but these discontinuities are nice in the sense that $C = 0$ for these arcs. It appears that our techniques would handle the cases of a finite numbers of arcs $A_n$ like $[0, u]$ with bad discontinuities, that is, the sets $C_n$ (like $C$) are nonempty and also $A_n - C_n$ is nonempty.

Finally, it is conjectured that $C$ contains $n$-cells, and that $C$ is closed under the operation of taking cones.

**Question.** Suppose that a compact metric continuum $S$ contains a subset $I$ such that (a) $S$ satisfies Conditions (1)-(6) for a ruled continuum and (b) $I$ admits a topological semigroup structure with zero $z$ and unit $u$ where $I$ and $u$ have the same meaning as in (1)-(6). Does $S$ admit the structure of a topological semigroup with zero and unit $u$?

The continuum $S$ above is a more general type of ruled continuum than that considered in Conditions (1)-(8).

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# On the lexicographic dimension of linearly ordered sets


by

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**1. Introduction.** In earlier papers from the theory of representation of linearly ordered sets chief interest was concentrated on finding so-called universal sets. Under an $\mathfrak{m}$-universal linearly ordered set (where $\mathfrak{m}$ is cardinality) we understand a linearly ordered set which contains a subset isomorphic with every linearly ordered set of cardinality $\leqslant \mathfrak{m}$. It was been shown that such universal sets are ordinal powers (in Birkhoff's sense) in which the base in any chain containing at least two elements and the exponent is a well-ordered set. Thus Hausdorff proved [1], p. 181) that every linearly ordered set of cardinality $\leqslant \aleph_\xi$ where $\aleph_\xi$ is a regular cardinal number is isomorphic with a certain set of sequences of type $\omega_\xi$ formed from three cyphers $0, 1, 2$, and ordered lexicographically. In other words, he proved that an ordinal power of type ${}^{\omega_\xi}3$ is an $\aleph_\xi$-universal linearly ordered set if $\aleph_\xi$ is regular. Sierpiński ([2]) improved his result in the following way: An ordinal power of type ${}^{\omega_\xi}2$ is an $\aleph_\xi$-universal linearly ordered set for every cardinal number $\aleph_\xi$.

Now it is clear that the type of base cannot be reduced. Hence interest has been concentrated on the problem if it is possible to reduce the type of the exponent. It has been shown, however, that in general this type cannot be reduced. In some cases it is possible, however, to map a given linearly ordered set of cardinality $\aleph_\xi$ isomorphically onto a subset of a power with the exponent of a lower type than $\omega_\xi$. Thus Novotný ([3]) proves that: Every $\aleph_\xi$-separable [1] linearly ordered set can be isomorphically mapped onto a subset of ordinal power of type ${}^{\omega_\xi}2$. This survey makes clear the effort to find the most economical representation, i.e. a representation in which both the base and the exponent are of the smallest possible types.

Now it is possible to pose this problem: Let the type of the base be constant. What is the smallest possible type of exponent such that the given linearly ordered set can be mapped isomorphically onto a subset of the corresponding ordinal power? This problem was partially

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(1) A linearly ordered set $G$ is called $\mathfrak{m}$-separable if it contains a dense subset $H$ of minimal possible cardinality $\mathfrak{m}$.

solved by I. Fleischer for a base of type $\lambda$ ($\lambda =$ the type of the set of real numbers). Fleischer ([5]) proved the necessary and sufficient conditions for the possibility of mapping a given linearly ordered set isomorphically onto a subset of ordinal power of type ${}^{\alpha}\lambda$, where $\alpha$ is a countable, respectively finite ordinal. An analogical problem has already been solved for (partially) ordered sets. The universal ordered sets are cardinal powers in which the base is a chain and the exponent is an antichain. It has been shown that the minimal cardinality of the exponent of a cardinal power on a suitable subset of which the given partially ordered set can be isomorphically mapped is very densely coherent with the so-called $\alpha$-dimension of set $G$ defined by H. Komm ([6]). In the paper [7] the so-called $\alpha$-pseudodimension of the ordered set $G$ is defined and shown to be equal to the minimal cardinality of the exponent of a cardinal power with base of type $\alpha$ which contains a subset isomorphic with $G$. This paper introduces the so-called *lexicographic* $\alpha$-dimension of a linearly ordered set $G$, which will be shown to be equal to the minimal type of exponent of an ordinal power with base of type $\alpha$ which contains a subset isomorphic with $G$. In this manner the problem of economy of representation of a given linearly ordered set with a given base is definitely solved. In the conclusion of the paper there is introduced a partition of type $\alpha$ of a linearly ordered set which is a generalization of the dyadic partition ([8], [3], [9]) and it is proved that the lexicographic $\alpha$-dimension of a linearly ordered set $G$ is equal to the minimum of all orders of all partitions of type $\alpha$ of set $G$.

In this paper we shall use the sign $\cong$ for an isomorphism of ordered sets. If $G$ is a set, then $\operatorname{card} G$ denotes the cardinality of $G$; if $G$ is a linearly ordered set, then $\bar{G}$ denotes the order type of this set. We shall use these synonymous in the whole paper: linearly ordered set = chain (= totally ordered set, i.e. a set $G$ with a non-symmetric transitive binary relation $<$ such that for every $x, y \in G$, $x \neq y$, there is $x < y$ or $y < x$). We shall use Birkhoff's symbolic (see [11]) such that, for instance, $A \oplus B$ denotes the ordinal sum and ${}^{N}M$ denotes the ordinal power. The set $G$ will be called non-trivial if $\operatorname{card} G \geqslant 2$. All sets in this paper are assumed to be non-trivial if the contrary is not stated, and all cardinalities and all order types are assumed to be cardinalities and respectively order types of non-trivial sets.

## 2. Lexicographic $\alpha$-realizer of a linearly ordered set.

DEFINITION 1. Let $N$ be a well-ordered set, let $M_\alpha$ ($\alpha \in N$) be an ordered set of every $\alpha \in N$. By the *ordinal product* $\underset{\alpha \in N}{\boldsymbol{P}} M_\alpha$ we understand the set for all functions $f$ defined on $N$ and such that $f(\alpha) \in M_\alpha$ for every $\alpha \in N$ ordered in the following way: $f < g \Leftrightarrow$ there exists an $\alpha_0 \in N$ so that $f(\alpha) = g(\alpha)$ holds for every $\alpha < \alpha_0$ whereas $f(\alpha_0) < g(\alpha_0)$. If all sets $M_\alpha$ are equal to the same set $M$, then we call the relevant ordinal product the *ordinal power* and denote it by ${}^{N}M$.



DEFINITION 2. Let $G$ be a linearly ordered set, let $N$ be a well-ordered set and let us assign to every $\nu \in N$ a chain $L_\nu$ and a mapping $f_\nu$ of $G$ into $L_\nu$ so that for $x, y \in G$, $x < y$ holds if and only if there exists $\nu_0 \in N$ so that $f_\nu(x) = f_\nu(y)$ for $\nu < \nu_0$ whereas $f_{\nu_0}(x) < f_{\nu_0}(y)$. Then we say that $\{L_\nu, f_\nu | \ \nu \in N\}$ is a *lexicographic realizer* (briefly: *l-realizer*) of set $G$. If all chains $L_\nu$ are of the same type $\alpha$, we call the corresponding realizer $\{L_\nu, f_\nu | \ \nu \in N\}$ a *lexicographic $\alpha$-realizer*.

THEOREM 1. *Let $G$ be a chain, let $N$ be a well-ordered set and let $L_\nu$ be a chain for every $\nu \in N$. Then the following statements are equivalent:*

(A) $G \cong G' \subseteq \underset{\nu \in N}{\boldsymbol{P}} L_\nu$.

(B) *For every $\nu \in N$ there exists a mapping $f_\nu$ of $G$ into $L_\nu$ such that $\{L_\nu, f_\nu | \ \nu \in N\}$ is a l-realizer of set $G$.*

Proof. I. Let (A) hold. Let $\varphi$ be an isomorphism of $G$ onto $G' \subseteq \underset{\nu \in N}{\boldsymbol{P}} L_\nu$. For any $x \in G$ and for any $\nu \in N$ let us put $\Phi(x, \nu) = [\varphi(x)](\nu)$. $\Phi$ is the mapping of the set $G \times N$ into the set $\bigcup_{\nu \in N} L_\nu$ with the property $\Phi(x, \nu_0) \in L_{\nu_0}$. $\Phi(x, \nu_0)$ is therefore the mapping of the set $G$ into $L_{\nu_0}$. Let us put $\Phi(x, \nu_0) = f_{\nu_0}(x)$. We shall show that the system $\{L_\nu, f_\nu | \ \nu \in N\}$ is an l-realizer of $G$. Indeed, let $x < y$ in $G$. Then $\varphi(x) < \varphi(y)$ so that there exists a $\nu_0 \in N$ so that for $\nu < \nu_0$ we have $[\varphi(x)](\nu) = [\varphi(y)](\nu)$ whereas $[\varphi(x)](\nu_0) < [\varphi(y)](\nu_0)$, i.e. $\Phi(x, \nu) = \Phi(y, \nu)$ for $\nu < \nu_0$ and $\Phi(x, \nu_0) < \Phi(y, \nu_0)$, i.e. $f_\nu(x) = f_\nu(y)$ for $\nu < \nu_0$ and $f_{\nu_0}(x) < f_{\nu_0}(y)$. Let us suppose, on the contrary, that $f_\nu(x) = f_\nu(y)$ for $\nu < \nu_0$ and $f_{\nu_0}(x) < f_{\nu_0}(y)$, i.e. $\Phi(x, \nu) = \Phi(y, \nu)$ for $\nu < \nu_0$ and $\Phi(x, \nu_0) < \Phi(y, \nu_0)$. Then $[\varphi(x)](\nu) = [\varphi(y)](\nu)$ for $\nu < \nu_0$ and $[\varphi(x)](\nu_0) < [\varphi(y)](\nu_0)$, so that $\varphi(x) < \varphi(y)$. Since $\varphi$ is an isomorphism, we have $x < y$. Hence $\{L_\nu, f_\nu | \ \nu \in N\}$ is indeed an l-realizer of $G$ and (B) holds.

II. Let (B) hold. Let us denote again $\Phi(x, \nu) = f_\nu(x)$ for any $x \in G$ and any $\nu \in N$. $\Phi$ is the mapping of the set $G \times N$ into the set $\bigcup_{\nu \in N} L_\nu$ with the property $\Phi(x_0, \nu) \in L_\nu$. Let us form the ordinal product $\underset{\nu \in N}{\boldsymbol{P}} L_\nu$ and let us put $\Phi(x_0, \nu) = [\varphi(x_0)](\nu)$. In this manner we define the mapping $\varphi$ of the set $G$ onto $G' \subseteq \underset{\nu \in N}{\boldsymbol{P}} L_\nu$, which will be shown to be isomorphism. Let $x < y$ in $G$. Then there exists a $\nu_0 \in N$ so that $f_\nu(x) = f_\nu(y)$ for $\nu < \nu_0$ whereas $f_{\nu_0}(x) < f_{\nu_0}(y)$, i.e. $\Phi(x, \nu) = \Phi(y, \nu)$ for $\nu < \nu_0$, whereas $\Phi(x, \nu_0) < \Phi(y, \nu_0)$. Therefore $[\varphi(x)](\nu) = [\varphi(y)](\nu)$ for $\nu < \nu_0$ whereas $[\varphi(x)](\nu_0) < [\varphi(y)](\nu_0)$, which implies $\varphi(x) < \varphi(y)$. If we suppose, on the contrary, that $\varphi(x) < \varphi(y)$, then there exists a $\nu_0 \in N$ so that $[\varphi(x)](\nu) = [\varphi(y)](\nu)$ for $\nu < \nu_0$ whereas $[\varphi(x)](\nu_0) < [\varphi(y)](\nu_0)$, i.e. $\Phi(x, \nu) = \Phi(y, \nu)$ for $\nu < \nu_0$

whereas $\Phi(x, \nu_0) < \Phi(y, \nu_0)$ and therefore $f_\nu(x) = f_\nu(y)$ for $\nu < \nu_0$ whereas $f_{\nu_0}(x) < f_{\nu_0}(y)$. Since $\{L_\nu, f_\nu |\ \nu \in N\}$ is an 1-realizer of $G$, we have $x < y$. Hence $\varphi$ is indeed an isomorphism and (A) holds.

THEOREM 2. *Let $G$ be a chain, let $N$ be a well-ordered set and let $L$ be a chain of type $\alpha$. Then the following statements are equivalent:*

(A) $G \cong G' \subseteq {}^N L$.

(B) *For every $\nu \in N$, there exists a mapping $f_\nu$ of $G$ into $L$ so that $\{L, f_\nu |\ \nu \in N\}$ is a lexicographic $\alpha$-realizer of the set $G$.*

The proof is clear from Theorem 1.

LEMMA. *Let $L_1, L_2$ be such chains that $L_1 \cong L_1' \subseteq L_2$. Let $G$ be a chain and let $N$ be a well-ordered set. If $G \cong G_1 \subseteq {}^N L_1$, then $G \cong G_2 \subseteq {}^N L_2$.*

Proof. If $L_1 \cong L_1'$, then clearly ${}^N L_1 \cong {}^N L_1'$. If $\varphi$ is a relevant isomorphism, then $G \cong G_1 \subseteq {}^N L_1$ and $G_1 \cong \varphi(G_1) = G_2 \subseteq {}^N L_2$ so that $G \cong G_2 \subseteq {}^N L_2$.

THEOREM 3. *Let $G, L$ be chains. Then there exists a well-ordered set $N$ such that $G \cong G' \subseteq {}^N L$.*

Proof. Let $B$ be a chain containing exactly two elements. According to Sierpiński's theorem ([2]) there exists a well-ordered set $N$ such that $G \cong G' \subseteq {}^N B$.

Our statement now follows from the preceding lemma.

CONSEQUENCE. *Let $G$ be a chain and let $L$ be a chain of type $\alpha$. Then there exists at least one lexicographic $\alpha$-realizer (and hence at least one 1-realizer) of the chain $G$.*

The proof follows from Theorems 2 and 3.

## 3. Lexicographic $\alpha$-dimension of a linearly ordered set.

DEFINITION 3. Let $G$ be a linearly ordered set and let $\alpha$ be a type of a chain $L$. The minimal ordinal type of the well-ordered set $N$ such that $\{L, f_\nu |\ \nu \in N\}$ is a lexicographic $\alpha$-realizer of the set $G$ is called a *lexicographic $\alpha$-dimension* of the set $G$. This ordinal is denoted by $\alpha\text{-ldim}\,G$.

THEOREM 4. *Let $G$ be a linearly ordered set. Let $L$ be a chain of type $\alpha$ and let $N$ be a well-ordered set of type $\nu$. Then the following statements are equivalent:*

(A) $$\alpha\text{-ldim}\,G \leqslant \nu,$$

(B) $$G \cong G' \subseteq {}^N L.$$

The proof is clear from Theorem 2.

THEOREM 5. *Let $G$ be a linearly ordered set and let $\alpha$ be the type of a chain $L$ such that $G \cong G' \subseteq L$. Then* $\alpha\text{-ldim}\,G = 1$.

The proof follows from Theorem 4.



THEOREM 6. *Let $G$ be a linearly ordered set and let $L_1, L_2$ be chains of types $\alpha$ and $\beta$, respectively such that $L_1 \cong L' \subseteq L_2$. Then $\alpha\text{-ldim}\,G \geqslant \beta\text{-ldim}\,G$.*

Proof. If the assumptions of the Theorem are true, then ${}^N L_1 \cong {}^N L_1' \subseteq {}^N L_2$ for every well-ordered set $N$. The statement now follows from Theorem 4.

Note. A linearly ordered set containing no gaps and no jumps is called a *continuous* set. A continuous set containing a smallest and a greatest element is called a *continuum.*

We now prove the following

LEMMA (²). *Let $G$ be a continuum, and let $P$ be a well-ordered set. Then ${}^P G$ is a continuum.*

Proof. Let us denote by $i$ the smallest and by $j$ the greatest element in $G$. Then the function $f_i$ such that $f_i(p) = i$ for every $p \in P$ is the smallest element in ${}^P G$, and the function $f_j$ such that $f_j(p) = j$ for every $p \in P$ is the greatest element in ${}^P G$. Suppose that there exists in ${}^P G$ a jump, i.e. such two elements $f, g$ that $f < g$ and that there exists no $h \in {}^P G$ such that $f < h$, $h < g$. Since $f < g$, there exists a $p_0 \in P$ such that $f(p) = g(p)$ for $p < p_0$, whereas $f(p_0) < g(p_0)$. Since $G$ is a continuum and $f(p_0) \in G$, $g(p_0) \in G$, there exists an $a \in G$ such that $f(p_0) < a < g(p_0)$.

Let us define the function $h$ in the following way: $h(p) = f(p)$ for $p < p_0$, $h(p) = a$ for $p \geqslant p_0$. Then $h \in {}^P G$, $f < h$, $h < g$ and this is a contradiction. Hence ${}^P G$ contains no jumps. We shall prove that ${}^P G$ contains no gaps. Let us suppose that in ${}^P G$ there exists a cut $(A, B)$ which is a gap. Let us denote $P' = \{p|\ p \in P$, there exists $f \in B$ such that $f(p) \neq j\}$. If $P'$ is an empty set then $B$ contains only the greatest element in ${}^P G$ and this is a contradiction. Hence $P' \neq \emptyset$ and let $p_0$ be the smallest element in $P'$. Then the following holds: for every $p < p_0$ and every $f \in B$ there is $f(p) = j$ while there exists $g \in B$ such that $g(p_0) \neq j$. In $A$ there necessarily exist functions $h$ such that $h(p) = j$ for $p < p_0$. If such a function $h$ did not exist, then the function $h_0$ defined as $h_0(p) = j$ for $p < p_0$, $h_0(p) = i$ for $p \geqslant p_0$ would be the smallest element in $B$, giving a contradiction. This implies that the given gap is also a gap in the set $F \subseteq {}^P G$ containing those functions $h \in A$ for which $h(p) = j$ for $p \neq p_0$ and those functions $f \in B$ for which $f(p) = i$ for $p > p_0$. But it is clear that $F \cong G$ and $G$ contains no gaps, which is a contradiction.

By an interval in a linearly ordered set $G$ we shall understand a subset $I \subseteq G$ such that $x, y \in I$, $z \in G$, $x \leqslant z \leqslant y$ implies $z \in I$. By a closed

(²) This lemma is a generalization of one of Hausdorff's theorems ([10]). Hausdorff proved this lemma for the case $\overline{P} < \omega_1$.

interval $[a, b]$ $(a, b \in G,\ a \leqslant b)$ we understand a subset $I \subseteq G$ containing those elements $x \in G$ for which $a \leqslant x \leqslant b$.

THEOREM 7. *Let $G$ be a continuum of type $a$ and let $N$ be a well-ordered set of type $\nu$. Then $a\text{-ldim}^N G = \nu$.*

Proof. According to Theorem 4, $a\text{-ldim}^N G \leqslant \nu$. The contrary statement will be proved if we show that ${}^N G \cong {}^{N_1} G$ does not hold for any segment [3] $N_1$ of the set $N$. We shall apply the theorem proved in [4]: *If $G$ is a continuum, then no system of disjoint closed intervals in $G$ ordered in a natural way is isomorphic with $G$.*

Now let $N_1$ be any segment of the set $N$. Let us form the ordinal power ${}^{N_1}G$ and let us denote $f(g) = \{f \mid f \in {}^N G,\ f(\nu) = g(\nu) \text{ for } \nu \in N_1\}$ for every $g \in {}^{N_1}G$. Then $\{f(g) \mid g \in {}^{N_1}G\}$ is a disjoint system of closed intervals in ${}^N G$ and it is isomorphic with ${}^{N_1}G$ with respect to natural order (the natural order of a disjoint system of intervals in a linearly ordered set will be defined in the beginning of section 4). As ${}^N G$ is a continuum, the relation ${}^N G \cong \{f(g) \mid g \in {}^{N_1}G\}$ is not valid and the proof is complete.

DEFINITION 4. Let $G$ be a linearly ordered set, let $L$ be a chain of type $a$ and let $N$ be a well-ordered set. Let $F \subseteq {}^N L$ and let $G \cong F$. Then $F$ will be called an *$a$-representation* of the set $G$. The order type $\nu$ of the set $N$ will be called the *$a$-economy* of this $a$-representation and will be denoted by $\nu = a\text{-ek}\{G, F\}$.

THEOREM 8. *Let $G$ be a linearly ordered set and let $a$ be an order type. Then* $\min a\text{-ek}\{G, F\} = a\text{-ldim}\, G$.

Proof. According to Theorem 4, $a\text{-ldim}\, G \leqslant a\text{-ek}\{G, F\}$ for every $a$-representation $F$ of the set $G$, so that $a\text{-ldim}\, G \leqslant \min a\text{-ek}\{G, F\}$. On the other hand, if $a\text{-ldim}\, G = \nu$, then according to Theorem 2 there exists such an $a$-representation $F$ of $G$ that $a\text{-ek}\{G, F\} = \nu$.

Now Fleischer's results can be formulated in the following way.

*If $G$ is a linearly ordered set, then $\lambda\text{-ldim}\, G < \omega_1$ [4] if and only if every gap in $G$ has character $c_{00}$ and every element in $G$ has character $(1, 1)$ or $(\omega, 1)$ or $(1, \omega^*)$ or $(\omega, \omega^*)$. If this condition is satisfied, then the sufficient condition for $\lambda\text{-ldim}\, G \leqslant n$ is that the equivalence $\varrho$ [5] applied successively to its own equivalence classes should yield after at most $n$ iteration to the universal equivalence.*

## 4. Partition of type $a$ of a linearly ordered set.

4.1. Let $\mathcal{S}$ be a system of intervals in a linearly ordered set $G$ such that any two distinct intervals of this system have at most one point in common. For $X, Y \in \mathcal{S}$ we put $X \leqslant Y$ if and only if $x \leqslant y$ for all $x \in X$, $y \in Y$. Relation $\leqslant$ is clearly the linear order of the set $\mathcal{S}$; this order will be called a *natural order*.

Let $a, \beta$ be order types. We put $a \subseteq \beta$, when there exist linearly ordered sets $A, B$ such that $\overline{A} = a$, $\overline{B} = \beta$, $A \subseteq B$ [6]. Note that, by the definition, a one-point set is also an interval. In the sequel, one-point sets are also counted as intervals but all other sets are assumed to be non-trivial.

Let $G$ be a linearly ordered set and let $a$ be an order type [7]. The system $\mathcal{S}$ of intervals in the set $G$ will be called a *partition* of type $a$ of the set $G$ if the following conditions are satisfied:

(1) $G \in \mathcal{S}$;

(2) $X, Y \in \mathcal{S} \Rightarrow X \cap Y = X$ or $X \cap Y = Y$ or $X \cap Y = \emptyset$;

(3) $X \in \mathcal{S}$, $X$ is non-trivial $\Rightarrow$ there exists a subsystem $\mathcal{S}(X) \subseteq \mathcal{S}$ containing disjoint intervals from $\mathcal{S}$ and having these properties: $X = \bigcup_{Y \in \mathcal{S}(X)} Y$, $\overline{\mathcal{S}(X)} = a(X) \subseteq a$ with respect to the natural order, $Y \in \mathcal{S}(X)$, $Z \in \mathcal{S}$, $Y \subseteq Z \subseteq X \Rightarrow Z = Y$ or $Z = X$.

(4) An intersection of any monotone subsystem $\mathcal{S}_1 \subseteq \mathcal{S}$ is empty or a one-point set or an interval belonging to $\mathcal{S}$.

Note that by definition every dyadic partition of set $G$ is a partition of type $a$.

In the system $\mathcal{S}$ we define the order $\precsim$ in this way: $X \precsim Y \Leftrightarrow X \not\supseteq Y$. Then we have the

LEMMA. *$\mathcal{S}$ does not contain any infinite decreasing chain with respect to order $\precsim$.*

Proof [8]. Suppose that $\mathcal{S}$ contains an infinite decreasing chain $X_1 \succsim X_2 \succsim \ldots \succsim X_n \succsim \ldots$ Then $\bigcup_{n=1}^{\infty} X_n = X$ is an interval in $G$. From (1) it follows that there exist intervals $Y \in \mathcal{S}$ such that $Y \supseteq X$. Let us denote the set of these intervals by $\mathcal{S}_1$. If $Y_1 \in \mathcal{S}_1$, $Y_2 \in \mathcal{S}_1$, then $Y_1 \cap Y_2 \supseteq X$ and, according to (2), $Y_1 \cap Y_2 = Y_1$ or $Y_1 \cap Y_2 = Y_2$, i.e. $Y_1 \subseteq Y_2$ or $Y_2 \subseteq Y_1$. Hence $\mathcal{S}_1$ is a monotone system of intervals belonging to $\mathcal{S}$ and, according to (4), $\bigcap_{Y \in \mathcal{S}_1} Y = Y_0 \in \mathcal{S}$ and clearly $Y_0 \in \mathcal{S}_1$. Now, according to (3), there exists a subsystem $\mathcal{S}(Y_0)$ such that $Y_0 = \bigcup_{Y \in \mathcal{S}(Y_0)} Y$ with the properties $\overline{\mathcal{S}(Y_0)} = a(Y_0) \subseteq a$; $Y \in \mathcal{S}(Y_0)$, $Z \in \mathcal{S}$, $Y \subseteq Z \subseteq Y_1 \Rightarrow Z = Y$ or $Z = Y_0$. As $Y_0 = \bigcap_{Y \in \mathcal{S}_1} Y$, the relation $Y \supseteq X$ is not valid for any

(3) A set $K$ is a segment of a chain $G$ if $G = K \oplus H$.

(4) $\lambda$ denotes the order type of the set of all real numbers.

(5) $\varrho$ is defined in following way: $x, y \in G$, $x \leqslant y$, $x \varrho y \Leftrightarrow$ the closed interval $[x, y]$ is isomorphic with a subset of a set of type $\lambda$.

(6) It is easy to see that relation $\subseteq$ is reflexive and transitive; therefore it is the quasiorder of any set of order types. Example: $\lambda \subseteq \lambda + 1 \subseteq \lambda$, $\lambda \neq \lambda + 1$ shows that this relation is not antisymmetric, therefore it is not an order.

(7) $a \neq 0$, $a \neq 1$ because all sets and all type are assumed to be non-trivial.

(8) This proof if accomplished in the same way as the proof of Lemma 1 in [8].

interval $Y \in \mathcal{S}(Y_0)$. Hence there exists an integer $n_0$ such that $Y \not\supseteq X_{n_0}$ for any $Y \in \mathcal{S}(Y_0)$. Hence there exist $Y_1 \in \mathcal{S}(Y_0)$, $Y_2 \in \mathcal{S}(Y_0)$, $Y_1 \neq Y_2$ such that $X_{n_0} \cap Y_1 \neq \emptyset$, $X_{n_0} \cap Y_2 \neq \emptyset$. This implies $X_{n_0} \cap Y_1 \neq X_{n_0}$, $X_{n_0} \cap Y_2 \neq X_{n_0}$ and according to (2), $X_{n_0} \cap Y_1 = Y_1$, $X_{n_0} \cap Y_2 = Y_2$ so that $Y_1 \subset X_{n_0}$. It follows, according to (3), that the relation $X_{n_0} \subset Y_0$ is not valid. As $X_{n_0} \subseteq Y_0$ this implies $X_{n_0} = Y_0$ so that $X_{n_0+1} \supset Y_0$ and this is a contradiction, because $Y_0 \subseteq X = \bigcup_{n=1}^{\infty} X_n$.

Let $A$ be an interval in $G$. Then the subsystem $\mathcal{S}_1 \subseteq \mathcal{S}$ containing those intervals that contain $A$ as a proper subset is, according to the lemma, a well-ordered set with respect to order $\precsim$. We define the order of $A$ as the ordinal which is the type of the set $\mathcal{S}_1$. The least ordinal $\gamma$ such that there exists no interval of order greater that $\gamma$ will be called the *order of the partition* $\mathcal{S}$.

4.2. *Let $G$ be a linearly ordered set. Let us assign to every non-trivial interval $I \subseteq G$ a system $\mathcal{S}_I$ of intervals having these properties*:

(1′) *any two intervals of the system $\mathcal{S}_I$ are disjoint,*

(2′) $\bigcup_{X \in \mathcal{S}_I} X = I$,

(3′) $\overline{\mathcal{S}_I} = a_I \subseteq a$ *with respect to the natural order.*

We shall construct a system $\mathcal{S}$ of intervals in $G$ and we shall assign to every interval $X \in \mathcal{S}$ a certain ordinal, the so-called order of this interval in the following manner: The only interval of order 0 is $G$. Let us suppose that we have constructed all intervalls of order $\gamma$ for every $\gamma < \delta$. Then we define the intervals of order $\delta$ in the following way:

1) If $\delta$ is an isolated ordinal and if $I \in \mathcal{S}$ is any non-trivial interval of order $\delta - 1$, then every interval $X \in \mathcal{S}_I$ will be called an interval of order $\delta$ and we shall put $X \in \mathcal{S}$.

2) If $\delta$ is a limit ordinal we shall put $\bigcap_{\nu<\delta} I_\nu = X$ where $I_\nu \in \mathcal{S}$ is any interval of order $\nu$. Then $X$ is an empty set or an interval. If $X$ is an interval, we shall put $X \in \mathcal{S}$ and we shall call $X$ the interval of order $\delta$.

We shall prove that there exists a certain ordinal $\varepsilon$ such that there exists no interval of order $\varepsilon$. If card $G \leqslant \aleph_i$ and if we suppose the existence of the interval $I_{\omega_{i+1}}$ of order $\omega_{i+1}$, then all sets $I_\nu - I_{\nu+1}$, where $\{I_\nu\}_{\nu<\omega_{i+1}}$ is a monotone sequence of intervals of order $\nu$, are non-void and disjoint so that card$G \geqslant \aleph_{i+1}$, and that is a contradiction.

Let $\gamma$ be the supremum of orders of all constructed intervals. It is not difficult to prove that the system $\mathcal{S}$ of all intervals so constructed forms a partition of type $a$ of the set $G$ and that the order of this partition is $\gamma$.



4.3. Theorem 9. *Let $G$ be a linearly ordered set and let $\beta$ be an ordinal. Then to every partition of type $a$ of the set $G$ of order $\beta$ there exists an $a$-representation of the set $G$ whose $a$-economy is $\beta$.*

Proof. Let $\mathcal{S}$ be a given partition of type $a$ of set $G$ of order $\beta$. Let $L$ be any linearly ordered set of type $a$, and let $B = \{b_\nu\}_{\nu<\beta}$ be any well-ordered set of type $\beta$. If $I \in \mathcal{S}$ is any non-trivial interval, then according to (3) there exists a subsystem $\mathcal{S}(I) \subseteq \mathcal{S}$ such that $I = \bigcup_{X \in \mathcal{S}(I)} X$ and $\overline{\mathcal{S}(I)} = a(I) \subseteq a$ with respect to the natural order. For every such interval $I$, let us choose any isomorphism of system $\mathcal{S}(I)$ into $L$ and let us denote it by $\varphi_I$.

Now let $x \in G$. Let us denote by $I(x)$ the system of all intervals from $\mathcal{S}$ which contain $\{x\}$ as a proper subset, $I(x)$ is a well-ordered set with respect to order $\precsim$ and it is of type $\beta(x) \leqslant \beta$. Hence $I(x)$ can be written in the form

$$I(x) = \{G = I_0(x) \precsim I_1(x) \precsim \ldots \precsim I_\lambda(x) \precsim \ldots \mid \lambda < \beta(x)\}.$$

If $\beta(x)$ is a limit ordinal, then $\{x\} = \bigcap_{\lambda<\beta(x)} I_\lambda(x)$. If $\beta(x)$ is an isolated ordinal, then clearly $\{x\} \in \mathcal{S}\big(I_{\beta(x)-1}(x)\big)$. We shall put in this case $I_{\beta(x)}(x) = \{x\}$.

We assign to $x \in G$ an element $\psi(x) \in {}^{B}L$ in this way:

$$\psi(x)(b_\lambda) = \begin{cases} \varphi_{I_\lambda(x)}[I_{\lambda+1}(x)] & \text{for} \quad \lambda < \beta(x), \\ l_0, & \text{for} \quad \beta(x) \leqslant \lambda < \beta, \end{cases}$$

where $l_0 \in L$ is any element for $\beta(x) \leqslant \lambda < \beta$. We shall show that $\psi$ is an isomorphism of $G$ onto a certain subset of ${}^{B}L$. Let $x, y \in G$, $x < y$. Then there exists an ordinal $\delta < \beta$ such that $I_\lambda(x) = I_\lambda(y)$ for $\lambda < \delta$ whereas $I_\delta(x) < I_\delta(y)$ with respect to the natural order. This ordinal $\delta$ is necessarily an isolated ordinal because if $\delta$ is a limit ordinal, then $I_\delta(x) = \bigcap_{\lambda<\delta} I_\lambda(x)$ so that $I_\lambda(x) = I_\lambda(y)$ for $\lambda < \delta$ implies $I_\delta(x) = I_\delta(y)$. This implies $\psi(x)(b_\lambda) = \varphi_{I_\lambda(x)}[I_{\lambda+1}(x)] = \varphi_{I_\lambda(y)}[I_{\lambda+1}(y)] = \psi(y)(b_\lambda)$ for $\lambda < \delta - 1$ whereas $\psi(x)(b_{\delta-1}) = \varphi_{I_{\delta-1}(x)}[I_\delta(x)] < \varphi_{I_{\delta-1}(y)}[I_\delta(y)] = \psi(y)(b_{\delta-1})$ so that $\psi(x) < \psi(y)$. It follows simultaneously that $\psi$ is one-one and therefore it is an isomorphism.

4.4. Theorem 10. *Let $G$ be a linearly ordered set and let $\beta$ be an ordinal. Then to every $a$-representation $F$ of the set $G$ such that $a$-ek$\{G, F\} = \beta$ there exists a partition of type $a$ of the set $G$ of order $\leqslant \beta$.*

Proof. We can suppose without loss of generality directly $G = F \subseteq {}^{B}L$ where $L$ is a chain of type $a$ and $B = \{b_\nu\}_{\nu<\beta}$ is a well-ordered set of type $\beta$. To every non-trivial interval $I \subseteq G$ there exists an ordinal

$\delta(I)$ such that for every two elements $f, g \in I$ we have $f(b_\nu) = g(b_\nu)$ for $\nu < \delta(I)$, whereas there exist two elements $k, l \in I$ such that $k(b_{\delta(I)}) < l(b_{\delta(I)})$. If $L_1 = \{f(b_{\delta(I)}) |\ f \in G,\ f \in I\}$, then let

$$I_i = \{f |\ f \in G,\ f \in I,\ f(b_{\delta(I)}) = i\} \quad \text{for any} \quad i \in L_1 .$$

It is easy to see that the system $\mathcal{S}_I = \{I_i |\ i \in L_1\}$ fulfils the conditions (1′)-(3′) so that this according to 4.2 determines a certain partition of type $\alpha$ of the set $G$. It is sufficient to prove that the order of this partition is $\leqslant \beta$.

Accordingly we shall prove that for any interval $I$ of order $\gamma$ of this partition we have $\gamma \leqslant \delta(I)$. The proof will be made by transfinite induction. For $\gamma = 0$ the relation is clear. Let $\gamma \leqslant \delta(I)$ for every $\gamma < \mu$ and for every non-trivial interval $I$ of order $\gamma$ of the given partition. If $\mu$ is an isolated ordinal then $\mu - 1 \leqslant \delta(I)$ for every non-trivial interval $I$ of order $\mu - 1$. Let $I$ be any such interval. Then $f(b_\nu) = g(b_\nu)$ for $\nu < \mu - 1$ and for any two elements $f, g \in I$, so that for the corresponding intervals $I_i \subseteq I$ of order $\mu$ we have $\delta(I_i) > \delta(I) \geqslant \mu - 1$, i.e. $\mu \leqslant \delta(I_i)$. If $\mu$ is a limit ordinal, then $\gamma \leqslant \delta(I)$ for every $\gamma < \mu$ and every non-trivial interval $I$ of order $\gamma$. Hence $\mu = \sup_{\gamma<\mu} \gamma \leqslant \sup \delta(I)$ ($I$ being of order $\gamma$) $\leqslant \delta(I)$ ($I$ being of order $\mu$).

Now for every non-trivial interval $I$ of the given partition we have $\delta(I) < \beta$, so that the order of every interval is $< \beta$. This implies that the order of the constructed partition is $\leqslant \beta$.

Conclusion. *Let $G$ be a linearly ordered set and let $\alpha$ be an order type. Then $\alpha$-$\operatorname{ldim} G = \min \beta$, where $\beta$ runs over the orders of the partitions of type $\alpha$ of the set $G$.*

**5. Two examples.** In this section $R$ denotes the set of all real numbers, $P$ denotes the set of all rational numbers, both with the natural order and $\lambda = \overline{R}$, $\eta = \overline{P}$.

1. Let $G$ be the set of type $\eta \cdot \lambda$ (⁹). $G$ is therefore the set of all pairs $[x, y]$ where $x$ is real and $y$ rational, or the set of all points of the open unit square with an arbitrary first coordinate and rational second coordinate and this set is ordered lexicographically.

We shall prove:

(a) $2$-$\operatorname{ldim} G = \omega \cdot 2$,

(b) $\eta$-$\operatorname{ldim} G = \omega + 1$,

(c) $\lambda$-$\operatorname{ldim} G = 2$.

(⁹) The product of order types is understood in Hausdorff's sense ([1]).



Proof. (a) We shall construct a dyadic partition of $G$ of order $\omega \cdot 2$ by sequential halving: we shall put

$$G_0 = G, \quad G_1^0 = \{[x, y] |\ 0 < x < \tfrac{1}{2}\}.$$
$$G_1^1 = \{[x, y] |\ \tfrac{1}{2} \leqslant x < 1\}, \quad G_2^{00} = \{[x, y] |\ 0 < x < \tfrac{1}{4}\},$$
$$G_2^{01} = \{[x, y] |\ \tfrac{1}{4} \leqslant x < \tfrac{1}{2}\}, \quad G_2^{10} = \{[x, y] |\ \tfrac{1}{2} \leqslant x < \tfrac{3}{4}\},$$
$$G_2^{11} = \{[x, y] |\ \tfrac{3}{4} \leqslant x < 1\}, \quad \ldots, \quad G_\omega^{i_1 i_2 \ldots} = \{[x, y] |\ x = \text{const}\},$$
$$G_{\omega+1}^{i_1 i_2 \ldots 0} = \{[x, y] |\ x = \text{const},\ 0 < y < \tfrac{1}{2}\},$$
$$G_{\omega+2}^{i_1 i_2 \ldots 1} = \{[x, y] |\ x = \text{const},\ \tfrac{1}{2} \leqslant y < 1\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$G_{\omega \cdot 1}^{i_1 i_2 \ldots j_1 j_2 \ldots} = \{[x, y] |\ x = \text{const},\ y = \text{const}\}.$$

It remains to prove that there exists no dyadic partition of $G$ of order less than $\omega \cdot 2$. Let $\mathcal{S}$ be any dyadic partition of $G$. Let us consider the subset $\mathcal{S}_1 \subseteq \mathcal{S}$ containing those intervals from $\mathcal{S}$ which are of order less than $\omega$. The set of all end-points of these intervals is countable; this implies that there exists at least one real $x$ such that the abscissa $[x, y]$, $0 < y < 1$ is a subset of some interval $I$ of order $\omega$ from $\mathcal{S}$. But then it is clear that at least $\omega$ steps are necessary for this interval $I$ to complete the given partition.

(b) It is not difficult to construct a partition of type $\eta$ of $G$ having order $\omega + 1$. The first $\omega$ steps are the same as an in (a); in the $(\omega+1)$st step each interval $[x, y]$: $x = \text{const}$, $0 < y < 1$, is divided into one-point intervals. The proof that there exists no partition of type $\eta$ of $G$ having order less than $\omega + 1$ is accomplished in the same way as in (a).

(c) As $\eta \cdot \lambda \subseteq {}^2\lambda$, we have $\lambda$-$\operatorname{ldim} G \leqslant 2$ according to Theorem 4. But $\lambda$-$\operatorname{ldim} G > 1$ because $\lambda$-$\operatorname{ldim} G = 1$ implies $G \cong G' \subseteq R$ and this is impossible because $G$ contains a non-denumerable system of disjoint non-trivial intervals.

2. Let $G$ be the set of type $\lambda \cdot \eta$, i.e. the set of all points of the open unit square with a rational first coordinate and an arbitrary second coordinate, ordered lexicographically. Then we have:

(a) $\lambda$-$\operatorname{ldim} G = 1$,

(b) $2$-$\operatorname{ldim} G = \omega$,

(c) $\eta$-$\operatorname{ldim} G = \omega$.

Proof. (a) As $R$ contains a subset $R'$ isomorphic with $G$ (for instance, Cantor's set), we have $\lambda$-$\operatorname{ldim} G = 1$ according to Theorem 5.

(b) As $2$-$\operatorname{ldim} R = \omega$ (Novotný [3]), we have $2$-$\operatorname{ldim} G = 2$-$\operatorname{ldim} R' \leqslant 2$-$\operatorname{l dim} R = \omega$. On the other hand, surely $2$-$\operatorname{ldim} G \geqslant \omega$ so that $2$-$\operatorname{ldim} G = \omega$.

(c) As $2$-$\operatorname{ldim} G = \omega$, we have $\eta$-$\operatorname{ldim} G \leqslant \omega$ according to Theorem 6.

Let us suppose that $\eta$-ldim $G < \omega$, i.e. $\eta$-ldim $G = n$ where $n < \omega$. Then $G \cong G' \subseteq {}^nP$ according to Theorem 4 and this is impossible because card $G = 2^{\aleph_0}$, card ${}^nP = \aleph_0$. Therefore $\eta$-ldim $G = \omega$.

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# Semigroups and clusters of indecomposability *

by

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In [4] and [9] we have generalized indecomposable continua in various ways; here we wish to consider these types of continua as topological semigroups. The examples in [4] and [9] are based upon Wilder's constructions for his Theorems 1 and 8 of [15], pp. 275-278, 290-292; these constructions and examples are complicated. However, we also give below simpler examples for which our definitions and theorems hold.

Below, $S$ is a topological semigroup, which we call a semigroup, such that there is a continuous mapping $m\colon S\times S\to S$, called multiplication, where $S$ is a Hausdorff space and $m$ is associative. For $x, y \in S$, we write $xy = m(x, y)$; and $AB = \{xy\colon x \in A,\ y \in B\}$. We let $u$ be the unit of $S$ and 0 be the zero, if these exist, where, for all $x \in S$, $xu = x = ux$ and $x0 = 0 = 0x$. We use $E$ to denote the set of idempotents of $S$, where for $e \in E$, $ee = e$. We recall that a non-null subset $A$ of $S$ is a *left ideal* if and only if $SA \subset A$ and it is a *right ideal* if $AS \subset A$; it is an *ideal* if and only if it is both a left and a right ideal. We denote the *minimal ideal* by $K$ and the null set by $\emptyset$.

Basic definitions and results concerning semigroups are in [14]; for topology they are in [6] and [16]. By a continuum, or a subcontinuum of $S$, we mean a connected subset of $S$ which is closed in $S$. We think of $S$ imbedded in another space, so that the connected semigroup $S$ need not be the same as its closure $\bar{S}$; but then the multiplication operation $m$ is extendable to $\bar{S}$; this is true for the examples of connected semigroups in [5] and [7].

Definitions. We say, for $A \subset B$, that $A$ is *region-containing* in $B$ if the interior of $A$ with respect to $B$ is non-null; that is if there exists a region (neighbourhood) $R$ such that $A \supset R \cap B$: if $x \in R \cap B$, we say that $A$ is region-containing at $x$. The connected set $S$ has an *$n$-fold set* $\bigcup Z_j$ $(j = 1, 2, \ldots, n)$ *of indecomposability* if and only if every region-containing connected subset $W$ of $S$ is such that $\overline{W}$, the closure of $W$ in $S$, contains some $Z_j$, and we take each $Z_j$ non-null: if $n = 1$, we let $Z = Z_1$ and say $S$ has a set $Z$ of indecomposability, and if $S$ is a con-

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