easily arrange that the sets $T^m A$ are disjoint, of positive measure, and cover X; and we define μ on $T^m A$ by

$$\mu B = \int_{T^{-n}B} \varphi_n(x)\,d\lambda(x) \qquad (B \subset T^n A).$$

In other words, the functions ω_n can be prescribed arbitrarily on any proper subset A of X, provided the transformation T is allowed to be compressible and the total measure μX is not required to be finite.

Thus, for example, we can arrange that $\omega_n \to 0$ but $\sum \omega_n = \infty$ throughout any such A; and so on.

As another application of this construction, suppose we start with an *incompressible* transformation T_1 on (X, λ) as above; then we can find a *compressible* transformation T_2 on (X, μ) such that μ and λ agree on A, and further $\omega_n(x; T_2) = \omega_n(x; T_1)$ on A for all n (these ω's being calculated in terms of μ, λ, respectively). Thus, it is impossible to tell, solely from the behavior of the functions ω_n on a proper subset A of X, whether or not the transformation T is compressible on X.

8.7. We conclude with one more corollary to Theorem 4 (8.2):

COROLLARY. *If T is incompressible, then for almost all $x \in X$*

$$\limsup_{n\to\infty} \omega_n(x) = \limsup_{n\to-\infty} \omega_n(x) = \sup_{-\infty<n<\infty} \omega_n(x) > 0 .$$

For the set where $\limsup_{n\to\infty} \omega_n(x) < \sup_n \omega_n(x)$ is just X_1, and so is null from Theorem 4. This proves one equality; the other follows from the first applied to T^{-1}.

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On products of sets in a locally compact group

by

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Introduction. Let G be a locally compact group, μ a left Haar measure on G, μ_* the corresponding inner measure. The group G is said to be *unimodular* if the left invariant measure μ is also right invariant; this happens, for instance, when G is either compact, or abelian, or discrete, or a semi-simple Lie group.

Let further A and B be given non-empty subsets of G. Then AB will denote the set of all elements $x \in G$ which admit at least one representation as a product $x = ab$ with $a \in A$ and $b \in B$.

THEOREM 1.1. *Suppose that G is unimodular and connected. Then*

$$\mu_*(AB) \geqslant \mu_*(A) + \mu_*(B), \tag{1.1}$$

unless $\mu(G) < \mu_*(A) + \mu_*(B)$, in which case G is compact and $AB = G$.

The special case, where G is abelian, is due to Kneser [6]. The further special case, that G is also compact and second countable, is due to Shields [9]. It remains to determine the class of pairs (A, B) such that (1.1) holds with the equality sign. For an abelian connected group, this problem was solved by Kneser [6].

THEOREM 1.2. *Suppose that G is unimodular, and further that there exists a pair of non-empty subsets A and B of G such that*

$$\mu_*(AB) < \mu_*(A) + \mu_*(B). \tag{1.2}$$

Assertion: *G contains at least one open and compact subgroup F of size $\mu(F) \leqslant \mu_*(AB)$.*

More precisely, the set AB is both open and compact, and the open and compact subgroup F can be chosen in such a way that

$$aFb \subset AB \quad \textit{whenever} \quad a \in A \textit{ and } b \in B. \tag{1.3}$$

Finally, if a subgroup F satisfies (1.3) *then $AB = A_1B_1$ as soon as*

$$A_1 \subset AF, \quad B_1 \subset FB, \quad \mu_*(A_1) + \mu_*(B_1) > \mu(AB). \tag{1.4}$$

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If G is connected then the only subgroup of G which is both open and closed is G itself, thus, Theorem 1.2 implies Theorem 1.1. A complete description of the special situation $\mu_*(AB)=\mu(B)$ (A and B measurable, $\mu(A)>0$, $\mu(B)<\infty$), was already given by Macbeath [7].

Given the compact open subgroup F of G, the largest closed and normal subgroup of G contained in F is given by

$$H=\bigcap_{x\in G} xFx^{-1}. \tag{1.5}$$

This group H is also open, provided that G is either abelian or compact. Using this remark one easily (section 4) obtains from Theorem 1.2 the following result. Here and in the sequel, if D is a set then $[D]$ denotes the number of elements in D.

THEOREM 1.3. *Suppose that G is either abelian or compact. Then the following construction yields precisely all the pairs of non-empty subsets A and B of G for which* (1.2) *holds.*

Let H be any open and compact normal subgroup of G. Choose A', B' as any pair of non-empty finite subsets of the (*discrete*) *group G/H such that*

$$[A'B']<[A']+[B']. \tag{1.6}$$

Let σ denote the quotient mapping $G\to G/H$. Now, choose A as any subset of $\sigma^{-1}A'$, B as any subset of $\sigma^{-1}B'$ in such a way that

$$[A'B']\mu(H)<\mu_*(A)+\mu_*(B). \tag{1.7}$$

We mention that in this case always $AB=\sigma^{-1}(A'B')$.

For the case where G is abelian, this result is due to Kneser [6]. His proof is quite different from ours.

In a natural way, Theorem 1.3 leads to the problem of determining the pairs (A', B') of finite subsets, of a given discrete group, for which (1.6) holds. For an abelian group this problem was solved by the author [5]. The general problem remains open and seems rather difficult; a few necessary properties of the pair (A', B') may be found in [4].

For a locally compact group which is not unimodular some results are given in section 5. It appears that the unimodular case is definitely more interesting.

Let us finally mention that in the present paper we also consider the case of a locally compact *semigroup* admitting an invariant measure. In fact, the main lemma (Theorem 3.1) in the proof of Theorem 1.2 is given in terms of this more general case.

2. An auxiliary result. In the sequel, if X is a locally compact Hausdorff space then a subset E of X is said to be *measurable* (or *Borel measurable*) if it belongs to the smallest σ-field of subsets of X containing all open (and thus all closed) subsets of X.

Let ν be a nonnegative regular Borel measure on X, in particular, $0\leqslant\nu(C)<\infty$ for each compact subset C of X. A subset E of X is said to be *ν-summable* if for each $\varepsilon>0$ there exists an open set U and a compact set C such that

$$C\subset E\subset U \quad \text{and} \quad \nu(U\cap\bar{C})<\varepsilon,$$

the bar denoting complementation. A set $E\subset X$ is said to be *ν-measurable* if $E\cap C$ is ν-summable for each compact subset C of X. In particular, each (Borel) measurable set is also ν-measurable [1]. Finally, a real-valued function f on X is said to be ν-measurable if $f^{-1}(B)$ is ν-measurable for each (Borel) measurable subset of the reals.

In this section, G denotes a fixed locally compact Hausdorff space. Further,

$$(x,y)\to xy$$

denotes a given mapping of $G\times G$ into G which is measurable in the sense that the inverse image of each measurable set is measurable; (later on, we shall make the stronger assumption that this mapping is continuous and associative). Finally, μ denotes a fixed nonnegative regular Borel measure on G.

Let Γ, Γ_1 and Γ_2 be given subsets of G such that Γ is (Borel) measurable and that Γ_1 and Γ_2 are μ-summable:

$$\mu(\Gamma_1)<\infty, \quad \mu(\Gamma_2)<\infty. \tag{2.1}$$

Denote by $\Pi=\Pi(\Gamma,\Gamma_1,\Gamma_2)$ the collection of all pairs (A,B) of μ-summable subsets A and B of G such that

$$\mu(A\cap\bar{\Gamma}_1)=0, \quad \mu(B\cap\bar{\Gamma}_2)=0, \tag{2.2}$$

and

$$(\mu\times\mu)\{(x,y):\ x\in A,\ y\in B,\ xy\notin\Gamma\}=0. \tag{2.3}$$

THEOREM 2.1. *Assume that Π is non-empty. Then there exists at least one pair (A_0,B_0) in Π with the property that for any other pair (A,B) in Π one has either*

$$\mu(A)+\mu(B)<\mu(A_0)+\mu(B_0), \tag{2.4}$$

or

$$\mu(A)+\mu(B)=\mu(A_0)+\mu(B_0) \quad \textit{and} \quad \mu(A)\leqslant\mu(A_0). \tag{2.5}$$

By the way, it is easily seen that Theorem 2.1 in turn implies the analogous result for the more general case, where we allow Γ_1 and Γ_2 to be arbitrary subsets of G having a finite inner measure.

Let $h(x,y)$ denote the function on $G\times G$ defined by

$$h(x,y)=\begin{cases}1 & \text{if} \quad xy\notin\Gamma,\\ 0 & \text{if} \quad xy\in\Gamma.\end{cases} \tag{2.6}$$

Note that the function $h(x, y)$ is (Borel) measurable and therefore $\mu\times\mu$-measurable. Further, condition (2.3) can be written as

$$\iint \chi_A(x)\chi_B(y)h(x, y)\mu(dx)\mu(dy) = 0\,, \tag{2.7}$$

where χ_A denotes the characteristic function of the set A, similarly, χ_B. The following result is implicit in the literature.

LEMMA 2.2. *For each number $\varepsilon > 0$ one can find finitely many bounded and continuous functions $\varphi_k(x), \psi_k(x)$ $(k = 1, \ldots, N)$ such that*

$$\iint_{\Gamma_1\times\Gamma_2}\Big|h(x, y) - \sum_{k=1}^{N}\varphi_k(x)\psi_k(y)\Big|\mu(dx)\mu(dy) < \varepsilon\,.$$

Proof. Let δ and η be given positive numbers. Choose U_i open and C_i compact such that

$$C_i\subset\Gamma_i\subset U_i\,,\quad \mu(U_i\cap\bar{C}_i) < \delta\quad (i = 1, 2)\,.$$

Let further V_i be an open set with compact closure such that $C_i\subset V_i\subset U_i$.

Because the function $h(x, y)$ is $\mu\times\mu$-measurable, there exists ([1], p. 180) a compact set

$$K\subset C_1\times C_2\quad\text{with}\quad (\mu\times\mu)\{(C_1\times C_2)\cap\bar{K}\} < \delta^2\,,$$

and such that the restriction of $h(x, y)$ to K is continuous. Noting that $h(x, y) = 0$ or 1, one easily ([3], p. 216) obtains a continuous function $h_1(x, y)$ on $G\times G$ which on K coincides with $h(x, y)$, while $0\leqslant h_1(x, y)\leqslant 1$ and $h_1(x, y) = 0$ for $(x, y)\notin V_1\times V_2$. In particular,

$$|h(x, y) - h_1(x, y)|\begin{cases} = 0 & \text{for}\quad (x, y)\in K\,,\\ \leqslant 1 & \text{for}\quad (x, y)\in G\times G\,.\end{cases}$$

Moreover ([1], p. 89), there exist finitely many continuous functions $\varphi_k(x), \psi_k(x)$ having a compact support (contained in V_1 and V_2, respectively), $k = 1, \ldots, N$, such that

$$\Big|h_1(x, y) - \sum_{k=1}^{N}\varphi_k(x)\psi_k(y)\Big| < \eta\,,$$

for each $(x, y)\in G\times G$. Choosing δ and η sufficiently small, one obtains the stated assertion.

Proof of Theorem 2.1. Let L_1 denote the real linear vector space of all real-valued μ-measurable functions f on G which have a finite norm:

$$\|f\| = \int|f(x)|\mu(dx)\,.$$

Here, we identify any two (so-called equivalent) functions f_1 and f_2 for which

$$\mu^*\{x\colon f_1(x)\neq f_2(x)\} = 0\,.$$

We shall supply L_1 with its *weak topology*. In particular, if $\varphi(x)$ is any bounded μ-measurable function on G then

$$\langle f, \varphi\rangle = \int (fx)\varphi(x)\mu(dx) \tag{2.8}$$

is a continuous function of $f\in L_1$.

Now, consider the subsets of L_1 defined by

$$K_i = \{f\in L_1\colon\ 0\leqslant f(x)\leqslant 1,\ f(x) = 0\quad\text{for}\quad x\notin\Gamma_i\}\,. \tag{2.9}$$

Here, $i = 1$ or 2. More precisely, if $f\in L_1$ then $f\in K_i$ if and only if at least one function in the equivalence class of f satisfies the stated conditions, that is, if each function in the equivalence class of f satisfies the stated conditions outside some set of μ-measure zero.

We claim that K_i is a closed subset of L_1. For, let i be fixed and let $\{f_a\}$ be a generalized sequence of elements $f_a\in K_i$ converging to $f_0\in L_1$. Applying (2.8) with φ as the characteristic function of any μ-measurable set E, one obtains

$$0\leqslant\int_E f_0\,d\mu\leqslant\mu(E);\qquad \int_{\bar{\Gamma}_i} f_0\,d\mu = 0\,.$$

Choosing E as the set of points x where $f_0(x)\leqslant -\varepsilon < 0$ (or $f_0(x)\geqslant 1+\varepsilon > 1$, respectively), it follows that $f_0\in K_i$.

Using (2.1) it follows that $\|f\|$ is bounded on K_i (namely, by $\mu(\Gamma_i) < \infty$) and further that

$$\lim_{n\to\infty}\int_{E_n} f(x)\mu(dx) = 0$$

holds uniformly for $f\in K_i$, whenever $\{E_n\}$ is a decreasing sequence of μ-measurable sets with empty intersection. This implies ([2], p. 292) that K_i is sequentially compact.

Next, consider $L_1\times L_2$ with the product topology having $K_1\times K_2$ as a subset which is both closed and sequentially compact. Clearly, the expression

$$\sum_{k=1}^{N}\iint_{\Gamma_1\times\Gamma_2} f(x)g(y)\varphi_k(x)\psi_k(y)\mu(dx)\mu(dy)$$

defines a continuous function of the pair $(f, g)\in L_1\times L_2$, whenever $\varphi_k(x)$ and $\psi_k(x)$ are bounded μ-measurable functions on G $(k = 1, \ldots, N;\ N < \infty)$. The limit of a uniformly converging sequence of continuous functions being continuous, it follows by Lemma 2.2 and the definitions of K_1 and K_2 that the integral

$$\iint f(x)g(y)h(x, y)\mu(dx)\mu(dy)$$

defines a function which is continuous on $K_1\times K_2$.

Now, consider the set M consisting of all $(f, g) \in K_1 \times K_2$ for which

$$\iint f(x) g(y) h(x, y) \mu(dx) \mu(dy) = 0 . \tag{2.10}$$

By the above remarks, we have that M is a subset of $L_1 \times L_2$ which is both closed and sequentially compact. Moreover, by the definitions of Π and M, see (2.7),

$$(A, B) \in \Pi \Rightarrow (\chi_A, \chi_B) \in M . \tag{2.11}$$

Hence, M is non-empty.

Clearly, the expression

$$\int f(x) \mu(dx) + \int g(y) \mu(dy)$$

defines a continuous function of $(f, g) \in L_1 \times L_2$. On M, it assumes its (finite) maximal value at at least one point. In the non-empty closed and sequentially compact subset of M, where this maximum is assumed, there in turn exists a point (f_0, g_0) where the continuous function

$$\int f(x) \mu(dx)$$

assumes its maximum value. We claim that the pair of sets

$$A_0 = \{x\colon f_0(x) > 0\}, \qquad B_0 = \{x\colon g_0(x) > 0\} \tag{2.12}$$

has all the required properties. That

$$(A_0, B_0) \in \Pi \tag{2.13}$$

follows easily from the definitions of M and Π.

Further, by $f_0 \in K_1$ and $g_0 \in K_2$, we have

$$\int f_0 d\mu \leqslant \mu(A_0) = \int \chi_{A_0} d\mu$$

and

$$\int g_0 d\mu \leqslant \mu(B_0) = \int \chi_{B_0} d\mu .$$

It follows, by (2.11), (2.13) and the maximal character of (f_0, g_0), that here the equality signs hold. By the same token, if $(A, B) \in \Pi$ then

$$\mu(A) + \mu(B) = \int \chi_A d\mu + \int \chi_B d\mu \leqslant \int f_0 d\mu + \int g_0 d\mu = \mu(A_0) + \mu(B_0) .$$

Finally, if here the equality signs hold then

$$\mu(A) = \int \chi_A d\mu \leqslant \int f_0 d\mu = \mu(A_0) .$$

This proves Theorem 2.1.

3. A locally compact semigroup. In this section, G will denote a locally compact semigroup, that is, a locally compact Hausdorff space together with a *continuous* mapping $(x, y) \to xy$ of $G \times G$ into G such that

$$x(yz) = (xy)z .$$

We shall further assume that there exists on G a nonnegative regular Borel measure μ (to be kept fixed) with the property that

$$\mu(xE) \geqslant \mu(E) \quad \text{and} \quad \mu(Ex) \geqslant \mu(E) , \tag{3.1}$$

for each measurable subset E of G and each element $x \in G$.

For instance, if G is discrete then (3.1) holds for the counting measure $\mu(E) = [E]$ if and only if G satisfies both cancellation laws.

As a further example, let H be any locally compact group with left Haar measure μ. Then $\mu(Ex) = \Delta(x)\mu(E)$, where $\Delta(x) > 0$ is continuous, $\Delta(xy) = \Delta(x)\Delta(y)$. Now, take G as the set of points x in H with $\Delta(x) \geqslant 1$.

The central result of the present paper is given by:

THEOREM 3.1. *Let C_0 be a given compact subset of G and let ϱ be a given number, $\varrho > 0$.*

Now, suppose that there exists a pair of measurable subsets A_0 and B_0 of C_0 for which

$$A_0 B_0 \subset C_0 \quad \textit{and} \quad \mu(A_0) + \mu(B_0) \geqslant \mu(C_0) + \varrho . \tag{3.2}$$

Assertion: *then there must exist a compact subset*

$$F \subset C_0 \tag{3.3}$$

which is a compact group of measure

$$0 < \varrho \leqslant \mu(F) \leqslant \mu(C_0) . \tag{3.4}$$

In particular, C_0 contains at least one idempotent; further, the restriction of μ to F is a Haar measure on F.

More precisely, one can find a pair of compact subsets A_0 and B_0 of C_0 satisfying (3.2), *such that*

$$F = A_0 \cap B_0$$

is a compact group, while

$$A_0 x = A_0, \qquad xB_0 = B_0 \quad \textit{for each } x \in F .$$

Proof. Let Π_1 denote the collection of all pairs (A, B) of measurable subsets of C_0 such that $AB \subset C_0$.

Then Π_1 is non-empty, in fact, it contains a pair (A_0, B_0) satisfying (3.2).

Let further Π stand for the collection of all pairs (A, B) of μ-measurable subsets of G satisfying

(3.5) $$\mu(A \cap \bar{C}_0) = 0, \quad \mu(B \cap \bar{C}_0) = 0,$$

and

(3.6) $$(\mu \times \mu)\{(x, y):\ x \in A,\ y \in B,\ xy \notin C_0\} = 0.$$

Clearly,

$$\Pi_1 \subset \Pi,$$

thus, Π is non-empty and, in fact, contains a pair (A, B) for which

(3.7) $$\mu(A) + \mu(B) \geqslant \mu(C_0) + \varrho.$$

Applying Theorem 2.1 with $\Gamma = \Gamma_1 = \Gamma_2 = C_0$, it follows that there exists a pair

$$(A_0, B_0) \in \Pi$$

with the property that, for any other pair $(A, B) \in \Pi$,

(3.8) $$\mu(A) + \mu(B) \leqslant \mu(A_0) + \mu(B_0);$$

moreover, if $(A, B) \in \Pi$ satisfies (3.8) with the equality sign then $\mu(A) \leqslant \mu(A_0)$. Observe that, by (3.7) and (3.8),

(3.9) $$\mu(A_0) + \mu(B_0) \geqslant \mu(C_0) + \varrho.$$

If D is a μ-measurable subset of G, we denote by $S(D)$ the support of D, that is, the (closed) set of all points $x \in G$ such that $\mu(D \cap U) > 0$ for *each* open neighbourhood U of x. It is easily seen that

(3.10) $$\mu\big(D \cap \overline{S(D)}\big) = 0.$$

On the other hand, it is quite possible that $\mu(D) < \mu\big(S(D)\big)$; (in $[0, 1]$, let D be the union of small intervals I_n about the nth rational number).

The pair (A_0, B_0) belongs to Π and thus satisfies (3.5) and (3.6). Using the fact that C_0 is closed and that the product xy is jointly continuous in x and y, it follows that

$$\big(S(A_0), S(B_0)\big) \in \Pi_1 \subset \Pi.$$

Hence, by (3.8),

$$\mu\big(S(A_0)\big) + \mu\big(S(B_0)\big) \leqslant \mu(A_0) + \mu(B_0),$$

thus, by (3.10),

$$\mu\big(S(A_0)\big) = \mu(A_0), \quad \mu\big(S(B_0)\big) = \mu(B_0).$$

Consequently, replacing A_0 by $S(A_0)$ and B_0 by $S(B_0)$, we may assume that $(A_0, B_0) \in \Pi_1$, that is,

(3.11) $$A_0 \subset C_0, \quad B_0 \subset C_0, \quad A_0 B_0 \subset C_0,$$

and further that

(3.12) $$A_0 = S(A_0) \quad \text{and} \quad B_0 = S(B_0).$$

In particular, the sets A_0 and B_0 are closed and hence compact (C_0 being compact).

Let us now introduce the compact set

(3.13) $$F = A_0 \cap B_0 \subset C_0.$$

By (3.9) and (3.11),

(3.14) $$0 < \varrho \leqslant \mu(F) \leqslant \mu(C_0),$$

showing that F is non-empty. We assert that

(3.15) $$d \in F \Rightarrow A_0 d = A_0 \quad \text{and} \quad dB_0 = B_0.$$

Let $d \in F$ be fixed, and consider the measurable sets

$$A' = \{a \in A_0:\ ad \in A_0\}, \qquad B' = \{b \in B_0:\ db \in B_0\},$$
$$A'' = \{a \in A_0:\ ad \notin A_0\}, \qquad B'' = \{b \in B_0:\ db \notin B_0\}.$$

Note that A', A'' and $A''d$ are pairwise disjoint, $A_0 = A' \cup A''$, similarly, B', B'' and dB''.

Using (3.11), $d \in A_0$, $d \in B_0$ and the associative law $(ad)b = a(db)$, it is easily seen that

(3.16) $$(A', B_0 \cup dB'') \in \Pi_1 \subset \Pi,$$

and

(3.17) $$(A_0 \cup A''d, B') \in \Pi_1 \subset \Pi.$$

By (3.8) and (3.16),

$$\mu(A') + \mu(B_0 \cup dB'') \leqslant \mu(A_0) + \mu(B_0),$$

that is, $\mu(dB'') \leqslant \mu(A'')$. By (3.8) and (3.17),

$$\mu(A_0 \cup A''d) + \mu(B') \leqslant \mu(A_0) + \mu(B_0),$$

that is, $\mu(A''d) \leqslant \mu(B'')$; moreover, the latter equality sign can only hold when $\mu(A''d) = 0$. Invoking (3.1), it follows that

(3.18) $$\mu(A'') = \mu(A''d) = \mu(B'') = \mu(dB'') = 0.$$

We assert that A'' is empty. For, let $a \in A'' \subset A_0$, thus, $ad \notin A_0$. But A_0 is compact. Thus, there exists a neighborhood U of a such that Ud is disjoint from A_0 and, hence, $A_0 \cap U \subset A''$. By $\mu(A'') = 0$ this would imply $a \notin S(A_0)$, which however contradicts (3.12).

The set A'' being empty, we have

(3.19) $$A_0 d \subset A_0.$$

If there were a point $a \in A_0$ with $a \notin A_0 d$, then an entire neighborhood U of a would be disjoint from the compact set $A_0 d$. But then (3.12) and (3.19) together would imply that $\mu(A_0) > \mu(A_0 d)$, which contradicts (3.1). It follows that $A_0 d = A_0$; similarly, $dB_0 = B_0$, proving (3.15).

In showing that the compact set F is a compact group, it suffices [10] to show that F is *algebraically* a group. Recall that $F = A_0 \cap B_0$.

In the first place, if $d_1 \in F$ and $d_2 \in F$ then

$$d_1 d_2 \in A_0 d_2 = A_0 \quad \text{and} \quad d_1 d_2 \in d_1 B_0 = B_0,$$

thus, $d_1 d_2 \in A_0 \cap B_0 = F$. Hence, F is a compact semigroup.

As is well known ([10], p. 99), each compact semigroup contains at least one idempotent. Let e be any idempotent in F, $e^2 = e$. By (3.15).

$$A_0 = A_0 e \subset Ge = \{x \in G:\ xe = x\},$$

thus, e acts as a right unit on A_0. Similarly, e acts as a left unit on B_0 and thus as a two-sided unit on $A_0 \cap B_0 = F$. This shows that e is unique.

Finally, let $d \in F$ be fixed and consider the compact semigroup $L = dF \subset F$. It has at least one idempotent, thus $e \in L$, hence, there exists an element $d^{-1} \in F$ such that $e = d \cdot d^{-1}$. This completes the proof of Theorem 3.1.

As is easily seen, two cosets $x_1 F$ and $x_2 F$ are either identical as sets or they are disjoint; moreover, the union of all left cosets xF is precisely Ge, e denoting the unit element of the group F. In particular, $Ge = F$ as soon as $\mu(F) > \frac{1}{2}\mu(G)$, similarly, $eG = F$.

Let us briefly consider the case that G is a compact semigroup, with μ as a nonnegative regular Borel measure satisfying (3.1), $0 < \mu(G) < \infty$. Applying Theorem 3.1 with $C_0 = G$, $\varrho = \mu(G)$ (thus, (3.2) holds with $A_0 = B_0 = G$) it follows that G contains a subset K which is a compact group such that

$$\mu(K) = \mu(G), \qquad Ge = eG = K. \tag{3.20}$$

Here, e denotes the unit of K. By $Kd = dK$ for each $d \in K$, K is a minimal left ideal ($GK \subset K$) and also a minimal right ideal, implying that K is unique.

That such a "kernel" K exists was first shown by Rosen ([8], p. 1078); (he assumed (3.1) with equality signs but his proof carries over). One has, for any pair of subsets A and B of G, that

$$\mu_*(A) + \mu_*(B) > \mu(G) \Rightarrow K \subset AB. \tag{3.21}$$

After all, if $x \in K$ then the subsets $A \cap K$ and $x(B \cap K)^{-1}$ of K must have a point in common.

The kernel K contains each compact set F which is a group and meets K. For, $K \cap F$ is a (non-empty) compact semigroup with at most one idempotent, hence, a subgroup of K with the same unit e as K, thus, $F = Fe \subset Ge = K$. By (3.20), a compact group F with $\mu(F) > 0$ necessarily meets K. Thus, the sets A_0, B_0 and F in the assertion of Theorem 3.1 are all subsets of K.

One has $K = G$ if and only if there are no idempotents in G outside of K, in particular, when G satisfies one of the cancellation laws; (for the right law: $xe = xe^2$ implies $x = xe$, thus, $G = Ge = K$).

Returning to the general case, we have as a further application of Theorem 3.1:

THEOREM 3.2. *If A and B are subsets of G then*

$$\mu_*(A \cup B \cup AB) \geqslant \operatorname{Min}\{\varrho_1,\ \mu_*(A) + \mu_*(B) - \varrho_0\}, \tag{3.22}$$

whenever $0 \leqslant \varrho_0 < \varrho_1 \leqslant \infty$ are real numbers such that G contains no subset F which is a compact group of measure $\varrho_0 < \mu(F) < \varrho_1$.

If $\mu_*(A) \geqslant \mu_*(B)$, say, then the left-hand side of (3.22) is not smaller than $\mu_*(A)$ and, thus, the assertion (3.22) is of interest only when $\varrho_0 < \mu_*(B) \leqslant \mu_*(A) < \varrho_1$.

If G is a unimodular locally compact group with Haar measure μ then (3.22) may be replaced by

$$\mu_*(AB) \geqslant \operatorname{Min}\{\varrho_1,\ \mu_*(A) + \mu_*(B) - \varrho_0\}, \tag{3.23}$$

provided that A and B are non-empty. For, apply (3.22) with A replaced by $a_0^{-1}A$ and B replaced by Bb_0^{-1}, where $a_0 \in A$, $b_0 \in B$.

Proof of Theorem 3.2. Put $A \cup B \cup AB = C$ and suppose that

$$\mu_*(C) < \mu_*(A) + \mu_*(B) - \varrho_0.$$

In particular, $\mu_*(A) < \infty$, $\mu_*(B) < \infty$. Thus, there exist compact subsets A_0 of A and B_0 of B such that

$$\mu(A_0) + \mu(B_0) \geqslant \mu_*(C) + \varrho,$$

for some $\varrho > \varrho_0$. Putting $C_0 = A_0 \cup B_0 \cup A_0 B_0$, we have $C_0 \subset C$, C_0 compact. It now follows from Theorem 3.1 that G contains a subset F which is a compact group of measure $\varrho \leqslant \mu(F) \leqslant \mu(C_0)$. But $\varrho > \varrho_0$, thus, $\mu(F) \geqslant \varrho_1$, therefore,

$$\mu_*(C) \geqslant \mu(C_0) \geqslant \varrho_1.$$

The above implies Theorem 1.1. For, let G be a connected unimodular group with Haar measure μ. Each closed subgroup F of G with $\mu(F) > 0$ is automatically open ([11], p. 50), hence, G contains no compact subgroup F of measure $0 < \mu(F) < \mu(G)$. It follows by (3.23) that

$$\mu_*(AB) \geqslant \operatorname{Min}\{\mu(G),\ \mu_*(A) + \mu_*(B)\},$$

whenever A and B are non-empty. The last assertion of Theorem 1.1 being obvious, see (3.21), this yields Theorem 1.1.

THEOREM 3.3. (i) *Let A and B be subsets of G such that*

$$C = A \cup B \cup AB$$

satisfies

$$\mu_*(C) < \mu_*(A) + \mu_*(B)\,. \tag{3.24}$$

Then C contains at least one idempotent e.

(ii) *Assume in addition that C contains at most one idempotent. Then*

$$e \in AB\,. \tag{3.25}$$

(iii) *Under these same assumptions,*

$$\mu^*\{a \in A\colon\ e \in aB\} \geqslant \varrho\,, \tag{3.26}$$

where

$$\varrho = \mu_*(A) + \mu_*(B) - \mu_*(C)\,.$$

Proof. Replacing A and B by slightly smaller compact sets A_0, B_0 and putting $C_0 = A_0 \cup B_0 \cup A_0B_0$, $C_0 \subset C$, assertion (i) is an immediate consequence of Theorem 3.1.

As to assertion (ii), assume that $e \notin AB$. Then either $e \in A$ or $e \in B$ but not both by $e = e^2$. Say, $e \in A$ and $e \notin B$. Now, consider the set A_1 obtained from A by deleting the single element e, and further

$$C_1 = A_1 \cup B \cup A_1B\,.$$

Clearly, $e \notin C_1$. But $C_1 \subset C$, thus, C_1 contains no idempotent at all. It follows from assertion (i) that

$$\mu_*(C_1) \geqslant \mu_*(A_1) + \mu_*(B)\,.$$

This however contradicts (3.24), by

$$\mu_*(C) \geqslant \mu_*(C_1) + \mu(\{e\}), \qquad \mu_*(A_1) + \mu(\{e\}) = \mu_*(A)\,.$$

In proving assertion (iii) (which in fact strengthens assertion (ii)), consider the sets

$$A_2 = \{a \in A\colon\ e \notin aB\}, \qquad C_2 = A_2 \cup B \cup A_2B\,.$$

The set C_2 is contained in C and thus contains at most one idempotent. On the other hand, $e \notin A_2B$ with e as the unique idempotent in C. It follows from assertion (ii) that

$$\mu_*(A_2) + \mu_*(B) \leqslant \mu_*(C_2) \leqslant \mu_*(C) = \mu_*(A) + \mu_*(B) - \varrho\,.$$

Hence,

$$\varrho \leqslant \mu_*(A) - \mu_*(A_2) \leqslant \mu^*(A \cap \bar{A}_2) = \mu^*\{a \in A\colon\ e \in aB\}\,.$$

4. The unimodular case. In this section, G denotes a unimodular locally compact group with a (two-sided) Haar measure μ. Clearly, all the results of section 3 apply equally well to this special case. Note that G has a unique idempotent, namely, the two-sided unit e of the group G.

Proof of Theorem 1.2. We are given a pair of non-empty subsets A and B of G satisfying

$$\mu_*(AB) < \mu_*(A) + \mu_*(B)\,; \tag{4.1}$$

thus, each of the three sets A, B and AB has a finite inner measure.

Consider a pair of sets A_1 and B_1 such that

$$A_1 \subset A, \quad B_1 \subset B, \quad \mu_*(A_1) + \mu_*(B_1) > \mu_*(AB)\,. \tag{4.2}$$

We assert that under these conditions

$$A_1B_1 = AB\,. \tag{4.3}$$

Clearly, $A_1B_1 \subset AB$. Let $a_0 \in A$ and $b_0 \in B$ be given; it suffices to prove that $a_0b_0 \in A_1B_1$. Consider the sets

$$A_2 = a_0^{-1}A_1, \qquad B_2 = B_1b_0^{-1}, \qquad C_2 = A_2 \cup B_2 \cup A_2B_2\,.$$

Then

$$a_0C_2b_0 = A_1b_0 \cup a_0B_1 \cup A_1B_1 \subset AB\,.$$

Hence,

$$\mu_*(C_2) \leqslant \mu_*(AB) < \mu_*(A_1) + \mu_*(B_1) = \mu_*(A_2) + \mu_*(B_2)\,.$$

It follows from assertion (ii) of Theorem 3.3 that $e \in A_2B_2$, thus,

$$a_0b_0 \in a_0A_2B_2b_0 = A_1B_1\,.$$

This proves (4.3).

By (4.1), there always exist compact sets A_1 and B_1 satisfying (4.2). Using (4.3), it follows that the set AB is compact.

Let A^* and B^* denote the closures of the sets A and B, respectively. If $x_0 \in A^*$, $y_0 \in B^*$ and $x_0y_0 \notin AB$ then $xy \notin AB$ for x near x_0, y near y_0, a contradiction. Therefore,

$$A \subset A^*, \quad B \subset B^*, \quad AB = A^*B^*\,. \tag{4.4}$$

The set AB being compact, we have that A^* and B^* are compact.

Let $\varrho > 0$ be defined by

$$\mu_*(A) + \mu_*(B) = \mu_*(AB) + \varrho\,. \tag{4.5}$$

Now consider a pair of elements

$$a \in A^*, \qquad b \in B^*,$$

and define

$$A_0 = a^{-1}A^*, \qquad B_0 = B^*b^{-1}, \qquad C_0 = a^{-1}ABb^{-1}\,.$$

One has, by (4.4) and (4.5), that

$$A_0 \subset C_0, \qquad B_0 \subset C_0, \qquad A_0B_0 = C_0\,,$$

and

$$\mu(A_0) + \mu(B_0) \geqslant \mu(C_0) + \varrho\,.$$

It follows by Theorem 3.1 that C_0 contains a subset $F_{a,b}$ which is a compact subgroup of G of measure

$$\mu(F_{a,b}) \geqslant \varrho > 0 . \tag{4.6}$$

In particular ([11], p. 50), $F_{a,b}$ is also an open subset of G. Finally, $aF_{a,b}b \subset aC_0b = AB$.

Thus, we have shown that to each pair $a \in A^*$, $b \in B^*$ there corresponds an *open and compact* subgroup $F_{a,b}$ of G satisfying (4.6) and

$$ab \in a \cdot F_{a,b} \cdot b \subset AB .$$

We conclude that AB is an open subset of G. The set $aF_{a,b}b$ being compact, there exists an open neighborhood $U_{a,b}$ of the unit element e such that

$$U_{a,b}(aF_{a,b}b)\,U_{a,b} \subset AB .$$

Hence, letting

$$V_{a,b} = \{(x, y)\colon\ x \in U_{a,b}\cdot a,\ y \in b \cdot U_{a,b}\} ,$$

we have

$$(x, y) \in V_{a,b} \Rightarrow x \cdot F_{a,b} \cdot y \in AB . \tag{4.7}$$

By $(a, b) \in V_{a,b}$, the system $\{V_{a,b}\colon\ a \in A^*,\ b \in B^*\}$ is an open covering of the compact subset $A^* \times B^*$ of $G \times G$. Therefore, there exists a finite set of pairs (a_i, b_i), $i = 1, \ldots, N$, in $A^* \times B^*$ such that

$$A^* \times B^* \subset \bigcup_{i=1}^{N} V_{a_i,b_i} .$$

Now, form

$$F = \bigcap_{i=1}^{N} F_{a_i,b_i} . \tag{4.8}$$

Clearly, F is an open and compact subgroup of G. We assert that

$$aFb \subset AB \quad \text{whenever} \quad a \in A^*,\ b \in B^* . \tag{4.9}$$

After all, given $a \in A^*$, $b \in B^*$, there exists an index $1 \leqslant i \leqslant N$ such that $(a, b) \in V_{a_i,b_i}$; hence, by (4.7) and (4.8),

$$aFb \subset a \cdot F_{a_i,b_i} \cdot b \subset AB .$$

In view of (4.4), (4.9) implies (1.3).

Note that (1.3) is equivalent to $AF \cdot FB = AB$. Hence, by (4.1), if A_1 and B_1 satisfy (1.4) then (4.1) and (4.2) hold with A and B replaced by AF and FB, respectively; hence, by (4.3),

$$A_1B_1 = AF \cdot FB = AB .$$

This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. Sufficiency. In fact, for any unimodular group G, the indicated construction yields a pair (A, B) of non-empty subsets of G satisfying (1.2). For,

$$\sigma(AB) = \sigma(A) \cdot \sigma(B) \subset A'B' ,$$

hence,

$$\mu_*(AB) \leqslant \mu\big(\sigma^{-1}(A'B')\big) = [A'B']\,\mu(H) < \mu_*(A) + \mu_*(B) ,$$

by (1.7). The last assertion of Theorem 1.3 follows from the last assertion of Theorem 1.2 (or from the assertion (4.3)), on replacing (A, B) and (A_1, B_1) by $(\sigma^{-1}A', \sigma^{-1}B')$ and (A, B), respectively.

Necessity. Suppose merely that the unimodular group G is such that any open and compact subgroup F contains an open and compact *normal* subgroup, equivalently, that for each open and compact subgroup F the compact group H defined by (1.5) is open. This is obviously true when G is abelian, and further when G is compact. For, in the latter case,

$$H = \bigcap_{i=1}^{N} x_i F x_i^{-1} ,$$

where $\{x_1, \ldots, x_N\}$ is such that each of the finitely many left cosets xF of F contains precisely one element x_i.

Now, consider a pair (A, B) of non-empty subsets of G satisfying (1.2). By Theorem 1.2 and the above assumption, there exists an open and compact normal subgroup H of G such that $aHb \subset AB$ whenever $a \in A$ and $b \in B$. In other words, AB is equal to the union of finitely many cosets of H. Let σ denote the quotient mapping $G \to G/H$, and put $\sigma A = A'$ and $\sigma B = B'$. Then

$$\sigma(AB) = A'B', \qquad A \subset \sigma^{-1}A', \qquad B \subset \sigma^{-1}B' .$$

Hence, using (1.2),

$$[A'B']\,\mu(H) = \mu(AB) < \mu_*(A) + \mu_*(B) \leqslant ([A'] + [B'])\,\mu(H) .$$

This implies (1.6) and (1.7).

5. The non-unimodular case. In this section, G denotes a locally compact group which is *not* unimodular, in other words, such that the left invariant Haar measure μ is not right invariant.

One has ([3], p. 264) for any measurable subset E of G

$$\mu(Ex) = \Delta(x)\,\mu(E) ,$$

where $\Delta(x) > 0$ is a continuous function on G such that

$$\Delta(xy) = \Delta(x)\,\Delta(y), \qquad \Delta(x) \not\equiv 1 .$$

Hence,

$$H = \{x \in G\colon\ \Delta(x) = 1\}$$

is a proper closed and normal subgroup of G.

Whether or not G is connected, there always exist pairs of non-empty subsets A and B of G such that

$$\mu_*(AB) < \mu_*(A) + \mu_*(B)\,. \tag{5.1}$$

For, let A and B_0 be compact subsets of G with $\mu(A) > 0$ and take $B = B_0 x$ with $\Delta(x)$ sufficiently small.

The proper parallel of Theorem 1.1 seems to be:

THEOREM 5.1. *Suppose that G is connected. Then one has for any pair of subsets A and B of G, such that $0 < \nu_*(A) < \infty$ and $0 < \mu_*(B) < \infty$, that*

$$\nu_*(A)/\nu_*(AB) + \mu_*(B)/\mu_*(AB) \leqslant 1\,. \tag{5.2}$$

Here, ν denotes a fixed right Haar measure on G for which we may take $\nu(E) = \mu(E^{-1})$, thus, $\nu(xE) = \Delta(x^{-1})\nu(E)$. It is an important feature of (5.2) that its left-hand side remains unchanged on replacing A by xA and B by By.

Let G be any non-unimodular locally compact group, and suppose that there exists a pair of subsets A and B such that

$$0 < \nu_*(A) < \infty, \qquad 0 < \mu_*(B) < \infty\,, \tag{5.3}$$

and

$$\nu_*(A)/\nu_*(AB) + \mu_*(B)/\mu_*(AB) > 1\,. \tag{5.4}$$

Replacing A and B by slightly smaller compact sets, we may assume that A and B are compact.

The continuous function $\Delta(x)$ assumes on A its smallest value at some point a_0 in A. Replacing A by $a_0^{-1}A$, we may assume that $e \in A$ and that $\Delta(x) \geqslant 1$ for each $x \in A$. Similarly, we may assume that $e \in B$ and that $\Delta(x) \leqslant 1$ for each $x \in B$.

Thus A is equal to the disjoint union of a set A_0 with

$$e \in A_0 \subset H$$

and a set A_+ on which $\Delta(x) > 1$, thus ([11], p. 40)

$$\mu(A_+) = \int_{A_+} \Delta(x)\,d\nu \geqslant \nu(A_+)\,,$$

while $\mu(A_0) = \nu(A_0)$. Further, B is equal to the disjoint union of a set B_0 with

$$e \in B_0 \subset H$$

and a set B_- on which $\Delta(x) < 1$, thus, $\mu(B_-) \leqslant \nu(B_-)$.

By (5.4), one has

$$\text{either} \quad \mu(AB) < \nu(A) + \mu(B) \quad \text{or} \quad \nu(AB) < \nu(A) + \mu(B)\,.$$

Observing that AB contains the disjoint union $A_+ \cup A_0 B_0 \cup B_-$, and using the above remarks, one obtains in either case that

$$\mu(A_0 B_0) < \mu(A_0) + \mu(B_0)\,; \tag{5.5}$$

(for subsets of H, μ and ν coincide).

It follows that the subsets A_0 and B_0 of H are of positive measure, hence, the group H is a proper *open* subgroup of G. This being impossible when G is connected, one obtains Theorem 5.1.

Assuming (5.3), (5.4), we actually proved more, namely, that there exists a pair of non-empty subsets A_0 and B_0 of H such that (5.5) holds. Then H is a unimodular open subgroup of G with μ as Haar measure. It follows from Theorem 1.2 that H in turn contains an open compact subgroup F (in fact, $F \subset A_0 B_0$). No such F can be a normal subgroup of G. For, if $x \notin H$ then the conjugate group $x^{-1}Fx$ has measure

$$\mu(x^{-1}Fx) = \Delta(x)\mu(F) \neq \mu(F)\,.$$

The following example seems to indicate that (unlike in the unimodular case) the inequality (5.4) does not imply any definite structure of the product set AB.

Assume (as we must) that H contains an open compact subgroup F. Choose A as a measurable subset of F, $\nu(A) > 0$, B as the disjoint union of F and Dx, where D is an arbitrary but fixed compact set, (while x satisfies $F \cap Dx = \emptyset$). One has $AB = F \cup ADx$, thus (5.4) is equivalent to

$$\frac{\nu(A)}{\nu(F) + \nu(AD)} + \frac{\mu(F) + \mu(D)\Delta(x)}{\mu(F) + \mu(AD)\Delta(x)} > 1\,.$$

Clearly, the latter inequality and the condition $F \cap Dx = \emptyset$ hold as soon as $\Delta(x)$ is sufficiently small.

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On quasi-translations in 3-space

by

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The topological translation (¹) of the plane was characterized by Kerékjártó [1] and Sperner [2] independently by apparently different conditions. To show that their conditions do not characterize the topological translation in 3-space is the purpose of the paper. Our example is naturally constructed from Fox-Artin's pathological one [3] (§ 3).

The notion of quasi-translation is due to Terasaka [4]. His condition is also apparently different from those of Kérekjártó and Sperner, but we shall prove that their three conditions are equivalent to each other for any n-sphere ($n \geqslant 1$) (§ 1).

In § 2 we shall be concerned with locally polyhedral 2-spheres with one singularity in 3-sphere; this may be of independent interest, even though it appears in the paper only as a preliminary to § 3.

The paper has an appendix, in which we shall prove that if an auto-homeomorphism of a certain kind that includes quasi-translations operates on a manifold, then that manifold must be an n-sphere.

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§ 1. Let h be an auto-homeomorphism of a compact metric space X. Then h is said to have *equi-continuous powers* at $x \in X$, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $d(x, y) < \delta$, $d\big(h^m(x), h^m(y)\big) < \varepsilon$ for every integer m. First we prove the following:

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(¹) Let g and h be two auto-homeomorphisms of a topological space X. Then g and h are said to be *topologically equivalent*, if there exists an auto-homeomorphism f of X such that $g = fhf^{-1}$. A *topological translation* means a transformation that is topologically equivalent to the ordinary translation.