$M/K-\pi(V)$ and $A \cup B_a$ is connected. Since $\nu$ is open on $(M-\pi^{-1}(W))/K_1$, there is a continuum containing each point $a' \in (\nu|T)^{-1}(a)$ and covering $B_a$. Let $B_1$ be the union of all such continua over all $a \in A \cap (M/K-W)$. The set $B_1$ is closed, for if $b \in B_1^-$, there is a net $\tilde{b} \to b$ where $\tilde{b}(i) \in B'_{\tilde{a}(i)}$, $B'_{\tilde{a}(i)}$ a continua covering $B_{\tilde{a}(i)}$, where $\tilde{a}(i)$ is a point in $A \cap (M/K-W)$. In the space of compact subsets of $(M-\pi^{-1}(W))/K_1$, there is a subnet $B'_{\tilde{a}\cdot\alpha}$ converging to a continua $B'$, and then $b \in B'$. Choose a subnet $\tilde{a} \circ \alpha \circ \beta$ converging to a point $a \in A \cap (M/K-W)$, and let $a' \in \pi^{-1}(a) \cap B'$. Then $B'$ is contained in the component of $a'$ in $(M-\pi^{-1}(W))/K_1$ and this set maps under $\nu$ onto $B_a$. Hence, $b \in B_1$. Since $A_1 = (\nu|T)^{-1}(A)$ is connected, $A_1 \cup B_1$ is then connected and meets $(M-V)/K_1$. Let $M_1 = \pi_1^{-1}(A_1 \cup B_1)$. Then $M_1 \subset M$, $M_1 \cap V \neq \emptyset$, $p \in M_1 \cap G(p) = K_1(p)$, and $M_1/K_1$ is connected. We must show that (*) is satisfied with respect to $M_1 \cap V$.

Let $Z$ be any neighborhood of $\pi_1(p)$ in $\pi_1(M_1)$. Choose a neighborhood $Y \subset W$ such that $(\nu|T)^{-1}(Y \cap \pi(M_1)) \subset T \cap Z$ and $Y$ satisfies (*) relative to $W$. Let $C$ be the component of $\pi(p)$ in $Y^-$, and $c \in C \cap (M/K-Y)$. By (*) the component of $c$ in $M/K-Y$ meets $M/K-\pi(V)$ and in particular the component $C'$ of $c$ in $W^- - Y$ meets $U^- - W$. Hence $C \cup C'$ is connected and therefore contained in $A$. Thus $(\nu|T)^{-1}(C)$ is connected and contained in the component of $Z$. Also, $(\nu|T)^{-1}(C \cup C')$ is connected and meets $\nu^{-1}(A \cap (U^- - W))$ and so meets $B_1$ whose every component meets $\pi_1(M_1 - V)$, which then proves (*).

In order to prove the theorem, we observe that $(G(P), G)$ satisfies the conditions of the new lemma 2 and so starts the induction. The rest of the proof remains unchanged.

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# Subdirect representations of relational systems

by

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**1. Introduction.** The representation of algebras as direct or subdirect products has been extensively investigated by Birkhoff [3], Hashimoto [7], Krull [9], and many others. Birkhoff establishes a necessary and sufficient condition for a universal algebra to be representable as a subdirect product. In this paper we give a generalization of the concept of subdirect product suitable for relational systems, and obtain representation conditions containing Birkhoff's result as a special case.

Adam [1] presents a counter example of L. Fuchs and G. Szász showing the invalidity of Birkhoff's conditions for an algebra to be represented as a direct product of finitely many algebras. We append to the subdirect representation conditions a third condition and prove the set both necessary and sufficient for the representation of a relational system as a direct product of finitely or infinitely many factors. This theorem is similar to a theorem of Hashimoto on the infinite direct product representations of an algebra.

Birkhoff also establishes the following representability theorem:

*Every algebra is representable as a subdirect product of subdirectly irreducible algebras.*

In section 5 we present a representability theorem for relational systems which yields Birkhoff's result as one special case, but also other more precise specializations to the algebraic case. Hindsight shows these specializations could have been obtained directly for algebras by Birkhoff's arguments alone.

Lyndon [10] and Pickert [12] have also generalized the concept of subdirect product to relational systems. The definition used by Pickert is slightly weaker than the definition presented here, and yields Birkhoff's condition for a relational system to be represented as a subdirect product. Pickert gives no direct product representation conditions. But the unity given to the two representation theorems by our definition makes it seem more natural. Pickert also fails to establish a significant subdirect product representability theorem, but it is clear that such a theorem could be established using his definition, following the development which we present. Lyndon's generalization seems both unnatural and unduly restrictive and the results obtained do not significantly parallel the material presented here.

**2. Preliminary notions and notations.** Let $X$ be a set of any cardinality. The set of all ordered $n$-tuples, $\{(x_1, \ldots, x_n)|$ each $x_i \in X\}$, is denoted by $X^n$ for finite $n$. An element $(x_1, \ldots, x_n)$ of $X^n$ is a *vector*, and will usually be denoted by $\bar{x}$. Its *rank* is $n$. The $j$th component of the subscribed vector $\bar{x}_i$ is denoted by $x_{ij}$, the lack of a bar signifying it is a component. A *relation* $R$, of rank $n$, is a subset of $X^n$. The null set $\emptyset$ is a relation, but is not assigned a rank. If $X_1 \subseteq X$, a relation $R$ is *on* $X_1$ iff $R \subseteq X_1^n$. The *restriction* of a relation $R$, of rank $n$, to $X_1$ is $R|X_1 = R \cap X_1^n$.

For $T$ a relation of rank 2, and $\bar{a}$, $\bar{b}$ vectors of rank $n$, the notation $(\bar{a}, \bar{b}) \in T$ indicates that $(a_i, b_i) \in T$ for each $i = 1, \ldots, n$. $(\bar{a}, \bar{b}) \notin T$ means some $(a_i, b_i) \notin T$. The *transform* of a relation $R$ by $T$ is the relation $R * T = \{\bar{b}|$ for some $\bar{a}$, $\bar{a} \in R$ and $(\bar{a}, \bar{b}) \in T\}$. A relation $T$ of rank 2 is *one-one* between $X_1 \subseteq X$ and $X_2 \subseteq X$ iff for each $x_1 \in X_1$ there is a unique $x_2 \in X_2$ such that $(x_1, x_2) \in T$, and conversely. Relations $R_1$ on $X_1 \subseteq X$ and $R_2$ on $X_2 \subseteq X$ are isomorphic iff there is a one-one relation $T$ between $X_1$ and $X_2$ such that $R_1 * T = R_2$, and $R_2 * T' = R_1$, where $T' = \{(x, y)| (y, x) \in T\}$. We will use the notations $X_1 \sim X_2$, $R_1 \sim R_2$, and infer the existence of $T$. For $T$ and $S$ both relations of rank 2, $T \circ S = \{(x, z)|$ for some $y$, $(x, y) \in T$ and $(y, z) \in S\}$ is the usual composition.

A relational *system* $\mathfrak{R} = (X, R_1, R_2, \ldots)$ is a sequence in which the first element is a set, called the *space of* $\mathfrak{R}$, and the succeeding elements are relations on $X$. The sequence $(n_1, n_2, \ldots)$ of ranks of the relations $R_1, R_2, \ldots$ is the *order type* of $\mathfrak{R}$. If each $n_i$ is finite, $\mathfrak{R}$ is finitary. With one exception, we shall discuss only finitary systems. A *universal algebra*, or *operational system*, is a relational system of order type $(n_1+1, n_2+1, \ldots)$ such that each relation $R_i$ has the property that for each sequence $x_1, \ldots, x_{n_i}$ from $X$ there is a unique $y$ in $X$ such that $(x_1, \ldots, x_{n_i}, y) \in R_i$. A *subsystem*, $\mathcal{S}$ of $\mathfrak{R}$, is a system $(Y, S_1, S_2, \ldots)$ where $Y \subseteq X$, and each $S_i = R_i|Y$. A subsystem of an algebra, itself an algebra, is a *subalgebra*. We shall have occasion to refer to various spaces $X_1, X_2$, etc., and adopt the convention that these spaces, together with all factor spaces, product spaces, etc., which we may generate, lie embedded in an unnamed overspace, preserving $X$ as a name for a subspace.

**3. On equivalence and factor relations.** An *equivalence relation* $E$ on $X$ is a relation of rank 2 satisfying, for every $x, y, z$ in $X$, (1) $(x, x) \in E$, (2) $(x, y) \in E$ implies $(y, x) \in E$, and (3) $(x, y) \in E$ and $(y, z) \in E$ imply $(x, z) \in E$. The relations $U = X^2$ and $I = \{(x, x)| x \in X\}$ are equivalence relations on $X$. A partition of $X$ is a family of nonempty, disjoint subsets of $X$, called *blocks*, whose union is $X$. It is well known that for any set $X$ there is a one-one correspondence between the family of equivalence relations on $X$ and the family of partitions of $X$. For a given relation $E$, and corresponding partition $P$, $(x, y) \in E$ iff $x$ and $y$ belong to the same block of $P$. We use the notation $X/E$ to stand for the partition of $X$ determined by $E$, and call it the *factor set* of $X$ with respect to $E$. Moreover, $x/E$ will denote the block of $X/E$ to which $x$ belongs, and for any vector $\bar{x} = (x_1, \ldots, x_n)$, $\bar{x}/E$ denotes the vector $(x_1/E, \ldots, x_n/E)$. For any relation $R$ on $X$, the relation $R/E = \{\bar{x}/E|\ \bar{x} \in R\}$ is called the *factor relation* of $R$ with respect to $E$.



If $\mathcal{E}$ is a class of equivalence relations on $X$ and $M \in \mathcal{E}$, the *factor class* of $\mathcal{E}$ with respect to $M$ is $\mathcal{E}/M = \{E/M|\ E \in \mathcal{E}$ and $E \supseteq M\}$. The use of this terminology is justified by the observation that $\mathcal{E}/M$ is nonempty, and the following lemma. The simple proof is omitted.

LEMMA. 1. *Each $E/M$ in $\mathcal{E}/M$ is an equivalence relation on $X/M$.*

2. *For each $E'$ in $\mathcal{E}/M$ there is a unique $E$ in $\mathcal{E}$ such that $E \supseteq M$, and $E' = E/M$.*

We will use later the following theorem, whose prototype is the Second Law of Isomorphism of group theory.

THEOREM 1. *Let $M$, $N$ be equivalence relations on $X$, with $M \supseteq N$. Then if $R$ is any relation on $X$, $R/M \sim (R/N)/(M/N)$. In particular, $X/M \sim (X/N)/(M/N)$.*

Proof. Each equivalence relation $E$ on a space $X$ defines a unique function, the *partition map* of $X$ with respect to $E$, which takes $X$ into $X/E$, and is $\{(x, x/E)|\ x \in X\}$. Let $F$, $G$, $H$ be the partition maps defined

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "H"'] & X/N \arrow[d, "G"] \\ X/M \arrow[r, "K"'] & (X/N)/(M/N) \end{tikzcd}$$

in the accompanying diagram and let $K = H' \circ F \circ G$. We denote elements in $X$ by $x, y$, etc., those of $X/M$ by $x^1, y^1$, etc., those of $X/N$ by $x_1, y_1$, etc., and finally those of $(X/N)/(M/N)$ by $x_2, y_2$, etc. Suppose $\bar{x}^1 \in R/M$. Then $\bar{x} \in \{\bar{x}^1\} * H'$ implies for some $\bar{u} \in R$ that $\bar{u} \in \{\bar{x}^1\} * H'$ and $(\bar{x}, \bar{u}) \in M$. We infer that $\bar{u}_1 = \bar{u}/N \in R/N$ and $(\bar{x}_1, \bar{u}_1) \in M/N$, where $\bar{x}_1 = \bar{x}/N$, and consequently for $\bar{x}_2 = \bar{x}_1/(M/N)$ and $\bar{u}_2 = \bar{u}_1/(M/N)$ that $\bar{x}_2 = \bar{u}_2$. Hence the map $K$ determines a unique image of $\bar{x}^1$ in $(R/N)/(M/N)$. Conversely, suppose $\bar{x}_2 \in (R/N)/(M/N)$. Then $\bar{x}_1 \in \{\bar{x}_2\} * G'$ implies for some $\bar{u}_1 \in R/N$ that $\bar{u}_1 \in \{\bar{x}_2\} * G'$ and $(\bar{x}_1, \bar{u}_1) \in M/N$. Let $\bar{x} \in \{\bar{x}_1\} * F'$. Then for some $\bar{u} \in \{\bar{u}_1\} * F'$, for some $\bar{y} \in \{\bar{x}_1\} * F'$, for some $\bar{z} \in \{\bar{u}_1\} * F'$, $\bar{u} \in R$

and $(\bar{x},\bar{y}) \in N \subseteq M$ and $(\bar{y},\bar{z}) \in M$ and $(\bar{z},\bar{u}) \in N \subseteq M$, and hence $(\bar{x},\bar{u}) \in M$. Setting $\bar{x}^1 = \bar{x}/M$ and $\bar{u}^1 = \bar{u}/M$ we conclude that $\bar{x}^1 = \bar{u}^1$, and therefore that $\bar{x}^1$ is the unique image of $\bar{x}_2$ under the map $K' = G' \circ F' \circ H$.

**4. On subdirect and direct products of relational systems.** By the *direct product* of a family of sets $X_\alpha$, $\alpha \in A$, is meant the set $X = \prod_\alpha X_\alpha$ of all functions $x$ on $A$ such that $x(\alpha) \in X_\alpha$, $\alpha \in A$. The direct product of a family of vectors $\bar{x}_\alpha \in X_\alpha^n$, $\alpha \in A$, is the vector $\bar{x} = \prod_\alpha \bar{x}_\alpha \in X^n$ whose $i$th component $x_i$ is the function satisfying $x_i(\alpha) = x_{\alpha i}$, $\alpha \in A$. The direct product of a family of relations, $R_\alpha \subseteq X_\alpha^n$, $\alpha \in A$, is the relation $R = \{\prod_\alpha \bar{x}_\alpha |\ \bar{x}_\alpha \in R_\alpha,\ \alpha \in A\} = \prod_\alpha R_\alpha \subseteq X^n$. A relation $R$ on a set $X$ is representable as a direct product of a family $R_\alpha$ on $X_\alpha$, $\alpha \in A$, iff there is a one-one mapping $T$ between $X$ and $\prod_\alpha X_\alpha$ such that $R \sim \prod_\alpha R_\alpha$. $R$ is *representable in a class* $\mathcal{E}$ of equivalence relations on $X$ iff there is a family $E_\alpha$, $\alpha \in A$, contained in $\mathcal{E}$ such that $X \sim \prod_\alpha (X/E_\alpha)$, $R \sim \prod_\alpha (R/E_\alpha)$. $X$ is a *subdirect product* of a family of sets $X_\alpha$, $\alpha \in A$, iff $X \subseteq \prod_\alpha X_\alpha$ and $X * P_\alpha = X_\alpha$, $\alpha \in A$, where $P_\alpha = \{(x, x(\alpha))|\ x \in \prod_\alpha X_\alpha\}$ is the $\alpha$th *projection map*. A relation $R \subseteq \prod_\alpha R_\alpha$, $R_\alpha$ on $X_\alpha$, $\alpha \in A$, is a subdirect product of the $R_\alpha$ iff there is a set $Y \subseteq \prod_\alpha X_\alpha$ such that (1) $R = (\prod_\alpha R_\alpha)|\ Y$, (2) $Y * P_\alpha = X_\alpha$, and (3) $R * P_\alpha = R_\alpha$, $\alpha \in A$. A relation $R$ on a set $X$ is *subdirectly representable* as a subdirect product of the family $R_\alpha$, $\alpha \in A$, iff $R \sim (\prod_\alpha R_\alpha)|\ Y$ and $X \sim Y$, where $(\prod_\alpha R_\alpha)|\ Y$ is a subdirect product. We write $X \simeq \prod_\alpha X_\alpha$, $R \simeq \prod_\alpha R_\alpha$. $R$ is *subdirectly representable in a class* $\mathcal{E}$ of equivalence relations on $X$ iff there is a family $E_\alpha$, $\alpha \in A$, contained in $\mathcal{E}$ such that $X \simeq \prod_\alpha (X/E_\alpha)$, $R \simeq \prod_\alpha (R/E_\alpha)$, $\alpha \in A$.

The above definitions require that both the relation $R$ and the set $X$ be represented as a product of corresponding factors. Thus, we are asking that the relational *system* $(X, R)$ be represented as a product of the relational *systems* $(X_\alpha, R_\alpha)$. Pickert's definition of a subdirect product of relational systems replaces condition (1) above by the weaker condition (1') $R \subseteq (\prod_\alpha R_\alpha)|\ Y$. However, it seems more natural to require a subdirect product to be a subsystem of the direct product system, as in the algebraic case. With the condition (1') theorem 2, below, which is a special case of Birkhoff's [3] theorem 9, p. 92, provides a necessary and sufficient condition for a relational system to be represented as a subdirect product.



THEOREM 2. (Birkhoff) *The representations of a set $X$ as a subdirect product correspond one-one to the families $E_\alpha$, $\alpha \in A$, of equivalence relations on $X$ satisfying the condition:*

$$\text{(C1)} \qquad \bigcap_\alpha E_\alpha = I\,.$$

Proof. Suppose $X \simeq \prod_\alpha X_\alpha$. Then there is a one-one map $K$ between $X$ and a subset $Y \subseteq \prod_\alpha X_\alpha$, and the mapping $K_\alpha = K \circ P_\alpha$ taking $X$ onto $X_\alpha$ is many-one for each $\alpha \in A$. For each $\alpha$, define $E_\alpha$ to be $\{(x, y)|\ xK_\alpha = yK_\alpha\}$. If $(x, y) \in \bigcap_\alpha E_\alpha$, then $(xK)P_\alpha = (yK)P_\alpha$ for each $\alpha$, so $xK = yK$ and hence $(x, y) \in I$. Conversely, let $E_\alpha$, $\alpha \in A$, be a family of equivalence relations on $X$ satisfying (C1), and let $X_\alpha = X/E_\alpha$, $\alpha \in A$. Let $M$ be the map carrying each $x \in X$ into the element $y \in \prod_\alpha X_\alpha$ such that $y(\alpha) = x/E_\alpha$, $\alpha \in A$. $M$ is many-one from $X$ to $X * M = Y \subseteq \prod_\alpha X_\alpha$ and, since (C1) is satisfied, $x_1 M = x_2 M$ implies that $x_1/E_\alpha = x_2/E_\alpha$, $\alpha \in A$, and thus that $x_1 = x_2$. Hence $M$ is one-one from $X$ onto $Y$, and $Y * P_\alpha = X_\alpha$, $\alpha \in A$.

The following theorem gives necessary and sufficient conditions for the stronger requirement (1) to be met [1].

THEOREM 3. (Subdirect representation theorem for relations). *The representations of a relation $R$ on $X$ as a subdirect product correspond one-one to the families $E_\alpha$, $\alpha \in A$, of equivalence relations on $X$ satisfying* (C1) *of theorem* 2 *and the further condition*:

$$\text{(C2)} \qquad R = \bigcap_\alpha (R * E_\alpha)\,.$$

Proof. (a) Suppose $X \simeq \prod_\alpha X_\alpha$, $R \simeq \prod_\alpha R_\alpha$, $\alpha \in A$. Denote by $M_\alpha$ the map which takes each $x \in X$ into the $\alpha$th component of its unique representative in $Y \subseteq \prod_\alpha X_\alpha$. By definition, $R * M_\alpha = R_\alpha$, $\alpha \in A$. For each $\alpha$, define $E_\alpha = \{(x, y)|\, xM_\alpha = yM_\alpha\}$. By theorem 2, condition (C1) holds for these equivalence relations. To prove (C2) we first observe that $R \subseteq \bigcap_\alpha (R * E_\alpha)$, since $R * E_\alpha \supseteq R * I = R$. Therefore, suppose $\bar{u} \in \bigcap_\alpha (R * E_\alpha)$.

[1] The referee has kindly brought to my attention the paper *On extending of models. V* by J. Łoś, J. Słomiński and R. Suszko, Fund. Math. 48 (1960), pp. 113-121. Theorem 4 of their paper is equivalent to theorem 3 below.

Then for each $\alpha$, for some $\bar{x}_\alpha \in R$, $(\bar{u}, \bar{x}_\alpha) \in E_\alpha$, and hence $\bar{u}M_\alpha = \bar{x}_\alpha M_\alpha \in R_\alpha$, where $\bar{x}M_\alpha$ indicates the unique element of $\{\bar{x}\} * M_\alpha$. This, since $\bar{u} \in X^n$, and therefore $\prod_\alpha \bar{u}M_\alpha \in Y^n$, implies $\prod_\alpha \bar{u}M_\alpha \in (\prod_\alpha R_\alpha)| Y$, and thus $\bar{u} \in R$.

(b) Conversely, suppose that $E_\alpha$, $\alpha \in A$, is a family of equivalence relations on $X$ satisfying (C1) and (C2), and set $X_\alpha = X/E_\alpha$, $R_\alpha = R/E_\alpha$, $\alpha \in A$. By theorem 2, $X \sim Y \subseteq \prod_\alpha X_\alpha$. As in part (a), let $M_\alpha$ be the map taking each $x \in X$ into the $\alpha$th component of its unique representative in $Y$. Then clearly $R * M_\alpha = R_\alpha$, for each $\alpha \in A$. Let $\bar{x} \in R$. Then $\prod_\alpha \bar{x} M_\alpha \in (\prod_\alpha R_\alpha)| Y$. Conversely, let $\prod_\alpha \bar{u}_\alpha \in (\prod_\alpha R_\alpha)| Y$, and let $\bar{u}$ be that unique member of $X^n$ such that for each $\alpha$, $\bar{u}M_\alpha = \bar{u}_\alpha$. Then for each $\alpha$, for some $\bar{x}_\alpha \in R$, $\bar{u}M_\alpha = \bar{x}_\alpha M_\alpha$, and hence $(\bar{u}, \bar{x}_\alpha) \in E_\alpha$. Since $R = \bigcap_\alpha (R * E_\alpha)$, it follows that $\bar{u} \in R$.

COROLLARY. *The subdirect representations of $R$ in a class $\mathcal{E}$ of equivalence relations on $X$ correspond one-one to the families $E_\alpha$, $\alpha \in A$, contained in $\mathcal{E}$ satisfying* (C1) *and* (C2) *above.*

The conditions of the following theorem are implicit in Adam [1], where the deficiency of the theorem of Birkhoff for the finite case is pointed out.

THEOREM 4. *The representations of a set $X$ as a direct product correspond one-one with the sets $E_\alpha$, $\alpha \in A$, of equivalence relations on $X$ satisfying condition* (C1) *of theorem 2 and the condition*:

(C3) *for each set $\{x_\alpha |\ x_\alpha \in X,\ \alpha \in A\}$ there exists an element $x$ in $X$ such that $(x, x_\alpha) \in E_\alpha$, $\alpha \in A$.*

(*Equivalently,* (C3) *says that the system of congruences, $x \equiv x_\alpha\ (E_\alpha)$ can always be solved.*)

Proof. Suppose $X = \prod_\alpha X_\alpha$ and, as in the proof of theorem 2, set $E_\alpha = \{(x, y) |\ xP_\alpha = yP_\alpha\}$, $\alpha \in A$. (C1) must hold, since a direct product of sets is a subdirect product. If $\{x_\alpha |\ \alpha \in A\}$ is contained in $X$, let $y$ be a member of $X$ such that $y(\alpha) = x_\alpha(\alpha)$, $\alpha \in A$. Then $x_\alpha P_\alpha = yP_\alpha$, or equivalently $(x_\alpha, y) \in E_\alpha$, for each $\alpha \in A$, and hence (C3) is satisfied. If, conversely, $E_\alpha$, $\alpha \in A$, is a family of equivalence relations on $X$ satisfying (C1) and (C3), there is by theorem 2 a one-one map $M$ taking $X$ onto a set $Y \subseteq \prod_\alpha X_\alpha$. $xM = y$ iff $y(\alpha) = x/E_\alpha$, $\alpha \in A$. Let $u \in \prod_\alpha X_\alpha$, so that $u(\alpha) = u_\alpha/E_\alpha$, where $u_\alpha \in X$, $\alpha \in A$. By (C3), for some $x \in X$, $x/E_\alpha = u_\alpha/E_\alpha$, $\alpha \in A$, and hence $u = xM \in Y$.

COROLLARY. *The representations of a relation $R$ on $X$ as a direct product correspond one-one to the families $E_\alpha$, $\alpha \in A$, of equivalence relations on $X$ satisfying conditions* (C1), (C2), *and* (C3) *above.*



Proof. A direct product is the subdirect product obtained by restriction to the whole product space.

COROLLARY. *The representations of $R$ in a class $\mathcal{E}$ of equivalence relations on $X$ correspond one-one to the families $E_\alpha$, $\alpha \in A$, contained in $\mathcal{E}$ satisfying conditions* (C1), (C2), *and* (C3) *above.*

We remark that if $(X, R)$ is an algebra and $E_\alpha$, $\alpha \in A$, is a family of congruence relations for $(X, R)$, then (C1) implies (C2), for if $(x_1, \ldots, x_n, y) \in R$ and $(x_1, \ldots, x_n, z) \in \bigcap_\alpha (R * E_\alpha)$, then $(y, z) \in E_\alpha$, $\alpha \in A$, which implies $(x_1, \ldots, x_n, z) \in R$.

**5. The subdirect representability theorem.** In view of the results of section 4, we are free to make the following definition, in which $\mathcal{E}^0$ is the class $\mathcal{E}$ with $I$ excluded. We say $X$ is *subdirectly irreducible* in a class $\mathcal{E}$ of equivalence relations on $X$ iff (C1) holds for no set of equivalence relations in $\mathcal{E}^0$, and hence in particular if it does not hold for the set of all equivalence relations in $\mathcal{E}^0$. Similarly, $R$ is subdirectly irreducible in $\mathcal{E}$ iff (C1) and (C2) do not simultaneously hold for $\mathcal{E}^0$.

THEOREM 5. (The subdirect product representability theorem.) *Let $X$ be a set, $R$ a relation of finite rank on $X$, and $\mathcal{E}$ a family of equivalence relations on $X$ satisfying* (1) *$I \in \mathcal{E}$, and* (2) *the union of the members of any nest* [2] *in $\mathcal{E}$ is a member of $\mathcal{E}$. Then $R$ may be represented in $\mathcal{E}$ as a subdirect product of factors, each of which is subdirectly irreducible in the corresponding factor family of equivalence relations. More generally, any finitary relational system whatsoever may be so represented.*

Proof. For each $a, b$ in $X$, $a \neq b$, let $\mathfrak{L}(a, b)$ be the family of all members of $\mathcal{E}$ which do not hold $(a, b)$. $\mathfrak{L}(a, b)$ is not empty since $I$ does not hold $(a, b)$. Let $\mathcal{N}$ be any nest in $\mathfrak{L}(a, b)$, and $N$ the union of all members of $\mathcal{N}$. $N$ is in $\mathcal{E}$, and $(a, b)$ is not in $N$ so $N$ is in $\mathfrak{L}(a, b)$. By Zorn's lemma there is a maximal element, say $L(a, b)$, in $\mathfrak{L}(a, b)$. Clearly, $\bigcap_{a \neq b} L(a, b) = I$. Since any member of $\mathcal{E}$ which properly contains $L(a, b)$ must hold $(a, b)$, $\bigcap [E/L(a, b)]$, the intersection being taken over all $E$ in $\mathcal{E}$ which properly contain $L(a, b)$, properly contains $I = L(a, b)/L(a, b)$. Hence $X/L(a, b)$ is irreducible in $\mathcal{E}/L(a, b)$.

Next, for a given $R$ in the relational system, and $\bar{a} \notin R$, let $\mathcal{M}(\bar{a})$ be the family of all members $E$ of $\mathcal{E}$ satisfying the condition

$$\text{(C)} \qquad \bar{x} \in R \quad \textit{implies} \quad (\bar{x}, \bar{a}) \notin E .$$

$\mathcal{M}(\bar{a})$ is not empty, since $I$ satisfies this condition. Let $\mathcal{N}$ be any nest in $\mathcal{M}(\bar{a})$ and $N$ the union of the members of $\mathcal{N}$. $N$ is in $\mathcal{E}$, and for each

[2] A *nest* is a family of sets simply ordered by $\subseteq$.

$\bar{x} \in R$, $(\bar{x}, \bar{a}) \notin N$, for if so then for some $N_1, N_2, \ldots, N_n$ in $\mathcal{N}$, where $n$ is the rank of $R$, $(x_i, a_i) \in N_i$, for each $i = 1, 2, \ldots, n$. Since one of them contains the others, $(\bar{x}, \bar{a}) \in K$ for some $K$ in $\mathcal{N}$. This contradiction establishes that $N$ is in $\mathcal{M}(\bar{a})$. By Zorn's lemma, $\mathcal{M}(\bar{a})$ has a maximal member, say $M(\bar{a})$.

Since $\bar{a} \notin R$, for no $\bar{x} \in R$ is $(\bar{x}, \bar{a}) \in M(\bar{a})$. Hence $\bar{a} \notin R * M(\bar{a})$. It follows that $R \supseteq \bigcap_{\bar{a} \notin R} [R * M(\bar{a})]$. Since always $R \subseteq \bigcap_{\bar{a} \notin R} [R * M(\bar{a})]$ we conclude that $R = \bigcap_{\bar{a} \notin R} [R * M(\bar{a})]$. Now for each $\bar{a} \notin R$, $\bar{a}/M(\bar{a}) \notin R/M(\bar{a})$, for this would imply that $(\bar{x}, \bar{a}) \in M(\bar{a})$ for some $\bar{x} \in R$. Because of the maximal character of $M(\bar{a})$ any member $E$ of $\mathcal{E}$ properly containing $M(\bar{a})$ has the property that for some $\bar{x} \in R$, $(\bar{x}, \bar{a}) \in E$, from which we conclude that $\bar{a}/M(\bar{a}) \in [R/M(\bar{a})] * [E/M(\bar{a})]$, i.e., that $R/M(\bar{a}) \neq \bigcap ([R/M(\bar{a})] * [E/M(\bar{a})])$, where the intersection runs over all members of $\mathcal{E}$ properly containing $M(\bar{a})$. Hence $R/M(\bar{a})$ is strictly subdirectly irreducible in $\mathcal{E}/M(\bar{a})$.

Let $\mathcal{F}$ be the family of all $L(a, b)$, $a \neq b$, and all $M(\bar{a})$, $\bar{a} \notin R$, $R$ in the relational system. It is clear that (1) $\bigcap F = I$, $F$ in $\mathcal{F}$, and (2) $\bigcap (R * F) = R$, $R$ in the relational system and $F$ in $\mathcal{F}$. Since by the corollary to theorem 3 these are necessary and sufficient conditions that each $R$ in the relational system be subdirectly represented in $\mathcal{E}$, and since the representation is irreducible, the theorem is proved.

With suitable restrictions on $\mathcal{E}$ the above theorem may be extended to nonfinitary systems as well. This is our only contact with nonfinitary systems.

THEOREM 6. *Let $X$ be a set, $\mathfrak{R} = (X, R_1, R_2, \ldots)$ any relational system whatsoever, and $\mathcal{E}$ a family of equivalence relations on $X$ satisfying* (1) $I \in \mathcal{E}$, *and* (2) *the ascending chain condition holds in $\mathcal{E}$. Then $\mathfrak{R}$ may be represented in $\mathcal{E}$ as a subdirect product of factors, each of which is subdirectly irreducible in the corresponding factor family of equivalence relations.*

Proof. We first observe that theorem 3 applies equally as well to non-finitary systems. The proof is then identical to the proof of theorem 5, except one has directly that for a nest $\mathcal{N}$ in $\mathcal{M}(\bar{a})$, the union $N$, of the members of $\mathcal{N}$, is a member of $\mathcal{N}$, and hence of $\mathcal{M}(\bar{a})$.

**6. On sentences possessing the nesting property.** The usefulness of theorem 5 comes in part from the following considerations. In representing an algebraic or relational system satisfying certain axioms as a subdirect product it is natural to desire the factors to be systems of the same kind; that is, to satisfy all of the axioms specified for the original system [3]. We will call a subdirect representation of

[3] I am indebted to Professor Bjarni Jonsson for the above remark, and for the application to partially ordered sets given in section 7.



a system a *proper* subdirect product if all of the factors are systems of the same kind, and we will say that a system is *properly* subdirectly irreducible if its only representations as a subdirect product of systems of the same kind are trivial, i.e., some factor of the representation is isomorphic to the original system.

Let $\mathfrak{R} = (X, R_1, R_2, \ldots)$ be a finitary relational system and let $S(R_1, \ldots, R_n; x_1, \ldots, x_m)$ be a finite first-order sentential function with unbound variables $x_1, \ldots, x_m$, in which the equality relation $I$, together with the primitive relations of $\mathfrak{R}$, may appear as constants in elementary positive sentences of the form "$\bar{x} \in R$", joined by the logical connectives $\bigwedge$, $\bigvee$, $\wedge$, $\vee$, $\rightarrow$, and $\sim$ [4]. An equivalence relation $E$ on $X$ *preserves* $S$ into the factor system $\mathfrak{R}/E$ providing the truth of $S(R_1, \ldots, R_n; x_1, \ldots, x_m)$, for a specific occurrence of the unbound variables $x_1, \ldots, x_m$, implies the truth of $S(R_1/E, \ldots, R_n/E; x_1/E, \ldots, x_m/E)$, that is, the sentential function obtained from $S$ by replacing each occurrence of a primitive relation $R$ by the relation $R/E$, each occurrence of an unbound variable $x$ by $x/E$, and changing the range of the bound variables from $X$ to $X/E$. Furthermore, $S$ has the *nesting property* iff for every nest $\mathcal{N}$ of equivalence relations on $X$, if $S(R_1/E, \ldots, R_n/E; x_1/E, \ldots, x_m/E)$ is true for each $E \in \mathcal{N}$, then $S(R_1/N, \ldots, R_n/N; x_1/N, \ldots, x_m/N)$ is also true, where $N = \sum \mathcal{N} (= \bigcup E, E \in \mathcal{N})$.

THEOREM 7. *A finitary relational system $\mathfrak{R} = (X, R_1, R_2, \ldots)$ which satisfies a (possibly infinite) set of finite first-order axioms, each possessing the nesting property, has a proper subdirect representation with properly subdirectly irreducible factors.*

Proof. Let $\mathcal{E}$ be the class of all equivalence relations on $X$ which preserve all of the axioms of the relational system $\mathfrak{R}$ into the corresponding factor systems. Then $I \in \mathcal{E}$, since $\mathfrak{R}$ satisfies all of the axioms, and if $\mathcal{N}$ is a nest in $\mathcal{E}$, then $\sum \mathcal{N} \in \mathcal{E}$. For let $\mathcal{E}_A$ be the family of equivalence relations preserving axiom A. Then $\mathcal{N} \subseteq \mathcal{E}_A$, and $\sum \mathcal{N} \in \mathcal{E}_A$, since A possesses the nesting property. This is true for each axiom A, so $\sum \mathcal{N} \in \mathcal{E} = \bigcap \mathcal{E}_A$. Thus each factor system is of the same kind, and $\mathcal{E}$ satisfies the two conditions of theorem 5. The factors of $\mathfrak{R}$ obtained by theorem 5 are not further reducible, for suppose some factor family $\mathcal{E}/E$ belonging to a factor could be augmented by an equivalence relation $\underline{D}$ on $X/E$ which preserves all of the axioms of $\mathfrak{R}$. Set $D = \{(x, y) | (x/E, y/E) \in \underline{D}\}$. $D$ is an equivalence relation on $X$, and $D \supseteq E$, so $D/E = \underline{D}$. By theorem 1, $R/D \sim (R/E)/(D/E) = (R/E)/\underline{D}$ for each primitive relation of $\mathfrak{R}$, and consequently any sentence $\underline{D}$ preserves is also preserved by $D$. Hence $\underline{D}$ is initially a member of $\mathcal{E}/E$.

[4] $\sim$ means negation when used within a sentential function.

A useful corollary of theorem 7 is that if the axioms of $\mathcal{R}$ can be separated into two sets, say $A$ and $B$, such that the axioms in $A$ have the nesting property, and such that each equivalence relation which preserves all of the axioms in $A$ also preserves all of the axioms in $B$, then the conclusion of theorem 7 still holds. For, the family of equivalence relations which preserve all of the axioms coincides with the family preserving the axioms in $A$.

Following theorem 8 we shall characterize recursively a large class of sentential functions which possess the nesting property. For conciseness we use $S$ or, to indicate a specific unbound variable, $S(x)$ as an abbreviation for the sentential function $S(R_1, \ldots, R_n;\ x_1, \ldots, x_m)$, and $S_E$ or $S(x)_E$ for the sentential function $S(R_1/E, \ldots, R_n/E;\ x_1/E, \ldots, x_m/E)$. The characterization is framed in terms of the following six properties, of which the third is the nesting property. $S$ has *nesting property* 1 (briefly, $\mathrm{np}_1(S)$) iff for every equivalence relation $E$ on $X$, and equivalence relation $D$ on $X$, $D \subseteq E$, if $S_E$ is true, then $S_D$ is also true. $S$ has *nesting property* 2 ($\mathrm{np}_2(S)$) iff for every nest $\mathcal{N}$ of equivalence relations on $X$, if for every $E \in \mathcal{N}$, there exists $D \in \mathcal{N}$, $D \supseteq E$, such that $S_D$ is true, then $S_{\sum\mathcal{N}}$ is also true, where, as above, $\sum\mathcal{N} = \bigcup E$, $E \in \mathcal{N}$. $S$ has *nesting property* 3 ($\mathrm{np}_3(S)$) iff for every nest $\mathcal{N}$ of equivalence relations on $X$, if for each $E \in \mathcal{N}$, $S_E$ is true, then $S_{\sum\mathcal{N}}$ is true. In an obviously dual fashion we say $S$ has the *reverse property* 1 ($\mathrm{rp}_1(S)$) iff for every equivalence relation $E$ on $X$, and equivalence relation $D$ on $X$, $D \supseteq E$, if $S_D$ is true, then also $S_E$ is true. $S$ has the *reverse property* 2 ($\mathrm{rp}_2(S)$) iff for every nest $\mathcal{N}$ of equivalence relations on $X$, if $S_{\sum\mathcal{N}}$ is true, then for some $E \in \mathcal{N}$, for every $D \in \mathcal{N}$, $D \supseteq E$, $S_D$ is true. $S$ has the *reverse property* 3 ($\mathrm{rp}_3(S)$) iff for every nest $\mathcal{N}$ of equivalence relations on $X$, if $S_{\sum\mathcal{N}}$ is true, then for some $E \in \mathcal{N}$, $S_E$ is true. Theorem 8 explores the interdependence of these six properties. It is understood that the subscript $i$ may have the values 1, 2, or 3.

THEOREM 8.

A. $\mathrm{np}_i(S)$ *is equivalent to* $\mathrm{rp}_i(\sim S)$.

B$_1$. a. $\mathrm{np}_1(S)$ *implies* $\mathrm{np}_2(S)$, *but not conversely.*
b. $\mathrm{np}_2(S)$ *is equivalent to* $\mathrm{np}_3(S)$.

C$_1$. *If $T$ is a logical consequence of $S$ then*
a. $\mathrm{np}_i(S)$ *implies* $\mathrm{np}_i(S \wedge T)$,
b. $\mathrm{np}_i(T)$ *implies* $\mathrm{np}_i(S \vee T)$.

D$_1$. a. $\mathrm{np}_i(S)$ *and* $\mathrm{np}_i(T)$ *imply* $\mathrm{np}_i(S \wedge T)$.
b. $\mathrm{np}_i(S)$ *and* $\mathrm{np}_i(T)$ *imply* $\mathrm{np}_i(S \vee T)$.
c. $\mathrm{np}_i\big(S(x)\big)$ *implies* $\mathrm{np}_i\big(\bigwedge x\, S(x)\big)$.
d. $\mathrm{np}_1\big(S(x)\big)$ *implies* $\mathrm{np}_1\big(\bigvee x\, S(x)\big)$.

E$_1$. $\mathrm{rp}_i(S)$ *and* $\mathrm{np}_i(T)$ *imply* $\mathrm{np}_i(S \to T)$.

B$_2$. a. $\mathrm{rp}_1(S)$ *implies* $\mathrm{rp}_2(S)$, *but not conversely.*
b. $\mathrm{rp}_2(S)$ *is equivalent to* $\mathrm{rp}_3(S)$.

C$_2$. *If $T$ is a logical consequence of $S$ then*
a. $\mathrm{rp}_i(S)$ *implies* $\mathrm{rp}_i(S \wedge T)$,
b. $\mathrm{rp}_i(T)$ *implies* $\mathrm{rp}_i(S \vee T)$.

D$_2$. a. $\mathrm{rp}_i(S)$ *and* $\mathrm{rp}_i(T)$ *imply* $\mathrm{rp}_i(S \wedge T)$.
b. $\mathrm{rp}_i(S)$ *and* $\mathrm{rp}_i(T)$ *imply* $\mathrm{rp}_i(S \vee T)$.
c. $\mathrm{rp}_i\big(S(x)\big)$ *implies* $\mathrm{rp}_2\big(\bigvee x\, S(x)\big)$.
d. $\mathrm{rp}_1\big(S(x)\big)$ *implies* $\mathrm{rp}_1\big(\bigwedge x\, S(x)\big)$.

E$_2$. $\mathrm{np}_i(S)$ *and* $\mathrm{rp}_i(T)$ *imply* $\mathrm{rp}_i(S \to T)$.

F. a. $\mathrm{np}_1(\bar{x} \in R)$ *and* $\mathrm{rp}_2(\bar{x} \in R)$.
b. $\mathrm{rp}_1(\sim \bar{x} \in R)$ *and* $\mathrm{np}_2(\sim \bar{x} \in R)$.
c. *If $S$ is either a tautology or a contradictory then* $\mathrm{np}_1(S)$ *and* $\mathrm{rp}_1(S)$.

Proof. A. This is evident from the definitions upon observing that $\sim S_E$ is true iff $S_E$ is false.

B$_1$. a. Let $\mathcal{N}$ be a nest such that for every $E \in \mathcal{N}$, for some $D \in \mathcal{N}$, $D \supseteq E$, $S_D$ is true. For such an equivalence relation $D$, $D \subseteq \sum\mathcal{N}$, and using $\mathrm{np}_1(S)$ we conclude $S_{\sum\mathcal{N}}$ is true. This establishes $\mathrm{np}_2(S)$. On the other hand, as a consequence of E$_1$ and F, proved below, it follows that the sentential function "$x \in R \to x \in S$", where $R$ and $S$ are relations of rank 1 on $X$, has nesting property 2. The following simple model shows it does not have nesting property 1. Take $X = \{1, 2, 3\}$, $R = \{1\}$, $S = \{3\}$, $E = I$, and $D$ the equivalence relation with block decomposition $\{\{1, 2\}, \{3\}\}$. Then "$2/E \in R/E \to 2/E \in S/E$" is true, but "$2/D \in R/D \to 2/D \in S/D$" is not.

B$_1$. b. That $\mathrm{np}_3(S)$ follows from $\mathrm{np}_2(S)$ is an obvious consequence of the definitions. Conversely, suppose $\mathrm{np}_3(S)$, and let $\mathcal{N}$ be a nest such that for each $E \in \mathcal{N}$, for some $D \in \mathcal{N}$, $D \supseteq E$, $S_D$ is true. The subnest $\mathcal{N}_1$, of all members $D$ of $\mathcal{N}$ such that $S_D$ is true, has $\sum\mathcal{N}_1 = \sum\mathcal{N}$. Using $\mathrm{np}_3(S)$ we conclude that $S_{\sum\mathcal{N}}$ is true, and obtain as a consequence $\mathrm{np}_2(S)$. We assume this equivalence throughout the remainder of the proof.

C$_1$. a. Suppose $\mathrm{np}_1(S)$, and let $E$ and $D \supseteq E$ be equivalence relations on $X$ such that $(S \wedge T)_E$ is true. Then $S_E$, and as a consequence of $\mathrm{np}_1(S)$ also $S_D$, are true. Since $T$ is a logical consequence of $S$, $T_D$ is true, whence $(S \wedge T)_D$ follows, verifying $\mathrm{np}_1(S \wedge T)$. Now suppose $\mathrm{np}_2(S)$, and let $\mathcal{N}$ be such a nest of equivalence relations that for each $E \in \mathcal{N}$, for some $D \in \mathcal{N}$, $D \subset E$, $(S \wedge T)_D$ is true. Then $S_D$ is true, and from $\mathrm{np}_2(S)$, also $S_{\Sigma\mathcal{N}}$ is true. As above, this entails $(S \wedge T)_{\Sigma\mathcal{N}}$, giving $\mathrm{np}_2(S \wedge T)$.

$\mathrm{C}_1$. b. Assume $\mathrm{np}_1(T)$ and let $E$ and $D \supseteq E$ be equivalence relations on $X$ such that $(S \wedge T)_E$ is true. Then $T_E$ is true, since $T$ is implied by $S$, and by virtue of $\mathrm{np}_1(T)$ we conclude that $T_D$, and therefore $(S \vee T)_D$, are true. From this $\mathrm{np}_1(S \vee T)$ follows. Next suppose $\mathrm{np}_2(T)$ and take a nest $\mathcal{N}$ of equivalence relations on $X$ such that for every $E \in \mathcal{N}$, for some $D \in \mathcal{N}$, $D \supseteq E$, $(S \vee T)_D$ is true. Because $S$ implies $T$, $T_D$ is true. Hence, from $\mathrm{np}_2(T)$, we conclude that $T_{\Sigma\mathcal{N}}$, and thus $(S \vee T)_{\Sigma\mathcal{N}}$, are true. From this follows $\mathrm{np}_2(S \vee T)$.

$\mathrm{D}_1$. a. If $\mathrm{np}_1(S)$, $\mathrm{np}_1(T)$, if $E$ and $D \supseteq E$ are equivalence relations on $X$, and if $(S \wedge T)_E$ is true, then we may conclude in sequence that $S_E$ and $T_E$ are true, that $S_D$ and $T_D$ are true, and that $(S \wedge T)_D$ is true, proving $\mathrm{np}_1(S \wedge T)$. In a corresponding fashion, if $\mathcal{N}$ is a nest such that for every $E \in \mathcal{N}$, for some $D \in \mathcal{N}$, $D \supseteq E$, $(S \wedge T)_D$ is true, then $S_D$ and $T_D$ are true, and $\mathrm{np}_2(S)$ and $\mathrm{np}_2(T)$ imply that $S_{\Sigma\mathcal{N}}$, $T_{\Sigma\mathcal{N}}$, and hence $(S \wedge T)_{\Sigma\mathcal{N}}$, are true. We conclude $\mathrm{np}_2(S \wedge T)$.

$\mathrm{D}_1$. b. If $D$ and $E$, with $D \supseteq E$, are equivalence relations on $X$, and if $(S \vee T)_E$ is true, then either $S_E$ or $T_E$ is true. From $\mathrm{np}_1(S)$ and $\mathrm{np}_1(T)$ follows either that $S_D$ is true or that $T_D$ is true. Hence $(S \vee T)_D$ is true, and we conclude $\mathrm{np}_1(S \vee T)$. Suppose $\mathrm{np}_2(S)$ and $\mathrm{np}_2(T)$. Let $\mathcal{N}$ be a nest of equivalence relations on $X$ such that for each $E \in \mathcal{N}$, for some $D \in \mathcal{N}$, $D \supseteq E$, $(S \vee T)_D$ is true. Let $\mathcal{N}_0$ be the subnest of $\mathcal{N}$ consisting of all $D \in \mathcal{N}$ such that $(S \wedge T)_D$ is true. It is clear that $\sum\mathcal{N} = \sum\mathcal{N}_0$. Now for each $E \in \mathcal{N}_0$ either $S_E$ or $T_E$ is true. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be respectively those subnests of $\mathcal{N}_0$ such that $S_E$ is true, $E \in \mathcal{N}_1$, and $T_E$ is true, $E \in \mathcal{N}_2$. Either $\sum\mathcal{N}_1 = \sum\mathcal{N}$, or $\sum\mathcal{N}_2 = \sum\mathcal{N}$. We suppose the former. Then for each $E \in \mathcal{N}_1$, for some $D \in \mathcal{N}_1$, $D \supseteq E$, $S_D$ is true. From $\mathrm{np}_2(S)$ we conclude that $S_{\Sigma\mathcal{N}}$ is true. Hence $(S \vee T)_{\Sigma\mathcal{N}}$ is true and $\mathrm{np}_2(S \vee T)$ follows.

$\mathrm{D}_1$. c. Suppose that $E$, and $D \supseteq E$, are equivalence relations on $X$, and suppose that $\big(\bigwedge \underline{x} S(\underline{x})\big)_E$ is true. Here $\underline{x}$ indicates a variable ranging over the factor space $X/E$. It is clear that $\big(\bigwedge \underline{x} S(\underline{x})\big)_E$ is true iff $\bigwedge x\,[S(x)_E]$ is true, where $x$ is a variable ranging over $X$. Therefore, for each $x \in X$, $S(x)_E$ is true, and by $\mathrm{np}_1(S)$ we have the consequence that $S(x)_D$ is true. Hence $\bigwedge x\,[S(x)_D]$, or equivalently, $\big(\bigwedge \underline{x} S(\underline{x})\big)_D$, is true, and thus $\mathrm{np}_1\big(\bigwedge x\, S(x)\big)$ follows. In a similar way, given $\mathrm{np}_2\big(S(x)\big)$ and a nest $\mathcal{N}$ of equivalence relations on $X$ such that for each $E \in \mathcal{N}$, for some $D \in \mathcal{N}$, $D \supseteq E$, $\big(\bigwedge \underline{x} S(\underline{x})\big)_D$ is true, we conclude first that for each $x \in X$, for each $E \in \mathcal{N}$, for some $D \in \mathcal{N}$, $D \supseteq E$, $S(x)_D$ is true, and therefore $S(x)_{\Sigma\mathcal{N}}$ is true. Hence, $\bigwedge x\,[S(x)_{\Sigma\mathcal{N}}]$, or equivalently, $\big(\bigwedge \underline{x} S(\underline{x})\big)_{\Sigma\mathcal{N}}$, is true and $\mathrm{np}_2\big(\bigwedge x\, S(x)\big)$ follows.



$\mathrm{D}_1$. d. Assume $\mathrm{np}_1\big(S(x)\big)$ and let $E$, $D$ be equivalence relations on $X$ such that $\big(\bigvee \underline{x} S(\underline{x})\big)_E$ is true and $D \supseteq E$. $\big(\bigvee \underline{x} S(\underline{x})\big)_E$ is true iff $\bigvee x\,[S(x)_E]$ is true, where $\underline{x}$, $x$ range over $X/E$ and $X$, respectively. Let $x_0 \in X$ be such that $S(x_0)_E$ is true. From $\mathrm{np}_1\big(S(x)\big)$ follows that $S(x_0)_D$ is true. Hence $\bigvee x\,[S(x)_D]$, or equivalently $\big(\bigvee \underline{x} S(\underline{x})\big)_D$, is true. We conclude $\mathrm{np}_1\big(\bigvee x\, S(x)\big)$. A counter example to show that $\mathrm{np}_2\big(S(x)\big)$ does not imply $\mathrm{np}_2\big(\bigvee x\, S(x)\big)$ is easily constructed. As a consequence of F, proved below, we have $\mathrm{np}_2(\sim x \in R)$, where $R$ is a relation of rank 1. Take $X$ to be the half-open interval $[0, 1)$, $R = \{0\}$, and $\mathcal{N}$ the nest of equivalence relations $E_a = [0, a) \times [0, a) \cup I$, $0 \leqslant a < 1$. It is clear that $(\bigvee x \sim x \in R)_E$ is true for each $E \in \mathcal{N}$, since $R/E_a = \{0/E_a\} = \{[0, a)\}$, but $(\bigvee x \sim x \in R)_{\Sigma\mathcal{N}}$ is false, since $\sum\mathcal{N} = [0, 1) \times [0, 1)$, and therefore $R/\sum\mathcal{N} = \{[0, 1)\}$.

$\mathrm{E}_1$. This can be obtained as a consequence of A and $\mathrm{D}_1$, part b, by replacing $S$ by $\sim S$. However, we shall give a direct proof.

Let $E$, and $D \supseteq E$, be equivalence relations on $X$ such that $(S \to T)_E$ is true. Either $T_E$ is true, in which case, by $\mathrm{np}_1(T)$, we obtain $T_D$, and therefore $(S \to T)_D$, true, or $S_E$ is false, and hence, by $\mathrm{rp}_1(S)$, $S_D$ is false, so that $(S \to T)_D$ is true. This shows that $\mathrm{np}_1(S \to T)$. Similarly, if $\mathcal{N}$ is a nest such that for each $E \in \mathcal{N}$, for some $D \in \mathcal{N}$, $D \supseteq E$, $(S \to T)_D$ is true, we have two cases to consider, assuming $\mathrm{rp}_2(S)$ and $\mathrm{np}_2(T)$. Either $S_{\Sigma\mathcal{N}}$ is false, so that $(S \to T)_{\Sigma\mathcal{N}}$ is true, vacuously, or for some $E_0$ in $\mathcal{N}$, for every $E \in \mathcal{N}$, $E \supseteq E_0$, $S_E$ is true, by $\mathrm{rp}_2(S)$. Let $\mathcal{N}_0$ be the subnest of all $E \supseteq E_0$, $E \in \mathcal{N}$. Then for every $E \in \mathcal{N}_0$, for some $D \in \mathcal{N}_0$, $T_D$ is true. Using $\mathrm{np}_2(T)$ we conclude $T_{\Sigma\mathcal{N}}$, and hence $(S \to T)_{\Sigma\mathcal{N}}$ are true. We may assert $\mathrm{np}_2(S \to T)$.

$\mathrm{B}_2$. This follows from A and $\mathrm{B}_1$ by substituting $\sim S$ for $S$. We give a direct proof showing $\mathrm{rp}_3(S)$ implies $\mathrm{rp}_2(S)$. By way of obtaining a contradiction, let $\mathcal{N}$ be a nest such that $S_{\Sigma\mathcal{N}}$ is true and assume $\mathrm{rp}_2(S)$ false. Then for every $E \in \mathcal{N}$, for some $D \in \mathcal{N}$, $D \supseteq E$, $S_D$ is false. Let $\mathcal{N}_0$ be the subnest of $\mathcal{N}$ consisting of those $D \in \mathcal{N}$ such that $S_D$ is false. Clearly $\sum\mathcal{N}_0 = \sum\mathcal{N}$. But by $\mathrm{rp}_3(S)$, if $S_{\Sigma\mathcal{N}}$ is true there exists $E \in \mathcal{N}_0$ such that $S_E$ is true. We have obtained our contradiction, showing that $\mathrm{rp}_2(S)$ follows from $\mathrm{rp}_3(S)$.

$\mathrm{C}_2$, $\mathrm{D}_2$, $\mathrm{E}_2$. These follow in an obvious fashion from A and $\mathrm{C}_1$, $\mathrm{D}_1$, and $\mathrm{E}_1$. Direct proofs are also easily given.

F. a. Suppose that $D$ and $E$ are equivalence relations on $X$, $D \supseteq E$, and $\bar{x}/E \in R/E$. Then for some $\bar{y}$, $\bar{y} \in R$ and $(\bar{x}, \bar{y}) \in E \subseteq D$, which implies

$\bar{x}/D \in R/D$. Hence, $\mathrm{np}_1(\bar{x} \in R)$. Now let $\mathcal{N}$ be any nest of equivalence relations on $X$ and assume $\bar{x}/\sum\mathcal{N} \in R/\sum\mathcal{N}$. Then for some $\bar{y} \in R$, $(\bar{x}, \bar{y}) \in \sum\mathcal{N}$. This implies for some $E_1, \ldots, E_r$ in $\mathcal{N}$, where $r$ is the rank of $R$, that $(x_i, y_i) \in E_i$, $i = 1, \ldots, r$. Set $E_0 = \bigcup_i E_i$. We have $E_0 \in \mathcal{N}$, and $(\bar{x}, \bar{y}) \in E_0$, which implies $\bar{x}/E_0 \in R/E_0$. Moreover, if $E \in \mathcal{N}$, and $E \supseteq E_0$, then $(\bar{x}, \bar{y}) \in E$ and hence $\bar{x}/E \in R/E$. Thus $\mathrm{rp}_2(\bar{x} \in R)$. Obviously, one does not have $\mathrm{rp}_1(\bar{x} \in R)$.

F. b. Suppose $D \supseteq E$ and $\sim \bar{x}/D \in R/D$. Then for every $\bar{y} \in R$, $(\bar{x}, \bar{y}) \notin D$. Then also, for every $\bar{y} \in R$, $(\bar{x}, \bar{y}) \notin E$, and hence $\sim \bar{x}/E \in R/E$. This shows that $\mathrm{rp}_1(\sim \bar{x} \in R)$. If $\mathcal{N}$ is a nest of equivalence relations on $X$ such that for each $E \in \mathcal{N}$, for some $D \in \mathcal{N}$, $D \supseteq E$, $\sim \bar{x}/D \in R/D$, then for every $\bar{y} \in R$, for every $E \in \mathcal{N}$, $(\bar{x}, \bar{y}) \notin E$, and hence $(\bar{x}, \bar{y}) \notin \sum\mathcal{N}$. Therefore $\sim \bar{x}/\sum\mathcal{N} \in R/\sum\mathcal{N}$, proving $\mathrm{np}_2(\sim \bar{x} \in R)$.

F. c. This result is immediate.

We remark on two consequences of theorem 8. A *positive* sentence has been defined by Lyndon [10] to have only the connectives $\bigwedge$, $\bigvee$, $\wedge$ and $\vee$. By $D_1$ and F, part a, every positive sentential function has nesting property 1. Hence, every positive *axiom* is preserved into *every* factor space, that is, is preserved by every equivalence relation. Secondly, every composition of functions or operators corresponds to a positive sentential function of a special type: all the quantifiers are existential, and precede the matrix of the sentence, and the matrix is a conjunction of elementary positive sentences. For example, $z = f(g(x))$ corresponds to $\bigvee y\big((x, y) \in g \wedge (y, z) \in f\big)$. Again by $D_1$ and F, part a, such compositions have nesting property 1 and reverse property 2.

It is easy now, using theorem 8, to obtain a large class of sentential functions having the nesting property. We define, on the family of all subsets of all sentential functions having the primitive relations of $\mathfrak{R}$ as constants, a binary operation $Im$, and unary operations $C$, $D$, $LC$, $LD$, $A$, $E$, and $N$, as follows:

$$Im(P, Q) = \{p \to q \mid p \in P \text{ and } q \in Q\},$$
$$C(P) = \{p \wedge q \mid p \in P \text{ and } q \in P\},$$
$$D(P) = \{p \vee v \mid p \in P \text{ and } q \in P\},$$
$$LC(P) = \{p \wedge q \mid p \in P \text{ and } q \text{ is a logical consequence of } p\},$$
$$LD(P) = \{p \vee q \mid p \in P \text{ and } p \text{ is a logical consequence of } q\},$$
$$A(P) = \{\textstyle\bigwedge x\, p(x) \mid p(x) \in P \text{ and } x \text{ is an unbound variable of } p\},$$
$$E(P) = \{\textstyle\bigvee x\, p(x) \mid p(x) \in P \text{ and } x \text{ is an unbound variable of } p\},$$
$$N(P) = \{\sim p \mid p \in P\}.$$



Next we let $P_{1,1}$ be the set of all elementary positive sentences, $Q_{1,1}$ the set of all elementary negative sentences, and set $P_{2,1} = Q_{2,1} = P_{1,1} \cup Q_{1,1}$. Then, for $i = 1, 2$, and, recursively, for $j = 1, 2, \ldots$ we have

$$P_{i,j+1} = P_{i,j} \cup Im(Q_{i,j}, P_{i,j}) \cup C(P_{i,j}) \cup D(P_{i,j}) \cup LC(P_{i,j}) \cup LD(P_{i,j}) \cup \cup A(P_{i,j}) \cup E(P_{1,j}) \cup N(Q_{i,j}),$$

$$Q_{i,j+1} = Q_{i,j} \cup Im(P_{i,j}, Q_{i,j}) \cup C(Q_{i,j}) \cup D(Q_{i,j}) \cup LC(Q_{i,j}) \cup LD(Q_{i,j}) \cup \cup A(Q_{1,j}) \cup E(Q_{i,j}) \cup N(P_{i,j}).$$

Finally, we set

$$P_i = \bigcup_j P_{i,j}, \ i = 1, 2, \quad \text{and} \quad Q_i = \bigcup_j Q_{i,j}, \ i = 1, 2.$$

It is clear from theorem 8 that each member of $P_{ij}$ has nesting property $i$, and hence each member of $P_i$ has nesting property $i$. In particular, each member of $P_2$ has the nesting property. Correspondingly, each member of $Q_{i,j}$ has reverse property $i$, and hence each member of $Q_i$ has reverse property $i$.

**7. Applications.** A relational system $\mathfrak{R}$ which is an algebra satisfies, for each primitive relation of $\mathfrak{R}$, two axioms; the axiom of closure, and the axiom of functionality. Formally, for a relation $R$ of rank $r+1$,

$$(C_R) \textstyle\bigwedge x_1 \ldots \bigwedge x_r \bigvee y\, [(x_1, \ldots, x_r, y) \in R],$$

$$(F_R) \textstyle\bigwedge x_1 \ldots \bigwedge x_r \bigwedge y \bigwedge z\, [(x_1, \ldots, x_r, y) \in R \wedge (x_1, \ldots, x_r, z) \in R \to (y, z) \in I].$$

$C_R$ is a positive sentence, and satisfies nesting property 1. $F_R$ satisfies nesting property 2, as a consequence of theorem 8; F, $E_1$, and $D_1$, part c. Furthermore, the congruence relations of an algebra are precisely the equivalence relations which preserve $F_R$ (every equivalence relation preserves $C_R$). Hence Birkhoff's representability theorem is a corollary of theorem 7. It may be obtained directly from theorem 5 by the observation that the congruence relations of an algebra form a complete lattice, of which $I$ is a member. We remark that the first part of the proof of theorem 5, pertaining to condition (C1), is enough to establish theorem 5 using Pickert's definition of subdirect product. The proof of this part is, in essence, the proof given by Birkhoff.

That the homomorphic images of a ring are rings, i.e., that the congruence relations of a ring preserve the ring axioms, is a classical result. Suppose that $\mathfrak{R}$ is a ring satisfying the additional axiom:

(ID) $$\textstyle\bigwedge x \bigwedge y\, [(x \times y = 0) \to (x = 0 \vee y = 0)].$$

Expressed in terms of elementary positive sentences this axiom reads:

$$\textstyle\bigwedge x \bigwedge y\, \big[(x, y, 0) \in \times \to \big((x, 0) \in I \vee (y, 0) \in I\big)\big].$$

The nesting property follows easily from theorem 8. Hence, making use of the remark following theorem 7, we have the theorem:

*An integral domain can be represented as a subdirect product of properly subdirectly irreducible integral domains.*

This theorem is not a consequence of Birkhoff's theorem because a homomorphic image of an integral domain need not be an integral domain.

Let $\mathfrak{R}$ be a ring without proper nilpotent elements. Then it satisfies, in addition to the ring axioms, the infinite set of axioms:

(NP) $\quad \bigwedge x[(x^n = 0) \to (x = 0)]$, for $\quad n = 1, 2, \ldots$

The composition "$x^n = 0$" has reverse property 2, as indicated in the discussion following theorem 8. Applying $E_1$, and $D_1$, part c, of theorem 8, we conclude that each of these axioms has the nesting property. Thus, as another consequence of the above sort, we have the theorem:

*A ring without proper nilpotent elements may be represented as a subdirect product of properly subdirectly irreducible rings without proper nilpotent elements.*

A similar observation establishes that

*A torsion free group may be represented as a subdirect product of properly subdirectly irreducible torsion free groups.*

Let $(X, \leqslant)$ be a partially ordered set. It then satisfies the axioms:

($PO_1$) $\quad \bigwedge x[x \leqslant x]$,

($PO_2$) $\quad \bigwedge x \bigwedge y[x \leqslant y \wedge y \leqslant x \to x = y]$,

($PO_3$) $\quad \bigwedge x \bigwedge y \bigwedge z[x \leqslant y \wedge y \leqslant z \to x \leqslant z]$.

Each of the axioms possesses the nesting property. A simply ordered set satisfies further

(SO) $\quad \bigwedge x \bigwedge y[x \leqslant y \vee y \leqslant x]$.

Partially ordered sets with upper bounds (directed sets) satisfy

(UB) $\quad \bigwedge x \bigwedge y \bigvee z[x \leqslant z \wedge y \leqslant z]$.

These too possess the nesting property, and theorem 7 can be applied. On the other hand, to have *least* upper bounds a partially ordered set must satisfy

(LUB) $\quad \bigwedge x \bigwedge y \bigvee z[x \leqslant z \wedge y \leqslant z \wedge \bigwedge w(x \leqslant w \wedge y \leqslant w \to z \leqslant w)]$.

Theorem 8 does not yield the nesting property for this axiom. However, it is a simple exercise to establish that every equivalence relation which preserves the order also preserves this axiom. A similar remark must clearly hold for greatest lower bounds. Hence, again making use of the corollary following theorem 7, *a partially ordered set with least upper and greatest lower bounds (a lattice) has a proper representation as a subdirect product with properly subdirectly irreducible factors.* Thus an apparent advantage which the algebraic representation of a lattice holds over the the representation as a partially ordered set, in virtue of Birkhoff's theorem, is hereby dispelled.



An abstract projective plane can be defined as a relational system $(X, P, L, D, ON)$, where $P$, $L$, $D$, and $ON$ have ranks 1, 1, 2, and 2, respectively, satisfying the following set of axioms:

($PP_1$) $\quad \bigwedge x[x \in P \vee x \in L]$,

($PP_2$) $\quad \bigwedge x \bigwedge y[x \in P \wedge y \in L \to \sim(x, y) \in I]$,

($PP_3$) $\quad \bigwedge x \bigwedge y[(x, y) \in D \to x \in P \wedge y \in P]$,

($PP_4$) $\quad \bigwedge x \bigwedge y[(x, y) \in D \to \sim(x, y) \in I]$,

($PP_5$) $\quad \bigwedge x \bigwedge y[(x, y) \in D \to (y, x) \in D]$,

($PP_6$) $\quad \bigvee x \bigvee y \bigvee z \bigvee w[(x, y) \in D \wedge (x, z) \in D \wedge (x, w) \in D \wedge (y, z) \in D \wedge \wedge (y, w) \in D \wedge (z, w) \in D]$,

($PP_7$) $\quad \bigwedge x \bigwedge y \bigwedge z \bigwedge w \bigwedge u\big[\sim\big((x, y) \in D \wedge (x, z) \in D \wedge (x, w) \in D \wedge \wedge (x, u) \in D \wedge (y, z) \in D \wedge (n, w) \in D \wedge (y, u) \in D \vee \wedge (z, w) \in D \wedge (z, u) \in D \wedge (w, u) \in D\big)\big]$,

($PP_8$) $\quad \bigwedge x \bigwedge y[(x, y) \in ON \to (x \in P \wedge y \in L) \vee (x \in L \wedge y \in P)]$,

($PP_9$) $\quad \bigwedge x \bigwedge y[(x, y) \in ON \to (y, x) \in ON]$,

($PP_{10}$) $\quad \bigwedge x \bigwedge y[x \in P \wedge y \in P \to \bigvee u((x, u) \in ON \wedge (y, u) \in ON)]$,

($PP_{11}$) $\quad \bigwedge x \bigwedge y \bigwedge u \bigwedge v[x \in P \wedge y \in P \wedge (x, u) \in ON \wedge (y, u) \in ON \wedge \wedge (x, v) \in ON \wedge (y, v) \in ON \to (u, v) \in I]$,

($PP_{12}$) $\quad \bigwedge x \bigwedge y[x \in L \wedge y \in L \to \bigvee u((x, u) \in ON \wedge (y, u) \in ON)]$,

($PP_{13}$) $\quad \bigwedge x \bigwedge y \bigwedge u \bigwedge v[x \in L \wedge y \in L \wedge (x, u) \in ON \wedge (y, u) \in ON \wedge \wedge (x, v) \in ON \wedge (y, v) \in ON \to (u, v) \in I]$,

($PP_{14}$) $\quad \bigwedge x \bigwedge y \bigwedge z \bigwedge u\big[(x, y) \in D \wedge (x, z) \in D \wedge (y, z) \in D \to \sim \sim\big((x, u) \in ON \wedge (y, u) \in ON \wedge (z, u) \in ON\big)\big]$.

Briefly, everything is either a point or a line ($PP_1$), ($PP_2$), $D$ distinguishes four points ($PP_3$)-($PP_7$), $ON$ is a symmetric relation between points and lines ($PP_8$), ($PP_9$), two points determine a unique line ($PP_{10}$), ($PP_{11}$), two lines determine a unique point ($PP_{12}$), ($PP_{13}$), and no three points distinguished by $D$ lie on a common line ($PP_{14}$). Applying theorem 8 it is seen that the nesting property holds for each of these axioms. Hence, *an abstract projective plane may be represented as a subdirect product of properly subdirectly irreducible abstract projective planes.*

Other systems to which theorem 7 may be applied include: algebraic systems with ordering relations, multilattices [2], multigroups [5], join systems [13], partitions of type $n$ [6], betweenness relations [11], finite state languages [4], Turing machines, and automata [8].

The usefulness of theorem 5 is not limited to the systems described in theorem 7. For example, a well-ordered set is a system $(X, \leqslant)$ satisfying

(WO$_1$) $(X_1 \leqslant)$ is simply ordered,

(WO$_2$) $\bigwedge Y \subseteq X \left[\bigvee x(x \in Y) \rightarrow \bigvee y \left(y \in Y \wedge \bigwedge z[z \in Y \rightarrow y \leqslant z]\right)\right]$.

Axiom WO$_1$ is equivalent to a set of sentences possessing the nesting property, as shown above, and axiom WO$_2$ is a *second-order* sentence preserved by every equivalence relation on $X$ which preserves the order. A fortiori conditions (1) and (2) of theorem 5 must hold. This implies that *a well-ordered set may be represented as a subdirect product of properly irreducible well-ordered sets.* One may show without difficulty that the irreducible factors are isomorphic to $(\{0, 1\}, \leqslant)$.

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# Closed subgroups of locally compact Abelian groups

by

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Let $G$ be an Abelian group, and let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be two topologies on $G$ such that $\mathfrak{D}_1 \supsetneq \mathfrak{D}_2$ and $(G, \mathfrak{D}_1)$ and $(G, \mathfrak{D}_2)$ are locally compact topological groups. E. Hewitt [1] has proved that there is an $\mathfrak{D}_1$-continuous character on $G$ that is $\mathfrak{D}_2$-discontinuous. We submit here an outline of a somewhat shorter proof of this result based on an observation about closed subgroups. We then make some further remarks about closed subgroups.

We first given an alternative proof of Lemma (2.1) in [1]:

*Let $R$ denote the additive group of real numbers with the usual topology, and let $(R, \mathfrak{D})$ be a locally compact group such that $\mathfrak{D}$ is strictly stronger than the usual topology of $R$. Then $\mathfrak{D}$ is the discrete topology.*

Proof. Let $\varphi$ denote the identity mapping of $(R, \mathfrak{D})$ onto $R$; $\varphi$ is clearly continuous. Let $C$ be the component of the identity in $(R, \mathfrak{D})$. If $C = R$, then $(R, \mathfrak{D})$ is $\sigma$-compact and (5.29) [2] shows that $\varphi$ is a homeomorphism, contrary to our hypothesis. Hence $\varphi(C)$ is a proper connected subgroup of $R$ in the usual topology. Therefore $\varphi(C) = \{0\}$, $C = \{0\}$, and $(R, \mathfrak{D})$ is totally disconnected. By Theorem (7.7) [2], $(R, \mathfrak{D})$ contains a compact open subgroup $H$. Since $\varphi(H)$ is a compact subgroup of $R$ in the usual topology, we have $\varphi(H) = \{0\}$ and $H = \{0\}$. Consequently, $\{0\}$ is open in $(R, \mathfrak{D})$ and $\mathfrak{D}$ is discrete.

Hewitt's theorem follows from the following lemma.

LEMMA 1. *Let $G$, $\mathfrak{D}_1$, and $\mathfrak{D}_2$ be as before. There exists a subgroup $H$ of $G$ that is $\mathfrak{D}_1$-closed but not $\mathfrak{D}_2$-closed.*

Proof. Let $\varphi$ be the [continuous] identity mapping of $(G, \mathfrak{D}_1)$ onto $(G, \mathfrak{D}_2)$. Arguing as in the proof of Theorem (3.3) [1] and noting that invoking Theorem (2.2) [1] is unnecessary, we find that there is a subgroup $J$ of $G$ such that the topology $\mathfrak{D}_1$ on $J$ is strictly stronger than the topology $\mathfrak{D}_2$ on $J$, and such that either

(1) $(J, \mathfrak{D}_2)$ is topologically isomorphic with $R$,

or

(2) $(J, \mathfrak{D}_2)$ is compact.