aleph_\alpha$" we say "there is a biunique correspondence between S and a subset of U". Furthermore, the collection of correspondences is replaced by one ternary relation which "parametrizes" these correspondences.

Definition (2.1). Let L^*_α be a language obtained from L_α by adding to the non-logical constants a new unary predicate U and a new ternary predicate F.

With every set Σ of sentences of L_α we associate a set Σ^* of first-order sentences of L^*_α. If Σ is finite, Σ^* will be finite.

First, with every formula φ of L_α we associate (by recursion) a first-order formula φ^* of L^*_α having the same free variables as φ.

(i) If φ is quantifier-free, then φ^* is φ;

(ii) If φ is of the form $\neg\psi$, then φ^* is $\neg\psi^*$;

(iii) If φ is of the form $\psi_1 \wedge \psi_2$, then φ^* is $\psi_1^* \wedge \psi_1^*$;

(iv) If φ is of the form $\bigvee v_n\psi$, then φ^* is $\bigvee v_n\psi^*$;

(v) If φ is of the form $Qv_n\psi$, then φ^* is

$$\bigvee v_k \bigwedge v_{k+1} \bigvee v_n [\psi^* \wedge F v_k v_n v_{k+1}]$$

where v_k is the first variable after all variables in ψ^*.

Σ^* is the set of all φ^* with φ in Σ, together with the following sentences:

(vi) $\bigwedge v_0 [\bigwedge v_1 \bigvee v_2 \bigwedge v_3 [F v_0 v_1 v_3 \to v_2 = v_3] \wedge \bigwedge v_1 \bigvee v_2 \bigwedge v_3 [F v_0 v_3 v_1 \to v_2 = v_3]]$;

(5) For a further Skolem-type normal form, see [1].

(vii) For every subformula of Σ of the form $Qv_n\psi$, the universal closure of

$$\bigvee v_k[\bigwedge v_{k+1}\bigvee v_n[\psi^*\wedge Fv_kv_nv_{k+1}]\vee\bigwedge v_n\bigvee v_{k+1}[\psi^*\to Uv_{k+1}\wedge Fv_kv_nv_{k+1}]]$$

where v_k is the first variable after all variables in ψ^*.

Note that Σ^* does depend on the particular α. $\Sigma^*\cup\{Qv_0v_0=v_0, \neg Qv_0Uv_0\}$ may be called a Skolem-type normal form of Σ.

THEOREM (2.2). *Assume that α is a successor ordinal, say $\alpha=\beta+1$. Let Σ be a set of sentences of L_α, and let Σ^* be the set of first-order sentences associated with Σ by* (2.1). *Under these assumptions, a relational system $\mathfrak{A}$ of power $\aleph_\alpha$ is a model of Σ if and only if for some unary relation X and some ternary relation Y over A $(\mathfrak{A}, X, Y)$ is a model of $\Sigma^*\cup\{\neg Qv_0Uv_0\}$. X may be chosen to be of power $\aleph_\beta$.*

Proof. Assume first that $\mathfrak{A}$ is a model of Σ of power $\aleph_\alpha$. Choose a subset X of A of power $\aleph_\beta$. Let $\mathcal{A}$ be the set of all subsets of A for which there is a subformula $Qv_n\psi$ of Σ and an assignment $\boldsymbol{a}$ over $\mathfrak{A}$ such that $S=\{x\in A|\boldsymbol{a}(n/x)$ satisfies ψ in $\mathfrak{A}\}$. $\mathcal{A}$ is at most of the power of A. Let J be a set of biunique correspondences such that for every S in $\mathcal{A}$ there either is a $j\in J$ with domain S and range A, or there is a $j\in J$ with domain S and range included in X. We can choose J to be at most of the power of A. Let f be a biunique function from J into A. Finally, set $Y=\{\langle f(j), x, j(x)\rangle|j\in J$ and $x\in\operatorname{dom}j\}$ and $\mathfrak{B}=(\mathfrak{A}, X, Y)$. We shall show that $\mathfrak{B}$ is a model of $\Sigma^*\cup\{\neg Qv_0Uv_0\}$.

Clearly, $\mathfrak{B}$ is a model of $\neg Qv_0Uv_0$. Since f is biunique, and since each $j\in J$ is biunique, the sentence (vi) of (2.1) is true in $\mathfrak{B}$, too.

Next we shall prove (by induction) that for every subformula φ of Σ and every assignment $\boldsymbol{a}$ over $\mathfrak{A}$

(*) *$\boldsymbol{a}$ satisfies φ in $\mathfrak{A}$ if and only if $\boldsymbol{a}$ satisfies φ^* in $\mathfrak{B}$.*

We shall only treat the case that φ is of the form $Qv_n\psi$.

Assume first that $\boldsymbol{a}$ satisfies φ in $\mathfrak{A}$. Thus $S=\{x\in A|\boldsymbol{a}(n_0/x)$ satisfies ψ in $\mathfrak{A}\}$ is of power at least $\aleph_\alpha$. Since S is in $\mathcal{A}$, there is a $j\in J$ with domain S and range A. (There can be no $j\in J$ with domain S and range included in X, the j's being biunique.) Thus for every $x\in A$ there is a $z\in S$ such that $\langle f(j), z, x\rangle\in Y$. The rest follows by inductive assumption.

Assume now that $\boldsymbol{a}$ satisfies φ^* in $\mathfrak{B}$. Since A has power $\aleph_\alpha$ and since fixing the first argument of Y yields a biunique function, there are at least $\aleph_\alpha$ elements $x\in A$ for which $\boldsymbol{a}(n/x)$ satisfies ψ^* in $\mathfrak{B}$. The rest follows again by inductive assumption.

From (*) it follows in particular that φ^* is true in $\mathfrak{B}$ for every φ in Σ. It remains to show that all sentences (vii) of (2.1) are true in $\mathfrak{B}$.

Let $Qv_n\psi$ be a subformula of Σ. Put $S=\{x\in A|\ \boldsymbol{a}(n/x)$ satisfies ψ in $\mathfrak{A}\}$. Since S is in $\mathcal{A}$, there is either a $j\in J$ with domain S and range A, or there is a $j\in J$ with domain S and range included in X. In the first case, using (*) and "taking $f(j)$ for v_k", the first disjunct in (vii) holds; in the second case, again using (*) and "taking $f(j)$ for v_k", the second disjunct in (vii) holds.

For the proof of the theorem from right to left, assume now that $(\mathfrak{A}, X, Y)$ is a model of $\Sigma^*\cup\{\neg Qv_0Uv_0\}$ of power $\aleph_\alpha$. Thus A has power $\aleph_\alpha$ and X has power less than $\aleph_\alpha$. We shall show that for every subformula φ of Σ and every assignment $\boldsymbol{a}$ over $\mathfrak{A}$

(**) *$\boldsymbol{a}$ satisfies φ in $\mathfrak{A}$ if and only if $\boldsymbol{a}$ satisfies φ^* in* $(\mathfrak{A}, X, Y)$.

Again we shall only treat the case that φ is of the form $Qv_n\psi$.

Assume that $\boldsymbol{a}$ satisfies φ in $\mathfrak{A}$. Thus $S=\{x\in A|\boldsymbol{a}(n/x)$ satisfies ψ in $\mathfrak{A}\}$ is of power at least $\aleph_\alpha$. Since (vii) of (2.1) is true in $(\mathfrak{A}, X, Y)$, we only have to rule out that $\boldsymbol{a}$ satisfies $\bigvee v_k\bigwedge v_n[\psi^*\to\bigvee v_{k+1}[Uv_{k+1}\wedge Fv_kv_nv_{k+1}]]$ in $(\mathfrak{A}, X, Y)$. But since S has power at least $\aleph_\alpha$ and X has power less than $\aleph_\alpha$, and since in view of (vi) fixing the first argument in Y yields a biunique function, the above is ruled out indeed.

The reverse implication follows similarly as in the proof of (*).

From (**) it follows in particular that $\mathfrak{A}$ is a model of Σ.

Remark. Definition (2.1) and Theorem (2.2) can be modified in various ways.

First, instead of using the universe of $\mathfrak{A}$ as the standard set of power $\aleph_\alpha$, we can choose an arbitrary subset of A of power $\aleph_\alpha$ for this purpose. This requires the addition of a new unary predicate V to L_α^* and a change in the definition of φ^*. In particular condition (v) of (2.1) has to be changed by replacing

$$\bigvee v_k\bigwedge v_{k+1}\bigvee v_n[\psi^*\wedge Fv_kv_nv_{k+1}]\quad\text{by}\quad\bigvee v_k\bigwedge v_{k+1}\bigvee v_n[Vv_{k+1}\to\psi^*\wedge Fv_kv_nv_{k+1}]$$

and a similar change in (vii). With this changed notion of normal form (2.2) can be stated for relational systems of power at least $\aleph_\alpha$ instead for those of power $\aleph_\alpha$ only [6].

Secondly, one can drop the requirement that the sets U and V serving as standards for the cardinality of sets are chosen from the subsets of A, thus admitting to enlarge the universe. This requires a further unary predicate W denoting the old universe. It also requires a further change in the definition of φ^*. In particular condition (iv) of (2.1) has to be changed by replacing $\bigvee v_n\psi^*$ by $\bigvee v_n[Wv_n\wedge\psi^*]$, and similar changes in (v) and (vii). With this changed notion of normal form (2.2) can be stated for arbitrary relational systems; $\mathfrak{A}$ is, however, no longer obtained as a reduct but as a relativized reduct. This change is also necessary if one wants to extend the results to languages with uncountably many

[6] This form was mentioned in [4].

non-logical constants and $\mathfrak{A}$ in the proof of (2.2) is no longer at most of the power of A.

The second Skolem-type normal form is designed for the study of L_α where α is any ordinal. The idea behind this normal form is as follows: Let $\mathfrak{A}$ be a model of power $\aleph_\alpha$. Choose a linear ordering of A such that every proper initial segment has power less than $\aleph_\alpha$. Instead of saying "the set S has power at least $\aleph_\alpha$" we shall say "there is a biunique correspondence between a subset of S and A"; instead of saying "the set S has power less than $\aleph_\alpha$" we say "there is a biunique correspondence between S and a proper initial segment of $\mathfrak{A}$". The collection of correspondences is again replaced by a ternary relation.

Definition (2.3). Let L_α^* be the language obtained from L_α by adding to the non-logical constants a binary predicate R and a ternary predicate F.

With every set Σ of sentences of L_α we associate a set Σ^* of first-order sentences of L_α^*. If Σ is finite, Σ^* will be finite.

First, with every formula φ of L_α we associate a first-order formula φ^* of L_α^* having the same free variables as φ.

(i)-(v) as in (2.1);

Σ^* is the set of all φ^* with φ in Σ together with the following sentences:

(vi) as in (2.1);

(vii) for every subformula of Σ of the form $Qv_n\psi$, the universal closure of

$$\bigvee v_k\big[\bigwedge v_{k+1}\bigvee v_n[\psi^*\wedge Fv_kv_nv_{k+1}]\vee \bigvee v_{k+2}\bigwedge v_n\bigvee v_{k+1}[\psi^*\to Rv_{k+1}v_{k+2}\wedge Fv_kv_nv_{k+1}]\big]$$

where v_k is the first variable after all variables in ψ^*;

(viii) a sentence saying that R is a linear ordering of the universe.

Note that Σ^* does not depend on the particular α. $\Sigma^*\cup\{Qv_0v_0=v_0, \bigwedge v_0\neg Qv_1Rv_1v_0\}$ may be called a *Skolem-type normal form of* Σ.

Theorem (2.4). *Let Σ be a set of sentences of L_α and Σ^* the set of first-order sentences associated with Σ by* (2.3). *A relational system $\mathfrak{A}$ of power $\aleph_\alpha$ is a model of Σ if and only if for some binary relation X and some ternary relation Y over A, $(\mathfrak{A}, X, Y)$ is a model of $\Sigma^*\cup\{\bigwedge v_0\neg Qv_1Rv_1v_0\}$. X may be even chosen to be a well-ordering.*

The proof of (2.4) is similar to the proof of (2.2).

Remark. First, the remarks following the proof of (2.2) can be repeated with suitable modifications. Secondly, if $\aleph_\alpha$ is regular, the definition of Σ^* can be simplified by using the observation that a subset of the corresponding ordered set is of power less than $\aleph_\alpha$ if and only if the subset is bounded.

3. Applications. Theorem (3.1). *Assume that $\aleph_\alpha$ is regular, and let Σ be a set of sentences of L_α. If Σ has a model, then Σ has a model when considered as a set of sentences of L_1.*

Proof. If Σ has a model of power less than $\aleph_\alpha$, then the assertion follows from (1.5) and the first-order Löwenheim-Skolem theorem. If Σ has a model of power at least $\aleph_\alpha$, then by (1.4) Σ has a model of power $\aleph_\alpha$. We may therefore assume that we have a model $\mathfrak{A}$ of Σ of power $\aleph_\alpha$.

Let Σ^* be the set of first-order sentences associated with Σ by (2.3). By (2.4) there are X and Y such that $(\mathfrak{A}, X, Y)$ is a model of Σ^* and X is a well-ordering of A of type ω_α. From (1.8) taking Σ^* for T, we obtain a model $(\mathfrak{B}, X', Y')$ of Σ^* of power $\aleph_1$ and where X' is a linear ordering of B every proper initial segment of which is countable. Applying (2.4) again we see that $\mathfrak{B}$ is a model of Σ considered as a set of sentences of L_1.

Remark. (3.1) can be regarded as a strengthened version of (1.6). Conversely, for the case α a successor ordinals (3.1) could have been obtained using (1.6) instead of (1.8).

Theorem (3.2). *Let Σ be a set of sentences of L_0. If Σ has a model, then Σ has a model when considered as a set of sentences of L_α, where α is any ordinal.*

Proof. If Σ has a model of power less than $\aleph_0$ we argue as in the beginning of the proof of (3.1). Otherwise, as there we shall assume that we have a model $\mathfrak{A}$ of Σ of power $\aleph_0$.

Let Σ^* be the set of first-order sentences associated with Σ by (2.3). By (2.4) there are X and Y such that $(\mathfrak{A}, X, Y)$ is a model of Σ^* and X is a linear ordering of A of type ω. Let T be the first-order theory of $(\mathfrak{A}, X, Y)$. By (1.9) T has a model $(\mathfrak{B}, X', Y')$ of power $\aleph_\alpha$ for which X' is a linear ordering of B every proper initial segment of which is of smaller power. $(\mathfrak{B}, X', Y')$ is in particular a model of Σ^*. Now considering Σ^* as obtained from a set of sentences of L_α and applying again (2.4) we obtain the required model $\mathfrak{B}$ of Σ.

Theorem (3.3) [7]. *Assume that α is a (weakly) inaccessible ordinal (greater than ω). Let Σ be a set of sentences of L_α. If Σ has a model, then there is an ordinal β, $\alpha>\beta$ and β cofinal with ω, and Σ has a model when considered as a set of sentence of L_β.*

The proof uses (2.3) and (2.4) as well as (1.10) and is similar to the preceding proofs.

Theorem (3.4). *Let Σ be a set of sentences of L_1. If every finite subset of Σ has a model, then Σ has a model.*

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[7] This theorem as well as its corollary 3.5 is due to Dana Scott.

Proof. If some finite subset of Σ has only countable models, then every finite subset of Σ has a countable model and in this case the assertion follows with the help of (1.5) from the first-order compactness theorem together with the Löwenheim-Skolem theorem. We shall therefore assume that every finite subset of Σ has an uncountable model and hence by (1.4) a model of power $\aleph_1$.

Let Σ^* be the set of first-order sentences associated with Σ by (2.1). We are going to apply (1.7) taking Σ^* for T. First, we observe that every finite subset T is contained in the * of some finite subset of Σ. Furthermore, every finite subset of Δ is contained in the corresponding set for some finite subset of T. Since by assumption every finite subset of Σ has a model $(\mathfrak{A}, X, Y)$ of power $\aleph_1$, by (2.2), every finite subset of T has a model $(\mathfrak{A}, X, Y)$ of power $\aleph_1$ with countable X, and hence by (1.7) every finite subset of $T \cup \Delta \cup \{\bigwedge v_0[Uv_0 \to Wv_0], \bigvee v_0 \neg Wv_0\}$ has a model. By the first-order compactness theorem the above set itself has a model, and hence by (1.7) Σ^* has a model $(\mathfrak{A}, X', Y')$ of power $\aleph_1$ with countable X. By (2.2) $\mathfrak{A}$ is a model of Σ.

Remark. (3.1) is a compactness theorem. Using ultraproducts one can show that *L_a is compact for every a for which $\mathfrak{m}_n < \aleph_a$, for each $n \in \omega$, implies $\prod \langle \mathfrak{m}_n | n \in \omega \rangle < \aleph_a$.* [8]

As corollaries of (3.1) and (3.2) we obtain immediately the assertions (1) and (2) of the introduction. (3.3) yields as a corollary

(3.5) *Assume that a is a (weakly) inaccessible ordinal (greater than ω). There is an ordinal β, $\beta < a$ and β cofinal with ω, such that $V_\beta \subseteq V_a$.*

To prove (3.5) from (3.3) one has to apply (3.3) not to L_a—the β would depend on the sentence to be falsified—but to a language obtained from L_a by dublication of the non-logical constants so that the sentences which are not logically valid "can be written with disjoint predicates" and thus can be falsified in one model.

From (3.4) one obtains as a further corollary that *the ordering of the natural numbers is not characterizable (up to isomorphism) in L_1 as a relativized reduct.*

4. Some counterexamples. We shall indicate, briefly, some examples of sentences of L_a which show the non-inclusions (3) to (6) from the introduction. None of the examples originates with the author. They are not the simplest ones; they are rather chosen because their critical properties are obvious.

Let R and G be binary predicates and F be a ternary predicate.

(4.1). Let φ be a first-order sentence saying that R is a linear ordering without last element. The sentence $\varphi \wedge \neg Qv_0Rv_6v_0$ is satisfiable as a sentence of L_a if and only if $a \neq 0$.

(8) Announced in [3].

(4.2). Let φ be a first-order sentence saying that R is a linear ordering of the universe. Let furthermore ψ be the conjunction of the following sentences:

(i) $\bigwedge v_0[\bigwedge v_1 \bigvee v_2 \bigwedge v_3[Fv_0v_1v_3 \to v_2 = v_3] \wedge \bigwedge v_1 \bigvee v_2 \bigwedge v_3[Fv_0v_3v_1 \to v_2 = v_3]]$;

(ii) $\bigwedge v_0 \bigwedge v_1 \bigvee v_2[Rv_1v_0 \to Fv_0v_1v_2]$;

(iii) $\bigvee v_0 \bigwedge v_1 \bigwedge v_2 \bigwedge v_3[Fv_1v_2v_3 \to Rv_3v_0]$.

The sentence $\varphi \wedge \psi \wedge Qv_0Rv_0v_0 \wedge \bigwedge v_0 \neg Qv_1Rv_1v_0$ is satisfiable as a sentence of L_a if and only if a is a succesor ordinal. (The sentence ψ says that fixing the first argument in F yields a biunique function (i) whose domain contains the predecessors of this element (ii) and whose range is uniformly bounded (iii).)

(4.3). Let φ be as in (4.2). Let furthermore ψ be the conjunction of the following sentences:

(i) $\bigwedge v_0 \bigvee v_1 \bigwedge v_2[Gv_0v_2 \to v_2 = v_1]$;

(ii) $\bigvee v_0 \bigwedge v_1 \bigwedge v_2[Gv_1v_2 \to Rv_1v_0]$;

(iii) $\bigwedge v_0 \bigvee v_1 \bigvee v_2[Rv_0v_1 \wedge Gv_2v_1]$.

The sentence $\varphi \wedge \psi \wedge Qv_0Rv_0v_0 \wedge \bigwedge v_0 \neg Qv_1Rv_1v_0$ is satisfiable as a sentence of L_a if and only if $\aleph_a$ is singular. (The sentence says that G is a function (i) whose domain is bounded (ii) while its range is cofinal in the universe (iii).)

(4.4). The sentence $\bigwedge v_0 \bigwedge v_1[\bigwedge v_2[Rv_2v_0 \leftrightarrow Rv_2v_1] \to v_0 = v_1] \wedge Qv_0 \bigvee v_1Rv_1v_0 \wedge \neg Qv_0 \bigvee v_1Rv_0v_1$ is satisfiable as a sentence of L_a if and only if, for some cardinal $\mathfrak{m} < \aleph_a$, $\aleph_a < 2^{\mathfrak{m}}$.

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The completeness of logic with the added quantifier "there are uncountably many"

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The language L_1 is obtained from an ordinary first order language L, having countably many symbols, by adding a new quantifier Q, to be read "there are uncountably many ... such that ...". Mostowski [10] (1) raised the completeness problem for L_1, i.e., the question whether the set V_1 of (semantically) logically valid sentences of L_1 is recursively enumerable.

The compactness of the language L_1 has been established in Theorem 3.4 of the preceding article by G. Fuhrken. The purpose of this note is to point out that the two results and the argument employed by Fuhrken to prove compactness also lead at once to a positive answer to Mostowski's question.

One of these two results is (2.2), Fuhrken's first normal form theorem for L_1. When modified as in the second (but not the first) part of the Remark following its proof, this theorem gives us the following information: Let L' be the first order language obtained from L by adding two new unary predicates U and W_1 and one new ternary predicate. We can define a recursive function correlating with each sentence σ of L_1 a sentence σ' of L' in such a way that:

(A) *σ has a model if and only if σ' has a model $\mathfrak{A}$ of power $\aleph_1$ in which $U^{(\mathfrak{A})}$ has power at most $\aleph_0$.*

The second result, (1.7), is easily derived from the proof of the author's 'Löwenheim-Skolem theorem for two cardinals' (Theorem 6.2 of [9]). (However, this fact for the case when T is incomplete and its significance were only observed recently, by Fuhrken.) Let L'' be the first order language obtained from L' by adding one new unary predicate W_2. (1.7) describes a certain recursive set Σ of sentences, such that, for any sentence δ of L':

(B) *δ has a model $\mathfrak{A}$ of power $\aleph_1$ in which $U^{(\mathfrak{A})}$ has power at most $\aleph_0$, if and only if $\Sigma \cup \{\delta\}$ is consistent.*

(1) The terminology of the preceding paper by G. Fuhrken will be used; numbers refer to its theorems, numbers in brackets to its bibliography.