On the non-existence of free complete Boolean algebras

by

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I. Introduction

In the study of algebraic systems it is often useful to investigate a "free" system of a given type, i.e. a system which has any other system of that type as a homomorphic image. Intuitively, one may look upon it as the "least constrained" system of the given type. Thus free groups, free Abelian groups, free lattices, and free Boolean algebras have been investigated extensively. In each of the above cases there is no difficulty in proving the existence of such a free system. The polynomial method of Birkhoff [1], p. viii, yields the desired result. If we consider algebraic systems which have operations applicable to arbitrarily many arguments, however, such as complete lattices and complete Boolean algebras, even a generalization of this polynomial method may not suffice to prove the existence of a free system. The difficulty in such cases arises from the fact that the "algebra" formed may have too many elements, i.e. may not be a set. Crawley and Dean [2] have in fact developed a technique which may be used to prove that there does not exist a free complete lattice on three complete generators. Such a proof, given in Chapter 3, depends upon an effective characterization of when two formally distinct polynomials are actually equal.

In 1951 Rieger [9] asked whether or not there exists a free complete Boolean algebra on ω complete generators. The technique of Crawley and Dean does not apply. Rieger's question is answered in the negative in Chapter 4 of this paper. The method involves constructing a class of examples which establish the inequality of certain formally distinct polynomials. This result was obtained independently by H. Gaifman [3, 4] and the author in the summer of 1961.

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In Chapter 5 the following more general result is proved in a similar fashion. Let γ be an infinite regular cardinal. Then there does not exist a free complete (γ, ∞) distributive Boolean algebra on γ complete generators.

II. Preliminaries

In this paper lattice (also Boolean algebra) unions and intersections are denoted by $\bigcup$ and $\bigcap$, respectively. Set unions and intersections are denoted by $\bigvee$ and $\bigwedge$, respectively. In either case inclusion and proper inclusion are denoted by $\leqslant$ and $<$, respectively. If A is a subset of a given set or an element of a Boolean algebra, its complement is denoted by A^c. Ordinal numbers are usually denoted by small Greek letters and small Latin letters. Cardinal numbers are identified with the corresponding initial ordinals.

Let α be an infinite cardinal. A lattice (or Boolean algebra) L is said to be *α-complete* if every subset of L with cardinality less than α has a least upper bound (union) and a greatest lower bound (intersection) in L [1]. A sublattice (subalgebra) L' of a lattice (Boolean algebra) L is said to be an *α-complete sublattice* (*subalgebra*) if unions and intersections in L of subsets of L' with cardinality less than α lie in L'. A sublattice (subalgebra) L' of a lattice (Boolean algebra) L is said to be an *α-regular sublattice* (*subalgebra*) if unions and intersections in L' of subsets of L' of cardinality less than α are also unions and intersections in L. A field of subsets of a set S (a collection of subsets of S closed under finite union, finite intersection, and complementation, and hence a Boolean algebra) is said to be an *α-complete field of sets* if it is an α-complete subalgebra of the Boolean algebra of all subsets of S.

If A is a subset of a lattice (Boolean algebra) L, then the sublattice (subalgebra) of L *α-generated* by A is the smallest α-complete sublattice (subalgebra) of L containing A. If S is a set and A is a collection of subsets of S, the field of sets *α-generated* by A is the subalgebra of the Boolean algebra of all subsets of S α-generated by A. The word "generated" is used in place of the word "ω-generated".

A homomorphism of a lattice (Boolean algebra) L is said to be an *α-complete homomorphism* if it preserves unions and intersections of subsets of L with cardinality less than α.

A lattice (Boolean algebra) which is α-complete for every α is called *complete*. A sublattice (subalgebra) of a lattice (Boolean algebra) which is an α-complete sublattice (subalgebra) for every α is called a *complete sublattice* (*subalgebra*). A sublattice (subalgebra) of a lattice (Boolean algebra) which is an α-regular sublattice (subalgebra) for every α is called a *regular sublattice (subalgebra)*. If A is a subset of a lattice (Boolean algebra) L, the smallest complete sublattice (subalgebra) of L containing A is called the *sublattice (subalgebra) completely generated by A*. A homomorphism of a lattice (Boolean algebra) which is an α-complete homomorphism for every α is called a *complete homomorphism*.

Any lattice L may be embedded as a regular sublattice of a complete lattice L^*, the "normal completion" of L. This was proved by MacNeille [6]. Stone [10] and Glivenko [5] have shown that if L is a Boolean algebra, then L^* is a Boolean algebra.

Let γ be a non-zero cardinal. A lattice (Boolean algebra) L is said to be a *free lattice* (*Boolean algebra*) on γ generators if L contains a subset A of cardinality equal to γ which generates L, and if any mapping f of A onto a subset A' of a lattice (Boolean algebra) L' which generates L' can be extended to a homomorphism f^* of L onto L'.

A free lattice on γ generators can also be defined in a manner which automatically establishes its existence. We denote the generators by $a_0, a_1, \ldots, a_i, \ldots$, where $0 \leqslant i < \gamma$. Then polynomials of rank r for each finite r are defined inductively as follows:

Definition 1. For each i with $0 \leqslant i < \gamma$, a_i is a polynomial of rank $r(a_i) = 0$. If $\mathcal{A}$ is a finite set of polynomials previously defined, and if $\mathcal{A}$ has cardinality n, then the symbols $\bigcup \mathcal{A}$ and $\bigcap \mathcal{A}$ are polynomials of rank $r(\bigcup \mathcal{A}) = r(\bigcap \mathcal{A}) = \max\big(n, \max_{A \in \mathcal{A}}[r(A)+1]\big)$.

The collection of all such polynomials is denoted by $L_\omega(\gamma)$. We now define a valuation of $L_\omega(\gamma)$ as a mapping f of $\{a_i\colon 0 \leqslant i < \gamma\}$ onto a subset of a lattice L which generates L. Such an f can be extended in a natural way ($f^*(\bigcup \mathcal{A}) = \bigcup_{A \in \mathcal{A}} f^*(A)$, etc.) to a mapping f^* from $L_\omega(\gamma)$ to L. If A_1 and A_2 lie in $L_\omega(\gamma)$, and if, for every valuation f of $L_\omega(\gamma)$, we have $f^*(A_1) = f^*(A_2)$, then A_1 and A_2 are said to be *equal*. Then $L_\omega(\gamma)$ is a lattice in the natural manner (under $\bigcup$ and $\bigcap$), and moreover it is a free lattice on the γ generators a_i.

We can extend Definition 1 as follows: if A is a previously defined polynomial, then the symbol A^c is a polynomial of rank $r(A^c) = r(A)+1$. We denote the extended collection of polynomials by $B_\omega(\gamma)$. A valuation of $B_\omega(\gamma)$ is defined as a mapping f of $\{a_i\colon 0 \leqslant i < \gamma\}$ onto a subset of a Boolean algebra B which generates B. Equality in $B_\omega(\gamma)$ is then defined analogously to that in $L_\omega(\gamma)$. $B_\omega(\gamma)$ thus becomes, in the natural way, a free Boolean algebra on the γ generators a_i.

A free Boolean algebra on γ generators can also be described as follows. Let $S = 2^\gamma$, i.e. the set of all functions from γ to $\{0, 1\}$. For each ϱ less than γ, let e_ϱ be the evaluation map corresponding to ϱ; i.e.,

[1] This and subsequent terminology differs from that of some other authors, who would replace "less than α" by "less than or equal to α" in this definition.

if f is in S, then $e_\varrho(f) = f(\varrho)$. Then the subsets of S of the form $e_\varrho^{-1}(1)$, for $\varrho < \gamma$, generate a field of subsets of S. This field of subsets is a free Boolean algebra on the γ generators $e_\varrho^{-1}(1)$.

In this thesis we shall be chiefly concerned with free complete lattices and free complete Boolean algebras, defined as follows. Let γ be a non-zero cardinal. A complete lattice (Boolean algebra) L is said to be a *free complete lattice* (*Boolean algebra*) *on* γ *complete generators* if L contains a subset A of cardinality equal to γ which completely generates L, and if any mapping f of A onto a subset A' of a complete lattice (Boolean algebra) L' which completely generates L' can be extended to a complete homomorphism f^* of L onto L'.

We can attempt as above to define a free complete lattice on γ complete generators in terms of (infinite) polynomials, thus establishing the existence of such a system. We denote the generators by $a_0, a_1, \ldots, a_i, \ldots$ where $0 \leqslant i < \gamma$. Then polynomials of rank r for each ordinal r are defined inductively as follows:

DEFINITION 2. For each i with $0 \leqslant i < \gamma$, a_i is a polynomial of rank $r(a_i) = 0$. If $\mathcal{A}$ is a set of polynomials previously defined, and if $\mathcal{A}$ has cardinality β, then the symbols $\bigcup \mathcal{A}$ and $\bigcap \mathcal{A}$ are polynomials of rank $r(\bigcup \mathcal{A}) = r(\bigcup \mathcal{A}) = \max\big(\beta, \sup_{A \in \mathcal{A}}[r(A)+1]\big)$.

The collection of all such polynomials is denoted by $L(\gamma)$. A valuation of $L(\gamma)$ is defined as a mapping f of $\{a_i\colon 0 \leqslant i < \gamma\}$ onto a subset of a complete lattice L which completely generates L. Such an f can be extended in a natural way ($f^*(\bigcup \mathcal{A}) = \bigcup_{A \in \mathcal{A}} f^*(A)$, etc.) to a mapping f^* from $L(\gamma)$ to L. If A_1 and A_2 lie in $L(\gamma)$, and if, for every valuation f of $L(\gamma)$, we have $f^*(A_1) = f^*(A_2)$, then A_1 and A_2 are said to be *equal*. At this point we are in trouble, however, for we cannot say that $L(\gamma)$ is a complete lattice in the natural manner. It may (after the identification of equal elements) not be a set, but a proper class instead; i.e., it may have too many elements.

If $L(\gamma)$ (after identifications) is a set, then in the natural manner (under $\bigcup$, $\bigcap$) it is a free complete lattice on γ complete generators. If there does exist a free complete lattice on γ complete generators, then its cardinality is an upper bound for the cardinality of any complete lattice with γ complete generators. Finally, suppose any complete lattice on γ complete generators has cardinality less than or equal to β. Let β' be the smallest cardinal greater than β, and let $L_{\beta'}(\gamma)$ be the collection of polynomials in $L(\gamma)$ with rank less than β'. It is easily shown, by induction on β', that $L_{\beta'}(\gamma)$ is a set. Then $L_{\beta'}(\gamma)$ (after identifications), in the natural manner, is a β'-complete lattice with γ complete generators. Its normal completion $L^*_{\beta'}(\gamma)$ is complete and has γ complete generators—hence the cardinality of $L^*_{\beta'}(\gamma)$ is less than or equal to β. But then $L_{\beta'}(\gamma)$ (after identifications) has cardinality less than or equal to β. This implies that any $\bigcup$ or $\bigcap$ of elements of $L_{\beta'}(\gamma)$ is equal to an element of $L_{\beta'}(\gamma)$. Thus, by induction on its rank, every polynomial in $L(\gamma)$ is equal to one in $L_{\beta'}(\gamma)$ so $L(\gamma)$ (after identifications) is a set.

We can extend Definition 2 in the same way we previously extended Definition 1: if A is a previously defined polynomial, then the symbol A^c is a polynomial of rank $r(A^c) = r(A)+1$. We denote the extended collection of polynomials by $B(\gamma)$. A valuation of $B(\gamma)$ is defined as a mapping f of $\{a_i\colon 0 \leqslant i < \gamma\}$ onto a subset of a complete Boolean algebra B which completely generates B. Equality in $B(\gamma)$ is then defined analogously to that in $L(\gamma)$. The conclusions of the preceding paragraph carry over, i.e. the following are equivalent: $B(\gamma)$ (after identifications) is a set; $B(\gamma)$ is a free complete Boolean algebra on γ complete generators; there exists a free complete Boolean algebra on γ complete generators; and the cardinality of complete Boolean algebras with γ complete generators is bounded.

To prove the non-existence of a free complete lattice (Boolean algebra) on γ complete generators we need only show that the cardinality of complete lattices (Boolean algebras) on γ complete generators is unbounded. Alternatively, we may exhibit an ordinal-indexed collection (hence not a set) of pairwise unequal polynomials in $L(\gamma)$ $\big(B(\gamma)\big)$, proving inequality by exhibiting an appropriate valuation for each pair. The actual method used is a combination of the two, in the more difficult case, for Boolean algebras (Chapters 4 and 5), the choice of the polynomials suggests the method of constructing complete Boolean algebras with γ complete generators of arbitrarily large cardinality. These complete Boolean algebras, in turn, establish the pairwise inequality of the chosen polynomials.

III. Free complete lattices

In this chapter we investigate the existence of a free complete lattice on γ complete generators. A free lattice on one generator is finite, consisting of one element (the polynomial a_0 in $L_\omega(1)$). A free lattice on two generators is also finite, consisting of four elements (the polynomials $\bigcap\{a_0, a_1\}$, a_0, a_1, and $\bigcup\{a_0, a_1\}$ in $L_\omega(2)$). Thus they coincide, respectively, with free complete lattices on one and two complete generators.

Crawley and Dean (2) have developed a technique which may be used to prove that there does not exist a free complete lattice on three complete generators. Such a proof is given here. The first step is to define inductively a partial order on $L(3)$.

DEFINITION 1. If A, B lie in $L(3)$, then $A \geqslant B$ if and only if one of the following conditions holds:

(1) $A \equiv B \equiv a_i$ for some $0 \leqslant i < 3$,

(2) $A \equiv \bigcup \mathcal{A}$ and $A' \geqslant B$ for some A' in $\mathcal{A}$,

(3) $A \equiv \bigcap \mathcal{A}$ and $A' \geqslant B$ for all A' in $\mathcal{A}$,

(4) $B \equiv \bigcup \mathfrak{B}$ and $A \geqslant B'$ for all B' in $\mathfrak{B}$,

(5) $B \equiv \bigcap \mathfrak{B}$ and $A \geqslant B'$ for some B' in $\mathfrak{B}$.

We establish some elementary properties of $\geqslant$.

LEMMA 1. *Let $A \equiv \bigcup \mathcal{A}$, $B \equiv \bigcap \mathfrak{B}$, and C be polynomials in $L(3)$. Then*

(1) *If $0 \leqslant i < 3$, $a_i \geqslant B$ implies $a_i \geqslant B'$ for some B' in $\mathfrak{B}$, and $A \geqslant a_i$ implies $A' \geqslant a_i$ for some A' in $\mathcal{A}$.*

(2) *$A \geqslant B$ implies either $A \geqslant B'$ for some B' in $\mathfrak{B}$ or $A' \geqslant B$ for some A' in $\mathcal{A}$.*

(3) *$C \geqslant A$ implies $C \geqslant A'$ for all A' in $\mathcal{A}$, and $B \geqslant C$ implies $B' \geqslant C$ for all B' in $\mathfrak{B}$.*

Proof. The proofs of (1) and (2) are clear from Definition 1 above. We prove the first half of (3) by induction on $r(C)$; the second half will follow by duality. If $r(C) = 0$, then $C \equiv a_i$ for some i with $0 \leqslant i < 3$, and the result follows from part 4 of Definition 1. Assume the result for all C' with $r(C')$ less than j, and suppose we have $C \geqslant A$ with $r(C) = j$. If $C \equiv \bigcup \mathfrak{C}$, then either part 2 or part 4 of Definition 1 applies. If part 4 applies, we are done. If part 2 applies, there is a C' in $\mathfrak{C}$ such that $C' \geqslant A$. But then $r(C')$ is less than j, so by induction hypothesis $C' \geqslant A'$ for all A' in $\mathcal{A}$. Thus $C \geqslant A'$ for all A' in $\mathcal{A}$ by part 2 of Definition 1. A similar proof applies if $C \equiv \bigcap \mathfrak{C}$.

We now prove that $\geqslant$ is a partial order.

LEMMA 2. *For A, B, C in $L(3)$, we have*

(1) *$A \geqslant A$*

and

(2) *$A \geqslant B$ and $B \geqslant C$ imply $A \geqslant C$.*

Proof. (1) If $A \equiv a_i$ with $0 \leqslant i < 3$, part 1 of Definition 1 gives the result. Suppose (1) is true for all A' with $r(A')$ less than j, and suppose $r(A) = j$. If $A \equiv \bigcup \mathcal{A}$, then $A' \geqslant A'$ for all A' in $\mathcal{A}$ by induction hypothesis. Parts 2 and 4 of Definition 1 then give $A \geqslant A$. A similar proof applies if $A \equiv \bigcap \mathcal{A}$.

(2) We prove this by induction on the ordered triples $\langle r(A), r(B), r(C)\rangle$, ordered lexicographically. When $\langle r(A), r(B), r(C)\rangle = \langle 0, 0, 0\rangle$, part 1 of Definition 1 applies. We assume that (2) is true for all triples less than $\langle j, j', j''\rangle$, and that $\langle r(A), r(B), r(C)\rangle = \langle j, j', j''\rangle$ with $A \geqslant B$ and $B \geqslant C$. There are ten cases to consider: (i) $j = j' = 0$, $C \equiv \bigcup \mathfrak{C}$; (ii) $j = j' = 0$, $C \equiv \bigcup \mathfrak{C}$; (iii) $j' = 0$, $A \equiv \bigcup \mathcal{A}$; (iv) $j' = 0$, $A \equiv \bigcap \mathcal{A}$; (v) $B \equiv \bigcup \mathfrak{B}$, $j'' = 0$; (vi) $B \equiv \bigcup \mathfrak{B}$, $C \equiv \bigcup \mathfrak{C}$; (vii) $B \equiv \bigcup \mathfrak{B}$, $C \equiv \bigcap \mathfrak{C}$; and the duals of (v), (vi), and (vii). We give proofs of (i) and (vii); the proofs of the other cases are very similar.

(i) $j = j' = 0$, $C \equiv \bigcup \mathfrak{C}$. Since $B \geqslant C$, it follows that $B \geqslant C'$ for all C' in C. But $\langle r(A), r(B), r(C')\rangle$ is less than $\langle j, j', j''\rangle$, so that $A \geqslant C'$ for all C' in $\mathfrak{C}$ by the induction hypothesis. Hence $A \geqslant C$.

(vii) $B \equiv \bigcup \mathfrak{B}$, $C \equiv \bigcap \mathfrak{C}$. Since $B \geqslant C$, we have either $B \geqslant C'$ for some C' in $\mathfrak{C}$ or $B' \geqslant C$ for some B' in $\mathfrak{B}$. In the first case, by the induction hypothesis, $A \geqslant C'$, and hence $A \geqslant C$. In the second case, Lemma 1 gives $A \geqslant B'$. Then, again by the induction hypothesis, $A \geqslant C$.

We have thus shown that $\geqslant$ is reflexive and transitive, and is therefore a partial order (where we identify A and B if and only if $A \geqslant B$ and $B \geqslant A$). If $\mathcal{A}$ is a subset of $L(3)$, then $\bigcup \mathcal{A}$ and $\bigcap \mathcal{A}$ are, respectively, the least upper bound and greatest lower bound of $\mathcal{A}$ under this partial order (this follows from Definition 1 and Lemma 2, Part 1). Thus, if β is an infinite regular cardinal, and $L_\beta(3)$ the subset of $L(3)$ consisting of polynomials of length less than β, $L_\beta(3)$ is a β-complete lattice with three complete generators (under the partial ordering). Its normal completion $L_\beta^*(3)$ is a complete lattice with three complete generators. Thus, if A, B in $L(3)$ are not identified by $\geqslant$ (i.e. not both $A \geqslant B$ and $B \geqslant A$), then, taking β greater than $\max\big(r(A), r(B)\big)$, A and B are unequal in the natural valuation of $L(3)$ in $L_\beta^*(3)$, and hence A and B are unequal in the sense of Chapter 2. Finally, by induction on the ordered pairs $\langle r(A), r(B)\rangle$, ordered lexicographically, it is easy to show that $A \geqslant B$ implies $f^*(A) \geqslant f^*(B)$ for all valuations f of $L(3)$. Thus elements in $L(3)$ are identified by $\geqslant$ if and only if they are equal in the sense of Chapter 2, i.e. coincide in every valuation. Thus Definition 1 gives us an *effective* method of deciding when two polynomials in $L(3)$ are equal.

For convenience we write, for A, B in $L(3)$, $A \bigcup B \equiv B \bigcup A \equiv \bigcup\{A, B\}$ and $A \bigcap B \equiv B \bigcap A \equiv \bigcap\{A, B\}$.

Now define polynomials x_i in $L(3)$ for all ordinals i as follows:

$$x_0 \equiv a_0\,,$$
$$x_{i+1} \equiv a_0 \bigcup \Big(a_1 \bigcap \Big(a_2 \bigcup \big(a_0 \bigcap \big(a_1 \bigcup (a_2 \bigcap x_i)\big)\big)\Big)\Big),$$

and, if i is a limit ordinal,

$$x_i \equiv \bigcup \{x_j : j < i\}\,.$$

We next prove that the x_i are a pairwise "unequal" collection of polynomials.

LEMMA 3. *If $i < j$, $x_j \geqslant x_i$ but x_i non $\geqslant x_j$.*

Proof. We first prove two preliminary statements:

(a) If $0 < i$, then x_i non $\geqslant a_1, a_2$ and $a_0, a_1, a_2 \geqslant x_i$. We easily verify that, for $0 < i$, $a_0 \bigcup \big(a_1 \bigcap (a_2 \bigcup a_0)\big) \geqslant x_i \geqslant a_0 \bigcup (a_1 \bigcap a_2)$. From this (a) follows.

(b) If A, B are in $L(3)$, then $a_0 \cup a_2 \geqslant A$, $B \geqslant a_2$ and $a_0 \cup (a_1 \cap A) \geqslant a_0 \cup (a_1 \cap B)$ imply $a_1 \cap A \geqslant a_1 \cap B$. Suppose $a_0 \cup (a_1 \cap A) \geqslant a_0 \cup (a_1 \cap B) \geqslant a_1 \cap B$. Then if $a_0 \cup (a_1 \cap A) \geqslant a_1$, we have $a_0 \cup a_2 \geqslant a_0 \cup A \geqslant a_1$, a contradiction. If $a_0 \cup (a_1 \cap A) \geqslant B$, we have $a_0 \cup a_1 \geqslant a_2$, a contradiction. If $a_0 \geqslant a_1 \cup B$, we have $a_0 \geqslant a_1 \cap a_2$, a contradiction. Thus the only remaining alternative is $a_1 \cap A \geqslant a_1 \cap B$. Note that (b) is valid if a_0, a_1, and a_2 are permuted.

Now from (a) it is clear that $x_i \geqslant x_0$, $x_0 \operatorname{non} \geqslant x_i$, for $0 < i$. Suppose that $j < k$ implies $x_j \leqslant x_k$ whenever $k < i$. If i is a limit ordinal then $x_i = \bigcup \{x_j \colon j < i\} \geqslant x_j$ for all $j < i$. If i and $i-1$ are not limit ordinals, $x_{i-1} \geqslant x_{i-2}$ implies $a_0 \cup \Big(a_1 \cap \Big(a_2 \cup \big(a_0 \cap (a_1 \cup (a_2 \cap x_{i-1}))\big)\Big)\Big) \geqslant a_0 \cup \Big(a_1 \cap \Big(a_2 \cup \big(a_0 \cap (a_1 \cup (a_2 \cap x_{i-2}))\big)\Big)\Big)$, i.e. $x_i \geqslant x_{i-1}$. If $i-1$ is a limit ordinal, then $x_i = a_0 \cup \Big(a_1 \cap \Big(a_2 \cup \big(a_0 \cap (a_1 \cup a_2 \cap x_{i-1})\big)\Big)\Big) \geqslant a_0 \cup \Big(a_1 \cap \Big(a_2 \cup \big(a_0 \cap (a_1 \cup (a_2 \cap x_j))\big)\Big)\Big) = x_{j+1}$ for all $j < i-1$, and hence $x_i \geqslant \bigcup \{x_j \colon j < i-1\} = x_{i-1} \geqslant x_j$ for all $j < i$. Thus, by induction, $x_j \leqslant x_k$ whenever $j < k$.

To show this inclusion is proper, assume the contrary. Then there is a smallest ordinal i such that $x_j \geqslant x_i$ for some $j < i$. Then i is not a limit ordinal, since otherwise $x_i = \bigcup \{x_k \colon k < i\} \leqslant x_j$ implies $x_j \geqslant x_{j+1}$, a contradiction to the minimality of i. Suppose j is not a limit ordinal. Then $x_j = a_0 \cup \Big(a_1 \cap \Big(a_2 \cup \big(a_0 \cap (a_1 \cup (a_2 \cap x_{j-1}))\big)\Big)\Big)$, and successive applications of (b) give $x_{j-1} \geqslant a_2 \cap x_{j-1} \geqslant a_2 \cap x_{i-1}$. If $j-1$ is not a limit ordinal, $x_{j-1} \operatorname{non} \geqslant a_2$ and $a_0, a_1 \operatorname{non} \geqslant a_2 \cap x_{i-1}$ imply that $x_{j-1} \geqslant x_{i-1}$, a contradiction. If $j-1$ is a limit ordinal, then $x_{j-1} \geqslant a_2, x_{i-1}$, and hence $x_k \geqslant a_2 \cap x_{i-1}$ for some $k < j-1$. Continuing in this way if k is a limit ordinal, we get a descending chain of ordinals which must end in a finite number of steps at an ordinal k', not a limit ordinal. But then, as above, $x_{k'} \geqslant a_2 \cap x_{i-1}$ implies $x_{k'} \geqslant x_{i-1}$, a contradiction. Finally, if j is a limit ordinal, we conclude from $x_j \operatorname{non} \geqslant a_1, a_2$ and $x_j = \bigcup \{x_k \colon k < j\} \geqslant x_i = a_0 \cup \Big(a_1 \cap \Big(a_2 \cup \big(a_0 \cap (a_1 \cup (a_2 \cap x_{i-1}))\big)\Big)\Big)$ that $x_k \geqslant x_i$ for some $k < j$. If k is a limit ordinal we repeat the process to obtain a descending chain of ordinals which must end in a finite number of steps at an ordinal k', not a limit ordinal. This is just the preceding case, however, and hence yields a contradiction. Thus $x_j \geqslant x_i$ but $x_i \operatorname{non} \geqslant x_j$ whenever $i < j$.

We are now in a position to prove

THEOREM 1. *There does not exist a free complete lattice on three complete generators.*

Proof. From Lemma 3 the x_i form an ordinal indexed collection (hence not a set) of pairwise unequal polynomials in $L(3)$. Alternatively, given any infinite regular cardinal β, $L^*_\beta(3)$ is a complete lattice with three complete generators and has cardinality greater than or equal to β (since $L^*_\beta(3)$ contains the β unequal elements x_i, for $i < \beta$). Thus the cardinality of complete lattices with three complete generators is unbounded.

IV. Free complete Boolean algebras

We now investigate the existence of a free complete Boolean algebra on γ complete generators. If n is finite, the free Boolean algebra on n generators is finite, and in fact is isomorphic to the collection of all subsets of a set of cardinality 2^n. It therefore coincides with the free complete Boolean algebra on n complete generators.

We now prove that there does not exist a free complete Boolean algebra on ω complete generators. To do this we would like to use a technique similar to that of Chapter 3, but this does not appear to be possible. In Chapter 3 we were able to give an effective method of deciding when two polynomials were equivalent. The presence of the distributive law in Boolean algebras appears to prevent this. We must therefore use a more subtle technique.

The first step is to choose an ordinal indexed collection of polynomials in $B(\omega)$ which we wish to prove pairwise unequal. To do this we must first establish how strong a distributivity condition holds in Boolean algebras. Theorem 1 is due to Tarski [11] and Von Neumann [7], Appendix, p. 7.

THEOREM 1. *A Boolean algebra is continuous; that is, whenever* $\bigcap_{i \in I} b_i$ *exists, we have* $\bigcap_{i \in I} (a \cup b_i)$ *exists, and*

$$a \cup \big(\bigcap_{i \in I} b_i\big) = \bigcap_{i \in I} (a \cup b_i) \qquad (\textit{and dually}).$$

Proof. Trivially we have $a \cup (\bigcap_{i \in I} b_i) \leqslant a \cup b_i$ for all i in I. Now assume $x \leqslant a \cup b_i$ for all i in I. Then $a^c \cap x \leqslant a^c \cap (a \cup b_i) = a^c \cap b_i$ for all $i \in I$. Then $a^c \cap x \leqslant \bigcap_{i \in I} (a^c \cap b_i) = a^c \cap (\bigcap_{i \in I} b_i)$. Thus $a \cup (a^c \cap x) \leqslant a \cup (a^c \cap (\bigcap_{i \in I} b_i))$, so $a \cup x \leqslant a \cup (\bigcap_{i \in I} b_i)$, so $x \leqslant a \cup (\bigcap_{i \in I} b_i)$. Thus $a \cup (\bigcap_{i \in I} b_i)$ is the greatest lower bound of $\{a \cup b_i \colon i \in I\}$, so $a \cup (\bigcap_{i \in I} b_i) = \bigcap_{i \in I} (a \cup b_i)$. The dual is proved similarly.

We now ask if, in a Boolean algebra, a stronger distributive law than continuity holds. The following theorem, which appears to be new, shows that the answer is no:

THEOREM 2. *A continuous lattice can be regularly embedded in a Boolean algebra.*

Proof. Let L be a continuous lattice. Since L is distributive, it is isomorphic to a collection C of subsets of a set S under finite set union and intersection (Birkhoff [1], p. 140). Adjoin φ (the null set) and S to C, obtaining C'. We now look at the field F of subsets of S generated by C'. Then F is a Boolean algebra in which C' (or C) is embedded. To show this embedding is regular, let $\{c_i\colon i\in I\}$ be a collection of elements in C with least upper bound c in C. Suppose f in F is an upper bound for $\{c_i\colon i\in I\}$. We may write f as $(f_1\vee f_2^c)\wedge(f_3\vee f_4^c)\wedge\ldots\wedge(f_{2n-1}\vee f_{2n}^c)$, where n is finite and each f_i is an element of C' (this follows from the way F was constructed). Then $f_1\vee f_2^c$ is an upper bound for $\{c_i\colon i\in I\}$. Now, since the lattice C is continuous, the lattice C' is also continuous, and we have $f_2\cap(\bigcup_{i\in I}c_i)=\bigcup_{i\in I}(f_2\cap c_i)$. But since $c_i\leqslant f_1\vee f_2^c$ for all $i\in I$, we have $f_2\cap c_i\leqslant f_2\cap(f_1\cup f_2^c)=f_2\cap f_1$ for all $i\in I$. Thus $\bigcup_{i\in I}(f_2\cap c_i)\leqslant f_2\cap f_1$. In other words, $f_2\cap(\bigcup_{i\in I}c_i)\leqslant f_2\cap f_1$. Thus $f_2^c\cup(f_2\cap(\bigcup_{i\in I}c_i))\leqslant f_2^c\cup(f_2\cap f_1)$, or $f_2^c\cup(\bigcup_{i\in I}c_i)\leqslant f_2^c\cup f_1$. This implies $c\leqslant f_2^c\vee f_1$. Similarly, for $1\leqslant k\leqslant n$, $c\leqslant f_{2k-1}\vee f_{2k}^c$. Thus $c\leqslant f$. We have therefore proved that c is still the least upper bound for $\{c_i\colon i\in I\}$ in F. A dual argument shows that all greatest lower bounds are preserved, so C is regularly embedded in F.

It is easy to show, using continuity and inducting on $r(A)$, that any polynomial A in $B(\omega)$ is equal to a polynomial of the form $(a_0\cap B)\cup(a_0^c\cap C)$, where neither a_0 nor a_0^c appear in either B or C. But if we define

$$x_0\equiv a_0,\quad x_1\equiv(x_0\cap B)\cup(x_0^c\cap C),\quad x_2\equiv(x_1\cap B)\cup(x_1^c\cap C),\ \ldots,$$

we obtain at most 2^{2^3} unequal x_i's, since 2^{2^3} is the size of a free Boolean algebra on three generators (a_0, B, C). Thus we cannot choose our polynomials in as simple a way as we did in Chapter 3. Roughly, we cannot build just one chain (or a finite number of chains), but must instead build ω chains simultaneously.

First, for notational convenience, let us relabel our ω generators. We thus suppose that the set of generators is $\{a_{0,j}\colon 0\leqslant j<\omega\}\vee\{b_{i,j}\colon i\neq j,\ 0\leqslant i,\ j<\omega\}$. We define polynomials $a_{i,j}$, where i is arbitrary and $j<\omega$, as follows:

$$a_{i+1,j}\equiv\bigcup\{a_{i,j}\cap a_{i,k}\cap b_{j,k}\colon 0\leqslant k<\omega,\ k\neq j\}$$

and, if i is a limit ordinal,

$$a_{i,j}\equiv\bigcap\{a_{k,j}\colon 0\leqslant k<i\}.$$

We wish to show that, for fixed j, the $a_{i,j}$ are pairwise unequal. To do this we construct, for each cardinal α, a field of sets containing elements $A_{i,j}$ (for $0\leqslant j<\omega$, $0\leqslant i\leqslant\alpha$) and $B_{i,j}$ (for $0\leqslant i,\ j<\omega,\ i\neq j$), with $A_{i,j}<A_{i',j}$ whenever $i>i'$. We take the normal completion of this field of sets. We then show, if f is the valuation such that $f(a_{0,j})=A_{0,j}$ for $0\leqslant j<\omega$ and $f(b_{i,j})=B_{i,j}$ for $0\leqslant i,\ j<\omega,\ i\neq j$, that $f^*(a_{i,j})=A_{i,j}$ for $0\leqslant j<\omega$, $0\leqslant i\leqslant\alpha$. This establishes that, for fixed j, the $a_{i,j}$ are pairwise unequal. We have also, in the process, constructed complete Boolean algebras with ω complete generators of arbitrarily large cardinality.

Our construction will, in fact, be more general, depending on a cardinal parameter γ (the number of complete generators) in addition to α. The case $\gamma=\omega$ is the relevant one for this Chapter. The general construction will be used in Chapter 5.

Let α, γ be infinite cardinals with γ regular. Let $\beta=\gamma^2+\alpha\cdot\gamma$, and $S=2^\beta$, i.e. the set of all functions from β to $\{0,1\}$. For each $\delta<\beta$, let e_δ be the evaluation map corresponding to δ; i.e., if f is in S, $e_\delta(f)=f(\delta)$. Note that each $\delta<\beta$ can be *uniquely* written in one of the following forms: $\gamma\cdot i+j$, where $0\leqslant i,j<\gamma$; or $\gamma^2+\alpha\cdot j+i$, where $0\leqslant j<\gamma$ and $0\leqslant i<\alpha$.

Define subsets $B_{i,j}$ $(0\leqslant i,j<\gamma,\ i\neq j)$ of S as follows:

$$B_{i,j}=e^{-1}_{\gamma\cdot i+j}(1).$$

Define subsets $A_{i,j}$ $(0\leqslant i\leqslant\alpha,\ 0\leqslant j<\gamma)$ of S as follows:

$$A_{0,j}=e^{-1}_{\gamma^2+\alpha\cdot j}(1),$$

$$A_{i+1,j}=\Big[\bigvee_{\substack{0\leqslant k<\gamma\\ k\neq j}}(A_{i,j}\wedge A_{i,k}\wedge B_{j,k})\Big]\vee\Big[\bigwedge_{0\leqslant k\leqslant i+1}e^{-1}_{\gamma^2+\alpha\cdot j+k}(1)\Big],$$

and, if i is a limit ordinal,

$$A_{i,j}=\bigwedge_{0\leqslant k<i}A_{k,j}.$$

Let F be the field of subsets of S γ-generated by $\{B_{i,j}\colon 0\leqslant i,j<\gamma,\ i\neq j\}\vee\{A_{i,j}\colon 0\leqslant i\leqslant\alpha,\ 0\leqslant j<\gamma\}$. Then F is a Boolean algebra in which we denote union and intersection by $\bigcup$ and $\bigcap$, respectively (where $\bigcup$ and $\bigcap$ coincide with $\bigvee$ and $\bigwedge$, respectively, when applied to finite collections).

LEMMA 1. *If $0\leqslant k<i\leqslant\alpha$, $0\leqslant j<\gamma$, then $A_{i,j}$ is properly contained in $A_{k,j}$.*

Proof. Suppose that $i<h$ implies $A_{i,j}\leqslant A_{k,j}$ for all $k\leqslant i$ (this is obviously true for $h=1$). Trivially $A_{h,j}\leqslant A_{h,j}$. If h is a limit

ordinal, then $A_{h,j} = \bigwedge_{k<h} A_{k,j} \leqslant A_{k,j}$ for all $k<h$. If h is not a limit ordinal, then

$$A_{h,j} = [\bigvee_{\substack{0\leqslant k<\gamma \\ k\neq j}} (A_{h-1,j} \wedge A_{h-1,k} \wedge B_{j,k})] \vee [\bigwedge_{0\leqslant k\leqslant h} e^{-1}_{\gamma^2+\alpha\cdot j+k}(1)] .$$

But it is obvious that

$$\bigvee_{\substack{0\leqslant k<\gamma \\ k\neq j}} (A_{h-1,j} \wedge A_{h-1,k} \wedge B_{j,k}) \leqslant A_{h-1,j} .$$

If $h-1$ is not a limit ordinal, then

$$\bigwedge_{0\leqslant k\leqslant h} e^{-1}_{\gamma^2+\alpha\cdot j+k}(1) \leqslant \bigwedge_{0\leqslant k\leqslant h-1} e^{-1}_{\gamma^2+\alpha\cdot j+k}(1) \leqslant A_{h-1,j} ,$$

so $A_{h,j} \leqslant A_{h-1,j}$. Finally, if $h-1$ is a limit ordinal, then

$$\bigwedge_{0\leqslant k\leqslant h} e^{-1}_{\gamma^2+\alpha\cdot j+k}(1) \leqslant \bigwedge_{0\leqslant k\leqslant k'+1} e^{-1}_{\gamma^2+\alpha\cdot j+k}(1)$$

for all $k'<h-1$, so

$$\bigwedge_{0\leqslant k\leqslant h} e^{-1}_{\gamma^2+\alpha\cdot j+k}(1) \leqslant A_{k'+1,j}$$

for all $k'<h-1$. But, if k' is a limit ordinal less than $h-1$, $A_{k'+1,j} \leqslant A_{k',j}$ by induction. Thus

$$\bigwedge_{0\leqslant k\leqslant h} e^{-1}_{\gamma^2+\alpha\cdot j+k}(1) \leqslant A_{k',j}$$

for all $k'<h-1$, so

$$\bigwedge_{0\leqslant k\leqslant h} e^{-1}_{\gamma^2+\alpha\cdot j+k}(1) \leqslant A_{h-1,j} .$$

Thus $A_{h,j} \leqslant A_{h-1,j}$. But by induction $A_{h-1,j} \leqslant A_{k,j}$ for all $k\leqslant h-1$, so $A_{h,j} \leqslant A_{k,j}$ for all $k<h$. We have thus proved, by induction, that $k<i$ implies $A_{i,j} \leqslant A_{k,j}$.

To show that the inclusion is proper, let k be given and define a function f from β to $\{0,1\}$ as follows:

$$f(\gamma^2+\alpha\cdot j+h) = 1 \quad \text{for} \quad 0\leqslant h\leqslant k ,$$
$$f = 0 \quad \text{otherwise.}$$

If k is not a limit ordinal, then f is an element of

$$\bigwedge_{0\leqslant h\leqslant k} e^{-1}_{\gamma^2+\alpha\cdot j+h}(1) ,$$

so f is an element of $A_{k,j}$. If k is a limit ordinal, then f is an element of

$$\bigwedge_{0\leqslant h\leqslant t+1} e^{-1}_{\gamma^2+\alpha\cdot j+h}(1)$$

for all $t<k$, so f is an element of $A_{t+1,j}$ for all $t<k$, and hence of $A_{t,j}$ for all $t<k$. Thus f is an element of $A_{k,j}$. But f is not an element of $B_{j,h}$ for any $h<\gamma$, since $f(\gamma\cdot j+h) = 0$, and f is not an element of

$$\bigwedge_{0\leqslant h\leqslant k+1} e^{-1}_{\gamma^2+\alpha\cdot j+h}(1) ,$$

since $f(\gamma^2+\alpha\cdot j+k+1) = 0$. Hence f is not an element of $A_{k+1,j}$, and therefore is not an element of $A_{i,j}$ for any $i\geqslant k+1$. Therefore $A_{k,j}$ properly contains $A_{i,j}$ for all $i>k$, and the lemma is proved.

LEMMA 2. *Every element of F can be written in the form*

$$\bigwedge_{y\in Y} (\bigvee_{z\in Z_y} x_{y,z}) ,$$

where the $x_{y,z}$ are chosen from a subset of $\{A_{i,j}\} \vee \{A^c_{i,j}\} \vee \{B_{i,j}\} \vee \{B^c_{i,j}\}$ which has cardinality less than γ (the subset varies with the element).

Proof. We must show that elements of the stipulated form are closed under intersection and unions of less than γ elements, and also closed under complementation. That they are closed under intersections follows from the regularity of γ. To show that they are closed under unions, note that by distributivity

$$\bigwedge_{y\in Y} (\bigvee_{z\in Z_y} x_{y,z}) = \bigvee_{\varphi\in \Pi Z_y} (\bigwedge_{y\in Y} x_{y,\varphi(y)}) ,$$

where the $x_{y,\varphi(y)}$ come from the same set as the $x_{y,z}$. But elements of the form $\bigvee_{y\in Y} (\bigwedge_{z\in Z_y} x_{y,z})$, with the same restriction on the $x_{y,z}$, are closed under unions, again by the regularity of γ. Then another application of the distributive law returns us to the original form. To show that elements of the stipulated form are closed under complementation, note that

$$[\bigwedge_{y\in Y} (\bigvee_{z\in Z_y} x_{y,z})]^c = \bigvee_{y\in Y} (\bigwedge_{z\in Z_y} x^c_{y,z}) .$$

An application of the distributive law returns us to the original form. Thus the lemma is proved. (Note that we have also proved that every element of F can be written as $\bigvee_{y\in Y} (\bigwedge_{z\in Z_y} x_{y,z})$, with the same restriction on the $x_{y,z}$.)

LEMMA 3. *If $0\leqslant i<\alpha$, $0\leqslant j<\gamma$, then*

$$A_{i+1,j} = \bigcup_{\substack{0\leqslant k<\gamma \\ k\neq j}} (A_{i,j} \cap A_{i,k} \cap B_{j,k}) .$$

Proof. We must show that $A_{i+1,j}$, obviously an upper bound, is the least upper bound of $\{(A_{i,j} \cap A_{i,k} \cap B_{i,k})\colon 0\leqslant k<\gamma,\ k\neq j\}$ in F. Suppose x in F is an upper bound for this collection. We have, from

Lemma 2, that $x = \bigwedge_{y\in Y}(\bigvee_{z\in Z_y} x_{y,z})$, with the stipulated restriction on the $x_{y,z}$. Then each $\bigvee_{z\in Z_y} x_{y,z}$ is an upper bound for this collection. If we can prove that each $\bigwedge_{z\in Z_y} x_{y,z}$ contains $A_{i+1,j}$, then x will contain $A_{i+1,j}$, and we are done. (Note that the stipulated restriction on the $x_{y,z}$ implies that Z_y may be assumed to have cardinality less than γ.)

Let us therefore assume that

$$A = \Big(\bigvee_{\varrho<\gamma_1} A_{m_\varrho,n_\varrho}\Big) \vee \Big(\bigvee_{\varrho<\gamma_2} A^c_{p_\varrho,q_\varrho}\Big) \vee \Big(\bigvee_{\varrho<\gamma_3} B_{r_\varrho,s_\varrho}\Big) \vee \Big(\bigvee_{\varrho<\gamma_4} B^c_{t_\varrho,u_\varrho}\Big),$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4 < \gamma$, is an upper bound for $\{(A_{i,j} \cap A_{i,k} \cap B_{j,k}\colon 0 \leqslant k < \gamma,\ k \neq j\}$. We wish to prove that $A \geqslant A_{i+1,j}$. Without loss of generality we may assume that $\gamma_2 > 0$, $p_0 = i+1$, and $q_0 = j$, since $A^c_{i+1,j}$ is disjoint from $A_{i+1,j}$.

Let

$$\lambda = \max(\sup_{\varrho<\gamma_1} n_\varrho, \sup_{\varrho<\gamma_2} q_\varrho, \sup_{\varrho<\gamma_3} s_\varrho, \sup_{\varrho<\gamma_4} t_\varrho, \sup_{\varrho<\gamma_4} u_\varrho)+1 .$$

Since γ is regular, λ is less than γ. Now define a function f from β to $\{0, 1\}$ as follows:

$$f(\gamma^2+\alpha\cdot q_\varrho+k) = 1 \quad \text{for} \quad 0 \leqslant k < \min(p_\varrho+1, \alpha),\ 0 \leqslant \varrho < \gamma_2;$$

$$f(\gamma\cdot t_\varrho+u_\varrho) = 1 \quad \text{for} \quad 0 \leqslant \varrho < \gamma_4;$$

$$f(\gamma^2+\alpha\cdot\lambda+k) = 1 \quad \text{for} \quad 0 \leqslant k \leqslant i;$$

$$f(\gamma\cdot j+\lambda) = 1;$$

and

$$f = 0 \quad \text{otherwise.}$$

Since $f(\gamma\cdot j+\lambda) = 1$, f is in $e^{-1}_{\gamma\cdot j+\lambda}(1)$, so f is in $B_{j,\lambda}$ (note that $\lambda > q_0 = j$). Since $f(\gamma^2+\alpha\cdot\lambda+k) = 1$ for $0 \leqslant k \leqslant i$, f is in $e^{-1}_{\gamma^2+\alpha\cdot\lambda+k}(1)$ for $0 \leqslant k \leqslant i$, and hence in $A_{i,\lambda}$. Since $f(\gamma^2+\alpha\cdot q_0+k) = 1$ for $0 \leqslant k \leqslant p_0$, where $p_0 = i+1$ and $q_0 = j$, f is in $e^{-1}_{\gamma^2+\alpha\cdot j+k}(1)$ for $0 \leqslant k \leqslant i+1$, so f is in $A_{i,j}$. Hence f is in $A_{i,j} \wedge A_{i,\lambda} \wedge B_{j,\lambda}$, and hence f must be in A.

Since $f(\gamma^2+\alpha\cdot q_\varrho+k) = 1$ for $0 \leqslant k < \min(p_\varrho+1, \alpha)$, $0 \leqslant \varrho < \gamma_2$, we see that f is in

$$\bigwedge_{0\leqslant k<\min(p_\varrho+1,\alpha)} e^{-1}_{\gamma^2+\alpha\cdot q_\varrho+k}(1)$$

for $0 \leqslant \varrho < \gamma_2$, so f is in A_{p_ϱ,q_ϱ} for $0 \leqslant \varrho < \gamma_2$. Thus f is not in $\bigvee_{0\leqslant\varrho<\gamma_2} A^c_{p_\varrho,q_\varrho}$. Since $f(\gamma\cdot t_\varrho+u_\varrho) = 1$ for $0 \leqslant \varrho < \gamma_4$, f is in $e^{-1}_{\gamma\cdot t_\varrho+u_\varrho}(1)$ for $0 \leqslant \varrho < \gamma_4$, so f is in B_{t_ϱ,u_ϱ} for $0 \leqslant \varrho < \gamma_4$. Thus f is not in $\bigvee_{0\leqslant\varrho<\gamma_4} B^c_{t_\varrho,u_\varrho}$.

There are two remaining possibilities: f is in $\bigvee_{0\leqslant\varrho<\gamma_3} B_{r_\varrho,s_\varrho}$, or f is in $\bigvee_{0\leqslant\varrho<\gamma_1} A_{m_\varrho,n_\varrho}$. Suppose f is in $\bigvee_{0\leqslant\varrho<\gamma_3} B_{r_\varrho,s_\varrho}$. Then, for some σ with $0 \leqslant \sigma < \gamma_3$,

f is in B_{r_σ,s_σ}. Thus $f(\gamma\cdot r_\sigma+s_\sigma) = 1$. But, since $\lambda > s_\sigma$, this implies that $r_\sigma = t_\tau$, $s_\sigma = u_\tau$, for some τ with $0 \leqslant \tau < \gamma_4$. But then we have $A \geqslant B_{r_\sigma,s_\sigma} \vee B^c_{t_\tau,u_\tau} = B_{r_\sigma,s_\sigma} \vee B^c_{r_\sigma,s_\sigma} = S \geqslant A_{i+1,j}$, and we are done.

Now suppose f is in $\bigvee_{0\leqslant\varrho<\gamma_1} A_{m_\varrho,n_\varrho}$. Then, for some σ with $0 \leqslant \sigma < \gamma_1$, f is in A_{m_σ,n_σ}. If $A \geqslant A^c_{m_\sigma,n_\sigma}$, then $A \geqslant A_{m_\sigma,n_\sigma} \vee A^c_{m_\sigma,n_\sigma} = S \geqslant A_{i+1,j}$ and we are done. Suppose A non $\geqslant A^c_{m_\sigma,n_\sigma}$. Then let π be the smallest ordinal such that there exists an ordinal $\theta \neq \lambda$ with f in $A_{\pi,\theta}$ but A non $\geqslant A^c_{\pi,\theta}$ (since $\lambda > n_\sigma$, m_σ is such an ordinal). Now π cannot be 0; if it were f in $A_{0,\theta}$ would imply $f(\gamma^2+\alpha\cdot\theta) = 1$. This, since $\theta \neq \lambda$, would imply that there exists a σ with $0 \leqslant \sigma < \gamma_2$ such that $q_\sigma = \theta$. But then $A \geqslant A^c_{p_\sigma,q_\sigma} \geqslant A^c_{0,\theta}$, a contradiction. Also π cannot be a limit ordinal; if it were, the minimality of π would imply that $A \geqslant A^c_{\pi',\theta}$ for all π' with $0 \leqslant \pi' < \pi$. But then $A \geqslant \bigvee_{0\leqslant\pi'<\pi} A^c_{\pi',\theta} = A^c_{\pi,\theta}$, a contradiction.

Thus $\pi = \pi'+1$. There are then two possibilities: f is in

$$\bigwedge_{0\leqslant k\leqslant\pi} e^{-1}_{\gamma^2+\alpha\cdot\theta+k}(1),$$

or f is in

$$\bigvee_{\substack{0\leqslant k<\gamma\\ k\neq\theta}} (A_{\pi',\theta} \wedge A_{\pi',k} \wedge B_{\theta,k}).$$

Suppose that f is in $\bigwedge_{0\leqslant k\leqslant\pi} e^{-1}_{\gamma^2+\alpha\cdot\theta+k}(1)$. Then $f(\gamma^2+\alpha\cdot\theta+\pi) = 1$. Since $\theta \neq \lambda$ this implies that, for some σ with $0 \leqslant \sigma < \gamma_2$, we have $q_\sigma = \theta$, $p_\sigma \geqslant \pi$. But then $A \geqslant A^c_{p_\sigma,q_\sigma} \geqslant A^c_{\pi,\theta}$, a contradiction.

Now suppose f is in

$$\bigvee_{\substack{0\leqslant k<\gamma\\ k\neq\theta}} (A_{\pi',\theta} \wedge A_{\pi',k} \wedge B_{\theta,k}).$$

Then, for some k' with $0 \leqslant k' < \gamma$, $k' \neq \theta$, we have that f is in $A_{\pi',\theta} \wedge A_{\pi',k'} \wedge B_{\theta,k'}$. We distinguish two further cases: $k' \neq \lambda$ and $k' = \lambda$. Suppose $k' \neq \lambda$. Since f is in $A_{\pi',\theta}$, and π was chosen minimal, we must have $A \geqslant A^c_{\pi',\theta}$. Since f is in $A_{\pi',k'}$, where $k' \neq \lambda$, and π was chosen minimal, we must have $A \geqslant A^c_{\pi',k'}$. Since f is in $B_{\theta,k'}$, we must have $f(\gamma\cdot\theta+k') = 1$. This implies, since $k' \neq \lambda$, that for some σ with $0 \leqslant \sigma < \gamma_4$ we have $t_\sigma = \theta$, $u_\sigma = k'$. But then $A \geqslant B^c_{t_\sigma,u_\sigma} = B^c_{\theta,k'}$. Thus $A \geqslant (A^c_{\pi',\theta} \vee A^c_{\pi',k'} \vee B^{\theta,k'}_c) \geqslant A^c_{\pi,\theta}$, a contradiction.

Finally suppose $k' = \lambda$. Since f is in $B_{\theta,k'} = B_{\theta,\lambda}$, we must have $f(\gamma\cdot\theta+\lambda) = 1$. But since $\lambda > \sup_{\varrho<\gamma_4} u_\varrho$, this implies that $\theta = j$. Thus A non $\geqslant A^c_{\pi,j}$. But we know that $A \geqslant A^c_{p_0,q_0} = A^c_{i+1,j}$. Thus $\pi > i+1$, or $\pi' \geqslant i+1$. Since f is in $A_{\pi',k'} = A_{\pi',\lambda}$, f must therefore be in $A_{i+1,\lambda}$. In other words, f is either in

$$\bigwedge_{0\leqslant k\leqslant i+1} e^{-1}_{\gamma^2+\alpha\cdot\lambda+k}(1),$$

or f is in

$$\bigvee_{\substack{0\leqslant k<\gamma\\ k\neq\lambda}} (A_{i,\lambda} \wedge A_{i,k} \wedge B_{\lambda,k}).$$

If f is in $\bigwedge_{0\leqslant k\leqslant i+1} e^{-1}_{\gamma^2+\alpha\cdot\lambda+k}(1)$, then $f(\gamma^2+\alpha\cdot\lambda+i+1)=1$. But, since $\lambda > \sup_{\varrho<\gamma_2} q_\varrho$, this is impossible. On the other hand, if f is in

$$\bigvee_{\substack{0\leqslant k<\gamma\\ k\neq\lambda}} (A_{i,\lambda} \wedge A_{i,k} \wedge B_{\lambda,k}),$$

then for some k'' with $0\leqslant k''<\gamma$, $k''\neq\lambda$, we know that f is in $B_{\lambda,k''}$. Thus $f(\gamma\cdot\lambda+k'')=1$. But, since $\lambda>\sup_{\varrho<\gamma_1} t_\varrho$, this is impossible. The existence of π has thus led to a contradiction, and the proof of Lemma 3 is complete.

We note that, if $0\leqslant i\leqslant\alpha$, $0\leqslant j<\gamma$, and i is a limit ordinal, then $A_{i,j} = \bigwedge_{0\leqslant k<i} A_{k,j} = \bigcap_{0\leqslant k<i} A_{k,j}$.

Let us write $F=F_{\gamma,\alpha}$ to indicate its dependence on γ and α. Then we denote the normal completion of $F_{\gamma,\alpha}$ by $F^*_{\gamma,\alpha}$.

THEOREM 3. *$F_{\gamma,\alpha}$, and hence $F^*_{\gamma,\alpha}$, is completely generated by* $\{A_{0,j}\colon 0\leqslant j<\gamma\}\vee\{B_{i,j}\colon 0\leqslant i,j<\gamma,\ i\neq j\}$, *and has cardinality at least* α.

Proof. From Lemma 3 it follows that $\{A_{0,j}\}\vee\{B_{i,j}\}$ completely generates $F_{\gamma,\alpha}$, and hence $F^*_{\gamma,\alpha}$. From Lemma 1 it follows that, for fixed j, the $A_{i,j}$ for $0\leqslant i\leqslant\alpha$ are pairwise unequal. Thus $F_{\gamma,\alpha}$, and hence $F^*_{\gamma,\alpha}$, have cardinality at least α.

THEOREM 4. *There is no free complete Boolean algebra on ω complete generators.*

Proof. $F^*_{\omega,\alpha}$ is completely generated by

$$\{A_{0,j}\colon 0\leqslant j<\omega\}\vee\{B_{i,j}\colon 0\leqslant i,j<\omega,\ i\neq j\}$$

a set of cardinality ω. Also $F^*_{\omega,\alpha}$ has cardinality at least α. Hence there exist complete Boolean algebras with ω complete generators of arbitrarily large cardinality.

Alternatively, consider the valuation f such that $f(a_{0,j})=A_{0,j}$ for $0\leqslant j<\omega$ and $f(b_{i,j})=B_{i,j}$ for $0\leqslant i,j<\omega$, $i\neq j$. Then, by Lemma 3, $f^*(a_{i,j})=A_{i,j}$ for $0\leqslant i\leqslant\alpha$, $0\leqslant j<\omega$. Hence, for fixed j, we conclude from Lemma 1 that the $a_{i,j}$ for all ordinals i are pairwise unequal.

V. (α,β) distributivity

In this chapter we generalize the results of Chapter 4. We shall be concerned with Boolean algebras in which a certain type of distributive law holds.

DEFINITION 1. A Boolean algebra B is said to be (α,β) *distributive*, where α and β are cardinals with $\alpha\geqslant\omega$ and $\beta\geqslant 3$, if the following identity is valid whenever Y has cardinality less then α, Z has cardinality less than β, and all the $\bigcup$'s and $\bigcap$'s exist in B:

$$\bigcap_{y\in Y}\Big(\bigcup_{z\in Z} x_{y,z}\Big) = \bigcup_{\varphi\in Z^Y}\Big(\bigcap_{y\in Y} x_{y,\varphi(y)}\Big).$$

Note that this identity implies its dual and vice versa.

If B is (α,β) distributive for all β, then it is said to be (α,∞) *distributive*.

A complete (α,β) distributive Boolean algebra B is said to be a *free complete (α,β) distributive Boolean algebra on γ complete generators* if B contains a subset A of cardinality γ which completely generates B, and if every mapding f of A onto a subset A' of a complete (α,β) distributive Boolean algebra B' which completely generates B' can be extended to a complete homomorphism f^* of B onto B'.

Replacing (α,β) by (α,∞) everywhere in the above definition, we obtain the definition of a *free complete (α,∞) distributive Boolean algebra on γ complete generators*.

We can define a new equality on polynomials in $B(\gamma)$ as follows: A_1, A_2 in $B(\gamma)$ are equal (α,β) if and only if, for every valuation f from $\{a_i\colon 0\leqslant i<\gamma\}$ into a complete (α,β) distributive Boolean algebra, $f^*(A_1) = f^*(A_2)$. Then the statements of Chapter 2 carry over, i.e. the following are equivalent: $B(\gamma)$ (after the identification of equal (α,β) elements is a set; $B(\gamma)$ (after the same identification) is a free complete (α,β) distributive Boolean algebra on γ complete generators; there exists a free complete (α,β) distributive Boolean algebra on γ complete generators; and the cardinality of complete (α,β) distributive Boolean algebras with γ complete generators is bounded.

Defining equal (α,∞) in the obvious way, the above statements carry over if (α,β) is replaced by (α,∞) everywhere.

We wish to investigate the existence of a free complete (α,β) distributive Boolean algebra on γ complete generators. Theorem 1, which was first proved by Tarski [12], settles the question for $\gamma<\alpha$.

THEOREM 1. *If $\gamma<\alpha$, the free complete (α,β) distributive Boolean algebra on γ complete generators is isomorphic to the collection of all subsets of a set of cardinality 2^γ.*

Proof. Let B be any complete (α,β) distributive Boolean algebra with the γ complete generators $\{x_i\colon 0\leqslant i<\gamma\}$. Then, applying the distributive law, we obtain

$$I = \bigcap_{0\leqslant i<\gamma}(x_i\cup x_i^c) = \bigcup_{\varphi\in 2^\gamma}\Big(\bigcap_{0\leqslant i<\gamma} x_{i,\varphi(i)}\Big),$$

where $x_{i,\varphi(i)} = x_i$ if $\varphi(i) = 0$, and $x_{i,\varphi(i)} = x_i^c$ if $\varphi(i) = 1$. Now choose any φ in 2^γ. Then $\bigcap\limits_{0 \leqslant i < \gamma} x_{i,\varphi(i)}$ is either contained in or disjoint to each x_i and each x_i^c. The collection of all elements in B which either contain or are disjoint to $\bigcap\limits_{0 \leqslant i < \gamma} x_{i,\varphi(i)}$ are easily seen to form a complete sub-algebra of B containing the x_i, and hence must include all of B. Thus $\bigcup\limits_{0 \leqslant i < \gamma} x_{i,\varphi(i)}$ is either 0 or an atom in B. Thus I is a union of not more than 2^γ atoms, so B is isomorphic to the collection of all subsets of a set of cardinality at most 2^γ. On the other hand, the Boolean algebra of all subsets of 2^γ is (α, β) distributive and is completely generated by $\{e_i^{-1}(1)\colon 0 \leqslant i < \gamma\}$. The mapping f taking $e_i^{-1}(1)$ to x_i for each $i < \gamma$ extends naturally to a complete homomorphism f^*, so Theorem 1 is proved.

We now suppose that $\gamma \geqslant \alpha$, and ask if there exists a free complete (α, ∞) distributive Boolean algebra on γ complete generators. It is easily seen that, if α is a singular cardinal, (α, ∞) distributivity implies (α^+, ∞) distributivity in a complete Boolean algebra, where α^+ is the smallest cardinal greater than α. We thus assume that α is regular. If we can show that, for $\gamma = \alpha$, such an algebra does not exist, the question will be settled for all $\gamma \geqslant \alpha$.

We therefore suppose that γ is an infinite regular cardinal, and prove that there does not exist a free complete (γ, ∞) distributive Boolean algebra on γ complete generators. Our method will be a direct extension of that of Chapter 4.

We first choose an ordinal indexed collection of polynomials in $B(\gamma)$ which we wish to prove pairwise unequal (γ, ∞). First we relabel the generators as $\{a_{0,j}\colon 0 \leqslant j < \gamma\} \vee \{b_{i,j}\colon 0 \leqslant i, j < \gamma,\ i \neq j\}$. Then we define polynomials $a_{i,j}$, where i is arbitrary and $j < \gamma$, as follows:

$$a_{i+1,j} \equiv \bigcup \{a_{i,j} \cap a_{i,k} \cap b_{j,k}\colon\ 0 \leqslant k < \gamma,\ k \neq j\},$$

and, if i is a limit ordinal,

$$a_{i,j} \equiv \bigcap \{a_{k,j}\colon\ 0 \leqslant k < i\}.$$

Thus the polynomials are obvious generalizations of those in Chapter 4.

Now, for any infinite cardinal α, consider the valuation f from $\{a_{0,j}\} \vee \{b_{i,j}\}$ to $F^*_{\gamma,\alpha}$ defined by $f(a_{0,j}) = A_{0,j}$ and $f(b_{i,j}) = B_{i,j}$. It follows from Lemma 3, Chapter 4, that $f^*(a_{i,j}) = A_{i,j}$ for $0 \leqslant i \leqslant \alpha$, $0 \leqslant j < \gamma$. Moreover, from Lemma 1, Chapter 4, the $A_{i,j}$ for fixed j and for $0 \leqslant i < \alpha$ are pairwise unequal. All that remains (to show the $a_{i,j}$ for fixed j are pairwise unequal (γ, ∞)) is to show that $F^*_{\gamma,\alpha}$ is (γ, ∞) distributive. This will also, of course, show that there exist complete (γ, ∞) distributive Boolean algebras on γ complete generators of arbitrarily large cardinality (namely the $F^*_{\gamma,\alpha}$).

To show that $F^*_{\gamma,\alpha}$ is (γ, ∞) distributive it is sufficient to show that $F_{\gamma,\alpha}$ is (γ, ∞) distributive. (Pierce [8]).

THEOREM 2. *$F_{\gamma,\alpha}$ is (γ, ∞) distributive.*

Proof. Let us assume the contrary. Then, for some cardinal $\gamma_0 < \gamma$, and for some choice of elements $x_{\varrho,y}$ in $F_{\gamma,\alpha}$, we have

(1) $$\bigcap_{\varrho<\gamma_0} \Big(\bigcup_{y \in Y} x_{\varrho,y}\Big) > \bigcup_{\varphi \in Y^{\gamma_0}} \Big(\bigcap_{\varrho<\gamma_0} x_{\varrho,\varphi(\varrho)}\Big),$$

where the $\bigcup$'s and $\bigcap$'s all exist in $F_{\gamma,\alpha}$.

By taking relative complements, we may assume that each $\bigcap\limits_{\varrho<\gamma_0} x_{\varrho,\varphi(\varrho)} = 0$, and that all the $\bigcup\limits_{y \in Y} x_{\varrho,y}$ are equal.

Now note that for $0 \leqslant i \leqslant \alpha$, $0 \leqslant j < \gamma$ we have

$$A_{i,j} = A_{\alpha,j} \vee \Big[\bigvee_{i \leqslant k < \alpha} (A_{k,j} \wedge A^c_{k+1,j})\Big].$$

Using this, the dual of Lemma 2 of Chapter 4, and the complete distributivity of $\bigvee$ and $\bigwedge$, we see that each element of $F_{\gamma,\alpha}$ is a set union of elements of the form

(2) $$\Big(\bigwedge_{\sigma<\gamma_1} A_{m_\sigma,n_\sigma}\Big) \wedge \Big(\bigwedge_{\sigma<\gamma_2} A^c_{p_\sigma,q_\sigma}\Big) \wedge \Big(\bigwedge_{\sigma<\gamma_3} B_{r_\sigma,s_\sigma}\Big) \wedge \Big(\bigvee_{\sigma<\gamma_4} B^c_{t_\sigma,u_\sigma}\Big),$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4 < \gamma$, and with the added condition (condition C) that for each $0 < \gamma_1$ such that $m_\sigma < \alpha$ there exists a $\tau < \gamma_2$ such that $m_\sigma + 1 = p_\tau$, $n_\sigma = q_\tau$. It is then easy to see that in (1) we may assume that each $\chi_{\varrho,y}$ is actually of the form (2), with condition C.

We thus assume that in (1) each $\bigcap\limits_{\varrho<\gamma_0} x_{\varrho,\varphi(\varrho)} = 0$, all the $\bigcup\limits_{y \in Y} x_{\varrho,y}$ are equal, and each $x_{\varrho,y}$ is of the form (2), with condition C.

For convenience assume Y is well ordered. Now define elements y_ϱ $(0 \leqslant \varrho < \gamma_0)$ in Y by induction as follows:

$y_0 =$ the first y in Y such that $x_{0,y} \neq 0$ (since $\bigcup\limits_{y \in Y} x_{0,y} > 0$, such a y_0 must exist),

$y_\varrho =$ the first y in Y such that $\Big(\bigcap\limits_{k<\varrho} x_{k,y_k}\Big) \cap x_{\varrho,y} \neq 0$, or, if no such y exists, $y_\varrho = y_0$.

The function φ such that $\varphi(\varrho) = y_\varrho$ for $0 \leqslant \varrho < \gamma_0$ lies in Y^{γ_0}. Thus $\bigcap\limits_{\varrho<\gamma_0} x_{\varrho,y_\varrho} = 0$. Let ϱ_0 be the smallest $\varrho \leqslant \gamma_0$ such that $\bigcap\limits_{k<\varrho} x_{k,y_k} = 0$.

Suppose ϱ_0 is not a limit ordinal, i.e. $\varrho_0 = \varrho' + 1$. Then, since $\varrho' < \varrho_0$, $\bigcap\limits_{k<\varrho'} x_{k,y_k} \neq 0$. But $\bigcap\limits_{k<\varrho'} x_{k,y_k} \leqslant \bigcup\limits_{y \in Y} x_{0,y} = \bigcup\limits_{y \in Y} x_{\varrho',y}$. Therefore, by continuity,

$$\bigcup_{y \in Y} \Big[x_{\varrho',y} \cap \Big(\bigcap_{k<\varrho'} x_{k,y_k}\Big)\Big] = \Big(\bigcap_{k<\varrho'} x_{k,y_k}\Big) \cap \Big(\bigcup_{y \in Y} x_{\varrho',y}\Big) = \bigcap_{k<\varrho'} x_{k,y_k} \neq 0.$$

Thus, for some y in Y, and hence for $y_{\varrho'}$, we have $x_{\varrho',y} \cap (\bigcap_{k<\varrho'} x_{k,y_k}) \neq 0$. Thus we have that $\bigcap_{k<\varrho_0} x_{k,y_k} \neq 0$, a contradiction.

Now suppose that ϱ_0 is a limit ordinal. We shall examine $\bigcup_{k<\varrho_0} x_{k,y_k}$ more closely. For each $k<\varrho_0$, x_{k,y_k} is of the form (2), with condition C. Thus, forming the intersection $\bigcap_{k<\varrho_0} x_{k,y_k}$ formally (remember $\varrho_0<\gamma$, so $\bigcap_{k<\varrho_0} x_{k,y_k} = \bigwedge_{k<\varrho_0} x_{k,y_k}$), and reindexing the terms, we can write

$$\bigcap_{k<\varrho_0} x_{k,y_k} = \Big(\bigwedge_{\sigma<\gamma_1} A_{m_\sigma,n_\sigma}\Big) \wedge \Big(\bigwedge_{\sigma<\gamma_2} A^c_{p_\sigma,q_\sigma}\Big) \wedge \Big(\bigwedge_{\sigma<\gamma_3} B_{r_\sigma,s_\sigma}\Big) \wedge \Big(\bigwedge_{\sigma<\gamma_4} B^c_{t_\sigma,u_\sigma}\Big)$$

with $\gamma_1, \gamma_2, \gamma_3, \gamma_4 < \gamma$, where each term occurs in the representation (in the form (2), with condition C) of x_{k,y_k} for some $k<\varrho_0$.

Now construct a function f in 2^β as follows:

$$\begin{aligned} f(\gamma^2+\alpha\cdot n_\sigma+x) &= 1 \quad \text{for} \quad 0\leqslant x<\min(m_\sigma+1,\alpha), \quad 0\leqslant\sigma<\gamma_1, \\ f(\gamma\cdot r_\sigma+s_\sigma) &= 1 \quad \text{for} \quad 0\leqslant\sigma<\gamma_3, \\ f &= 0 \quad \text{otherwise.} \end{aligned}$$

Since, for $0\leqslant\sigma<\gamma_1$, f is in $\bigwedge_{0\leqslant x<\min(m_\sigma+1,\alpha)} e^{-1}_{\gamma^2+\alpha\cdot n_\sigma+x}(1)$, f is in $\bigwedge_{\sigma<\gamma_1} A_{m_\sigma,n_\sigma}$. Since for $0\leqslant\sigma<\gamma_3$, f is in $e^{-1}_{\gamma\cdot r_\sigma+s_\sigma}(1)$, f is in $\bigwedge_{\sigma<\gamma_3} B_{r_\sigma,s_\sigma}$. Then, since $\bigcap_{k<\varrho_0} x_{k,y_k}=0$, f cannot lie in both $\bigwedge_{\sigma<\gamma_4} B^c_{t_\sigma,u_\sigma}$ and $\bigwedge_{\sigma<\gamma_2} A^c_{p_\sigma,q_\sigma}$. Thus f is either in $\bigvee_{\sigma<\gamma_4} B_{t_\sigma,u_\sigma}$ or in $\bigvee_{\sigma<\gamma_2} A_{p_\sigma,q_\sigma}$.

Suppose f is in $\bigvee_{\sigma<\gamma_4} B_{t_\sigma,u_\sigma}$. Then, for some σ with $0\leqslant\sigma<\gamma_4$, $f(\gamma\cdot t_\sigma+u_\sigma)=1$. Therefore there exists a τ with $0\leqslant\tau<\gamma_3$ such that $r_\tau=t_\sigma$, $s_\tau=u_\sigma$. But B_{r_τ,s_τ} occurs as a term in the representation (in the form (2), with condition C), of x_{k,y_k} for some $k<\varrho_0$, say $k=k_1$. Likewise $B^c_{t_\sigma,u_\sigma}$ occurs as a term in the representation of x_{k,y_k} for some $k<\varrho_0$, say $k=k_2$. Then

$$\bigcap_{k<\max(k_1,k_2)+1} x_{k,y_k} \leqslant B_{r_\tau,s_\tau}\wedge B^c_{t_\sigma,u_\sigma}=0,$$

a contradiction, since $\max(k_1,k_2)+1<\varrho_0$, and ϱ_0 was chosen minimal.

Finally suppose f is in $\bigvee_{\sigma<\gamma_2} A_{p_\sigma,q_\sigma}$. Then, for some σ with $0\leqslant\sigma<\gamma_2$, f is in A_{p_σ,q_σ}. Suppose, for some $\varrho'<\varrho_0$, $\bigcap_{k<\varrho'} x_{k,y_k}\leqslant A_{p_\sigma,q_\sigma}$. We know that $A^c_{p_\sigma,q_\sigma}$ occurs as a term in the representation (in the form (2), with condition C) of x_{k,y_k} for some $k<\varrho_0$, say $k=k_1$. But then

$$\bigcap_{k<\max(\varrho',k_1)+1} x_{k,y_k}\leqslant A_{p_\sigma,q_\sigma}\wedge A^c_{p_\sigma,q_\sigma}=0.$$

This is a contradiction, since $\max(\varrho',k_1)+1<\varrho_0$, and ϱ_0 was chosen minimal. Thus, for all $\varrho'<\varrho_0$, $\bigcap_{k<\varrho'} x_{k,y_k}\ \mathrm{non}\leqslant A_{p_\sigma,q_\sigma}$.

Now let i be the smallest ordinal less than or equal to α such that there exists a j, $0\leqslant j<\gamma$, with f in $A_{i,j}$ but, for all $\varrho'<\varrho_0$, $\bigcap_{k<\varrho'} x_{k,y_k}\ \mathrm{non}\leqslant A_{i,j}$ (p_σ is such an ordinal).

Since f is in $A_{i,j}$, f is in $A_{0,j}$, and hence $f(\gamma^2+\alpha\cdot j)=1$. Thus there exists a $\tau<\gamma_1$ such that $n_\tau=j$. Then A_{m_τ,n_τ} must occur as a term in the representation (in the form (2), with condition C) of x_{k,y_k} for some $k<\varrho_0$, say $k=k_1$. If $i=0$, then

$$\bigcap_{k<k_1+1} x_{k,y_k}\leqslant A_{m_\tau,n_\tau}\leqslant A_{0,j},$$

a contradiction, since $k_1+1<\varrho_0$. Suppose i is a limit ordinal. Since $\bigcap_{k<k_1+1} x_{k,y_k}\ \mathrm{non}\leqslant A_{i,j}$, we must have $m_\tau<i$. Moreover, it follows from condition C that $A^c_{m_\tau+1,n_\tau}$ is a term in $x_{k_1,y_{k_1}}$. But $m_\tau+1<i$ and f is in $A_{m_\tau+1,n_\tau}$. Hence, by the minimality of i, there exists a $\varrho'<\varrho_0$ such that $\bigcap_{k<\varrho'} x_{k,y_k}\leqslant A_{m_\tau+1,n_\tau}$. But then

$$\bigcap_{k<\max(\varrho',k_1)+1} x_{k,y_k}\leqslant A_{m_\tau+1,n_\tau}\wedge A^c_{m_\tau+1,n_\tau}=0,$$

a contradiction, since $\max(\varrho',k_1)+1<\varrho_0$.

The only remaining alternative is that $i=i'+1$. There are then two possibilities: f is in $\bigwedge_{k\leqslant i} e^{-1}_{\gamma^2+\alpha\cdot j+k}(1)$, or f is in

$$\bigvee_{\substack{0\leqslant k<\gamma \\ k\neq j}} (A_{i',j}\wedge A_{i',k}\wedge B_{j,k}).$$

If f is in $\bigwedge_{k\leqslant i} e^{-1}_{\gamma^2+\alpha\cdot j+k}(1)$, then $f(\gamma^2+\alpha\cdot j+i)=1$. Thus there exists a $\delta<\gamma_1$ such that $n_\delta=j$, $m_\delta\geqslant i$. But A_{m_δ,n_δ} occurs as a term in the representation (in the form (2) with condition C) of x_{k,y_k} for some $k<\varrho_0$, say $k=k_2$. Thus $\bigcap_{k<k_2+1} x_{k,y_k}\leqslant A_{m_\delta,n_\delta}\leqslant A_{i,j}$, a contradiction, since $k_2+1<\varrho_0$.

Now suppose f is in

$$\bigvee_{\substack{0\leqslant k<\gamma \\ k\neq j}} (A_{i',j}\wedge A_{i',k}\wedge B_{j,k}).$$

Then there is a k' with $0\leqslant k'<\gamma$, $k'\neq j$, such that f is in $A_{i',j}\wedge A_{i',k'}\wedge B_{j,k'}$. Since i was minimal and $i'<i$, there exist ϱ'', $\varrho'''<\varrho_0$ such that $\bigcap_{k<\varrho''} x_{k,y_k}\leqslant A_{i',j}$ and $\bigcap_{k<\varrho'''} x_{k,y_k}\leqslant A_{i',k'}$. Since f is in $B_{j,k'}$, $f(\gamma\cdot j+k')=1$. Thus there exists a $\lambda<\gamma_3$ such that $r_\lambda=j$, $s_\lambda=k'$.

Then B_{r_λ, s_λ} occurs as a term in the representation (in the form (2), with condition C) of x_{k,y_k} for some $k < \varrho_0$, say $k = k_3$. But then

$$\bigcap_{k<\max(\varrho'',\varrho''',k_3)+1} x_{k,y_k} \leqslant (A_{i',j} \wedge A_{i',k'} \wedge B_{j,k'}) \leqslant A_{i,j} ,$$

a contradiction, since $\max(\varrho'', \varrho''', k_3)+1 < \varrho_0$.

Thus our original assumption, that $F_{\gamma,a}$ is not (γ, ∞) distributive, has led to a contradiction, and Theorem 2 is proved.

THEOREM 3. *If γ is an infinite regular cardinal, then there does not exist a free complete (γ, ∞) distributive Boolean algebra on γ complete generators.*

Proof. Theorem 3 follows from the remarks preceding Theorem 2 and Theorem 2 itself.

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On the Lebesgue measurability and the axiom of determinateness

by

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It is the purpose of this paper to show that the axiom of determinateness (A) (see [2], [3]) implies that all linear sets are Lebesgue measurable. We will use (A) in the following form: *every infinite positional game with perfect information and a denumerable set of positions is determined.* (Let us recall another form (see [2]) which does not use notions of the theory of games: *for every set P of sequences of natural numbers there exists a function f defined on all finite (or empty) sequences of natural numbers, taking natural values and such that for every sequence $n_1, n_2, \ldots$*

$$\big(n_1, f(n_1), n_2, f(n_1, n_2), n_3, f(n_1, n_2, n_3), \ldots\big) \in P$$

or for every sequence $n_1, n_2, \ldots$

$$\big(f(\emptyset), n_1, f(n_1), n_2, f(n_1, n_2), n_3, \ldots\big) \notin P.\,)$$

(A) implies also the property of Baire of every linear set (see [2]) and the proof of the result of this paper, although more complicated, is based on an analogous idea as the proof of this fact. Let us mention that the development of the theory of measure, e.g. the denumerable additivity, is based on a weak form of the axiom of choice which is a consequence of (A) (see [2], prop. $\mathcal{C}$). Of course our result could be formulated as follows: the existence of a non-measurable set implies the existence of non-determined games of the prescribed form (and the existence of sets P without the above mentioned property). Clearly the axiom of choice is not used in this paper.

1. THEOREM. (A) *implies the Lebesgue measurability of every linear set* (1).

First we note that it is enough to show for every subset X of the closed interval $\langle 0, 1\rangle$ the following proposition:

(P) $\qquad$ (A) *implies* $|X|_i > 0$ *or* $|cX|_i > 0$. (2)

(1) For a generalization of this result, see section 2.

(2) $|\cdot|_i$ denotes the interior measure and cX the complement of X in $\langle 0, 1\rangle$.