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Boundary behavior of temperatures I

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Section 0. Introduction

In this work, we will study solutions to the heat equation in upper half spaces. In particular, we will be interested in conditions on these solutions or their derivatives under which certain limits, which we will call parabolic limits, exist for a set E in the boundary hyperplane $t = 0$.

We will denote by E_{n+1}^{+} the upper half space $\{(x, t): x = (x_1, x_2, \ldots, x_n) \in E_n, t > 0\}$, where E_n is the n-dimensional Euclidean space. We consider E_n to be imbedded in E_{n+1}^{+} as the set $\{(x, 0): x \in E_n\}$, and $|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$. A continuous function u which satisfies the heat equation

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0$$

on the domain D will be called a *temperature*, or a *solution to the heat equation*, on D.

Let $P(x; a) = \{(z, t): z \in E_n, \ |z-x|^2 < a^2 t\}$ be the paraboloid with vertex x and aperture a, $P(x; a, h) = P(x; a) \cap \{(z, t) \in E_{n+1}^{+}: 0 < t < h\}$.

Definition. Let $g(z, t)$ be a function on E_{n+1}^{+}. We say that g has a *parabolic limit* (p. lim.) $g(x, 0)$ at $x \in E_n$ if $g(z, t) \to g(x, 0)$ as (z, t) tends to $(x, 0)$ along any curve in $P(x; a)$ for each $a > 0$. We similarly say that g is *parabolically bounded* (p. bdd.) at $x \in E_n$ if there exist M, $h > 0$, and $a > 0$ such that $|g(z, t)| \leqslant M$ for $(z, t) \in P(x; a, h)$.

With these definitions and certain preliminary results, we will show the following connection between these two concepts in section 1:

THEOREM 1. *Let $u(x,t)$ be a temperature in E_{n+1}^{+}. If u is parabolically bounded on a set $E \subset E_n$, then u has parabolic limits almost everywhere (a. e.) in E.*

The converse statement is trivially true, i. e., if u has parabolic limits on a set $E \subset E_n$, then u is parabolically bounded on E.

The proof of this theorem depends on utilizing the known limit properties of a special class of solutions of the heat equation, the Weierstrass transforms. We will develop these properties in the preliminary results in section 1. The essence of the proof then is to approximate our function by one Weierstrass transform and then dominate their difference by another.

The central result of this work is another property which is both necessary and sufficient for parabolic limits to exist almost everywhere on a set $E \subset E_n$. This result will be proved in section 3 as

THEOREM 2. *Let u be a temperature on E_{n+1}^{+}.*

(a) *If u has parabolic limits on a set $E \subset E_n$, then*

$$\int_{P(x_0;a,h)} t^{-n/2}\left\{|\nabla_s u|^2 + t\left|\frac{\partial u}{\partial t}\right|^2\right\} dx\,dt$$

is finite for almost every $x_0 \in E$, $a > 0$, $h > 0$.

(b) *If, for every $x_0 \in E \subset E_n$,*

$$\int_{P(x_0;a,h)} t^{-n/2}\left\{|\nabla_s u|^2 + t\left|\frac{\partial u}{\partial t}\right|^2\right\} dx\,dt$$

is finite, for some $a > 0$, $h > 0$, then u has parabolic limits a. e. in E

$$\left[|\nabla_s u|^2 = \sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^2\right].$$

In the proof of this theorem, we rely heavily on being able to obtain the value of a temperature at a given point by means of taking the integral of the temperature with a kernel function, over a surface which "surrounds" the point. This is similar to the case of harmonic functions where we obtain the value at a point interior to a sphere by integrating over the surface of the sphere with the Poisson kernel. Deriving this representation for temperatures and developing some of its properties will be the content of section 2.

In section 4, we will show an application of theorem 2. If two solutions to the heat equation are related in a given way, we will prove that p. lim. for the second on a set $E \subset E_n$ imply p. lim. for the first almost everywhere on E; more precisely:

THEOREM 4. *Let $u(x,t)$, $v(x,t)$ be, respectively, k- and m-dimensional vector valued solutions to the heat equation on E_{n+1}^{+}, i. e., k- and m-dimensional vector valued functions on E_{n+1}^{+} such that each component is a temperature on E_{n+1}^{+} in the usual sense. Let $P(D)$ be a $k\times m$ matrix, each of whose entries is a homogeneous differential polynomial in x of degree $2r$ with constant coefficients. Suppose that $\partial^r u/\partial t^r = P(D)v$. Then, if v has parabolic limits on a set $E \subset E_n$, u has parabolic limits almost everywhere on E.*

We prove this theorem by applying theorem 2, a lemma relating certain integrals (Lemma 12), and theorem 3, which states that the two terms of the integral in theorem 2 are essentially equivalent, i. e.,

THEOREM 3. *Let u be a temperature on $P(x_0;\beta,k)$.*

(a) *If*

$$\int_{P(x_0;\beta,k)} t^{1-(n/2)}\left|\frac{\partial u}{\partial t}\right|^2 dx\,dt < \infty,$$

then, for $0 < a < \beta$, $0 < h < k$,

$$\int_{P(x_0;a,h)} t^{n/2}\left|\frac{\partial u}{\partial x_i}\right|^2 dx\,dt < \infty \quad \text{for} \quad i = 1, 2, \ldots, n.$$

(b) *If*

$$\int_{P(x_0;\beta,k)} t^{-n/2}\left|\frac{\partial u}{\partial x_i}\right|^2 dx\,dt < \infty \quad \text{for} \quad i = 1, 2, \ldots, n,$$

then

$$\int_{P(x_0;a,h)} t^{1-(n/2)}\left|\frac{\partial u}{\partial t}\right|^2 dx\,dt < \infty \quad \text{for} \quad 0 < a < \beta,\ 0 < h < k.$$

These results are analogous to results for harmonic functions which were proved by Calderón [2, 3] and Stein [13].

Let $\Gamma(x;a) = \{(z,t)\colon |z-x| < at\}$ be the cone of vertex x and aperture a. Then, we define non-tangential limits and non-tangential boundedness as we defined p. lim. and p. bdd., replacing paraboloids with cones.

Calderón proved in [2] that if a harmonic function u on E_{n+1}^{+} (i. e., a solution to the equation $\sum_{i=1}^{n}\partial^2 u/\partial x_i^2 + \partial^2 u/\partial t^2 = 0$) is non-tangentially bounded on a set $E \subset E_n$, then u has non-tangential limits a. e. in E.

In [3] he showed that if a harmonic function u on E_{n+1}^{+} has non-tangential limits on a set $E \subset E_n$, then $\int_{\Gamma(x_0;a,h)} t^{1-n}|\nabla u|^2 dx\,dt$ is finite for almost every $x_0 \in E$,

$$|\nabla u|^2 = \sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^2 + \left|\frac{\partial u}{\partial t}\right|^2.$$

Stein showed that this condition was sufficient to guarantee non-tangential limits a. e. in E, and also proved theorems for harmonic functions analogous to theorems 3 and 4.

The powers of t appearing in our integrals seem more natural if we consider the following

Definition. Let $W(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$ for $x \in E_n$, $t > 0$, be the Weierstrass kernel. For $f \in L_p(E_n)$, we define the *Weierstrass transform* $f(x,t)$ of f by

$$f(x,t) = \int_{E_n} W(x-z,t) f(z)\,dz, \qquad x \in E_n,\ t > 0.$$

We note that the Fourier transform $\hat{W}$ of $\hat{W}$ is given by $\hat{W}(x,t) = e^{-4\pi^2|x|^2 t}$.

For $f \in L_2(E_n)$, let

$$S(f)(x) = \left[\int_{P(x;a)} t^{-n/2}\left\{|\nabla_s f(z,t|^2 + t\left|\frac{\partial}{\partial t} f(z,t)\right|^2\right\} dx\,dt\right]^{1/2}$$

where $a > 0$. We note that the indicated powers of t are precisely those which give us $\|S(f)\|_2 = A\|f\|_2$, where A depends only on a and n.

We will show this for the case $a = 1$. Throughout this argument, the A's and B's will be constants depending only on a and n. They may vary from step to step.

Let $\chi(z,t)$ be the characteristic function of $P(0;a)$. Then,

$$[S(f)(x)]^2 = \int_0^\infty \left\{\int_{E_n} \chi(z,t) t^{-n/2}\left[|\nabla_s f(x-z,t)|^2 + t\left|\frac{\partial}{\partial t} f(x-z,t)\right|^2\right] dz\right\} dt.$$

Letting $\hat{f}$ be the Fourier transform of f and using Fubini's and Plancherel's theorems, we obtain:

$$\begin{aligned}
\|S(f)\|_2^2 &= \int_0^\infty t^{-n/2}\left\{\int_{E_n} \chi(z,t)\left[\int_{E_n} |\nabla_s f(x-z,t)|^2 dx\right] dz\right\} dt + \\
&\quad + \int_0^\infty t^{1-(n/2)}\left\{\int_{E_n} \chi(z,t)\left[\int_{E_n}\left|\frac{\partial}{\partial t} f(x-z,t)\right|^2 dx\right] dz\right\} dt \\
&= A\int_0^\infty t^{-n/2}\left\{\int_{E_n} \chi(z,t)\left[\int_{E_n} |x|^2 e^{-8\pi^2|x|^2 t} |\hat{f}(x)|^2 dx\right] dz\right\} dt + \\
&\quad + B\int_0^\infty t^{1-(n/2)}\left\{\int_{E_n} \chi(z,t)\left[\int_{E_n} |x|^4 e^{-8\pi^2|x|^2 t} |\hat{f}(x)|^2 dx\right] dz\right\} dt \\
&= A\int_0^\infty \left[\int_{E_n} |x|^2 e^{-8\pi^2|x|^2 t} |\hat{f}(x)|^2 dx\right] dt + \\
&\quad + B\int_0^\infty t\left[\int_{E_n} |x|^4 e^{-8\pi^2|x|^2 t} |\hat{f}(x)|^2 dx\right] dt,
\end{aligned}$$

since $\int_{E_n} \chi(z,t)\,dz = At^{n/2}$.

Thus,

$$\begin{aligned}
\|S(f)\|_2^2 &= A\int_{E_n}\left\{|x|^2 \int_0^\infty e^{-8\pi^2|x|^2 t} dt\right\} |\hat{f}(x)|^2 dx + \\
&\quad + B\int_{E_n}\left\{|x|^4 \int_0^\infty t e^{-8\pi^2|x|^2 t} dt\right\} |\hat{f}(x)|^2 dx \\
&= A\int_{E_n} |\hat{f}(x)|^2 dx = A\|f\|_2^2.
\end{aligned}$$

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Section 1. Parabolic boundedness and parabolic limits

We will first introduce a few necessary results after which we will prove theorem 1.

LEMMA 1. *If $f \in L_p(E_n)$, $1 \leqslant p \leqslant \infty$, then the Weierstrass transform $f(x,t)$ of f is a temperature on E_{n+1}^+, and $f(x,t)$ converges to $f(x)$ almost everywhere as $t \to 0$* [1, 8, 9, 12].

Proof. Since $W(x,t)$ is a temperature on E_{n+1}^+, so is $f(x,t)$.

W has the following properties, which we will use in the proof:

(a) $\int_{E_n} W(x,t)\,dx = 1$ for all $t > 0$.

(b) $W(x,t) \geqslant 0$ for $x \in E_n$, $t > 0$.

(c) For a fixed $\eta > 0$,

$$\left\{\int_{|x|\geqslant\eta} |W(x,t)|^q dx\right\}^{1/q} \to 0$$

as $t \to 0+$, $1 \leqslant q < \infty$; $\sup_{|x|\geqslant\eta} |W(x,t)| \to 0$ as $t \to 0+$ for $q = \infty$.

(d) $\frac{\partial}{\partial r} W(rx',t) \leqslant 0$, $|x'| = 1$, $r > 0$.

(e) $\int_0^\infty r^n \left|\frac{\partial}{\partial r} W(rx',t)\right| dr \leqslant c < \infty$, where c is independent of $t > 0$.

These properties follow directly from the definition.

Let x_0 be a point of E_n such that

$$s^{-n}\int_{|z|<s} [f(x_0-z) - f(x_0)]\,dx \to 0 \quad \text{as} \quad s \to 0+.$$

Such points occur almost everywhere in E_n. By property (a),

$$f(x_0,t)-f(x_0) = \int_{E_n}[f(x_0-z)-f(x_0)]W(z,t)\,dz$$
$$= \Big\{\int_{|z|\leqslant\eta}+\int_{|z|>\eta}\Big\}[f(x_0-z)-f(x_0)]W(z,t)\,dz.$$

Let $r=|z|$, $z=rz'$. Then

$$\int_{|z|\leqslant\eta}[f(x_0-z)-f(x_0)]W(z,t)\,dz$$
$$=\int_{\Sigma}\Big\{\int_0^{\eta}[f(x_0-z)-f(x_0)]W(rz',t)r^{n-1}\,dr\Big\}dz'$$
$$=\int_0^{\eta}g_{x_0}(r)W(rz',t)r^{n-1}\,dr,$$

where Σ is the unit sphere in E_n, and $g_{x_0}(r)=\int_{\Sigma}[f(x_0-z)-f(x_0)]\,dz'$.

Integrating by parts gives us:

$$\int_{|z|\leqslant\eta}[f(x_0-z)-f(x_0)]W(z,t)\,dz$$
$$=[G_{x_0}(r)W(rz',t)]_{r=0}-\int_0^{\eta}G_{x_0}(r)\frac{\partial}{\partial r}W(rz',t)\,dr,$$

where

$$G_{x_0}(r)=\int_0^{r}g_{x_0}(\varrho)\varrho^{n-1}\,d\varrho$$
$$=\int_{\Sigma}\int_0^{r}[f(x_0-\varrho z')-f(x_0)]^{n-1}\,d\varrho\,dz'=\int_{|z|<r}f(x_0-z)-f(x_0)\,dz=o(r^n).$$

Given $\varepsilon>0$, choose $\eta>0$ small enough so that, if $0<r\leqslant\eta$, $|G_{x_0}(r)|<\varepsilon r^n$. Then,

$$\Big|\int_{|z|\leqslant\eta}[f(x_0-z)-f(x_0)]W(z,t)\,dz\Big|$$
$$\leqslant\varepsilon\eta^n(4\pi t)^{-n/2}e^{-\eta^2/4t}+\varepsilon\int_0^{\eta}r^n\Big|\frac{\partial}{\partial r}W(rz',t)\Big|\,dr$$
$$\leqslant\varepsilon\Big[\eta^{n/2}e^{-\sqrt{2n}}\Big(\frac{4\pi}{\sqrt{2n}}\Big)^{-n/2}+c\Big],$$

$$\Big|\int_{|z|>\eta}[f(x_0-z)-f(x_0)]W(z,t)\,dz\Big|$$
$$\leqslant\int_{|z|>\eta}|f(x_0-z)|W(z,t)\,dz+|f(x_0)|\int_{|z|>\eta}W(z,t)\,dz=\mathrm{I}+\mathrm{II}.$$

Now, $\mathrm{II}\to 0$ as $t\to 0$ by property (c) with $Q=1$;

$$\mathrm{I}\leqslant\|f\|_p\Big[\int_{|z|>\eta}|W(z,t)|^q\,dz\Big]^{1/q},\quad (1/p)+(1/q)=1,$$

which tends to zero as $t\to 0$ for $f\in L_p(E_n)$, $1\leqslant p\leqslant\infty$.

COROLLARY. *The Weierstrass transform $f(z,t)$ of $f(x)\in L_p(E_n)$, $1\leqslant p\leqslant\infty$, has parabolic limit $f(x)$ a. e. in E_n.*

Proof. Note that, for $(z,t)\in P(x;a)$, $|x-u|^2\leqslant 2|z-u|^2+2|x-z|^2\leqslant 2|z-u|^2+2at$, whence $e^{-a/2}e^{-(|z-u|^2/2t)}\leqslant e^{-(|x-u|^2/4t)}$. Thus,

$$\int_{E_n}|f(x)-f(u)|W(z-u\tfrac{1}{2}t)\,du\leqslant 2^{n/2}e^{a/2}\int_{E_n}|f(x)-f(u)|W(x-u,t)\,du,$$

which tends to zero for almost every $x\in E_n$ as $t\to 0+$.

In his paper, Nirenberg [11] proved a maximum principle for the solutions to a fairly general class of parabolic equations. We will state his results appropriately specialized to the solutions of the heat equation.

Definition. Let T be a bounded domain in E_{n+1}. For $P\in T$, we define $S(P)$ as follows: $(x,t)\in S(P)$ if $(x,t)\in T$ and there exists a simple curve $f(y)$, $0\leqslant y\leqslant 1$, such that $f(0)=(x,t)$, $f(1)=P$, and, if $y_1\geqslant y_2$, then the t coordinate of $f(y_1)$ is greater than or equal to the t coordinate of $f(y_2)$.

THEOREM. *Let u be a function defined on T. Assume that, for some point P of T, the maximum or minimum of u in the set $S(P)$ is attained at P. If u is a temperature on $S(P)$, then u is identically equal to $u(P)$ in $S(P)$.*

COROLLARY. *If u is a temperature in T and is continuous on the closure of T, then, for any point P of T, the maximum of u in the closure of $S(P)$ is attained at a point on the boundary of T. Similarly, the minimum is taken on the boundary.*

We are now ready to prove theorem 1.

THEOREM 1. *Let $u(x,t)$ be a temperature on E_{n+1}^+. If u is parabolically bounded on a set $E\subset E_n$, then u has parabolic limits almost everywhere in E.*

Proof. For each $x\in E$, there exist $a>0$, $h>0$ such that $|u(z,t)|\leqslant M$ for $(z,t)\in P(x;a,h)$, M, a, h depending on x. Changing the bounds, if necessary, we can choose $h=2$ for all $x\in E$. By considering only rational bounds and apertures, we can break E up into a countable number of subsets, each of which corresponds to a uniform aperture and bound. We can further split each of these into a countable number of sets, each of which is contained in a hypercube with side of length 1. Thus, it suffices to show that we have parabolic limits a. e. in each of these sets. Multiplying u by a constant, if necessary, we need only consider $|u(z,t)| \leqslant 1$ for $(z,t)\in P(x;\alpha,2)$, α fixed, for $x\in E$, and E contained in a hypercube with side of length 1.

Let $A = \bigcup_{x\in E} P(x;\alpha,2)$, $D = A \cap \{(x,t): 0 < t \leqslant 1\}$; B, the boundary of D; D_n, the translation of D by $-(1/n)$, i. e., $(x,t)\in D_n$ if and only if $(x, t+(1/n))\in D$; $G_n = D_n \cap E_n$; $u_n(x,t) = u(x, t+(1/n))$, and χ_n, the characteristic function of G_n.

Let $\varphi_n(x,t)$ be the Weierstrass transform of $\chi_n(x)u_n(x,0) = f_n(x)$, and $\psi_n(x,t) = u_n(x,t)-\varphi_n(x,t)$. Since G_n is contained in a hypercube C of side at most $1+2\alpha$,

$$\|\chi_n\|_2 \leqslant \Big\{\int_C 1^2 dx\Big\}^{1/2} = (1+2\alpha)^{n/2}.$$

Therefore, $\|f_n\|_2 \leqslant (1+2\alpha)^{n/2}$ since $|f_n(x)| \leqslant |\chi_n(x)|\,|u_n(x,0)| \leqslant 1$. Because the elements of the sequence $\{f_n\}$, $n = 1, 2, \ldots$, are uniformly bounded in $L_2(E_n)$ norm, there exits a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges weakly to a function $f\in L_2(E_n)$, i. e., a subsequence such that, if h is a continuous function which vanishes at infinity, then

$$\int_{E_n} f_{n_k}(v)h(v)\,dv \to \int_{E_n} f(v)h(v)\,dv \quad \text{as} \quad k\to\infty.$$

If we let $h(v) = W(x-v,t)$, $(x,t)\in E_{n+1}^+$, we have

$$\varphi_{n_k}(x,t) \to \varphi(x,t) = \int_{E_n} W(x-v,t)f(v)\,dv.$$

Since $u_n(x,t)\to u(x,t)$ as $n\to\infty$ by the continuity of u, we see that

$$\psi(x,t) = \lim_{k\to\infty}\psi_{n_k}(x,t) = \lim_{k\to\infty}[u_{n_k}(x,t)-\varphi_{n_k}(x,t)] = u(x,t)-\varphi(x,t)$$

exists for every $(x,t)\in E_{n+1}^+$. As Φ is the Weierstrass transform of a function in $L_2(E_n)$, it suffices to show that ψ has parabolic limits almost everywhere in E. (We will show that the limit is zero.)

We note that, for each n, (a) $|\psi_n(x,t)| \leqslant 2$ for $(x,t)\in D$, and (b) $|\psi_n(x,t)|\to 0$ as $(x,t)\to z\in E$, $(x,t)\in D$.

Let us suppose that we have a temperature w on E_{n+1}^+ with the following properties: (a′) $w(x,t)\geqslant 0$ on E_{n+1}^+; (b′) $w(x,t)\geqslant 2$ on $B-E$; (c′) w has parabolic limit zero almost everywhere in E. Then, we claim that we are finished. We see immediately that $w(x,t)\pm\psi_n(x,t)\geqslant 0$ for $(x,t)\in B-E$ and, by property (b) of ψ_n,

$$\liminf_{\substack{(x,t)\to z\in E\\ (x,t)\in D}} [w(x,t)\pm\psi_n(x,t)] \geqslant \liminf_{\substack{(x,t)\to z\in E\\ (x,t)\in D}} w(x,t) \geqslant 0.$$

But these two statements imply that $w(x,t)\pm\psi_n(x,t)\geqslant 0$ for all $(x,t)\in D$. For, if not, then for some point $(x_0,t_0)\in D$, $w(x_0,t_0)\pm\psi_n(x_0,t_0) \leqslant a$, $a<0$. Consider the sequence of domains $D^k = D\cap\{(x,t): (1/k) < t\leqslant 1\}$. For all k greater than some sufficiently large N, (x_0,t_0) will be in the domain D^k. Applying the minimum principle due to Nirenberg to that component of D^k which contains (x_0,t_0), we see that a minimum not greater than a must occur on the boundary of D^k. Since this cannot occur on $B-E$, this minimum must be taken at a point of the form $(x_k, 1/k)$. As k tends to infinity, this gives us an infinite set of points at which $w(x,t)\pm\psi_n(x,t)$ is less than or equal to a. This set of points must have a point of E as a limit point, but this contradicts the statement that

$$\liminf_{\substack{(x,t)\to z\in E\\ (x,t)\in D}} [w(x,t)\pm\psi_n(x,t)] \geqslant 0.$$

Thus, as claimed, $w(x,t)\pm\psi_n(x,t)\geqslant 0$ for all $(x,t)\in D$, i. e., $|\psi_n(x,t)| \leqslant w(x,t)$ for all $(x,t)\in D$ and for all n. Letting $n\to\infty$, we obtain $|\psi(x,t)| \leqslant w(x,t)$ for $(x,t)\in D$. Property (c′) then shows that $\psi(x,t)$ has parabolic limit zero almost everywhere in E.

All that remains to be done is to find a function w with the desired properties.

Let $\chi(z)$ be the characteristic function of $C_1 \dot{-} E$, where $C_1\subset E_n$ is a hypercube with the same center as the hypercube C and sides of length $1+6\alpha$; $q\in E_n$ be a point exterior to C_1; and $w(x,t) = KW(x-q,t)+ +M\int_{E_n}\chi(x)W(x-z,t)\,dz$.

Properties (a′) and (c′) are obvious for this function. To show (b′) for $(x,1)$ in the upper part of $B-E$, we need only make K large enough to make $w(x,1)\geqslant 2$. On the other hand, if $(x,t)\in B-E$, $t<1$, consider the inverted paraboloid of aperture α and vertex (x,t). Let S be the intersection of this paraboloid with E_n. If v is interior to S, then v is not in E, since, if it were, (x,t) would be interior to D, and, thus, not in B. Therefore,

$$w(x,t)\geqslant M\int_S W(x-z,t)\,dz \geqslant cM\int_0^{\alpha t^{1/2}} (4\pi t)^{-n/2}e^{-r^2/4t}r^{n-1}dr$$

$$\geqslant cM^{-n/2}\int_0^{\alpha/2} e^{-s^2}s^{n-1}ds \geqslant 2 \quad \text{for} \quad M\geqslant 2\Big[\pi^{-n/2}c\int_0^{\alpha/2} e^{-s^2}s^{n-1}ds\Big]^{-1},$$

where c is the surface area of the unit sphere in E_n.

This completes the proof of the theorem.

Section 2. An integral representation for temperatures

In this section we will obtain an integral representation of a temperature in the following sense: For a certain domain $R\subset E_{n+1}^+$ with boundary K, we will show that if u is a temperature on R and continuous on $R \cap K$, then

$$u(p) = \int_K H(p,q)u(q)\,dq, \qquad p \in R.$$

We will also indicate some of the properties of the kernel H which will be of use to us later.

Our procedure will be to obtain the results in one dimension, and then extend them to n dimensions. The technique will be similar in both cases, i. e., we will show that if a kernel H has certain properties we obtain the desired representation and then we will exhibit a kernel which has these properties.

a. One-dimensional results. This material is essentially due to Hartman and Wintner [7] with notational changes made to facilitate extension to higher dimension.

Throughout this discussion, x will denote a real number, i. e., $x \in E_1$. Let $R = \{p = (x,t): 0 < x < 1,\ t > 0\}$. Then $K = \{q = (v,s): v = 0, s > 0\} \cup \{q = (v,s): 0 \leqslant v \leqslant 1,\ s = 0\} \cup \{q = (v,s): v = 1,\ s > 0\}$. Let dq be the Lebesgue measure on K oriented so that the positive direction is downward on the left hand side and upward on the right.

Let us suppose that we have a kernel with the following properties for $p \in R$:

(a) $\int_K H\,dq = 1$;

(b) $\int_K |H|\,dq < \infty$ (in our case, it is bounded independently of p);

(c) For a given $a > 0$, (i) $\int_{T_a} |H|\,dq \to 0$ as $x \to 0$, independently of $t > 0$, where $T_a = K - \{(0,s): 0 \leqslant t-s \leqslant a\}$; (ii) $\int_{U_a} |H|\,dq \to 0$ as $t \to 0$, independently of $0 < x < 1$, where $U_a = K - \{(v,0): |x-v| \leqslant a\}$; (iii) $\int_{V_a} |H|\,dq \to 0$ as $x \to 1$, independently of $t > 0$, where $V_a = K - \{(1,s: 0 \leqslant t-s \leqslant a\}$;

(d) $H(p,q) = 0$ when $s \geqslant t$, where $p = (x,t)$, $q = (v,s)$;

(e) For each $q \in K$, H is a temperature as a function of $p \in R$;

(f) $\int_K |\partial^k H/\partial x^k|\,dq \leqslant A_\varepsilon < \infty$, where A_ε is independent of p when $p = (x,t)$ satisfies $0 < \varepsilon \leqslant x \leqslant 1-\varepsilon < 1$, $t \geqslant \varepsilon > 0$.

We note that properties (a)-(c) are the ones of a quasi-positive kernel, see [15]. This guarantees that H is a function of p, q so that $u(p)$ defined by

$$u(p) = \int_K H(p,q)f(q)\,dq$$

will approach $f(q)$ (in some sense to be defined presently) as $p \to q$ along a path orthogonal to K at q. More precisely:

LEMMA 2. *Let $f(q)$ be a continuous function of $q = (v,s) \in K$. Then,*

$$u(p) = \int_K H(p,q)f(q)\,dq$$

is a temperature on R, and (i) α. $\lim_{x\to 0+} u(x,t) = f(0,t)$ *and* β. $\lim_{x\to 1-} u(x,t) = f(1,t)$, $t > 0$; (ii) $\lim_{t\to 0+} u(x,t) = f(x,0)$ *for* $0 < x < 1$.

Moreover, for $t_0 > 0$ fixed, $u(p)$ is uniformly continuous on $\{p \in R: 0 < t < t_0\} = R_{t_0}$.

Proof. We will prove the lemma under the assumption that H satisfies (a)-(f). We will later exhibit an H having these properties.

Since, for $t \geqslant 0$ fixed, $f(q)$ is bounded on the set where $s \leqslant t$, $\int_K |H|\,dq < \infty$, and H is non-zero only for $s \leqslant t$, $u(p)$ exists. From (e) and (f) we see that u is a temperature.

Let $t_0 > 0$, $M = \sup_{0\leqslant s\leqslant t_0} |f(v,s)|$. On the set $\{q \in K: 0 \leqslant s \leqslant t_0\}$, f is uniformly continuous, i. e., given $\varepsilon > 0$, there exists $a_0 > 0$ such that $|f(q_1)-f(q_2)| < \varepsilon/(6\int_K |H|\,dq)$ for $d(q_1,q_2) < a_0$, where $d(q_1,q_2)$ is the distance from q_1 to q_2 along K. To show (i) α, using (a), we have

$$|u(x,t)-f(0,t)| = \Big|\int_K H(x,t,v,s)[f(v,s)-f(0,t)]\,dq\Big|$$

$$\leqslant \Big|\int_t^{t-a_0} H(x,t,0,s)[f(0,s)-f(0,t)]\,ds\Big| + \Big|\int_{T_{a_0}} H(p,q)[f(q)-f(0,t)]\,dq\Big|$$

$$= |I_1| + |I_2|.$$

Clearly, $|I_2| \leqslant 2M\int_{T_{a_0}} |H(p,q)|\,dq \to 0$ as $x \to 0$ independently of $t > 0$;

$$|I_1| \leqslant \Big[\varepsilon/(6\int_K |H|\,dq)\Big]\int_t^{t-a_0} |H(x,t,0,s)|\,ds < \varepsilon/6.$$

Note that the proof shows the existence of a $\delta_1 > 0$ depending only on t_0 such that $|u(x,t)-f(0,t)| < \varepsilon/3$ for $|x| < \delta_1$, $0 < t \leqslant t_0$. We similarly obtain a $\delta_2 > 0$ and $\delta_3 > 0$ depending only on t_0 such that $|u(x,t) - f(1,t)| < \varepsilon/3$ for $|x-1| < \delta_2$, $0 < t \leqslant t_0$ and $|u(x,t)-f(x,0)| < \varepsilon/3$ for $0 < t < \delta_3$, $0 < x < 1$. Hence, letting $\delta' = \min(a_0/4, \delta_1, \delta_2, \delta_3)$, we have: if the distance from p to K is less than δ', $p \in R_{t_0}$, and $q \in K$ such that $|p-q| =$ distance (p,K), then $|u(p)-f(q)| < \varepsilon/3$, and, further, if $p_1, p_2 \in R_{t_0}$, $|p_1-p_2| < \delta'$, distance $(p_i,K) < \delta'$, $i = 1,2$, then $|u(p_1) - u(p_2)| \leqslant |u(p_1)-f(q_1)| + |f(q_1)-f(q_2)| + |f(q_2)-u(p_2)| < (\varepsilon/3)+(\varepsilon/6)+(\varepsilon/3) < \varepsilon$. Consider the set S of points $p \in R_{t_0}$ such that distance $(p,K) \geqslant \delta'/2$. u is clearly continuous on S and, hence, uniformly continuous there, i.e., there exists a $\delta''>0$ such that, if $|p_1-p_2|<\delta''$, $p_1,p_2\in S$, then $|u(p_1)-u(p_2)|<\varepsilon$. Let $\delta=\min(\delta'',\delta'/3)$. Then, $p_1,p_2\in R_{t_0}$ and $|p_1-p_2|<\delta$, then $|u(p_1)-u(p_2)|<\varepsilon$. This is obvious if both are in S. If one point is not in S, then the distances of both from K are less than δ', and we are done.

An immediate consequence of this lemma is the desired representation, i.e.,

COROLLARY. *If $u(p)$ is a temperature on R and continuous on $R\cup K$, then $u(p)=\int_K H(p,q)u(q)dq$.*

Proof. Since the integral and u are both temperatures on R and have the same boundary values on K, the corollary follows from the maximum principle stated earlier.

To complete the one dimensional results, we need only to show that there exists an $H(p,q)$ which satisfies conditions (a)-(f).

We will define H in terms of the following periodization of the Weierstrass kernel: Let

$$\theta_1(x,t)$$

$$\text{(A)}\qquad =\sum_{k=-\infty}^{\infty}(4\pi t)^{-1/2}\exp\left(-(x+2k)^2/4t\right)$$

$$\text{(B)}\qquad =1+2\sum_{n=1}^{\infty}\exp(-n^2\pi^2t)\cos n\pi x$$

$$\text{(C)}\qquad =\sum_{n=-\infty}^{\infty}\exp(-n^2\pi^2t)e^{in\pi x}$$

$$\text{(D)}\qquad =\prod_{n=1}^{\infty}(1-e^{-2\pi^2nt})\prod_{n=1}(1+e^{(-4n-2)\pi^2t})\prod_{n=1}\left(1+[\cos\pi x/\cosh(2n-1)\pi^2t]\right)$$

for $t>0$.

The second and third representations of θ_1 are just the Fourier series of the first with respect to an interval of length two. The fourth representation may be found in Magnus and Oberhettinger [10] and a proof of its validity is given in Whittaker and Watson [14]. In both of these references, our $\theta_1(x,t)$ corresponds to their $\theta_3(z,q)$, $z=x/2$, $q=e^{i\pi\tau}$, $\tau=i\pi t$, $t>0$. Let

$$\theta(x,t)=\begin{cases}\theta_1(x,t) & \text{for}\quad t>0,\\ 0 & \text{otherwise;}\end{cases}$$

$$G_1(p,q)=\tfrac12[\theta(x-v,t-s)-\theta(x+v,t-s)]$$

for $p=(x,t)\in R$, $q=(v,s)\in K$,

$$G_2(p,q)=\begin{cases}-\dfrac{\partial}{\partial v}G_1(p,q) & \text{for}\quad v=0,1\\ 0 & \text{otherwise;}\end{cases}$$

and $H(p,q)=G_1(p,q)+G_2(p,q)$, for $p\in R$, $q\in K$.

Straightforward computations from representation (C) show:

$$\text{(1)}\qquad \int_t^0 G_2(x,t,0,s)ds=(1/\pi)\sum_{n=-\infty}^{\infty}{}'(1-e^{-n^2\pi^2t})e^{in\pi x}/in$$

$$=f_1(x)-[f_1(\cdot)*\theta_1(\cdot,t)](x),\qquad (x,t)\in R,$$

where $f_1(x)=1-x$ for $0\leqslant x<2$, $f_1(x)=f_1(x-2k)$ for $2k\leqslant x<2(k+1)$, k an integer, and

$$[f_1(\cdot)*\theta_1(\cdot,t)](x)=\int_{-1}^{1}f_1(z)\theta_1(x-z,t)dz$$

is the convolution of f_1 with θ_1, since, for $0<x<1$, $t>0$, the Fourier series involved converge to the respective functions. Hence, using (A) and the fact that θ_1 is just the periodization of the Weierstrass kernel to evaluate the L_1 norm of θ_1,

$$\left|\int_t^0 G_2(x,t,0,s)ds\right|\leqslant\|f_1\|_\infty+\|f_1*\theta_1\|_\infty$$

$$\leqslant 1+\|f_1\|_\infty\|\theta_1\|_1\leqslant 1+1=2.$$

Similarly,

$$\text{(2)}\qquad \int_0^t G_2(x,t,1,s)ds=(1/\pi)\sum_{n=-\infty}^{\infty}{}'(-1)^{n+1}(1-e^{-n^2\pi^2t})e^{in\pi x}/in$$

$$=f_2(x)-[f_2(\cdot)*\theta_1(\cdot,t)](x),$$

where $f_2(x)=x$ for $-1\leqslant x<1$, $f_2(x)=f_2(x-2k)$ for $2k-1\leqslant x<2k+1$, k an integer. As above,

$$\left|\int_0^t G_2(x,t,1,s)ds\right|\leqslant 2\qquad\text{for}\qquad (x,t)\in R.$$

In the same way,

$$(*)\qquad \int_0^1 G_1(x,t,v,s)dv=(1/\pi)\sum_{n=-\infty}^{\infty}{}'\{[1+(-1)^{n+1}]e^{-n^2\pi^2(t-s)}e^{in\pi x}/in\}$$

$$=[f_1(\cdot)*\theta_1(\cdot,t-s)](x)+[f_2(\cdot)*\theta_1(\cdot,t-s)](x)$$

for $0 \leqslant s < t$, $0 < x < 1$, and

$$\left|\int_0^1 G_1(x,t,v,s)\,dv\right| \leqslant 2, \qquad (x,t)\in R.$$

Thus, for $p\in R$,

$$\int_K H(p,q)\,dq = \int_t^0 G_2(p,0,s)\,ds + \int_0^1 G_1(p,v,0)\,dv + \int_0^t G_2(p,1,s)\,ds$$

$$= f_1(x)+f_2(x) = 1,$$

which shows that H satisfies (a).

For $0\leqslant x\leqslant 1$, $0\leqslant v\leqslant 1$, $\cos\pi(x-v)-\cos\pi(x+v) = 2\sin\pi x\sin\pi v \geqslant 0$. Thus, by using representation (D) of $\theta_1(x,t)$, we see that $G_1(p,q) \geqslant 0$ for $p,q\in R\cup K$. This, together with the fact that $G_1(p,0,s) = G_1(p,1,s) = 0$ for $p\in R$, shows that $G_2(p,0,s)\leqslant 0$ and $G_2(p,1,s)\geqslant 0$.

From this we have:

$$\int_K |H(p,q)|\,dq$$

$$= \int_t^0 G_2(p,0,s)\,ds + \int_0^1 G_1(p,v,0)\,dv + \int_0^t G_2(p,1,s)\,ds \leqslant 6 < \infty$$

by (1), (2), and (*), for $p\in R$. This shows (b).

Using (B), we see

$$\int_{t-a}^0 G_2(x,t,0,s)\,ds = (1/\pi)\sum_{n=1}^{\infty}\{(\sin n\pi x/n)(e^{-n^2\pi^2 a}-e^{-n^2\pi^2 t})\},$$

$$\int_0^1 G_1(x,t,v,0)\,dv = (1/\pi)\sum_{n=1}^{\infty}\{[1+(-1)^{n+1}](\sin n\pi x/n)e^{-n^2\pi^2 t}\},$$

and

$$\int_0^t G_2(x,t,1,s)\,ds$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}[\sin n\pi x/n]-(1/\pi)\sum_{n=1}^{\infty}(-1)^{n+1}(\sin n\pi x/n)e^{-n^2\pi^2 t}.$$

Hence,

$$\int_{T_a} H(p,q)\,dq = \int_{t-a}^0 G_2(x,t,0,s)\,ds + \int_0^1 G_1(x,t,v,0)\,dv + \int_0^t G_2(x,t,1,s)\,ds$$

$$= (1/\pi)\sum_{n=1}^{\infty}(\sin n\pi x/n)e^{-n^2\pi^2 a} + (1/\pi)\sum_{n=1}^{\infty}(-1)^{n+1}(\sin n\pi x/n)$$

which tends to zero as $x\to 0$, independently of $t>0$, since the first series is uniformly convergent in a neighborhood of $x=0$, and the second is the Fourier series of $f_2(x)$, and thus converges to x in a neighborhood of $x=0$. This shows (c i).

(c ii) is shown in a similar manner.

Since the L_1 norm on K of G_2 is bounded independently of $p\in R$, we have that both $\int_t^0 G_2(p,0,s)\,ds$ and $\int_0^t G_2(p,1,s)\,ds$ tend to zero as $t\to 0$ independently of x. To obtain (c iii), we thus need only show that for $1 > x\geqslant a$,

(′) $\int_0^{x-a} G_1(p,v,0)\,dv$ tends to zero as $t\to 0$ independently of x,

and, for $1-a\geqslant x>0$,

(″) $\int_{x+a}^1 G_1(p,v,0)\,dv$ tends to zero as $t\to 0$, independently of x.

Let

$$\theta_x(x,t) = \frac{\partial}{\partial x}\theta_1(x,t) = -\pi^{-1/2}\sum_{k=-\infty}^{\infty}\{(x+2k/2t^{3/2})\exp[-(x+2k)^2/4t]\}$$

(we use (A)). This converges uniformly for $0<x_0\leqslant x\leqslant x_1<2$, $0<t\leqslant t_1<\infty$. Since each term tends to zero as $t\to 0$, we have $\theta_x(x,t)\to 0$ as $t\to 0$, $0<x_0\leqslant x\leqslant x_1<2$. By the mean value theorem, for $v>0$, we have $G_1(x,t,v,s)/v = \theta_x(x+hv,t-s)$, $h = h(x,t,v,s)$, $|h|<1$. Letting $x_0 = a>0$, $x_1 = 2-a<2$, we have, given $\varepsilon>0$, $|G_1(x,t,v,s)|/v<\varepsilon$ for $t-s$ small enough, $0\leqslant s<t$. Thus,

$$\int_0^{x-a}|G_1(x,t,v,s)|\,dv < \varepsilon\int_0^{1-a} v\,dv.$$

We have made this integral small independently of $1>x\geqslant a$, which shows (′) if we let $s=0$. Similarly, for $v<1$, $G_1(x,t,v,s)/(1-v) = \theta_x(x+1-h(1-v),\,t-s)$, and letting x_0, x_1 be as above, for a given $\varepsilon>0$, we obtain

$$\int_{x+a}^1 |G_1(x,t,v,s)|\,dv \leqslant \varepsilon\int_a^1 (1-v)\,dv.$$

Letting $s=0$, this shows (″) and completes (c iii).

(e) is just a matter of direct computation.

In the proof of (f), we will make use of the following observation:

LEMMA 3. *Let $W(x,t)$ be the one dimensional Weierstrass kernel. Then,*

(1) $\dfrac{\partial^m}{\partial x^m}W(x,t)$ *has at most m zeros;*

(2) *these zeros occur at points of the form* $Kt^{1/2}$, *where* K *depends only on* m;

(3) $\int\limits_{-\infty}^{\infty}\left|\frac{\partial^m}{\partial x^m}W(x,t)\right|dx \leqslant Mt^{-m/2}$, $t>0$, M *not depending on* t.

Proof.

$$\frac{\partial^m}{\partial x^m}W(x,t)=\begin{cases}(') \ (M/t^{(m+1/2)})P(x^2/t)e^{-x^2/4t}, & m \text{ odd},\\ ('') \ (M/t^{(m/2)+1})xP(x^2/t)e^{-x^2/4t}, & m \text{ even},\end{cases}$$

where P is a polynomial of degree $m/2$ and $(m-1/2)$, respectively, and M is independent of x, t. It is obvious that this holds for $m=0$ or 1. Assume that it holds for an even m, i. e.,

$$\frac{\partial^m}{\partial x^m}W(x,t)=(M/t^{(m+1/2)})P(x^2/t)e^{-x^2/4t}.$$

Differentiating, we have

$$\begin{aligned}\frac{\partial^{m+1}}{\partial x^{m+1}}W(x,t) &= (M/t^{(m+1/2)})[(2x/t)P'(x^2/t)-(2x/4t)P(x^2/t)]e^{-x^2/4t}\\ &= (M/t^{(m+1/2)+1})x[2P'(x^2/t)-\tfrac{1}{2}P(x^2/t)]e^{-x^2/4t}\end{aligned}$$

which is $('')$ for $m+1$. Assume this representation holds for m odd,

$$\frac{\partial^m}{\partial x^m}W(x,t)=(M/t^{(m/2)+1})xP(x^2/t)e^{-x^2/4t}.$$

Differentiating, we see

$$\frac{\partial^{m+1}}{\partial x^{m+1}}W(x,t)$$

$$=e^{-x^2/4t}(M/t^{((m+2)/2)})[P(x^2/t)+(2x^2/t)P'(x^2/t)-(2x^2/4t)P(x^2/t)],$$

which is $(')$ for $m+1$. Thus, the representation holds for all m by induction.

(1) and (2) follow immediately from this representation, since the zeros of $\partial^mW/\partial x^m$ will be the zeros of $P(x^2/t)$, with one at $x=0$ if m is odd.

Suppose $-\infty=x_0<x_1<\ldots<x_k<x_{k+1}=\infty$, where $x_1,\ldots,x_k$ are the distinct zeros of $\partial^mW/\partial x^m$, $k\leqslant m$. Then,

$$\int\limits_{-\infty}^{\infty}\left|\frac{\partial^mW}{\partial x^m}\right|dx=\sum_{i=0}^{k}\left|\int\limits_{x_i}^{x_{i+1}}\frac{\partial^mW}{\partial x^m}dx\right|$$

$$=\sum_{i=0}^{k}\left|\left[\frac{\partial^{m-1}W}{\partial x^{m-1}}\right]_{x_i}^{x_{i+1}}\right|\leqslant\sum_{i=0}^{k}\left[\left|\frac{\partial^{m-1}}{\partial x^{m-1}}W(x_{i+1},t)\right|+\left|\frac{\partial^{m-1}}{\partial x^{m-1}}W(x_i,t)\right|\right].$$

Now $x_i=At^{1/2}$, where A depends only on m. Thus,

$$\frac{\partial^{m-1}}{\partial x^{m-1}}W(x_i,t)=(M/t^{m/2})P(A^2)e^{-A^2/4},$$

whence

$$\int\limits_{-\infty}^{\infty}\left|\frac{\partial^mW}{\partial x^m}\right|dx\leqslant M_1t^{-m/2},$$

M_1 depending only on m.

Property (f) is contained in the following

LEMMA 4.

(I) $\int\limits_0^1\left|\frac{\partial^m}{\partial x^m}G_1(x,t,v,s)\right|dv\leqslant M(t-s)^{-m/2}$, *where* M *is independent of* s *and* t, *and* $0\leqslant s<t$.

(II) $\int\limits_0^t\left|\frac{\partial^m}{\partial x^m}G_2(x,t,\delta,s)\right|ds<\infty$ *for* $\delta=0,1$, $t>0$, $0<\varepsilon\leqslant x\leqslant 1-\varepsilon<1$.

(III) $\int\limits_0^t(t-s)^{-k/2}\left|\frac{\partial^m}{\partial x^m}G_2(x,t,\delta,s)\right|ds<\infty$ *for* $\delta=0,1$, $t>0$, $0<\varepsilon\leqslant x\leqslant 1-\varepsilon<1$.

Proof. (II) and (III) are obvious if we realize that, for x restricted in the indicated way, the integrands are finite. For (I) we have

$$\int\limits_0^1\left|\frac{\partial^m}{\partial x^m}G_1(x,t,v,s)\right|dv\leqslant\int\limits_0^1\left|\frac{\partial^m}{\partial x^m}\theta(x-v,t-s)\right|dv+\int\limits_0^1\left|\frac{\partial^m}{\partial x^m}\theta(x+v,t-s)\right|dv$$

$$\leqslant\int\limits_0^1\left|\frac{\partial^m}{\partial x^m}\theta(x-v,t-s)\right|dv-\int\limits_0^{-1}\left|\frac{\partial^m}{\partial x^m}\theta(x-v,t-s)\right|dv$$

$$\leqslant\int\limits_{-1}^1\left|\frac{\partial^m}{\partial x^m}\theta(x-v,t-s)\right|dv\leqslant\int\limits_{-\infty}^{\infty}\left|\frac{\partial^m}{\partial x^m}W(x,t-s)\right|dx\leqslant M(t-s)^{-m/2},$$

by part 3 of lemma 3. This completes the one dimensional discussion.

b. n-dimensional results. Throughout this discussion we shall use the following notation: given $x=(x_1,\ldots,x_n)\in E_n$, we put $x^0=(x_1^0,\ldots,x_n^0)$, where $x_k^0=x_k$, $k\neq i$, and $x_i^0=0$, $x^1=(x_1^1,\ldots,x_n^1)$, with $x_k^1=x_k$, $k\neq i$, $x_i^1=1$; $C=\{x\in E_n: 0<x_i<1,\ i=1,\ldots,n\}$; $R=\{p=(x,t): x\in C,\ t>0\}$; $K=$ boundary of $R=\bigcup\limits_{i=1}^{n}\{q=(v,s): s>0,\ v\in E_n,\ v_i=0,1,\ 0\leqslant v_k\leqslant 1,\ k\neq 1\}\cup\{q=(v,s): s=0,\ 0\leqslant v_i\leqslant 1,\ i=1,\ldots,n\}$. Let dq be the Lebesgue measure on K oriented so that the positive direction is "downward" on the surfaces $\{v_i = 0\}$ and "upward" on the surfaces $\{v_i = 1\}$ (1). We also define $T_a^{(i)} = K - \{(v, s): v_i = 0,\ t-s < a,\ |v_k - x_k| < a,\ k \neq i\}$, $V_a^{(i)} = K - \{(v, s): v_i = 1,\ t-s < a,\ |v_k - x_k| < a,\ k \neq i\}$, $U_a = K - \{(v, 0): |v_i - x_i| < a,\ i = 1, \ldots, n\}$ for $a > 0$.

We will define on $R \times K$ a kernel $H(p, q)$, $p \in R$, $q \in K$, such that the following properties are satisfied:

(a′) $\int_K H(p, q)\,dq = 1$, $p \in R$;

(b′) $\int_K |H(p, q)|\,dq < \infty$ (as in one dimension, it is bounded independently of p);

(c′) For a given $a > 0$, (i) $\int_{T_a^{(i)}} |H|\,dq \to 0$ as $x_i \to 0$ independently of $t > 0$, $0 < x_k < 1$, $k \neq i$, (ii) $\int_{U_a} |H|\,dq \to 0$ as $t \to 0$ independently of $x \in C$, (iii) $\int_{V_a^{(i)}} |H|\,dq \to 0$ as $x_i \to 1$ independently of $t > 0$, $0 < x_k < 1$, $k \neq i$;

(d′) $H(p, q) = 0$ for $s \geqslant t$, $p = (x, t)$, $q = (v, s)$;

(e′) H is a temperature as a function of $p \in R$;

(f′) $\int_K |\partial^k H / \partial x^k|\,dq < \infty$ independently of $p = (x, t)$ for $0 < \varepsilon \leqslant x_i \leqslant 1 - \varepsilon < 1$, $t \geqslant \varepsilon > 0$, k a multi-index, i. e., $k = (k_1, \ldots, k_n)$, k_i a non-negative integer, and

$$\frac{\partial^k}{\partial x^k} = \frac{\partial^{k_1}}{\partial_{x_1}^{k_1}} \frac{\partial^{k_2}}{\partial_{x_2}^{k_2}} \cdots \frac{\partial^{k_n}}{\partial_{x_n}^{k_n}}.$$

Then, in precisely the same way as in part a, we have

LEMMA 2′. *Let $f(q)$ be a continuous function on K. Then $u(p) = \int_K H(p, q) f(q)\,dq$ is a temperature on R, and*

(i) α. $\lim_{x_i \to 0+} u(p) = f(p^0)$, $t > 0$, $p = (x, t)$, $p^0 = (x^0, t)$ *and* β. $\lim_{x_i \to 1-} u(p) = f(p^1)$, $t > 0$, $p = (x, t)$, $p^1 = (x^1, t)$;

(ii) $\lim_{t \to 0+} u(p) = f(x, 0)$, $p = (x, t)$, $x \in C$.

Moreover, for $t_0 > 0$ fixed, $u(p)$ is uniformly continuous on $R_{t_0} = \{p \in R: 0 < t < t_0\}$.

COROLLARY. *If $u(p)$ is a temperature on R and continuous on $R \cup K$, then $u(p) = \int_K H(p, q) u(q)\,dq$, for $p \in R$.*

To find an H with the desired properties and to demonstrate those properties, we make the following definitions.

(1) By this we mean that on the surface $v_i = 0$, $dq = -dxdt$, while on the surface $v_i = 1$, $dq = dxdt$.

Let $G_1(x_i, t, v_i, s)$, $G_2(x_i, t, v_i, s)$ be as in part a;

$$G_3(x_i, t, v_i, s) = -\frac{\partial^2}{\partial v_i^2} G_1(x_i, t, v_i, s) = \frac{\partial}{\partial_s} G_1(x_i, t, v_i, s);$$

$$H_i(x_i, t, v_i, s) = G_1(x_i, t, v_i, s) + G_2(x_i, t, v_i, s), \qquad 1 \leqslant i \leqslant n;$$

and

$$H(x, t, v, s) = \prod_{i=1}^{n} H_i(x_i, t, v_i, s),$$

$x = (x_1, \ldots, x_n)$, $v = (v_1, \ldots, v_n)$, $0 \leqslant v_i \leqslant 1$.

We note that

$$\int_E G_2(x_i, t, v_i, s)\,dv_i = 0 \quad \text{for} \quad E \subset \{v_i: 0 \leqslant v_i \leqslant 1\},$$

and that due to this, the only terms in H which contribute to the integral over a subset of K are those containing at most one factor of the form $G_2(x_i, t, v_i, s)$.

We will now show explicitly that properties (a′)-(f′) are satisfied by this H in the case $n = 2$. This case will serve to make the ideas clear since the arguments are virtually the same for general n and the notation is much more cumbersome.

Here $H = H_1 H_2 = G_1(x_1, t, v_1, s) G_1(x_2, t, v_2, s) + G_1(x_1, t, v_1, s) G_2(x_2, t, v_2, s) + G_2(x_1, t, v_1, s) G_1(x_2, t, v_2, s) + G_2(x_1, t, v_1, s) G_2(x_2, t, v_2, s)$. As we previously noted, the fourth term contributes nothing to our integrals.

Direct differentiation shows that if $g_1(x_1, t)$ and $g_2(x_2, t)$ are temperatures for (x_1, t) and (x_2, t) respectively, then $g_1(x_1, t) g_2(x_2, t)$ is a temperature in (x_1, x_2, t). Thus, (e′) follows from (e).

From the estimates on $\int_0^1 G_1(x, t, v, s)\,dv$ and $\int_0^t |G_2(x, t, \delta, s)|\,ds$, $\delta = 0, 1$, in the proof of (a), part a, and from estimates (I)-(III), we obtain (f′).

Let us observe the following:

$$(1) \qquad \int_{K \cap \{s=0\}} H(p, q)\,dq = \Big\{\int_0^1 G_1(x_1, t, v_1, 0)\,dv_1\Big\}\Big\{\int_0^1 G_1(x_2, t, v_2, 0)\,dv_2\Big\};$$

$$(2) \qquad \int_{K \cap \{v_1 = 0\}} H(p, q)\,dq = \int_t^0 \int_0^1 G_2(x_1, t, 0, s) G_1(x_2, t, v_2, s)\,dv_2\,ds;$$

$$(3) \qquad \int_{K \cap \{v_1 = 1\}} H(p, q)\,dq = \int_0^t \int_0^1 G_2(x_1, t, 1, s) G_1(x_2, t, v_2, s)\,dv_2\,ds.$$

Reversing the order of integration in (2), we combine (2) and (3) to obtain:

(4)
$$\int\limits_{K\cap[\{v_1=0\}\cup\{v_1=1\}]} H(p,q)\,dq$$
$$=\int_0^t\int_0^1 G_1(x_2,t,v_2,s)\,[G_2(x_1,t,1,s)-G_2(x_1,t,0,s)]\,dv_2\,ds$$
$$=\int_0^t\int_0^1 G_1(x_2,t,v_2,s)\Big\{\int_0^1 G_3(x_1,t,v_1,s)\,dv_1\Big\}dv_2\,ds$$
$$=\int_0^t\Big\{\int_0^1 G_1(x_2,t,v_2,s)\,dv_2\Big\}\Big\{\frac{\partial}{\partial s}\int_0^1 G_1(x_1,t,v_1,s)\,dv_1\Big\}ds.$$

Similarly,

(5)
$$\int\limits_{K\cap[\{v_2=0\}\cup\{v_2=1\}]} H(p,q)\,dq$$
$$=\int_0^t\Big[\int_0^1 G_1(x_1,t,v_1,s)\,dv_1\Big]\Big[\frac{\partial}{\partial s}\int_0^1 G_1(x_1,t,v_1,s)\,dv_2\Big]ds.$$

Hence,

$$\int\limits_{K\cap\{s>0\}} H(p,q)\,dq=\int_0^t\Big[\frac{\partial}{\partial s}\int_0^1 G_1(x_1,t,v_1,s)\,dv_1\Big]\Big[\int_0^1 G_1(x_2,t,v_2,s)\,dv_2\Big]ds$$
$$=\lim_{r\to t}\Big[\int_0^1 G_1(x_1,t,v_1,r)\,dv_1\int_0^1 G_1(x_2,t,v_2,r)\,dv_2\Big]-$$
$$-\int_0^1 G_1(x_1,t,v_1,0)\,dv_1\int_0^1 G_1(x_2,t,v_2,0)\,dv_2.$$

Combining with (1), we have:

$$\int_K H(p,q)\,dq=\lim_{r\to t}\int_0^1 G_1(x_1,t,v_1,r)\,dv_1\int_0^1 G_1(x_2,t,v_2,r)\,dv_2=1$$

by (*), part a. This shows (a').

$$\int\limits_{K\cap\{s=0\}} H(p,q)\,dq=\int_0^1 G_1(x_1,t,v_1,0)\,dv_1\int_0^1 G_1(x_2,t,v_2,0)\,dv_2\leqslant 4.$$
$$\int\limits_{K\cap\{v_1=0\}} |H(p,q)|\,dq=\int_0^t\int_0^1 |G_2(x_1,t,0,s)|\,|G_1(x_2,t,v_2,s)|\,dv_2\,ds$$
$$=\int_0^t |G_2(x_1,t,0,s)|\Big[\int_0^1 G_1(x_2,t,v_2,s)\,dv_2\Big]ds$$
$$\leqslant 2\int_0^t |G_2(x_1,t,0,s)|\,ds\leqslant 4.$$

Similar arguments on the remaining sides give us (b').

Fix a $t_0>0$. Then

$$\int_0^1 G_1(x,t,v,0)\,dv\leqslant\int_{T_\alpha}|H(p,q)|\,dq$$

which tends to zero as $x\to 0$ independently of $0<t<t_0$, x, $v\in E_1$. Hence,

$$\int_0^1 G_1(x,t,v,s)\,dv=\int_0^1 G_1(x,t-s,v,0)\,dv$$

tends to zero as $x\to 0$ independently of $0<t-s<t_0$, i.e., $0\leqslant s<t<t_0$, $x,v\in E_1$.

Similarly, given $\varepsilon>0$, we can choose $\alpha\geqslant\eta\geqslant 0$ so that

$$\int\limits_{|x-v|\geqslant\alpha} G_1(x,t,v,s)\,dv<\varepsilon/2\quad\text{for}\quad t-s<\eta.$$

$$\int\limits_{T_\alpha^{(1)}}|H|\,dq\leqslant\int\limits_{T_\eta}|H_1(x_1,t,v_1,s)|\Big\{\int_0^1 G_1(x_2,t,v_2,s)\,dv_2\Big\}dq_1+$$
$$+\int\limits_{K\cap[\{v_2=0\}\cup\{v_2=1\}]}|H(p,q)|\,dq+\int_{t-\eta}^t\int\limits_{|x_2-v_2|\geqslant\alpha}|G_2(x_1,t,0,s)|\,G_1(x_2,t,v_2,s)\,dv_2\,ds$$
$$=\mathrm{I}+\mathrm{II}+\mathrm{III}.$$

$\mathrm{I}\leqslant 2\int_{T_\eta}|H_1(x_1,t,v_1,s)|\,dq_1<0$ as $x_1\to 0$ independently of t.

$\mathrm{II}\leqslant\int_0^t|G_2(x_2,t,0,s)|\{\int_0^1 G_1(x_1,t,v_1,s)\,dv_1\}\,ds+\int_0^t|G_2(x_2,t,1,s)|\{\int_0^1 G_1(x_1,t,v_1,s)\,dv_1\}\,ds$ which tends to zero as $x_1\to 0$ independently of t by Lebesgue's dominated convergence theorem and the first remark made above.

$\mathrm{III}\leqslant\int_{t-\eta}^t|G_2(x_1,t,0,s)|\{\int_{|x_2-v_2|\geqslant\alpha} G_1(x_2,t,v_2,s)\,dv_2\}\,ds<(\varepsilon/2)\int_{t-\eta}^t G_2(x_1,t,0,s)\,ds\leqslant\varepsilon$, independently of t by the second remark above.

Similar arguments give the remainder of (c') i and (c') iii.

$$\int\limits_{U_\alpha}|H(p,q)|\,dq\leqslant\sum_{\substack{\delta=0,1\\ i=1,2}}\int_0^t|G_2(x_{2-i},t,\delta,s)|\int_0^1 G_1(x_1,t,v_i,s)\,dv_i\,ds+$$
$$+\sum_{i=1,2}\int_0^1 G_1(x_{2-i},t,v_{2-i},0)\,dv_{2-i}\int\limits_{|x_i-v_i|\geqslant\alpha}G_1(x_i,t,v_i,0)\,dv_i$$
$$2\sum_{\substack{\delta=0,1\\ i=1,2}}\int_0^1|G_2(x_i,t,\delta,s)|\,ds+2\sum_{i=1,2}\int\limits_{|x_i-v_i|\geqslant\alpha}G_1(x_i,t,v_i,0)\,dv_i$$

which tends to zero as $t\to 0$ independently of $x\in C$. This completes (c') and this part.

c. A mean value type result. Here we will obtain a special modification of the kernel in the preceding part, and obtain from it a bound on the value of a temperature at a point in terms of the integral of its square over a related solid body.

Definition. Let $H_r(x_0, t_0, x, t) = H(\tfrac{1}{2}I, 1, (x-x_0+\tfrac{1}{2}rI)/r, (t-t_0+r^2)/r^2)$, where $x, x_0 \in E_n$, $t_0-t \geqslant 0$, $I = (1, 1, \ldots, 1) \in E_n$, and (x, t) satisfies $r = (t_0-t)^{1/2}$ if $4|x_i-x_{0i}|^2 < t_0-t$ for all $i = 1, 2, \ldots, n$, $r = 2|x_k-x_{0k}|$ otherwise, where $|x_k-x_{0k}| = \max\limits_{1\leqslant i\leqslant n}|x_i-x_{0i}|$.

We see intuitively that the points with a fixed value of r lie on the surfaces other than the top of a rectangular prism of width r and height r^2 with the point (x_0, t_0) located at the center of the top of the prism.

Let $R_\varrho = \{(x, t): |x_i-x_{0i}| < \varrho/2, 0 \leqslant t_0-t < \varrho^2\}$ be a rectangular prism, K_ϱ its boundary other than the top, and dq the measure induced on K by dq on K under the natural mapping of $R \cap \{t \leqslant 1\}$ onto R with $(\tfrac{1}{2}I, 1)$ mapping onto (x_0, t_0). Then, if u is a temperature on the closure of R_ϱ,

$$u(x_0, t_0) = \int\limits_{K_\varrho} H_\varrho(x_0, t_0, x, t)\, u(x, t)\, dq_\varrho .$$

This is obvious from the definition and the corollary to lemma 2′.

Let $S_{\varrho_0} = R_{\varrho_0} - \{R_{\varrho_0/2} \cup K_{\varrho_0/2}\}$. Let $\mathrm{I}_k = \{(x, t) \in S_{\varrho_0}: r = 2(x_{0k}-x_k)\}$, $\mathrm{II}_k = \{(x, t) \in S_{\varrho_0}: r = 2(x_k-x_{0k})\}$, $\mathrm{III} = \{(x, t) \in S_{\varrho_0}: r = (t_0-t)^{1/2}\}$, $k = 1, \ldots, n$. Intuitively, III is the set of bottoms, I_k the set of left hand sides in the different directions, and II_k the set of right hand sides in the different directions of K_ϱ, $\varrho_0/2 < \varrho < \varrho_0$. Obviously, $S_{\varrho_0} = \bigcup\limits_{k=1}^{n} \mathrm{I}_k \bigcup\limits_{k=1}^{n} \mathrm{II}_k \cup \mathrm{III}$.

Lemma 5. *If $u(x, t)$ is a temperature on the closure of R_{ϱ_0}, then*

$$|u(x_0, t_0)|^2 \leqslant (M/\varrho_0^{n+2}) \int\limits_{S_{\varrho_0}} |u(x, t)|^2\, dx\, dt,$$

where M is independent of u, ϱ_0, x_0, t_0.

Proof. We have

$$u(x_0, t_0) = \int\limits_{K_\varrho} H_\varrho(x_0, t_0, x, t)\, u(x, t)\, dq_\varrho, \quad 0 < \varrho < \varrho_0,$$

from which we have

$$u(x_0, t_0) = (2/\varrho_0) \int\limits_{\varrho_0/2}^{\varrho_0} \Big\{\int\limits_{K_\varrho} H(x_0, t_0, x, t)\, u(x, t)\, dq_\varrho\Big\} d\varrho$$
$$= (2/\varrho_0) \int\limits_{S_{\varrho_0}} H_\varrho(x_0, t_0, x, t)\, u(x, t)\, dq_\varrho\, d\varrho .$$

Applying Schwartz's inequality, we have

$$|u(x_0, t_0)|^2 \leqslant \Big\{(4/\varrho_0^2) \int\limits_{S_{\varrho_0}} |H_\varrho(x_0, t_0, x, t)|^2 dq_\varrho\, d\varrho\Big\}\Big\{\int\limits_{S_{\varrho_0}} |u(x, t)|^2 dq_\varrho\, d\varrho\Big\}.$$

We will show:

1) $\int\limits_{S_{\varrho_0}} |H(x_0, t_0, x, t)|^2 dq_\varrho\, d\varrho \leqslant M\varrho_0$,

2) $\int\limits_{S_{\varrho_0}} |u(x, t)|^2 dq_\varrho\, d\varrho \leqslant (M/\varrho_0^{n+1}) \int\limits_{S_{\varrho_0}} |u(x, t)|^2 dx\, dt$,

which will prove the lemma.

$$\int\limits_{S_{\varrho_0}} |H_\varrho(x_0, t_0, x, t)|^2 dq_\varrho\, d\varrho = \int\limits_{\varrho_0/2}^{\varrho_0} \Big\{\int\limits_{K_\varrho} |H_\varrho(x_0, t_0, x, t)|^2 dq_\varrho\Big\} d\varrho$$
$$= \int\limits_{\varrho_0/2}^{\varrho_0} \Big\{\int\limits_{K} |H(\tfrac{1}{2}I, 1, q)|^2 dq\Big\} d\varrho \leqslant M\varrho_0,$$

since $H(\tfrac{1}{2}I, 1, q)$ is bounded. This shows 1).

Let $dX_k' = dx_1 \ldots dx_{k-1} dx_{k+1} \ldots dx_n$, i. e., dx with dx_k deleted. Then, on I_k or II_k, $dq_\varrho = dX_k'(dt/\varrho^{n+1})$, $d\varrho = -dx_k$ on I_k, $d\varrho = dx_k$ on II_k; on III, $dq_\varrho = (dx/\varrho^n)$, $d\varrho = -\tfrac{1}{2}(dt/\varrho)$. Thus,

$$\int\limits_{\mathrm{I}_k} |u(x, t)|^2 dq_\varrho\, d\varrho = \int\limits_{x_{0k}-(\varrho_0/2)}^{x_{0k}-(\varrho_0/4)} (1/\varrho^{n+1}) \Big\{\int\limits_{\substack{x_{0i}-(\varrho/2)\\ i\neq k}}^{x_{0i}+(\varrho/2)} \int\limits_{t_0-\varrho^2}^{t_0} |u(x, t)|^2 dt\, dX_k'\Big\} dx_k$$
$$\leqslant (2/\varrho_0)^{(n+1)} \int\limits_{\mathrm{I}_k} |u(x, t)|^2 dx\, dt.$$

Similarly,

$$\int\limits_{\mathrm{II}_k} |u(x, t)|^2 dq_\varrho\, d\varrho \leqslant (2/\varrho_0)^{(n+1)} \int\limits_{\mathrm{II}_k} |u(x, t)|^2 dx\, dt.$$

$$\int\limits_{\mathrm{III}} |u(x, t)|^2 dq_\varrho\, d\varrho = \tfrac{1}{2} \int\limits_{t_0-\varrho_0^2}^{t_0-(\varrho_0^2/4)} (1/\varrho^{n+1}) \Big\{\int\limits_{\substack{x_{0i}-(\varrho/2)\\ i=1,\ldots,n}}^{x_{0i}+(\varrho/2)} |u(x, t)|^2 dx\Big\} dt$$
$$\leqslant \tfrac{1}{2}(2/\varrho_0)^{(n+1)} \int\limits_{\mathrm{III}} |u(x, t)|^2 dx\, dt.$$

This completes 2), the lemma, and the section.

Section 3. A condition on certain integrals

In this section, we will derive a necessary and sufficient condition for the existence of parabolic limits almost everywhere on a set $E \subset E_n$ of a temperature u defined on E_{n+1}^+. In parts a and b, we will derive a regularization of a pertinent domain and prove some preliminary lemmas. Then, in part c, we will state and prove the main theorem.

a. A regularization of a given domain. In this paragraph, we will obtain a regularization of a class of domains we will use. The technique was suggested by the previously mentioned paper by Stein [13].

If $E \subset E_n$ is closed and bounded, a, h positive, let $R = \bigcup_{x_0 \in E} P(x_0; h)$. The boundary B of R is $B^1 \cup B^2$, where $B^1 = \{(x, t): t = [a^{-1} d(x, E)]^2, 0 \leqslant t \leqslant h\}$ and $B^2 = \{(x, t): t = h, [a^{-1} d(x, E)]^2 \leqslant h\}$, where $d(x, E)$ is the distance from x to E.

Lemma 6. *There exists a sequence of regions R_k such that* 1) $R_k \subset R$, 2) $R_k \subset R_i$ *if* $k < i$, 3) $\bigcup_{k=1}^{\infty} R_k = R$, 4) *The boundary B_k of R_k is at a positive distance from E_n, and* 5) $B_k = B_k^1 \cup B_k^2$, *where* $B_k^2 = B_k \cap \{(x, t): t = h\}$, *and B_k^1 is a portion of* $\{(x, t): t = [a^{-1}\delta_k(x)]^2\}$, *where* $\delta_k \in C^\infty$, *and* $\left|\frac{\partial}{\partial x_j} \delta_k(x)\right| \leqslant 1$, $j = 1, \ldots, n$.

Proof. Let

$$\delta(x) = \begin{cases} d(x, E), & d(x, E) \leqslant h^{1/2}, \\ h^{1/2} & \text{otherwise}. \end{cases}$$

Let $\Phi(x) \in C^\infty$ be such that $\Phi(x) \geqslant 0$, $\Phi(x) = 0$ for $x \geqslant 1$, and $\int_{E_n} \Phi(x) dx = 1$. Let $\Phi_\eta(x) = \eta^{-n} \Phi(x/\eta)$, and $f_\eta(x) = \int_{E_n} \delta(x - z) \Phi_\eta(z) dz$. Then, $f_\eta(x) \in C^\infty$ and $f_\eta(x) \to \delta(x)$ uniformly as $\eta \to 0$. Let η_m be chosen so that $|f_{\eta_m}(x) - \delta(x)| < 1/m$ and set $\delta_m(x) = f_{\eta_m}(x) + (2/m)$. If $m_1 > 3m_2$, we see that $\delta_{m_1} \leqslant \delta_{m_2}$, since $\delta_{m_1}(x) = f_{\eta_{m_1}}(x) + (2/m_1) \leqslant [\delta(x) + (1/m_1)] + (2/m_1) \leqslant \delta(x) + (1/m_2) \leqslant (\delta(x) - (1/m_2)) + (2/m_2) \leqslant f_{\eta_{m_2}}(x) + (2/m_2) = \delta_{m_2}(x)$. Hence, we can take a subsequence $\delta_k(x)$ of $\delta_m(x)$ so that: a) $\delta_k(x) > \delta(x)$, b) $\delta_k(x) \leqslant \delta_i(x)$ if $k > i$, c) $\delta_k(x) \to \delta(x)$ as $k \to \infty$. Define $R_k = \{(x, t): |\delta_k(x)|^2 < a^2 t, 0 < t < h\}$. Then, a), b), and c) give us 1), 2), 3), and 4). We need only show that

$$\left|\frac{\partial}{\partial x_j} \delta_k(x)\right| = \left|\frac{\partial}{\partial x_j} f_\eta(x)\right| \leqslant 1$$

to complete the proof. $f(x_1) - f(x_2) = \int_{E_n} [\delta(x_1 - z) - \delta(x_2 - z)] \Phi_\eta(z) dz$ implies $|f_\eta(x_1) - f_\eta(x_2)| \leqslant |x_1 - x_2|$, which gives the desired result.

b. Preliminary lemmas. Lemma 7. *Let u be a bounded temperature on $P(x_0; \beta, h)$, i. e., $|u(x, t)| \leqslant N$ for $(x, t) \in P(x_0; \beta, h)$. Then, for $(x, t_1) \in P(x_0; a, h)$, $0 < a < \beta$, we have*

a) $$|\nabla_s u(x, t_1)| \leqslant (MN/t_1^{1/2}),$$

b) $$\left|\nabla_s \frac{\partial u}{\partial t} u(x, t_1)\right| \leqslant (MN/t_1^{(3/2)}),$$

and

c) $$\left|\frac{\partial}{\partial t} u(x, t_1)\right| \leqslant (MN/t_1),$$

where, in each case, M depends only on a and β.

Proof. We will write out the proof of a). The proofs of b) and c) are virtually identical.

1) If u is a temperature on $R \cap \{(x, t): t \leqslant 1\}$, continuous on the closure, and $|u| \leqslant N$ there, then $|\nabla_s u(\frac{1}{2}I, 1)| \leqslant M'N$, where M' is independent of u. This follows immediately from the representation of u given in section 2b, since $\partial H(x, t, v, s)/\partial x_i$ is integrable for (x, t) bounded away from K.

2) If u is a temperature on the paraboloid $P(0; \beta, \beta^{-2})$ and $|u| \leqslant N$ there, then $|\nabla_s u(0, \beta^{-2})| < M''N$, where M depends only on β. Let $k = \min(2^{-n}, 2^{-n}/\beta)$. Then, the rectangular prism $\{(x, t): (-k/2) \leqslant x_i \leqslant (k/2), (1/\beta^2) - k^2 \leqslant t \leqslant (1/\beta^2)\} \subset P(0; \beta, \beta^{-2})$. Consider the mapping $x_i \to x_i' = (x_i/k) + \frac{1}{2}$, $t \to t' = (t/k^2)$. Then $u(x', t')$ satisfies the conditions of 1) in x', t'. Thus, $|\nabla_s u(\frac{1}{2}I', 1)| \leqslant M'N$, whence $|\nabla_s u(0, \beta^{-2}) \leqslant (M'N/k) = M''N$, where M'' depends only on k and, hence, only on β.

3) If u is a temperature on $P(0; \beta, y)$ and $|u| \leqslant N$ there, then $|\nabla_s u(0, y) \leqslant M''' N/y^{1/2}$, where M''', depends only on β. If we make the change of variable, $x_i \to x_i' = (x_i/\beta y^{1/2})$, $t \to t' = (t/\beta^2 y)$, we find that $u(x', t')$ satisfies the conditions of 2). Thus, we have 3).

4) Let us note that along the hyperplane $t = t_1$, the distance between the boundaries of the paraboloids $P(x_0; a, h)$ and $P(x_0; \beta, h)$ is $[(1/a) - (1/\beta)]t_1^{1/2}$. Consider the paraboloid with aperture β, diameter $[(1/a) - (1/\beta)]t_1^{1/2}$ in the hyperplane $t = t_1$, and vertex at the point $(x, (1-b)t_1)$, $(x, t_1) \in P(x_0; a, h)$ and $b = 1 - \{[(1/a - (1/\beta)]/4\beta^2\}$. This paraboloid satisfies the conditions of 3) up to a translation with $y = bt_1$ since it is contained in $P(x_0; \beta, h)$. Hence, we have $|\nabla_s u(x, t_1)| \leqslant (M'''N/(bt_1)^{1/2}) = (MN/t_1^{1/2})$, where M depends only on β and b, and, thus, only on a and β.

We shall also require the following

Lemma 8. *If $u(x, t)$ is a temperature on $P(x_0; \beta, h)$ satisfying*

$$\int_{P(x_0;\beta,h)} t^{-n/2} \left\{|\nabla_s u(x, t)^2 + t\left|\frac{\partial u}{\partial t}\right|^2\right\} dx\, dt,$$

then a) $t^{1/2} |\nabla_s u(x, t)|$ *and* b) $t|\partial u(x, t)/\partial t|$ *are bounded for* $(x, t) \in P(x_0; a, h)$ *where the bound depends only on a and β.*

Proof. We will use the notation of lemma 5. For $(x_1, t_1) \in P(x_0; \alpha, h)$, there exists a $k > 0$, depending only on α and β, such that, for $\varrho_0 = kt^{1/2}$, $R_{\varrho_0}(x_1, t_1) \subset P(x_0; \beta, h)$. Since $t < t_1$ in S_{ϱ_0}, we have, by lemma 5,

$$\left|\frac{\partial}{\partial x_i} u(x_1, t_1)\right|^2 \leqslant (M/\varrho_0^{n+2}) \int\limits_{S_{\varrho_0}} \left|\frac{\partial u}{\partial x_i}\right|^2 dx\,dt$$

$$\leqslant (M/t_1^{(n+2)/2)}) \int\limits_{S_{\varrho_0}} \left|\frac{\partial u}{\partial x_i}\right|^2 dx\,dt \leqslant (M/t_1) \int\limits_{S_{\varrho_0}} t^{-n/2} \left|\frac{\partial u}{\partial x_i}\right|^2 dx\,dt,$$

$i = 1, 2, \ldots, n$. Adding, we have

$$|\nabla_s u(x_1, t_1)|^2 \leqslant (M/t_1) \int\limits_{S_{\varrho_0}} t^{-n/2} |\nabla_s u|^2 dx\,dt$$

$$\leqslant (M/t_1) \int\limits_{P(x_0;\beta,h)} t^{-n/2} |\nabla_s u|^2 dx\,dt \leqslant (M/t_1).$$

Similarly,

$$\left|\frac{\partial}{\partial t} u(x_1, t_1)\right|^2 \leqslant (M/t_1^2) \int\limits_{S_{\varrho_0}} t^{1-(n/2)} \left|\frac{\partial u}{\partial t}\right|^2 dx\,dt$$

$$\leqslant (M/t_1^2) \int\limits_{P(x_0;\beta,h)} t^{1-(n/2)} \left|\frac{\partial u}{\partial t}\right|^2 dx\,dt \leqslant (M/t_1^2),$$

which completes the proof of the lemma.

c. The main theorem. We may state the theorem as follows:

THEOREM 2. *Let u be a temperature on E_{n+1}^+. Then,*

a) *if u has parabolic limits on a set $E \subset E_n$,*

$$\int\limits_{P(x_0)} t^{-n/2} \left\{ |\nabla_s u|^2 + t \left|\frac{\partial u}{\partial t}\right|^2 \right\} dx\,dt$$

is finite for almost every $x_0 \in E$;

b) *if, for every $x_0 \in E \subset E_n$,*

$$\int\limits_{P(x_0)} t^{n/2} \left\{ |\nabla_s u|^2 + t \left|\frac{\partial u}{\partial t}\right|^2 \right\} dx\,dt$$

is finite, u has parabolic limits almost everywhere in E.

Proof. By theorem 1, and by arguments similar to those used at the beginning of the proof of theorem 1, we may reduce the hypotheses of part a) to the following: $u(x, t)$ is uniformly bounded on the region

$$\tilde{R} = \bigcup_{x_0 \in E} P(x_0; \beta, h),$$

where β, h are fixed, and E is a closed and bounded set.

We will show that

$$\text{(i)} \qquad A_1(x_0) = \int\limits_{P(x_0;\alpha,h)} t^{-n/2} |\nabla_s u|^2 dx\,dt,$$

$$\text{(ii)} \qquad A_2(x_0) = \int\limits_{P(x_0;\alpha,h)} t^{1-(n/2)} \left|\frac{\partial u}{\partial t}\right|^2 dx\,dt$$

are finite for almost every $x_0 \in E$, where $\alpha < \beta$ is fixed. It is clearly sufficient to show that

$$\int\limits_E A_i(x_0)\,dx_0 < \infty, \qquad i = 1, 2.$$

Let $R = \bigcup_{x_0 \in E} P(x_0; \alpha, h)$. Then $R \subset \tilde{R}$, and u is uniformly bounded on R. Let $\chi(x_0, x, t)$ be the characteristic function of $P(x_0; \alpha, h)$.

Then, to show (i), it suffices to show that

$$\int\limits_R \left\{ \int\limits_E \chi(x_0, x, t)\,dx_0 \right\} t^{-n/2} |\nabla_s u|^2 dx\,dt < \infty$$

since then Fubini's theorem will give the desired result. Further, since

$$\int\limits_E \chi(x_0, x, t)\,dx_0 \leqslant \int\limits_{|x_0 - x|^2 < \alpha^2 t} dx_0 = ct^{n/2},$$

it is enough to show $\int_R |\nabla_s u|^2 dx\,dt < \infty$. To show this, we will prove

$$(*) \qquad \int\limits_{R_k} |\nabla_s u|^2 dx\,dt \leqslant c < \infty,$$

where c is independent of k, and R_k is as in part a, lemma 6.

Similarly, to show (ii), we need only prove

$$(**) \qquad \int\limits_{R_k} t \left|\frac{\partial u}{\partial t}\right|^2 dx\,dt \leqslant c < \infty,$$

where R_k and c are as above.

We will denote by $s_k(t_1)$ the intersection of R_k with the hyperplane $t = t_1$, and by $\Gamma_k(t_1)$ the boundary of $s_k(t_1)$ in the hyperplane. Then

$$\int\limits_{R_k} |\nabla_s u|^2 dx\,dt = \int\limits_0^h \left\{ \int\limits_{s_k(t)} |\nabla_s u|^2 dx \right\} dt.$$

Let us apply Green's formula on $s_k(t)$, i. e.,

$$\int\limits_{\Gamma_k(t)} [G(\nabla_s F \cdot \eta_k) - F(\nabla_s G \cdot \eta_k)]\,d\tau_k = \int\limits_{s_k(t)} [G\Delta_s F - F\Delta_s G]\,dx,$$

where η_k is the outwardly directed normal to $\Gamma_k(t)$ in the hyperplane, $\Delta_s g = \sum_{i=1}^{n} \partial^2 g/\partial x_i^2$, and $d\tau_k$ is the element of area of $\Gamma_k(t)$. Letting $F = u^2$, $G = 1$, we obtain

$$\int_{\Gamma_k(t)} [\nabla_s(u^2)\cdot\eta_k]\, d\tau_k = 2\int_{s_k(t)} [|\nabla_s u|^2 + u\Delta_s u]\, dx.$$

It then suffices to show that

$$\int_0^h \Big\{\int_{\Gamma_k(t)} 2u(\nabla_s u\cdot\eta_k)\, d\tau_k\Big\}\, dt - 2\int_0^h \Big\{\int_{s_k(t)} u\Delta_s u\, dx\Big\}\, dt$$

is finite independently of k.

$$\Big|\int_0^h \Big[\int_{s_k(t)} u\Delta_s u\, dx\Big]\, dt\Big| = \Big|\int_{R_k} u\frac{\partial u}{\partial t}\, dx\, dt\Big|$$

$$= \Big|\int_{F_k}\Big[\int_{\tau_x}^h u\frac{\partial u}{\partial t}\, dt\Big]\, dx \leqslant 2M^2 m(F),$$

where M is the bound on u over R, $(x, \tau_x)\in B_k^1$, F_k and F, the respective projections of R_k and R on E_n, and $m(F)$, the area of F. Thus, the second integral is bounded independently of k.

To prove $(*)$, we need only show that

$$\int_0^h \Big[\int_{\Gamma_k(t)} 2u(\nabla_s u\cdot\eta_k)\, d\tau_k\Big]\, dt$$

is bounded independently of k.

By lemma 7, $|\nabla_s u\cdot\eta_k| \leqslant |\nabla_s u| \leqslant Nt^{-1/2}$, and u is bounded on R by assumption. So, it suffices to show that

$$\int_0^h \Big[\int_{\Gamma_k(t)} t^{-1/2}\, d\tau_k\Big]\, dt$$

is bounded independently of k. To show this we make the following observations:

The angle θ between the normal to the surface B_k^1 and the positive t axis satisfies $\cos\theta = 1/D$, where

$$D = [1+(4/a^4)\,\delta_k(x)^2\,|\nabla_s\delta_k(x)|^2]^{1/2} = [1+(4t/a^2)\,|\nabla_s\delta_k|^2]^{1/2}$$
$$\leqslant [1+(4nt/a^2)]^{1/2}$$

by property 5 of lemma 6, the regularization of R.

Let γ be the angle between the normal to the surface B_k^1 and the hyperplane $t = t_1$. Then, $-\sin\gamma = \cos\theta = -1/D \leqslant -[(4nt/a^2)+1]^{-1/2}$. Thus,

$\cos\gamma = (1-\sin^2\gamma)^{1/2} \leqslant \{1-[(4nt/a^2)+1]^{-1}\}^{1/2} = [(4nt/a^2)/(4nt/a^2)+1]^{1/2} \leqslant (2n^{1/2}/a)t^{1/2}$, and $\sec\gamma \geqslant (a/2n^{1/2})t^{-1/2}$.

We then see that

$$\int_0^h \Big[\int_{\Gamma_k(t)} t^{-1/2}\, d\tau_k\Big]\, dt \leqslant c\int_0^h \Big[\int_{\Gamma_k(t)} \sec\gamma\, d\tau_k\Big]\, dt.$$

The latter integral is just the surface area of B_k^1. By property 5, lemma 6, and by the boundedness of E, this surface area is bounded independently of k. This completes the proof of $(*)$.

Let F_k, τ_x, $s_k(t)$, $\Gamma_k(t)$, $d\tau_k$ be as above. Then,

$$\int_{R_k} t\Big|\frac{\partial u}{\partial t}\Big|^2 dx\, dt = \int_{F_k}\int_{\tau_x}^h t\Big|\frac{\partial u}{\partial t}\Big|^2 dt\, dx$$

$$= \int_{F_k} tu\frac{\partial u}{\partial t} - 1/2u^2\Big]_{t=\tau_x}^h dx - \int_{F_k}\Big\{\int_{\tau_x}^h ut\frac{\partial^2 u}{\partial t^2}\, dt\Big\}\, dx.$$

Since the integrand in the first integral is bounded, (by lemma 7c and hypothesis), this integral is bounded independently of k. It thus suffices to show that

$$\int_{F_k}\int_{\tau_x}^h ut\frac{\partial^2 u}{\partial t^2}\, dt\, dx = \int_{R_k} ut\frac{\partial^2 u}{\partial t^2}\, dx\, dt \leqslant c < \infty.$$

Let us use Green's formula on $s_k(t)$ in the form:

$$\int_{s_k(t)} (F\Delta_s G + \nabla_s F\cdot\nabla_s G)\, dx = \int_{\Gamma_k(t)} F(\nabla_s G\cdot\eta_k)\, d\tau_k.$$

Letting $F = ut$, $G = \partial u/\partial t$, we have:

$$\int_{R_k} ut\frac{\partial^2 u}{\partial t^2}\, dx\, dt = \int_0^h \Big\{\int_{s_k(t)} ut\Delta_s\Big[\frac{\partial u}{\partial t}\Big]\, dx\Big\}\, dt$$

$$= \int_0^h \Big\{\int_{\Gamma_k(t)} ut\Big[\nabla_s\frac{\partial u}{\partial t}\cdot\eta_k\Big]\, d\tau_k - \int_{s_k(t)} \nabla_s(ut)\cdot\nabla_s\frac{\partial u}{\partial t}\, dx\Big\}\, dt.$$

Since

$$\Big|t\nabla_s\frac{\partial u}{\partial t}\cdot\eta_k\Big| \leqslant t\Big|\nabla_s\frac{\partial u}{\partial t}\Big| \leqslant Mt^{-1/2}$$

by lemma 7b, we have

$$\left|\int_0^h \int_{\Gamma_k(t)} ut\left[\nabla_s \frac{\partial u}{\partial t}\cdot \eta_k\right] d\tau_k\, dt\right| \leqslant M \int_0^h \int_{\Gamma_k(t)} t^{-1/2} d\tau_k\, dt \leqslant M'$$

as we saw in the proof of (*). It then is sufficient to show that

$$\left|\int_{R_k} t\left[\nabla_s u\cdot \nabla_s \frac{\partial u}{\partial t}\right] dx\, dt\right| \leqslant c < \infty .$$

However,

$$\int_{R_k} t\left[\nabla_s u\cdot \nabla_s \frac{\partial u}{\partial t}\right] dx\, dt$$

$$= \int_{F_k}\left\{\sum_{i=1}^n \int_{\tau_x}^h t \frac{\partial u}{\partial x_i}\left[\frac{\partial}{\partial x_i}\frac{\partial u}{\partial t}\right] dt\right\} dx$$

$$= \int_{F_k}\sum_{i=1}^n \left\{\int_{\tau_x}^h t \frac{\partial u}{\partial x_i}\frac{\partial}{\partial t}\frac{\partial u}{\partial x_i} dt\right\} dx$$

$$= \frac{1}{2}\int_{F_k}\sum_{i=1}^n \left\{t\left|\frac{\partial u}{\partial x_i}\right|^2\Big]_{t=\tau_x}^h - \int_{\tau_x}^h \left|\frac{\partial u}{\partial x_i}\right|^2 dt\right\} dx$$

$$= \frac{1}{2}\int_{F_k} t\,|\nabla_s u|^2]_{t=\tau_x}^h dx - \frac{1}{2}\int_{F_k}\int_{\tau_x}^h |\nabla_s u|^2 dt\, dx$$

$$= \frac{1}{2}\int_{F_k} t|\nabla_s u|^2]_{t=\tau_x}^h dx - \frac{1}{2}\int_{R_k} |\nabla_s u|^2 dx\, dt.$$

The first integral is bounded since its integrand is, lemma 7a. The second integral is just (*). Hence, we have proved (**) and thus part a) of our theorem.

For the proof of b), let us temporarily denote the set where the integral is finite by E_0. As in the first part, we can reduce our hypotheses to:

A) $\int_{P(x_0;\beta,k)} t^{-n/2}|\nabla_s u|^2 dx\, dt$ is uniformly bounded as x_0 ranges over E_0, β, k fixed;

B) $\int_{P(x_0;\beta,k)} t^{1-(n/2)}\left|\frac{\partial u}{\partial t}\right|^2 dx\, dt$ is uniformly bounded as x_0 ranges over E_0, β, k fixed;

C) E_0 is a bounded set.

Given $\varepsilon > 0$, we now choose a closed set $E \subset E_0$, $m(E) > 0$, such that

D) $m(E_0 - E) < \varepsilon$;

E) there exists a fixed ϱ_ε such that $m(\{y: |x-y| \leqslant \varrho\} \cap E_0) \geqslant \frac{1}{2}m(\{y: |x-y| \leqslant \varrho\})$ for $x \in E$, $0 < \varrho < \varrho_\varepsilon$.

To see that this set E may be so chosen, consider a point of density x of E_0. Then

$$\lim_{\varrho\to 0}\frac{m(\{y: |x-y| \leqslant \varrho\} \cap E_0)}{m(\{y: |x-y| \leqslant \varrho\})} = 1.$$

Hence, there exists a $\tilde{\varrho}_x$ such that

$$\frac{m(\{y: |x-y| \leqslant \varrho\} \cap E_0)}{m(\{y: |x-y| \leqslant \varrho\})} \geqslant \frac{1}{2}$$

for all $0 < \varrho < \tilde{\varrho}_x$. Let $\varrho_x = \sup \tilde{\varrho}_x$. Let E^* be the set of points of density of E_0. Then $m(E_0 - E^*) = 0$. The function $x \to \varrho_x$ is measurable on E^*. Let $B_k = \{x \in E^*: (1/k) \leqslant \varrho_x < (1/k-1)\}$, $k = 2, 3, \ldots, B_1 = \{x \in E^*: 1 \leqslant \varrho_x\}$. Then, $E^* = \bigcup_{k=1}^\infty B_k$ and the B_k are disjoint. $m(E^*) = \sum_{k=1}^\infty m(B_k)$. Choose k_0 large enough that $\sum_{k=k_0}^\infty m(B_k) < \varepsilon$. Let $E = \bigcup_{k=1}^{k_0-1} B_k$. Then,

$$m(E_0 - E) \leqslant m(E_0 - E^*) + m(E^* - E) \leqslant \sum_{k=k_0}^\infty m(B_k) < \varepsilon,$$

and, letting $\varrho_\varepsilon = (1/k_0)$, we have, $m(\{y: |x-y| \leqslant \varrho\} \cap E_0) \geqslant \frac{1}{2}m(\{y: |x-y| \leqslant \varrho\})$ for $x \in E$, $0 < \varrho < \varrho_\varepsilon$.

We fix the set E which we have obtained. Since it is sufficient to prove the existence of parabolic limits on a subset of positive measure, we need only show their existence almost everywhere in E.

Let $R = \bigcup_{x_0 \in E} P(x_0; \alpha, h)$, $\alpha < \beta$, $h < k$ fixed, and R_k, B_k^1, B_k^2, F_k, and τ_x be as in the proof of a).

i) We will first show that

$$\int_{F_k} |u(x, \tau_x)|^2 dx \leqslant c < \infty,$$

where c is independent of k.

By hypothesis,

$$\int_{P(x_0;\beta,k)} t^{-n/2}|\nabla_s u|^2 dx\, dt \leqslant K < \infty, \qquad x_0 \in E.$$

Let χ_{E_0} be the characteristic function of E_0 and $\psi(x_0, x, t)$ the characteristic function of $P(x_0; \beta, k)$. Then, integrating over E_0, we obtain:

$$\int_{E_{n+1}^+\times E_n} \chi_{E_0}(x_0)\psi(x_0, x, t)t^{-n/2}|\nabla_s u(x, t)|^2 dx\,dt\,dx_0 < \infty .$$

Since $R = \bigcup_{z\in E} P(z; h)$, $(x, t)\in R$ implies that there exists a $z\in E$ such that $|x-z|^2 < a^2 t$, $0 < t < h$. Moreover, $\psi(x_0, x, t)$ is the characteristic function of the set where $|x-x_0|^2 < \beta^2 t$, $0 < t < k$. Hence,

$$\int_{E_0} \psi(x_0, x, t)\,dx_0 \geqslant \int_{E_0\cap\{|x_0-z|^2<(\beta^2-a^2)t\}} dx_0 .$$

Since $z\in E$, using E) with $\varrho = (\beta^2-\alpha^2)^{1/2}t^{1/2}$, α chosen close enough to β so that $(\beta^2-\alpha^2)^{1/2}t^{1/2} < \varrho_\varepsilon$ for $0 < t < k$, we see that this integral exceeds $ct^{n/2}$, $0 < t < h$, for an appropriate $c > 0$. This gives us

$$\int_R |\nabla_s u|^2 dx\,dt < \infty .$$

Hence,

$$\int_{R_k} |\nabla_s u|^2 dx\,dt \leqslant c < \infty ,$$

where c is independent of h, which implies that

$$\int_0^h \Big\{\int_{s_k(t)} |\nabla_s u|^2 dx\Big\}\,dt \leqslant c < \infty ,$$

where $s_k(t)$ is as in a).

Applying Green's formula, we have

$$\int_0^h \Big\{\int_{\Gamma_k(t)} 2u(\nabla_s u\cdot\eta_k)\,d\tau_k - \int_{s_k(t)} u\Delta_s u\,dx\Big\}\,dt \leqslant c ,$$

Γ_k, η_k, $d\tau_k$ as in a). Thus,

$$\int_0^h \Big\{\int_{\Gamma_k(t)} 2ut^{1/2}(\nabla_s u\cdot\eta_k)t^{-1/2}d\tau_k\Big\}\,dt \leqslant \int_{R_k} u\frac{\partial u}{\partial t}\,dx\,dt + c .$$

By lemma 8,

$$-\int_0^h \Big\{\int_{\Gamma_k(t)} |u|\,t^{-1/2}d\tau_k\Big\}\,dt$$

$$\leqslant c_1 \int_{F_k}\Big\{\int_{\tau_x}^h u\frac{\partial u}{\partial t}\,dt\Big\}\,dx + c_2 \leqslant -c_1\int_{F_k} |u(x, \tau_x)|^2 dx + c_2 .$$

Thus,

$$c_1\int_{F_k} |u(x, \tau_x)|^2 dx \leqslant \int_0^h \Big\{\int_{\Gamma_k(t)} |u(x, t)|\,t^{-1/2}d\tau_k\Big\}\,dt + c_2 \leqslant c_3 \int_{B_k^1} |u(x, t)|\,d\sigma + c_2 ,$$

as in the proof of (**), where $d\sigma$ is the surface area measure on B_k^1. By property 5 of the regularization, lemma 6, we have that $d\sigma \leqslant K\,dx$. Thus,

$$\int_{F_k} |u(x, \tau_x)|^2 dx \leqslant c_1\int_{F_k} |u(x, \tau_x)|\,dx + c_2 \leqslant c_1\Big\{\int_{F_k} |u(x, \tau_x)|^2 dx\Big\}^{1/2} + c_2 ,$$

by Schwartz's inequality, and the boundedness of E. Letting

$$J_k = \Big\{\int_{F_k} |u(x, \tau_x)|^2 dx\Big\}^{1/2},$$

we have, $J_k^2 \leqslant c_1 J_k + c_2$, c_1, c_2 independent of k. Hence, $J_k \leqslant c < \infty$, c independent of k, which proves i).

ii) We will next majorize $u(x, t)$ in R by a function $v(x, t)$ whose boundary behavior is known.

Let $f_k(x) = u(x, \tau_k)$ for $x\in F_k$, $f_k(x) = 0$, otherwise. Then, by i),

$$\int_{E_n} |f_k(x)|^2 dx \leqslant c < \infty ,$$

c not depending on k. Let

$$v_k(x, t) = \int_{E_n} W(x-z, t)\,|f_k(z)|\,dz ,$$

where $W(x, t)$ is the Weierstrass kernel of section 1. We will show that there exist constants c_1, c_2 independent of k, such that $u(x, t) \leqslant c_1 v_k(x, t) + c_2$ for $(x, t)\in R_k$. By the maximum principle (11), it suffices to show this on B_k^1.

Since $R = \bigcup_{x_0\in E} P(x_0; \alpha, h)$, letting β^*, k^* be such that $\alpha < \beta^* < \beta$, $h < k^* < k$, we can find a $c > 0$, depending only on α, β^*, h, k^* such that, for $\sigma = (x, t)\in R$, the rectangular prism $C(\sigma)$, of height $2c^2 t$, center at σ, with hypercubic base of side $ct^{1/2}$, is contained in $\bigcup_{x_0\in E} P(x_0; \beta^*, k^*)$.

By hypothesis,

$$\int_{P(x_0;\beta,k)} t^{-n/2}\Big\{|\nabla_s u|^2 + t\Big|\frac{\partial u}{\partial t}\Big|^2\Big\}\,dx\,dt \leqslant K < \infty \quad \text{for} \quad x_0\in E .$$

By lemma 8, this implies $t^{1/2}|\nabla_s u|$, and $t|\partial u/\partial t|$ are bounded for $(x, t)\in \bigcup_{x_0\in E} P(x_0; \beta^*, k^*)$. Let $\sigma = (x_1, t_1)\in B_k^1 \subset R$; $C(\sigma)$, the associated prism; $\sigma' = (x_2, t_2)\in C(\sigma)$; $\sigma_1 = (x_2, t_1)$; s_1, the line segment joining σ and σ_1; s_2, the line segment joining σ_1 and σ'. Then,

$$\begin{aligned}|u(\sigma)-u(\sigma')| &\leqslant |u(\sigma)-u(\sigma_1)| + |u(\sigma_1)-u(\sigma')|\\ &\leqslant |\sigma-\sigma_1|\sup_{s_1}|\nabla_s u| + |\sigma_1-\sigma'|\sup_{s_2}|\partial u/\partial t|\\ &\leqslant ct^{1/2}t^{-1/2}K_1 + c^2 t t^{-1} K_2 = M < \infty .\end{aligned}$$

Let D be the intersection of B_k^1 with $C(\sigma)$, $|D|$, its area. Since B_k^1 is given by $F = [\delta_k(x)]^2 - \alpha^2 t = 0$, and, by part 5 of lemma 6,

$$|\partial F/\partial x_m| = \left|2\delta_k(x)\frac{\partial \delta_k}{\partial x_m}\right| \leqslant 2|\delta_k(x)| = 2\alpha t^{1/2},$$

a straightforward geometrical argument shows that $|D| \geqslant kt_1^{n/2}$, for an appropriately chosen $k > 0$. Letting $d\sigma'$ be the surface area measure on B_k^1, we have

$$u(\sigma) \leqslant (1/|D|)\int_D |u(\sigma')|\,d\sigma' + A$$

$$\leqslant Bt_1^{-n/2} \int_{\{|x_1-z|<ct_1^{1/2}n^{1/2}\}} |f_k(z)|\,dz + A, \qquad \sigma = (x_1, t_1).$$

For $|x| < kt^{1/2}$, $W(x, t) \geqslant k't^{-n/2}$ for appropriate $k' > 0$. Therefore, for $(x, t) \in B_k^1$, we have

$$u(x, t) \leqslant c_1 \int_{E_n} W(x-z, t)|f_k(z)|\,dz + A \leqslant c_1 v_k(x, t) + c_2,$$

c_1, c_2 independent of k.

Since we have a uniform bound on the L_2 norms of the $|f_k(x)|$, we can find a subsequence $|f_{k_j}(x)|$ which converges weakly to a function $|f(x)| \in L_2(E_n)$. Let

$$v(x, t) = \int_{E_n} W(x-z, t)|f(z)|\,dz.$$

Then, for $(x, t) \in E_{n+1}^+$, $v_{k_j}(x, t) \to v(x, t)$. Since $\bigcup_{j=1}^{\infty} R_{k_j} = R$, we have $|u(x, t)| \leqslant c_1 v(x, t) + c_2$ for all $(x, t) \in R$.

We know that $v(x, t)$ is parabolically bounded a. e. in E_n. Hence, $u(x, t)$ is parabolically bounded almost everywhere in E. By theorem 1, $u(x, t)$ has parabolic limits a. e. in E, which completes the theorem.

Section 4. An application of the Main Theorem

The primary result of this chapter will be to show that if two vector valued functions whose components are temperatures are related in a particular way, then parabolic limits on a set $E \subset E_n$ for the one will imply parabolic limits almost everywhere in E for the other.

This result (theorem 4) will follow easily from a lemma and theorem relating the boundedness of certain integrals.

We will first develop certain results which will be needed in the proof of these theorems.

a. Preliminary results. Let us consider the equation

$$(*) \qquad (1/2\tau)\frac{\partial u}{\partial \tau} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0, \qquad (x, \tau) \in E_{n+1}^+.$$

This equation is obtained from the heat equation by the change of variable $\tau^2 = t$. Hence, we see that, for solutions to $(*)$ in the upper half space, we have maximal and uniqueness results similar to those for solutions to the heat equation.

In particular, if G_1, G_2 are as in section 2, part a, and we define

$$H_k^*(x_k, \tau, s_k, \eta) = G_1(x_k, \tau^2, s_k, \eta^2) + 2\eta G_2(x_k, \tau^2, s_k, \eta^2)$$

and

$$H^*(x, \tau, s, \eta) = \prod_{k=1}^{n} H_k^*(x_k, \tau, s_k, \eta),$$

we obtain the integral representation

$$u(x, \tau) = \int_K H^*(x, \tau, s, \eta)u(s, \eta)\,d\sigma(s, \eta)$$

which gives us solutions to $(*)$ on R in terms of their boundary values $u(s, \eta)$ on K.

By the properties of G_1, G_2, we know that, for (x, τ) bounded away from K, this integral, as well as the integrals of $[P(D)H^*(x, \tau, s, \eta)]\cdot u(s, \eta)$, converge absolutely, where $P(D)$ is a differential polynomial with respect to the variables $x_1, x_2, \ldots, x_n$.

We proceed as in section 2, part c.

Definition. Let $H_r^*(x_0, \tau_0, x, \tau) = H^*(\frac{1}{2}I, 1, (x-x_0+\frac{1}{2}rI)/r, (\tau-\tau_0+r)/r)$, where $x, x_0 \in E_n$, $\tau_0 - \tau \geqslant 0$, I is as before, and (x, τ) satisfies $r = \tau_0 - \tau$ if $2|x_i - x_{0i}| \leqslant \tau_0 - \tau$ for all $i = 1, 2, \ldots, n$, and $r = 2|x_k - x_{0k}|$ otherwise, where $|x_k - x_{0k}| = \max_{1\leqslant i\leqslant n}|x_i - x_{0i}|$.

Intuitively, as in 2, part c, the set of points with a fixed r are the points lying on the surfaces other than the top of a rectangular prism of width and height r with the point (x_0, τ_0) at the center of the top of the prism.

Analogously to section 2, part c, let $R_\varrho^* = \{(x, \tau): |x_i - x_{0i}| < \varrho/2, 0 \leqslant \tau_0 - \tau < \varrho\}$, K_ϱ^*, the boundary other than the top of R_ϱ^*, $d\sigma_\varrho^*$ the measure induced on K_ϱ^* by Lebesgue measure on K under the natural mapping of R onto R^* with $(\frac{1}{2}I, 1)$ mapping onto (x_0, τ_0). Then, if u is a solution to $(*)$ on $R_\varrho^* \cup K_\varrho^*$,

$$u(x_0, \tau_0) = \int_{K_\varrho^*} H_\varrho^*(x_0, \tau_0, x, \tau)u(x, \tau)\,d\sigma_\varrho^*.$$

Let $S^*_{\varrho_0} = R^*_{\varrho_0} - \{R^*_{\varrho_0/2} \cup K^*_{\varrho_0/2}\}$; $\mathrm{I}^*_k = \{(x, \tau) \in S^*_{\varrho_0} : r = 2(x_{0k} - x_k)\}$, $\mathrm{II}^*_k = \{(x, \tau) \in S^*_{\varrho_0} : r = 2(x_k - x_{0k})\}$, $\mathrm{III}^* = \{(x, \tau) \in S^*_{\varrho_0} : r = \tau_0 - \tau\}$. Then

$$S^*_{\varrho_0} = \bigcup_{k=1}^{n} \mathrm{I}^*_k \bigcup_{k=1}^{n} \mathrm{II}^*_k \cup \mathrm{III}.$$

The sets I^*_k, II^*_k, III^* have the same intuitive meanings as I_k, II_k, III, respectively.

Similar to lemma 5, we have

LEMMA 9. *If $u(x, \tau)$ satisfies $(*)$ on $R^*_{\varrho_0} \cup K^*_{\varrho_0}$, and $P(D)$ is a homogeneous differential polynomial of order m in $x_1, \ldots, x_n$, with constant coefficients, then*

$$|P(D)u(x_0, \tau_0)|^2 \leqslant M\varrho_0^{-(n+1+2m)} \int_{S^*_{\varrho_0}} |u(x, \tau)|^2 dx\, d\tau,$$

where M is independent of u, ϱ_0, x_0, τ_0,

Proof. We have

$$\big(P(D)u\big)(x_0, \tau_0) = \int_{K^*_\varrho} \{[P(D)H^*_\varrho(x_0, \tau_0, x, \tau)]u(x, \tau)\} d\sigma^*_\varrho$$

for all $0 < \varrho < \varrho_0$. Hence,

$$[P(D)u](x_0, \tau_0) = (2/\varrho_0) \int_{\varrho_0/2}^{\varrho_0} \int_{K^*_\varrho} \{[P(D)H^*_\varrho(x_0, \tau_0, x, \tau)]u(x, \tau)\} d\sigma^*_\varrho\, d\varrho$$

$$= \int_{S^*_{\varrho_0}} (2/\varrho_0)[P(D)H^*_\varrho(x_0, \tau_0, x, \tau)]u(x, \tau) d\sigma^*_\varrho\, d\varrho.$$

By Schwartz's inequality,

$$|\big(P(D)u\big)(x_0, \tau_0)|^2$$
$$\leqslant \Big\{(4/\varrho_0^2) \int_{S^*_{\varrho_0}} |P(D)H^*_\varrho(x_0, \tau_0, x, \tau)|^2 d\sigma^*_\varrho\, d\varrho\Big\}\Big\{\int_{S^*_{\varrho_0}} |u(x, \tau)|^2 d\sigma^*_\varrho\, d\varrho\Big\}.$$

To complete the proof, we need only show:

1) $\int_{S^*_{\varrho_0}} [P(D)H^*_\varrho(x_0, \tau_0, x, \tau)]^2 d\sigma^*_\varrho\, d\varrho \leqslant M\varrho_0^{1-2m}$,

2) $\int_{S^*_{\varrho_0}} |u(x, \tau)|^2 d\sigma^*_\varrho\, d\varrho \leqslant (M/\varrho_0^n) \int_{S^*_{\varrho_0}} |u(x, \tau)|^2 dx\, d\tau$,

where in each case M is independent of u, ϱ_0, x_0, τ_0.

To show 1):

$$\int_{S^*_{\varrho_0}} |P(D)H^*_\varrho(x_0, \tau_0, v, \tau)|^2 d\sigma^*_\varrho\, d\varrho$$
$$= \int_{\varrho_0/2}^{\varrho_0} \Big\{\int_{K^*_\varrho} |P(D)H^*_\varrho(x_0, \tau_0, x, \tau)|^2 d\sigma^*_\varrho\, d\varrho$$
$$\leqslant M \int_{\varrho_0/2}^{\varrho_0} \varrho^{-2m} \Big\{\int_K |P(D)H^*(\tfrac{1}{2}\mathrm{I}, 1, v, s)|^2 d\sigma\Big\} d\varrho \leqslant M\varrho_0^{1-2m},$$

since the integral over K is bounded.

To show 2): Let dX'_k be as in section 2, part c. Then on I^*_k and II^*_k, $d\sigma^*_\varrho = dX'_k d\tau/\varrho^n$, $d\varrho = -dx_k$ on I^*_k, $d\varrho = dx_k$ on II^*_k, on III^*, $d\sigma^*_\varrho = dx/\varrho^n$, $d\varrho = -d\tau$. Hence,

$$\int_{\mathrm{I}^*_k} |u(x, \tau)|^2 d\sigma^*_\varrho\, d\varrho \leqslant (2^n/\varrho_0^n) \int_{\mathrm{I}^*_k} |u(x, \tau)|^2 dx\, d\tau,$$

$$\int_{\mathrm{II}^*_k} |u(x, \tau)|^2 d\sigma^*_\varrho\, d\varrho \leqslant (2^n/\varrho_0^n) \int_{\mathrm{II}^*_k} |u(x, \tau)|^2 dx\, d\tau,$$

$$\int_{\mathrm{III}^*} |u(x, \tau)|^2 d\sigma^*_\varrho\, d\varrho \leqslant (2^n/\varrho_0^n \int_{\mathrm{III}^*} |u(x, \tau)|^2 dx\, d\tau,$$

which completes the proof.

LEMMA 10. *Let $T = (1/\tau)(\partial/\partial\tau)$. Assume that, for $0 \leqslant k \leqslant m$. $T^k u$ exists and is continuous in E^+_{n+1}. Then,*

$$u(x, y) = \big(-1/(m-1)!2^{m-1}\big)\int_y^h (y^2 - \tau^2)^{m-1} T^m u(x, \tau)\tau\, d\tau + A_m,$$

where A_m is a polynomial in $T^k u$, $k = 0, 1, \ldots, m-1$, evaluated at $t = h$.

Proof. By the fundamental theorem of calculus,

$$u(x, y) = -\int_y^h (Tu)\tau\, d\tau + A_1, \quad A_1 = u(x, h).$$

Integrating by parts, we obtain:

$$u(x, y) = -\Big\{[(\tau^2/2)Tu]^h_{\tau=y} - \int_y^h (\tau^2/2)\frac{\partial Tu}{\partial\tau} d\tau\Big\} + A_1$$

$$= (y^2/2)\Big[(1/y)\frac{\partial u}{\partial\tau}\Big] + \int_y^h (\tau^2/2)\frac{\partial Tu}{\partial\tau} d\tau + A'_1$$

$$= -(y^2/2)\int_y^h \frac{\partial Tu}{\partial\tau} d\tau + \int_y^h (\tau^2/2)[T^2 u]\tau\, d\tau + A_2$$

$$= -\frac{1}{2}\int_y^h (y^2 - \tau^2)[T^2 u]\tau\, d\tau + A_2,$$

where $A_1' = u(x,h)-(h^2/2)Tu$, and $A_2 = u(x,h)-(h^2/2)Tu(x,h)+$ $+(y^2/2)Tu(x,h)$. This shows the lemma for $m=2$.

Assume that the lemma holds for $m=k$, i. e.,

$$u(x,y) = \big(-1/(k-1)!2^{k-1}\big)\int_y^h (y^2-\tau^2)^{k-1}[T^k u]\tau\,d\tau + A_k.$$

Again integrating by parts, we have:

$$u(x,y) = (-1/(k-1)!2^{k-1})\left\{\left[-T^k u\frac{(y^2-\tau^2)^k}{2k}\right]_{\tau=y}^{h} + \right.$$

$$\left. +(1/2k)\int_y^h (y^2-\tau^2)^k \frac{\partial}{\partial\tau} T^k u\,d\tau\right\} + A_k.$$

Since the integrated term vanishes at the lower limit, and can be incorporated into A_{k+1} at the upper limit, we have;

$$u(x,y) = (-1/2^k k!)\int_y^h (y^2-\tau^2)^k[T^{k+1}u]\tau\,d\tau + A_{k+1}.$$

Thus, we have shown the lemma for $m=k+1$. By mathematical induction, we are finished.

We shall also use the following

LEMMA 11. *Let* $0<a_0\leqslant a<\infty$, *and*

$$F(s) = \int_{as}^b f(t)\,dt.$$

Then

$$\int_0^b |F(s)|^2 s\,ds \leqslant 4a_0^{-2}\int_0^b |f(t)|^2 t^3\,dt.$$

Proof. An inequality due to Hardy [6] states that if

$$\Phi(s) = \int_s^\infty \varphi(t)/t\,dt,$$

then

$$\int_0^\infty |\Phi(s)|^2 ds \leqslant 4\int_0^\infty |\varphi(t)|^2 dt.$$

A change of variables shows that

(′) if

$$\Phi(s) = \int_{as}^\infty \varphi(t)/t\,dt,$$

then

$$\int_0^\infty |\Phi(s)|^2 ds \leqslant 4a^{-1}\int_0^\infty |\varphi(t)|^2 dt.$$

Now,

$$|s^{1/2}F(s)| = \left|s^{1/2}\int_{as}^b f(t)\,dt\right| \leqslant a^{-1/2}\int_{as}^b t^{1/2}|f(t)|\,dt \leqslant a^{-1/2}\int_{as}^b t^{3/2}|f(t)|\,dt/t.$$

Applying (′), we have the lemma.

b. The theorems.

Definitions. Let $\Gamma(x_0; a, h) = \{(x,\tau)\colon |x-x_0|<a\tau,\ \ 0<\tau<h\}$. Throughout this section, unless otherwise stated, $u(x,t)$, $v(x,t)$ will denote vector valued solutions to the heat equation, i. e., $u(x,t) = \big(u_1(x,t), u_2(x,t), \ldots, u_r(x,t)\big)$, $v(x,t) = \big(v_1(x,t), v_2(x,t), \ldots, v_s(x,t)\big)$, and $u_1(x,t)$, $v_j(x,t)$, $i=1,2,\ldots,r$, $j=1,2,\ldots,s$, are temperatures in the usual sense. We will refer to u and v as temperatures. We will denote by

$$|u(x,t)| = \Big[\sum_{i=1}^r |u_i(x,t)|^2\Big]^{1/2}, \qquad |v(x,t)| = \Big[\sum_{j=1}^s |v_j(x,t)|^2\Big]^{1/2}.$$

LEMMA 12. *Let $u(x,t)$, $v(x,t)$ be temperatures in $P(x_0;\beta,k)$; let $P(D)$ be a $r\times s$ matrix, each of whose entries is a homogeneous differential polynomial of degree q in $x_1, x_2, \ldots, x_n$ with constant coefficients.*

1) *If* $q=2m+1$, $m=0,1,\ldots,$

a) $\dfrac{\partial^m u}{\partial t^m} = P(D)v,$

b) $\displaystyle\int_{P(x_0;\beta,k)} t^{-n/2}|v|^2\,dx\,dt<\infty,$

then, for $0<a<\beta$, $0<h<k$,

$$(') \qquad \int_{P(x_0;a,h)} t^{1-(n/2)}|u|^2\,dx\,dt<\infty.$$

2) *If* $q=2m-1$, $m=1,2,\ldots,$

a) $\dfrac{\partial^m u}{\partial t^m} = P(D)v,$

b) $\displaystyle\int_{P(x_0;\beta,k)} t^{1-(n/2)}|v|^2\,dx\,dt<\infty,$

then, for $0<a<\beta$, $0<h<k$,

$$('') \qquad \int_{P(x_0;a,h)} t^{-n/2}|u|^2\,dx\,dt<\infty.$$

Proof. Without loss of generality, we will consider x_0 to be the origin throughout the proof.

We will do the case $m = 0$ of part 1) separately.

Under the change of variable $t \to \tau^2$, u and v become vector solutions of (*) and satisfy:

a*) $u(x, \tau^2) = P(D)v(x, \tau^2)$,

and

b*) $\int\limits_{\Gamma(\beta,k^{1/2})} \tau^{1-n}|v(x, \tau^2)|^2 dx\,d\tau < \infty$, $\Gamma(\beta, k^{1/2}) = \Gamma(0; \beta, k^{1/2})$.

Also, $P(\alpha, h) = P(0; \alpha, h)$ maps into $\Gamma(\alpha, h^{1/2})$.

By lemma 9,

$$|P(D)v(\sigma)| \leqslant M\varrho_0^{-1/2(n+3)}\Big\{\int\limits_{S^*_{\varrho_0}(\sigma)} |v|^2 dx\,d\eta\Big\}^{1/2},$$

where $S^*_{\varrho_0}(\sigma)$ is situated so that $\sigma = (x_1, \tau_1)$ is the center of the upper face.

For $(x, \tau)\in\Gamma(\alpha, h^{1/2})$, there exists a $c > 0$ such that $S^*_{c\tau}(x, \tau) \subset \Gamma(\beta, k^{1/2})$. Hence,

$$u(x, \tau^2) \leqslant B\tau^{-1/2(n+3)}\Big\{\int\limits_{S^*_{c\tau}(x,\tau)} |v|^2 dx\,d\eta\Big\}^{1/2},$$

B a constant independent of u, v, x, τ.

Let L_τ be the layer through the cone $\Gamma(\beta, k^{1/2})$ between τ and $\tau - c$, i. e., $L_\tau = \{(x, \eta)\in\Gamma(\beta, k^{1/2}): (1-c)\tau < \eta < \tau\}$. Then,

$$\Big\{\int\limits_{S^*_{c\tau}(x,\tau)} |v|^2 dx\,d\eta\Big\}^{1/2} \leqslant \Big\{\int\limits_{L_\tau} |v|^2 dx\,d\eta\Big\}^{1/2} = J_\tau^{1/2},$$

for $(x, \tau)\in\Gamma(\alpha, h^{1/2})$, and $|u(x, \tau^2)| \leqslant B\tau^{-1/2(n+3)}J_\tau^{1/2}$.

Let $u_\varrho(s)$ be the restriction of u to the ray ϱ coming from the origin with the distance from the origin given by s. If θ is the angle made by ϱ with the positive τ axis, $\tau = s\cos\theta$, $1 \geqslant \cos\theta \geqslant a_0 = (1+\alpha^2)^{-1/2} > 0$. Then,

$$\int\limits_0^{h^{1/2}} s^3|u_\varrho(s)|^2 ds \leqslant B\int\limits_0^{h^{1/2}} \tau^{-n}J_\tau\,d\tau.$$

Now,

$$\int\limits_0^{h^{1/2}} \tau^{-n}J_\tau\,d\tau = \int\limits_0^{h^{1/2}} \tau^{-n}\Big\{\int\limits_{L_\tau}|v|^2 dx\,d\eta\Big\}\,d\tau \int\limits_{\Gamma(\beta,k^{1/2})} |v|^2\Big\{\int \tau^{-n}\chi(\tau; x, \eta)\,d\tau\Big\}\,dx\,d\eta,$$

where $\chi(\tau; x, \eta)$ is the characteristic function of L_τ.

$$\int \tau^{-n}\chi(\tau; x,\eta)\,d\tau = \int\limits_{\substack{\tau(1-c)<\eta<\tau\\ 0<\eta<k^{1/2}}} \tau^{-n}d\tau \leqslant \int\limits_{\eta}^{\eta/(1-c)} \tau^{-n}d\tau = c_1\eta^{1-n}.$$

Hence,

$$\int\limits_0^{h^{1/2}} s^3|u_\varrho(s)|^2 ds \leqslant B\int\limits_{\Gamma(\beta,k^{1/2})} \eta^{1-n}|v|^2 dx\,d\eta < \infty.$$

Let $\tilde{\Gamma}$ be the intersection of $\Gamma(\alpha, h^{1/2})$ with the sphere of radius $h^{1/2}$ about the origin. Then, integrating across the rays ϱ in $\tilde{\Gamma}$, we have

$$\int\limits_{\tilde{\Gamma}} \tau^{3-n}|u|^2 dx\,d\tau < \infty.$$

Since the remainder of the cone is at a positive distance from $\tau = 0$, and u is continuous, we have

$$\int\limits_{\Gamma(\alpha,h^{1/2})} \tau^{3-n}|u|^2 dx\,d\tau < \infty.$$

By the change of variable $t = \tau^2$, we have,

$$\frac{1}{2}\int\limits_{P(\alpha,h)} t^{1-(n/2)}|u|^2 dx\,dt < \infty,$$

which completes the proof of this case.

For the remainder, the change of variables $t = \tau^2$ gives us $u(x, \tau^2)$, $v(x, \tau^2)$ as solutions to (*), and

a*) $T^m u = P(D)v$, T as in lemma 10.

In part 1), b) gives us

b*) $\int\limits_{\Gamma(\beta,k^{1/2})} \tau^{1-n}|v|^2 dx\,d\tau < \infty$.

By lemma 10,

$$u(x, y^2) = \big(-1/(m-1)!\,2^{m-1}\big)\int\limits_y^{h^{1/2}} \{(y^2-\tau^2)^{m-1}[T^m u(x, \tau^2)]\tau\,d\tau\} + A_m$$

$$= \big(-1/(m-1)!\,2^{m-1}\big)\int\limits_y^{h^{1/2}} (y^2-\tau^2)^{m-1}[P(D)v(x, \tau^2)]\tau\,d\tau + A_m.$$

By lemma 9,

$$|P(D)v(\sigma)| \leqslant M\varrho_0^{-1/2(n-4m+3)}\Big\{\int\limits_{S^*_{\varrho_0}(\sigma)} |v|^2 dx\,d\eta\Big\}^{1/2},$$

$S^*_{\varrho_0}(\sigma)$ is as in the above case. For $(x, \tau)\in\Gamma(\alpha, h^{1/2})$, there exists a $c > 0$ such that $S^*_{c\tau}(x, \tau) \subset \Gamma(\beta, k^{1/2})$. Hence,

$$|u(x, y^2)| \leqslant B\int\limits_y^{h^{1/2}} \tau^{2m-1}\tau^{-1/2(n-4m+3)}\Big\{\int\limits_{S^*_{c\tau}(x,\tau)} |v|^2 dx\,d\eta\Big\}^{1/2} d\tau + A.$$

Letting J_τ be as before, we have,

$$|u(x, y^2)| \leqslant B \int_0^{h^{1/2}} \tau^{-1/2(n+5)} J_\tau^{1/2} d\tau + A.$$

Letting ϱ and $u_\varrho(s)$ be as before, we have

$$|u_\varrho(s)| \leqslant B \int_{s\cos\theta}^{h^{1/2}} \tau^{-1/2(n+5)} J_\tau^{1/2} d\tau + A,$$

whence,

$$s|u_\varrho(s)| \leqslant B \int_{s\cos\theta}^{h^{1/2}} \tau^{-1/2(n+3)} J_\tau^{1/2} d\tau + A.$$

By lemma 11,

$$\int_0^{h^{1/2}} s^3 |u_\varrho(s)|^2 ds \leqslant B \int_0^{h^{1/2}} \tau^{-n} J_\tau d\tau + A.$$

Proceeding as in the case $m = 0$, we obtain part 1).

Upon the usual change of variables, 2b) gives us

$$\int_{\Gamma(\beta, k^{1/2})} \tau^{3-n} |v|^2 dx\, d\tau < \infty.$$

Using the same representation as in part 1) and the estimate

$$|P(D)v(\sigma)| \leqslant M\varrho_0^{-1/2(n+4m-1)} \Big\{ \int_{S^*_{\varrho_0}(\sigma)} |v|^2 dx\, d\eta \Big\}^{1/2}$$

from lemma 9, we obtain,

$$|u(x, y^2)| \leqslant B \int_y^{h^{1/2}} \tau^{-1/2(n+1)} J_\tau^{1/2} d\tau + A,$$

and thus,

$$|u_\varrho(s)| \leqslant B \int_{s\cos\theta}^{h^{1/2}} \tau^{-1/2(n+1)} J_\tau^{1/2} d\tau + A.$$

By lemma 11,

$$\int_0^{h^{1/2}} s|u_\varrho(s)|^2 ds \leqslant B \int_0^{h^{1/2}} \tau^{2-n} J_\tau d\tau + A.$$

Similar to before,

$$\int_0^{h^{1/2}} \tau^{2-n} J_\tau d\tau = \int_0^{h^{1/2}} \tau^{2-n} \Big\{ \int_{L_\tau} |v|^2 dx\, d\eta \Big\} d\tau \quad \int_{\Gamma(\beta, k^{1/2})} |v|^2 \Big\{ \int \tau^{2-n} \chi(\tau; x, \eta)\, d\tau \Big\} dx\, d\eta,$$

$\chi(\tau; x, \eta)$ as before.

$$\int \tau^{2-n} \chi(\tau; x, \eta)\, d\tau = \int_{\substack{(1-c)\tau < \eta < \tau \\ 0 < \eta < h^{1/2}}} \tau^{2-n} d\tau \leqslant \int_\eta^{\eta/(1-c)} \tau^{2-n} d\tau = c_2 \eta^{3-n}.$$

Thus,

$$\int_0^{h^{1/2}} s|u_\varrho(s)|^2 ds \leqslant B \int_{\Gamma(\beta, k^{1/2})} \eta^{3-n} |v|^2 dx\, d\eta + A < \infty.$$

This implies that

$$\int_{\Gamma(a, h^{1/2})} \tau^{1-n} |u|^2 dx\, d\tau < \infty,$$

and by the change of variables $t = \tau^2$,

$$('') \qquad \int_{P(a,h)} t^{-n/2} |u|^2 dx\, dt < \infty.$$

Utilizing this lemma, we can now show that the two terms in the integral in theorem 2 are essentially equivalent in their finiteness, i. e.,

THEOREM 3. *Let u_1 be a solution to the heat equation on $P(x_0; \beta, k)$.*

1) *If*

$$\int_{P(x_0; \beta, k)} t^{1-(n/2)} \left| \frac{\partial u_1}{\partial t} \right|^2 dx\, dt < \infty,$$

then

$$\int_{P(x_0; a, h)} t^{-n/2} \left| \frac{\partial u_1}{\partial x_i} \right|^2 dx\, dt < \infty$$

for $0 < a < \beta$, $0 < h < k$, $i = 1, 2, \ldots, n$.

2) *If each of the integrals*

$$\int_{P(x_0; \beta, k)} t^{-n/2} \left| \frac{\partial u_1}{\partial x_i} \right|^2 dx\, dt < \infty$$

for $i = 1, 2, \ldots, n$, then

$$\int_{P(x_0; a, h)} t^{1-(n/2)} \left| \frac{\partial u_1}{\partial t} \right|^2 dx\, dt < \infty,$$

$0 < a < \beta$, $0 < h < k$.

Proof. 1) In part 2 of lemma 12, $m = 1$, let

$$u(x,t) = \frac{\partial}{\partial x_i} u_1(x,t), \quad v(x,t) = \frac{\partial}{\partial t} u_1(x,t), \quad P(D) = \frac{\partial}{\partial x_i}.$$

2) In part 1 of lemma 12, $m = 0$, let

$$u(x,t) = \frac{\partial}{\partial t} u_1(x,t),$$

$$v(x,t) = \left(\frac{\partial}{\partial x_1} u_1(x,t), \frac{\partial}{\partial x_2} u_1(x,t), \ldots, \frac{\partial}{\partial x_n} u_1(x,t)\right),$$

and

$$P(D) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right).$$

Making use of both lemma 12 and theorem 3, we have the following application of theorem 2:

THEOREM 4. *Let u, v be solutions to the heat equation in E_{n+1}^{+} of respective dimension r and s; let $P(D)$ be as in lemma 12 with $q = 2m$, $m \geq 1$. Suppose that they satisfy $\partial^m u/\partial t^m = P(D)v$. Then, if $v(x,t)$ has parabolic limits on a set $E \subset E_n$, $u(x,t)$ has parabolic limits almost everywhere on E.*

Proof. By theorem 2, and part 1 of theorem 3, it is enough to show that

$$\int_{P(x_0;\alpha,h)} t^{1-(n/2)} \left|\frac{\partial u}{\partial t}\right|^2 dx\,dt < \infty$$

for almost every $x_0 \in E$.

By theorem 2,

$$\int_{P(x_0;\beta,k)} t^{-n/2} |\nabla_s v|^2 dx\,dt < \infty$$

for almost every $x_0 \in E$.

We let $U = \partial u/\partial t$, $V = \nabla_s v$, and $\tilde{P}(D)$ be the $r \times sn$ matrix

$$\tilde{P}(D) = \left(P\left(\frac{\partial}{\partial x_1} D\right) P\left(\frac{\partial}{\partial x_2} D\right) \ldots P\left(\frac{\partial}{\partial x_n} D\right)\right).$$

Then $\partial^m U/\partial t^m = \tilde{P}(D)V$, and the differential polynomial entries of $P(D)$ will be homogeneous of order $2m+1$.

By part 1 of lemma 12, the finiteness of

$$\int_{P(x_0;\beta,k)} t^{-n/2} |V|^2 dx\,dt = \int_{P(x_0;\beta,k)} t^{-n/2} |\nabla_s v|^2 dx\,dt$$

implies the finiteness of

$$\int_{P(x_0;\alpha,h)} t^{1-(n/2)} |U|^2 dx\,dt = \int_{P(x_0;\alpha,h)} t^{1-(n/2)} \frac{\partial u^2}{\partial t} dx\,dt,$$

which completes the proof.

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