On non-equivalent bases and conditional bases in Banach spaces

by

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Introduction

A basis $\{x_n\}$ of a Banach space E is called *equivalent* (cf. Banach [4]) to a basis $\{y_n\}$ of a Banach space F provided that $\sum_{i=1}^{\infty} a_i x_i$ is convergent in E if and only if $\sum_{i=1}^{\infty} a_i y_i$ is convergent in F. This happens if and only if there exists an isomorphism A of E onto F such that $A(x_n) = y_n$ for $n = 1, 2, \ldots$ ([2], [7]). A basis $\{x_n\}$ is said to be *conditional* (*unconditional*) if there exists (if there does not exist) a series $\sum_{i=1}^{\infty} a_i x_i$ which is convergent but not unconditionally convergent. A basis $\{x_n\}$ is said to be *normalized* if $\|x_n\| = 1$ $(n = 1, 2, \ldots)$.

In finite-dimensional spaces all normalized bases are equivalent and unconditional. In the present paper we shall show that each of these properties characterizes finite-dimensional spaces among Banach spaces with a basis, by proving the following

THEOREM. *In every infinite-dimensional Banach space with a basis there exist two non-equivalent normalized bases, one of which is conditional* (1).

In the usual concrete infinite-dimensional separable Banach spaces, excepting Hilbert spaces, it is easy to give examples of non-equivalent normalized bases and conditional bases (see e. g. [24], [14]). In the case of Hilbert spaces the problem becomes considerably more difficult. The first example of a normalized conditional basis (and hence non-equivalent to an orthogonal basis) in a separable Hilbert space has been given by Babenko [3] (see also [15], [1], [13]).

Since every infinite-dimensional Banach space contains a basic sequence (2), i. e. a sequence $\{z_n\}$ which is a basis of its closed linear hull

(1) Added in proof. This theorem substantiates a conjecture of Bonnice and Klee [31], p. 26.

(2) This result has been given without proof by Banach [5], p. 238; for various proofs see [7], [16], [8].

$[z_n]$, it follows from the above theorem as an immediate consequence that every Banach space contains a conditional basic sequence and two non-equivalent normalized basic sequences. The first of these results has been obtained by Gurariĭ [19] with the aid of a profound result of Dvoretzky [10], and, as has been remarked by C. Bessaga, the second result may be obtained by a similar method.

We prove our theorem by the following method: we reduce the problem to symmetric spaces (see definition 1 below) and then from symmetric spaces to Hilbert spaces, where we apply the result of Babenko [3]. For the second step, in symmetric spaces we introduce analogues of the classical function systems of Haar and Rademacher [22] and prove a certain abstract analogue of the Khinchin inequality ([22], p. 131-132) which may also be of some interest for other applications.

In the last part of the paper we make some remarks and formulate some unsolved problems.

We wish to express here our gratitude to Dr C. Bessaga for valuable discussions and critical remarks.

§ 1. Symmetric Banach spaces

A basis $\{x_j\}$ of a Banach space E is called [27] ([3]) *symmetric*, if

$$(1)\qquad |||x||| = \sup_{\sigma\in P(N)} \sup_{\substack{|\delta_i|\leqslant 1\\ 1\leqslant n<+\infty}} \Big\|\sum_{i=1}^{n} \delta_i x_i^*(x)\, x_{\sigma(i)}\Big\| < +\infty \quad \text{for all} \quad x\in E,$$

where $P(N)$ denotes the set of all permutations of the set $N = \{1, 2, \ldots\}$ and $\{x_j^*\}$ is the sequence of continuous linear functionals biorthogonal to $\{x_j\}$. For a symmetric basis $\{x_j\}$, formula (1) defines a new norm on E, equivalent to the original norm and "*symmetric* with respect to $\{x_j\}$", i. e. such that

$$(2)\qquad \Big|\Big|\Big|\sum_{i=1}^{\infty} \varepsilon_i x^*_{\varrho(i)}(x)\, x_{\tau(i)}\Big|\Big|\Big| = \Big|\Big|\Big|\sum_{i=1}^{\infty} x_i^*(x)\, x_i\Big|\Big|\Big|$$

for all $x\in E$, $\varrho, \tau\in P(N)$ and $|\varepsilon_i| = 1$, $i = 1, 2, \ldots$ (see [27], theorem 1).

Definition 1. We shall call a *symmetric space* any couple $(E, \{x_j\})$, where E is a Banach space with a symmetric basis and $\{x_j\}$ a symmetric basis of E, such that the original norm of E is symmetric with respect to $\{x_j\}$.

In the sequel we shall denote by n an arbitrary positive integer.

([3]) See also [28], [23].

Definition 2. Let $(E_{2^n}, \{x_j\})$ be a 2^n-dimensional symmetric space. We shall call the *Haar system* the sequence $\{y_j\}_{j=1}^{2^n}$ defined by ([4])

$$(3)\qquad y_1 = \sum_{i=1}^{2^n} x_i, \qquad y_{2^k+l} = \sum_{i=1}^{2^n} \beta_i^{(k,l)} x_i$$
$$(l = 1, 2, \ldots, 2^k;\ k = 0, 1, \ldots, n-1),$$

where

$$(4)\qquad \beta_i^{(k,l)} = \begin{cases} 1 & \text{for} \quad (2l-2)2^{n-k-1}+1 \leqslant i \leqslant (2l-1)2^{n-k-1}, \\ -1 & \text{for} \quad (2l-1)2^{n-k-1}+1 \leqslant i \leqslant 2l\cdot 2^{n-k-1}, \\ 0 & \text{for} \quad 1\leqslant i \leqslant (2l-2)2^{n-k-1} \text{ and } 2l\cdot 2^{n-k-1}+1 \leqslant i \leqslant 2^n. \end{cases}$$

Definition 3. Let $(E_{2^n}, \{x_j\})$ be a 2^n-dimensional symmetric space. We shall call the *Rademacher system* the sequence $\{r_k\}_{k=1}^{n}$ defined by

$$(5)\qquad r_k = \sum_{l=1}^{2^{k-1}} y_{2^{k-1}+l} \qquad (k = 1, 2, \ldots, n),$$

where $\{y_j\}$ is the Haar system in $(E_{2^n}, \{x_j\})$.

Proposition 1. *Let $(E_{2^n}, \{x_j\})$ be a 2^n-dimensional symmetric space. Then the Haar system $\{y_i\}$ is a monotone basis* ([5]) *of E_{2^n}.*

Proof. Let m be an arbitrary integer such that $1\leqslant m\leqslant 2^n-1$, and let $a_1, a_2, \ldots, a_{m+1}$ be arbitrary scalars. Then, since $\{x_j\}$ is a basis of E_{2^n}, there exists a sequence of scalars $\{b_j\}_{j=1}^{2^n}$ such that

$$(6)\qquad \sum_{j=1}^{m} a_j y_j = \sum_{i=1}^{2^n} b_i x_i.$$

Let (k, l) be the couple of non-negative integers determined by the following properties: $1\leqslant l\leqslant 2^k$, $2^k+l = m+1$. Then, by (6), (3) and (4),

$$\Big\|\sum_{j=1}^{m+1} a_j y_j\Big\| = \Big\|\sum_{i=1}^{2^n} b_i x_i + a_{m+1}\sum_{i=1}^{2^n} \beta_i^{(k,l)} x_i\Big\|$$
$$= \Big\|\sum_{i=1}^{(2l-2)2^{n-k-1}} b_i x_i + \sum_{i=(2l-2)2^{n-k-1}+1}^{(2l-1)2^{n-k-1}} (b_i + a_{m+1}) x_i +$$
$$+ \sum_{i=(2l-1)2^{n-k-1}+1}^{2l\cdot 2^{n-k-1}} (b_i - a_{m+1}) x_i + \sum_{i=2l\cdot 2^{n-k-1}+1}^{2^n} b_i x_i\Big\|.$$

([4]) A similar construction of Haar system for certain function spaces has been made by Ellis and Halperin [12]. More general definition than our Definition 3, see Rutovitz [32].

([5]) A basis $\{x_j\}$ in a Banach space E is said to be *monotone* (cf. [9], p. 67) provided that $\|t_1x_1+t_2x_2+\ldots+t_kx_k\| \leqslant \|t_1x_1+t_2x_2+\ldots+t_kx_k+t_{k+1}x_{k+1}\|$ for every scalars $t_1, t_2, \ldots, t_{k+1}$ $(k = 1, 2, \ldots)$.

Since $(E_{2^n}, \{x_j\})$ is a symmetric space, this number is equal to

$$\Big\| \sum_{i=1}^{(2l-2)2^{n-k-1}} b_i x_i + \sum_{i=(2l-2)2^{n-k-1}+1}^{(2l-1)2^{n-k-1}} (b_i - a_{m+1}) x_i + \sum_{i=(2l-1)2^{n-k-1}+1}^{2l\cdot 2^{n-k-1}} (b_i + a_{m+1}) x_i + \sum_{i=2l2^{n-k-1}+1}^{2^n} b_i x_i \Big\|.$$

Adding these equalities and multiplying by $\frac{1}{2}$, we obtain

$$\Big\| \sum_{j=1}^{m+1} a_j y_j \Big\| \geqslant \Big\| \sum_{i=1}^{2^n} b_i x_i \Big\| = \Big\| \sum_{j=1}^{m} a_j y_j \Big\|,$$

which completes the proof.

COROLLARY. *Let $(E_{2^n}, \{x_j\})$ be a 2^n-dimensional symmetric space. Then the Rademacher system $\{r_k\}_{k=1}^{n}$ is a monotone block basic sequence* [6] *with respect to the Haar system $\{y_j\}_{j=1}^{2^n}$.*

We shall now prove the following abstract analogue of the Khinchin inequality:

PROPOSITION 2. *Let $(E_{2^n}, \{x_j\})$ be a 2^n-dimensional symmetric space and let $\{r_k/\|r_k\|\}_{k=1}^{n}$ be the normalized Rademacher system in this space. Then for any scalars $a_1, a_2, \ldots, a_n$ we have*

$$\Big\| \sum_{k=1}^{n} a_k \frac{r_k}{\|r_k\|} \Big\| \geqslant \frac{1}{8} \sqrt{\sum_{k=1}^{n} |a_k|^2}. \tag{7}$$

Proof. Since $\{x_j\}$ is a basis of E_{2^n}, there exists for each integer k $(1 \leqslant k \leqslant n)$ a unique sequence of scalars $\{r_{kj}\}_{j=1}^{2^n}$ such that

$$r_k = \sum_{i=1}^{2^n} r_{ki} x_i \qquad (k = 1, 2, \ldots, n). \tag{8}$$

Moreover, it is easy to compute that

$$r_{ki} = \begin{cases} 1 & \text{for } (2l-2)2^{n-k}+1 \leqslant i \leqslant (2l-1)2^{n-k}, \\ -1 & \text{for } (2l-1)2^{n-k}+1 \leqslant i \leqslant 2l\cdot 2^{n-k} \end{cases} \tag{9}$$

$(l = 1, 2, \ldots, 2^{k-1};\ k = 1, 2, \ldots, n)$.

[6] Let us recall that if $\|x_j\|$ is a basis of a Banach space E, any sequence $\{z_j\} \subset E$ of the form $z_j = \sum_{i=m_{j-1}+1}^{m_j} a_i x_i$, $z_j \neq 0$ $(j = 1, 2, \ldots)$, where $\{m_j\}$ is an increasing sequence of positive integers, $m_0 = 0$, and where $\{a_j\}$ is a sequence of scalars, is called [7] a *block basic sequence with respect to* $\{x_j\}$; it is necessarily [7] a basic sequence.

Let $r_k(\cdot)$, $k = 1, 2, \ldots, n$, be the usual Rademacher functions on $[0, 1]$. We claim that for any scalars $a_1, a_2, \ldots, a_n$ we have

$$\int_0^1 \Big| \sum_{k=1}^{n} a_k r_k(t) \Big| dt = \frac{1}{2^n} \sum_{i=1}^{2^n} \Big| \sum_{k=1}^{n} a_k r_{ki} \Big|. \tag{10}$$

In fact, let us denote by $(l^1_{2^n}, \{e_j\})$ the 2^n-dimensional symmetric space in which the norm is defined by $\|\sum_{i=1}^{2^n} b_i e_i\| = \sum_{i=1}^{2^n} |b_i|$ and by $\chi_i(\ )$ the characteristic function of the interval $\big(2^{-n}(i-1); 2^{-n}i\big)$ for $i = 1, 2, \ldots, 2^n$. Then the mapping φ, with $\varphi\big(\sum_{i=1}^{2^n} a_i \chi_i\big) = 2^{-n} \sum_{i=1}^{2^n} a_i e_i$ is obviously a linear isometry of the 2^n-dimensional subspace of $L^1([0;1])$ spanned by the characteristic functions $\chi_i(\ )$ $(i = 1, 2, \ldots, 2^n)$ onto the space $l^1_{2^n}$. Since $r_k(t) = \sum_{i=1}^{2^n} r_{ki} \chi_i(t)$, it follows that

$$\varphi\Big[\sum_{k=1}^{n} a_k r_k(\cdot)\Big] = \sum_{k=1}^{n} a_k \varphi[r_k(\cdot)] = \sum_{k=1}^{n} \frac{a_k}{2^k} \sum_{i=1}^{2^n} r_{ki} e_i,$$

whence, since φ is an isometry, we infer (10).

By (10) and the usual Khinchin inequality ([22], p. 131-132) we have, for any scalars $a_1, a_2, \ldots, a_n$,

$$\frac{1}{2^n} \sum_{i=1}^{2^n} \Big| \sum_{k=1}^{n} a_k r_{ki} \Big| \geqslant \frac{1}{8} \sqrt{\sum_{k=1}^{n} |a_k|^2}. \tag{11}$$

On the other hand, since $(E_{2^n}, \{x_j\})$ is a symmetric space, we have, by (8) and (9),

$$\|r_1\| = \|r_2\| = \ldots = \|r_n\| = \Big\| \sum_{i=1}^{2^n} x_i \Big\|. \tag{12}$$

Let us denote this common value by A_n and let

$$B_n = \Big\| \sum_{i=1}^{2^n} x_i^* \Big\|,$$

where $\{x_i^*\} \subset E_{2^n}^*$, $x_i^*(x_j) = \delta_i^j$ $(i, j = 1, 2, \ldots, 2^n)$. Then we have (see lemma 1 below)

$$A_n B_n = 2^n. \tag{13}$$

Now let $a_1, a_2, \ldots, a_n$ be arbitrary scalars and let $\varepsilon_i = \operatorname{sign}\sum_{k=1}^{n} a_k r_{ki}$ $(i = 1, 2, \ldots, 2^n)$. Then, taking into account that $(E^*_{2^n}, \{x^*_i\})$ is a symmetric space and (8), (11), (12), (13), we obtain

$$\Big\|\sum_{k=1}^{n} a_k \frac{r_k}{\|r_k\|}\Big\| = \frac{1}{A_n}\Big\|\sum_{k=1}^{n} a_k r_k\Big\| = \frac{B_n}{2^n}\Big\|\sum_{k=1}^{n} a_k r_k\Big\|$$

$$= \frac{1}{2^n}\Big\|\sum_{i=1}^{2^n} \varepsilon_i x^*_i\Big\|\,\Big\|\sum_{k=1}^{n} a_k r_k\Big\| \geqslant \frac{1}{2^n}\Big|\Big(\sum_{i=1}^{2^n}\varepsilon_i x^*_i\Big)\Big(\sum_{k=1}^{n} a_k r_k\Big)\Big|$$

$$= \frac{1}{2^n}\Big|\sum_{i=1}^{2^n}\varepsilon_i \sum_{k=1}^{n} a_k r_{ki}\Big| = \frac{1}{2^n}\sum_{i=1}^{2^n}\Big|\sum_{k=1}^{n} a_k r_{ki}\Big| \geqslant \frac{1}{8}\sqrt{\sum_{k=1}^{n}|a_k|^2},$$

which is nothing else but (7). Thus, in order to complete the proof of proposition 2, we have only to present the proof of the following lemma:

LEMMA 1. *Let $(E_{2^n}, \{x_j\})$ be a 2^n-dimensional symmetric space, and let $A_n = \|\sum_{i=1}^{2^n} x_i\|$, $B_n = \|\sum_{i=1}^{n} x^*_1\|$, where $\{x^*_i\} \subset E^*_{2^n}$, $x^*_i(x_j) = \delta_{ij}$. Then we have* (13).

Proof. We have, obviously,

$$A_n B_n \geqslant \Big(\sum_{i=1}^{2^n} x^*_i\Big)\Big(\sum_{j=1}^{2^n} x_j\Big) = 2^n. \tag{14}$$

On the other hand, let $x^* = \sum_{i=1}^{2^n} x^*_i$ and let $x = \sum_{j=1}^{2^n} a_j x_j \in E$ be such that $x^*(x) = B_n$, $\|x\| = 1$. Then

$$B_n = x^*(x) = \Big(\sum_{i=1}^{2^n} x^*_i\Big)\Big(\sum_{j=1}^{2^n} a_j x_j\Big) = \sum_{j=1}^{2^n} a_j. \tag{15}$$

Now, let Π_n denote the set of all permutations of the set $\{1, 2, \ldots, 2^n\}$ and let

$$x_\sigma = \sum_{j=1}^{2^n} a_{\sigma(j)} x_j \qquad (\sigma \in \Pi_n), \tag{16}$$

$$x_0 = \frac{1}{(2^n)!}\sum_{\sigma\in\Pi_n} x_\sigma. \tag{17}$$

Then, since $(E_{2^n}, \{x_j\})$ is a symmetric space, we have $\|x_\sigma\| = \|x\| = 1$ for all $\sigma \in \Pi_n$, whence

$$\|x_0\| \leqslant 1. \tag{18}$$

On the other hand, by (16) and (17) we have

$$x_0 = \frac{1}{2^n}\sum_{j=1}^{2^n} a_j \sum_{i=1}^{2^n} x_i$$

(since the coefficient of x_i is

$$\frac{1}{(2^n)!}\sum_{\sigma\in\Pi_n} a_{\sigma(i)}$$

and since for any couple of integers i, j with $1 \leqslant i, j \leqslant 2^n$ there are exactly $(2^n-1)!$ permutations $\sigma \in \Pi_n$ such that $\sigma(i) = j$), whence, by (15) and the definition of A_n,

$$\|x_0\| = \frac{B_n A_n}{2^n}. \tag{19}$$

Comparing (14), (18) and (19), we obtain (13), which completes the proof.

§ 2. Block perturbations of bases

Definition 4. Let $\{x_j\}$ be a normalized basis of a Banach space E. We shall call *block-perturbation* (7) of $\{x_j\}$ any sequence $\{v_k\} \subset E$ of the form

$$v_k = \begin{cases} x_k & \text{for} \quad k \neq p_n, \\ x_{p_n} + u_n & \text{for} \quad k = p_n \end{cases} \qquad (n = 1, 2, \ldots), \tag{20}$$

where

$$u_n = \sum_{i=m_{n-1}+1}^{p_n-1} a_i x_i + \sum_{i=p_n+1}^{m_n} a_i x_i, \quad \|u_n\| \leqslant M < +\infty \quad (n = 1, 2, \ldots), \tag{21}$$

and where $\{m_n\}$, $\{p_n\}$ are increasing sequences of positive integers such that $m_0 = 0$, $m_{n-1}+1 \leqslant p_n \leqslant m_n$ $(n = 1, 2, \ldots)$.

LEMMA 2. *Let $\{x_j\}$ be a normalized basis of a Banach space E. Then every block perturbation $\{v_k\}$ of $\{x_j\}$ is a basis of E.*

Proof. Let $\{v_k\}$ be of the form (20) with $\{u_n\}$ satisfying (21). Then $\{v_k\}$ admits a biorthogonal sequence $\{x^*_k\} \subset E^*$ given by

$$v^*_k = \begin{cases} x^*_k - a_k x^*_{p_n} & \text{for} \quad k \neq p_n,\ m_{n-1}+1 \leqslant k \leqslant m_n, \\ x^*_{p_n} & \text{for} \quad k = p_n \end{cases} \qquad (n = 1, 2, \ldots),$$

(7) Let us mention that V. G. Vinokurov has considered perturbations of the form $z_{2j-1} = x_{2j-1}$, $z_{2j} = a_{2j-1}x_{2j-1} + a_{2j}x_{2j}$ $(j = 1, 2, \ldots)$, where $\sup_j |a_{2j-1}| < +\infty$, $\inf_j |a_{2j}| > 0$, and has established that they constitute a basis of E ([30], theorem 4).

where $x_n^*(x_m) = \delta_n^m$ $(n, m = 1, 2, \ldots)$. Hence, for all $x \in E$,

$$\sum_{k=1}^{l} v_k^*(x) v_k = \begin{cases} \sum_{j=1}^{l} x_j^*(x) x_j - x_{p_n}^*(x) \sum_{i=m_{n-1}+1}^{l} a_i x_i & \text{for } m_{n-1}+1 \leqslant l \leqslant p_n - 1, \\ \sum_{j=1}^{l} x_j^*(x) x_j + x_{p_n}^*(x) \sum_{i=l+1}^{m_n} a_i x_i & \text{for } p_n \leqslant l \leqslant m_n \end{cases} \quad (n = 1, 2, \ldots).$$

Since $\{x_n\}$ is a basis of E, there exists a constant $K \geqslant 1$ such that

$$\Big\| \sum_{i=m_{n-1}+1}^{l} a_i x_i \Big\| \leqslant K \|u_n\| \leqslant KM \quad (m_{n-1}+1 \leqslant l \leqslant p_n - 1;\ n = 1, 2, \ldots),$$

$$\Big\| \sum_{i=l+1}^{m_n} a_i x_i \Big\| \leqslant \|u_n\| + K\|u_n\| \leqslant (1+K)M \quad (p_n \leqslant l \leqslant m_n;\ n = 1, 2, \ldots).$$

Since the basis $\{x_n\}$ is normalized, we also have

$$\lim_{n\to\infty} x_{p_n}^*(x) = 0 \quad \text{for all} \quad x \in E.$$

Consequently, for every $\varepsilon > 0$ and $x \in E$, there exists an integer $N(\varepsilon, x) > 0$ such that

$$\Big\| \sum_{k=1}^{l} v_k^*(x) v_k - \sum_{j=1}^{l} x_j^*(x) x_j \Big\| < \varepsilon \quad \text{for} \quad l > N(\varepsilon, x),$$

whence $x = \sum_{k=1}^{\infty} v_k^*(x) v_k$ for all $x \in E$, which completes the proof.

PROPOSITION 3. *Let $\{x_n\}$ be a normalized non-symmetric unconditional basis of a Banach space E. Then there exists a block perturbation of a suitable permutation of $\{x_n\}$, which is a conditional basis of E.*

Proof. We claim that there exists a permutation of the basic sequence $\{x_{2j}\}$ which is not equivalent to the basic sequence $\{x_{2j-1}\}$. In fact, assume that all permutations of $\{x_{2j}\}$ are equivalent to $\{x_{2j-1}\}$. Then, by [28], $\{x_{2j}\}$ is a symmetric basic sequence, whence, again by [28], $\{x_{2j}\}$ is equivalent to its subsequences $\{x_{4j-2}\}$ and $\{x_{4j}\}$. We shall show that the mapping $x_{2j-1} \to x_{4j-2}$, $x_{2j} \to x_{4j}$ defines an equivalence of the basis $\{x_n\}$ with its subsequence $\{x_{2j}\}$, which is a contradiction since $\{x_n\}$ is non-symmetric. In fact, since $\{x_n\}$ is unconditional, $\sum_{i=1}^{\infty} a_i x_i$ is convergent if and only if $\sum_{i=1}^{\infty} a_{2i-1} x_{2i-1}$ and $\sum_{i=1}^{\infty} a_{2i} x_{2i}$ are convergent. Since $\{x_{2j-1}\}$, $\{x_{2j}\}$, are equivalent to $\{x_{4j-2}\}$ and $\{x_{4j}\}$ respectively, this happens if and only if $\sum_{i=1}^{\infty} a_{2i-1} x_{4i-2}$ and $\sum_{i=1}^{\infty} a_{2i} x_{4i}$ are convergent, i. e. (since $\{x_{2j}\}$ is unconditional) if and only if $\sum_{i=1}^{\infty} a_i x_{2i}$ is convergent.

Thus, let $\{x_{\tau(2j)}\}$ be a permutation of $\{x_{2j}\}$ such that $\{x_{2j-1}\}$ and $\{x_{\tau(2j)}\}$ are not equivalent. Let $\{x_{\sigma(n)}\}$ be the permutation of $\{x_n\}$ defined by

$$(22) \qquad x_{\sigma(n)} = \begin{cases} x_n & \text{for } n = 2j-1, \\ x_{\tau(n)} & \text{for } n = 2j \end{cases} \qquad (j = 1, 2, \ldots)$$

and let $\{v_k'\}$, $\{v_k''\}$ be the following two block perturbations of the basis $\{x_{\sigma(n)}\}$:

$$(23) \qquad v_k' = \begin{cases} x_{\sigma(k)} & \text{for } k = 2n-1, \\ x_{\sigma(k)} + x_{\sigma(k-1)} & \text{for } k = 2n, \end{cases} \qquad (n = 1, 2, \ldots),$$

$$(24) \qquad v_k'' = \begin{cases} x_{\sigma(k)} & \text{for } k = 2n, \\ x_{\sigma(k)} + x_{\sigma(k+1)} & \text{for } k = 2n-1 \end{cases} \qquad (n = 1, 2, \ldots).$$

By lemma 2, $\{v_k'\}$ and $\{v_k''\}$ are bases of the space E. We shall complete the proof by showing that at least one of these bases must be conditional.

Assume that both $\{v_k'\}$ and $\{v_k''\}$ are unconditional bases of E. Then, since $\{v_k'\}$ is unconditional, there exists a constant $K_1 \geqslant 1$ such that we have, for any scalars $a_1, a_2, \ldots, a_n$,

$$\Big\| \sum_{i=1}^{n} a_i x_{\sigma(2i)} \Big\| = \Big\| \sum_{i=1}^{n} a_i x_{\sigma(2i-1)} - \sum_{i=1}^{n} a_i (x_{\sigma(2i)} + x_{\sigma(2i-1)}) \Big\|$$
$$= \Big\| \sum_{i=1}^{n} a_i v_{2i-1}' - \sum_{i=1}^{n} a_i v_{2i}' \Big\| \geqslant K_1 \Big\| \sum_{i=1}^{n} a_i v_{2i-1}' \Big\|.$$
$$= K_1 \Big\| \sum_{i=1}^{n} a_i x_{\sigma(2i-1)} \Big\|.$$

Similarly, since $\{v_k''\}$ is unconditional, there exists a constant $K_2 \geqslant 1$ such that we have, for any scalars $a_1, a_2, \ldots, a_n$,

$$\Big\| \sum_{i=1}^{n} a_i x_{\sigma(2i-1)} \Big\| \geqslant K_2 \Big\| \sum_{i=1}^{n} a_i x_{\sigma(2i)} \Big\|.$$

Hence the basic sequences $\{x_{\sigma(2j-1)}\}$ and $\{x_{\sigma(2j)}\}$ are equivalent, which contradicts the construction (22) of the permutation $\{x_{\sigma(n)}\}$ and completes the proof of proposition 3.

§ 3. Proof of the theorem

We shall first prove the following proposition:

PROPOSITION 4. *Let E be a Banach space with a basis, in which all normalized bases are equivalent, and let $\{x_n\}$ be a normalized basis of E. Then*

(a) *$\{x_n\}$ is a symmetric basis.*

(b) *If $\{z_n\}$ is a normalized block basic sequence with respect to $\{x_n\}$, then we have the following implication:*

$$\sum_{i=1}^{\infty} a_i x_i \text{ is convergent} \Rightarrow \sum_{i=1}^{\infty} a_i z_i \text{ is convergent.}$$

(c) *We have the following implication:*

$$\sum_{i=1}^{\infty} a_i x_i \text{ is convergent} \Rightarrow \sum_{i=1}^{\infty} |a_i|^2 < +\infty.$$

(d) *E is reflexive.*

Proof. (a) Every sequence $\{\varepsilon_n x_n\}$ with $|\varepsilon_n| = 1$ $(n = 1, 2, \ldots)$ is a normalized basis of E. Hence, by our hypothesis, $\{x_n\}$ is equivalent to every sequence $\{\varepsilon_n x_n\}$ with $|\varepsilon_n| = 1$ $(n = 1, 2, \ldots)$. Consequently, $\sum_{i=1}^{\infty} a_i x_i$ is convergent if and only if $\sum_{i=1}^{\infty} \varepsilon_i a_i x_i$ is convergent for all $\{\varepsilon_n\}$ with $|\varepsilon_n| = 1$ $(n = 1, 2, \ldots)$, i. e. if and only if $\sum_{i=1}^{\infty} a_i x_i$ is unconditionally convergent. Thus $\{x_n\}$ is a normalized unconditional basis of E, whence every permutation $\{x_{\sigma(n)}\}$ of $\{x_n\}$ is a normalized basis of E. Since by our hypothesis the bases $\{x_n\}$ and $\{x_{\sigma(n)}\}$ must be equivalent, it follows by [28], that $\{x_n\}$ is a symmetric basis of E.

(b) Let

$$z_j = \sum_{i=m_{j-1}+1}^{m_j} a_i x_i, \quad \|z_j\| = 1 \quad (m_0 = 0;\ j = 1, 2, \ldots)$$

be an arbitrary normalized block basic sequence with respect to $\{x_n\}$ and let

$$v_k = \begin{cases} x_k & \text{for} \quad k \neq m_n \\ x_{m_n} + u_n & \text{for} \quad k = m_n \end{cases} \quad (n = 1, 2, \ldots),$$

where

$$u_n = \sum_{i=m_{n-1}+1}^{m_n - 1} a_i x_i = z_n - a_{m_n} x_{m_n} \quad (n = 1, 2, \ldots).$$

Then, since $\{x_n\}$ is a basis, there exists a constant $M \geqslant 1$ such that $\|u_n\| \leqslant M \|z_n\| \leqslant M$ $(n = 1, 2, \ldots)$, i. e. $\{v_k\}$ is a block perturbation of $\{x_n\}$, whence, by lemma 2, $\{v_k\}$ is a basis of the space E. Consequently, by our hypothesis, $\{v_k/\|v_k\|\}$ is equivalent to $\{x_n\}$, whence, by the assertion (a) proved above, $\{v_k/\|v_k\|\}$ is a symmetric basis. Since $\{v_k\}$ is equivalent to $\{v_k/\|v_k\|\}$ (because $1 \leqslant \|v_k\| \leqslant 1+M$, $k = 1, 2, \ldots$ and $\{v_k/\|v_k\|\}$ is unconditional), it follows that $\{v_k\}$ is a symmetric basis, equivalent to $\{x_n\}$.

Now, let $\{a_n\}$ be a sequence of scalars such that $\sum_{i=1}^{\infty} a_i x_i$ is convergent. Then, since $\{v_k\}$ is equivalent to $\{x_n\}$, $\sum_{i=1}^{\infty} a_i v_i$ is convergent, whence, since $\{v_k\}$ is a symmetric basis,

$$\sum_{i=1}^{\infty} a_i v_{m_i} = \sum_{i=1}^{\infty} a_i (x_{m_i} + u_i)$$

is convergent. On the other hand, since $\{x_n\}$ is a symmetric basis, $\sum_{i=1}^{\infty} a_i x_{m_i}$ is convergent. Consequently, $\sum_{i=1}^{\infty} a_i u_i$ is convergent. Furthermore, since

$$|a_{m_i}| = \|a_{m_i} x_{m_i}\| = \|z_i - u_i\| \leqslant 1+M \quad (i = 1, 2, \ldots),$$

and since $\{x_n\}$ is unconditional, the series $\sum_{i=1}^{\infty} a_{m_i} a_i x_i$ is convergent. Consequently, the series

$$\sum_{i=1}^{\infty} a_i z_i = \sum_{i=1}^{\infty} a_i u_i + \sum_{i=1}^{\infty} a_{m_i} a_i x_i$$

is convergent.

(c) By the assertion (a) proved above, $\{x_n\}$ is a symmetric basis. We may assume without loss of generality that $(E, \{x_n\})$ is a symmetric space (by introducing, if necessary, the equivalent norm $|||x|||$ defined by (1); the basis $\{x_n\}$ will remain normalized in this new norm).

Assume now that there exists a sequence of scalars $\{a_n\}$ for which $\sum_{i=1}^{\infty} a_i x_i$ is convergent but $\sum_{i=1}^{\infty} |a_i|^2 = +\infty$. Then there exists an increasing sequence of positive integers $\{m_n\}$ such that

$$\sum_{i=m_{n-1}+1}^{m_n} |a_i|^2 \geqslant 1 \quad (m_0 = 0;\ n = 1, 2, \ldots). \tag{25}$$

Let

$$p_n = m_n - m_{n-1} \quad (n = 1, 2, \ldots),$$

$$q_0 = 0, \quad q_n = \sum_{j=1}^{n} 2^{p_j} \quad (n = 1, 2, \ldots),$$

and let $E_{2^{p_n}}$ denote the 2^{p_n}-dimensional subspace of E spanned by $x_{q_{n-1}+1}, x_{q_{n-1}+2}, \ldots, x_{q_n}$ $(n = 1, 2, \ldots)$. Furthermore, let $\{y_j\}_{j=q_{n-1}+1}^{q_n}$ denote the Haar system and $\{r_i\}_{i=m_{n-1}+1}^{m_n}$ the Rademacher system in the symmetric space $(E_{2^{p_n}}, \{x_j\}_{j=q_{n-1}+1}^{q_n})$.

Since $\{x_n\}$ is a basis of E, we have (8) $E = \bigoplus_{n=1}^{\infty} E_{2^{p_n}}$. On the other hand, by proposition 1, $\{y_j\}_{j=q_{n-1}+1}^{q_n}$ is a monotone basis of $E_{2^{p_n}}$. Consequently, by [18], theorem 5, and [19], theorem 2, the sequence

$$\{y_j\} = \bigcup_{n=1}^{\infty} \{y_j\}_{j=q_{n-1}+1}^{q_n}$$

is a basis (9) of the space E. Since by our hypothesis the normalized bases $\{x_n\}$ and $\{y_n/\|y_n\|\}$ of E are equivalent and since $\sum_{i=1}^{\infty} a_i x_i$ is convergent, the series

$$\sum_{i=1}^{\infty} a_i \frac{y_i}{\|y_i\|}$$

is convergent. Let us put $z_i = r_i/\|r_i\|$ $(i = 1, 2, \ldots)$. Since by the corollary of lemma 1, the sequence $\{z_i\}$ is a normalized block basic sequence with respect to the normalized basis $\{y_n/\|y_n\|\}$, from the assertion (b) proved above it follows that the series $\sum_{i=1}^{\infty} a_i z_i$ is convergent, whence

$$\lim_{n\to\infty} \sum_{i=m_{n-1}+1}^{m_n} a_i z_i = 0. \tag{26}$$

On the other hand, by proposition 2 and by (25) we have

$$\Big\| \sum_{i=m_{n-1}+1}^{m_n} a_i z_i \Big\| \geqslant \tfrac{1}{8} \sqrt{\sum_{i=m_{n-1}+1}^{m_n} |a_i|^2} \geqslant \tfrac{1}{8},$$

which contradicts (26). This proves (c).

(d) Assume that E is non-reflexive. Then, since $\{x_n\}$ is an unconditional basis of E, there exists in E, by the results of R. C. James ([20], theorem 2, and [21], the proof of theorem 2) and A. Sobczyk ([29], theorem 5), either a complemented subspace isomorphic to c_0 or a complemented subspace isomorphic to l. Let F denote an arbitrary complementary subspace of such a subspace. Then we have either the isomorphisms $E \oplus c_0 \cong F \oplus c_0 \oplus c_0 \cong F \oplus c_0 \cong E$ or, similarly, the isomorphism $E \oplus l \cong E$. Hence, taking a conditional basis of c_0, respectively of l, we obtain a conditional basis of E, whence also a normalized conditional basis of E, which contradicts the assumption that all normalized bases in E are equivalent. This completes the proof of proposition 4.

(8) We recall that E is called (see e. g. [18]) the *direct sum* of its subspaces E_n, in symbols $E = \bigoplus_{n=1}^{\infty} E_n$, if for every $x \in E$ there exists a unique sequence w_n with $w_n \in E_n$ $(n = 1, 2, \ldots)$ such that $x = \sum_{n=1}^{\infty} w_n$.

(9) One can also give a simpler direct proof of this assertion.

Proof of the theorem. 1° Let E be an infinite-dimensional Banach space with a basis, such that all normalized bases of E are equivalent. Then, by proposition 4 (d), E is reflexive, whence E^* has a basis. Let $\{y_n^*\}$, $\{z_n^*\}$ be two normalized bases of E^*. Since E is reflexive, there exist bases $\{y_n\}$, $\{z_n\}$ of E such that $y_i^*(y_j) = z_i^*(z_j) = \delta_i^j$. Since $1 \leqslant \|y_j\|$, $\|z_j\| \leqslant M < +\infty$ $(j = 1, 2, \ldots)$ and since, by proposition 4 (a), $\{y_n\}$, $\{z_n\}$ are unconditional bases of E, $\{y_n\}$, $\{z_n\}$ are equivalent to $\{y_n/\|y_n\|\}$ and $\{z_n/\|z_n\|\}$ respectively. Since by our hypothesis $\{y_n/\|y_n\|\}$ and $\{z_n/\|z_n\|\}$ are equivalent, it follows that $\{y_n\}$ and $\{z_n\}$ are equivalent, whence $\{y_n^*\}$ and $\{z_n^*\}$ are also equivalent. Thus all normalized bases in E^* are equivalent.

Now, let $\{x_n\}$ be a normalized basis of E and let $\{x_n^*\} \subset E^*$, $x_i^*(x_j) = \delta_i^j$. Then, by the above arguments, $\{x_n^*\}$ is a basis of E^*, equivalent to the normalized basis $\{x_n^*/\|x_n^*\|\}$. Since all normalized bases in E^* are equivalent, it follows, by proposition 4 (c) applied to E^* and $\{x_n^*/\|x_n^*\|\}$, that we have the following implication:

$$\sum_{i=1}^{\infty} b_i x_i^* \text{ is convergent} \Rightarrow \sum_{i=1}^{\infty} |b_i|^2 < +\infty. \tag{27}$$

We shall now prove that $\{x_n\}$ is equivalent to the unit vector basis of l^2. By proposition 4 (c), it is sufficient to prove the implication

$$\sum_{i=1}^{\infty} |a_i|^2 < +\infty \Rightarrow \sum_{i=1}^{\infty} a_i x_i \text{ is convergent}. \tag{28}$$

Let $\{a_n\}$ be a sequence of scalars such that $\sum_{i=1}^{\infty} |a_i|^2 < +\infty$, and let $x^* = \sum_{i=1}^{\infty} b_i x_i^*$ be an arbitrary element of E^*. Then, by (27), we also have $\sum_{i=1}^{\infty} |b_i|^2 < +\infty$, whence we infer by the Schwartz inequality that the limit

$$\lim_{n\to\infty} x^*\Big(\sum_{i=1}^{n} a_i x_i\Big) = \lim_{n\to\infty} \sum_{i=1}^{n} a_i b_i$$

exists. Since E is reflexive, it follows that there exists an element $x \in E$ such that

$$\lim_{n\to\infty} x^*\Big(\sum_{i=1}^{n} a_i x_i\Big) = x^*(x) \quad \text{for all} \quad x^* \in E^*.$$

Hence, for $x^* = x_j^*$ $(j = 1, 2, \ldots)$, we obtain $a_j = x_j^*(x)$ $(j = 1, 2, \ldots)$. Consequently, since $\{x_n\}$ is a basis, the series

$$\sum_{i=1}^{\infty} a_i x_i = \sum_{i=1}^{\infty} x_i^*(x) x_i$$

is convergent, which proves (28).

Now, since $\{x_n\}$ is equivalent to the unit vector basis of l^2, the space E is isomorphic to l^2, whence, by the theorem of Babenko [3], E has a normalized conditional basis. However, this contradicts our hypothesis that in E all normalized bases are equivalent.

$2°$ Assume that all bases in E are unconditional and let $\{x_n\}$ be a normalized basis of E. Then all bases of the subspace $[x_{2j}]$ of E spanned by the sequence $\{x_{2j}\}$ are unconditional. Hence, by part $1°$ proved above, there exists a normalized unconditional basis $\{y_{2j}\}$ of $[x_{2j}]$ which is not equivalent to $\{x_{2j}\}$. Thus, either the basis $\{x_n\}$, or the basis $\{z_n\}$ of E, defined by

$$z_{2j-1} = x_{2j-1}, \qquad z_{2j} = y_{2j} \qquad (j = 1, 2, \ldots),$$

is a normalized non-symmetric unconditional basis of E. Therefore, by proposition 3, the space E has a conditional basis. However, this contradicts our hypothesis that in E all bases are unconditional. This completes the proof of the theorem.

§ 4. Remarks and unsolved problems

4.1. Remark 1. *In every infinite-dimensional Banach space E with a basis there exist a continuum of mutually non-equivalent normalized conditional bases.*

Proof. According to the theorem proved above, there exists in E a normalized conditional basis $\{x_n\}$. Then, since $\{x_n\}$ is conditional, there exist a sequence of scalars $\{a_n\}$ and an $x^* \in E^*$ such that $\sum_{i=1}^{\infty} a_i x_i$ is convergent but

$$\sum_{i=1}^{\infty} |x^*(a_i x_i)| = +\infty.$$

Let $\varepsilon_n = \operatorname{sign} x^*(a_n x_n)$ $(n = 1, 2, \ldots)$. Then

$$\Big\|\sum_{i=1}^{n} \varepsilon_i a_i x_i\Big\| \geqslant \frac{1}{\|x^*\|} \sum_{i=1}^{n} |x^*(a_i x_i)| \to +\infty \quad \text{for} \quad n \to +\infty.$$

Hence there exists an increasing sequence of positive integers $\{m_n\}$ with the following properties:

$$(29) \qquad \Big\|\sum_{i=m_{n-1}+1}^{l} a_i x_i\Big\| \leqslant \frac{1}{2^n} \quad (m_0 = 0;\ m_{n-1}+1 \leqslant l \leqslant m_n;\ n = 1, 2, \ldots),$$

$$(30) \qquad \Big\|\sum_{i=m_{n-1}+1}^{m_n} \varepsilon_i a_i x_i\Big\| \geqslant 1 \quad (n = 1, 2, \ldots).$$

Now, for each increasing sequence of positive integers $\{p_i\}$ let us define a normalized conditional basis $\{y_j^{\{p_i\}}\}$ of E by

$$(31) \qquad y_j^{\{p_i\}} = \begin{cases} \varepsilon_j x_j & \text{for } m_{p_n-1}+1 \leqslant j \leqslant m_{p_n}, \\ x_j & \text{for other } j \end{cases} \qquad (n = 1, 2, \ldots).$$

We claim that for $\{p_i'\}$ and $\{p_i''\}$ such that the set $(\{p_i'\}\setminus\{p_i''\}) \cup (\{p_i''\}\setminus\{p_i'\})$ is infinite ([10]), the bases $\{y_j^{\{p_i'\}}\}$ and $\{y_j^{\{p_i''\}}\}$ are not equivalent. In fact, assume that, say, $\{p_i'\}\setminus\{p_i''\} = \{p_{i_k}'\}$ is infinite (the treatment of the case where $\{p_i''\}\setminus\{p_i'\}$ is infinite is similar) and let

$$(32) \qquad \beta_j = \begin{cases} \varepsilon_j a_j & \text{for } m_{p_{i_k}'-1}+1 \leqslant j \leqslant m_{p_{i_k}'}, \\ 0 & \text{for other } j \end{cases} \qquad (k = 1, 2, \ldots).$$

Then, by (32) and (31) we have

$$\sum_{j=1}^{\infty} \beta_j y_j^{\{p_i'\}} = \sum_{j=1}^{\infty} \gamma_j x_j, \quad \text{where} \quad \gamma_j = \begin{cases} a_j & \text{for } m_{p_{i_k}'-1}+1 \leqslant j \leqslant m_{p_{i_k}'}, \\ 0 & \text{for other } j \end{cases} \quad (k = 1, 2 \ldots),$$

and, by (29), this series is convergent. On the other hand, by (32), (31) and the definition of $\{p_{i_k}'\}$, we have

$$\sum_{j=1}^{\infty} \beta_j y_j^{\{p_i''\}} = \sum_{k=1}^{\infty} \delta_j x_j, \quad \text{where} \quad \delta_j = \begin{cases} \varepsilon_j a_j & \text{for } m_{p_{i_k}'-1}+1 \leqslant j \leqslant m_{p_{i_k}'}, \\ 0 & \text{for other } j \end{cases} \quad (k = 1, 2, \ldots),$$

whence, by (30) and since $\{p_{i_k}'\}$ is infinite, this series is divergent. Thus

([10]) The symbol $\{p_i\}$ denotes the set of all elements of the sequence $\{p_i\}$.

the bases $\{y_j^{(p_i')}\}$ and $\{y_j^{(p_i'')}\}$ are not equivalent. Since there exists a continuum of increasing sequences of positive integers $\{p_i\}$ such that for $\{p_i'\} \neq \{p_i''\}$ even both $\{p_i'\} \setminus \{p_i''\}$ and $\{p_i''\} \setminus \{p_i'\}$ are infinite ([11]), the corresponding collection of bases $\{y_j^{(p_i)}\}$ of E has the property required in remark 1. This completes the proof.

4.2. For conditional bases the usual condition of being equivalent is too strong, since a transformation of the form $y_n = \lambda_n x_n$ $(n = 1, 2, \ldots)$, where

$$0 < \inf_n |\lambda_n| \leqslant \sup_n |\lambda_n| < +\infty, \tag{33}$$

leads to a basis $\{y_n\}$, which in general is not equivalent to the basis $\{x_n\}$. Therefore, the following less restrictive condition of "affine equivalence" seems to be useful:

Definition 5. We shall say that a basis $\{x_n\}$ of a Banach space E is *affinely equivalent* to a basis $\{y_n\}$ of a Banach space F if there exists a sequence of scalars $\{\lambda_n\}$, $\lambda_n \neq 0$ $(n = 1, 2, \ldots)$ such that $\{x_n\}$ is equivalent to the basis $\{\lambda_n y_n\}$ of the space F in the usual sense, i. e. such that

$$\sum_{i=1}^{\infty} a_i x_i \text{ is convergent if and only if } \sum_{i=1}^{\infty} \lambda_i a_i y_i \text{ is convergent.}$$

Problem 1. Do there exist, in every infinite-dimensional Banach space with a basis, two bases which are not affinely equivalent?

4.3. Bari [6] and Gelfand [17] have proved that in the space l^2 all normalized unconditional bases (and hence all normalized unconditional basic sequences) are equivalent.

Remark 2. *Let E be an infinite-dimensional Banach space with an unconditional basis, in which all normalized unconditional basic sequences are equivalent. Then E is isomorphic to l^2.*

Proof. Let $\{x_n\}$ be a normalized unconditional basis of E. Then, by a theorem of Dvoretzky [10], there exist an increasing sequence of positive integers $\{m_n\}$, a sequence $\{E_n\}$ of subspaces of E with

$$\dim E_n = n, \qquad E_n \subset [x_j]_{j=m_{n-1}+1}^{m_n} \qquad (m_0 = 0;\ n = 1, 2, \ldots) \tag{34}$$

([11]) In fact, let φ be a one to one mapping of $N = \{1, 2, 3, \ldots\}$ onto the set of all rational numbers. Take, for each real number a a sequence of rational numbers $\{q_n^{(a)}\}$ such that $\lim_{n\to+\infty} q_n^{(a)} = a$ and let $p_n^{(a)} = \varphi^{-1}(q_n^{(a)})$ $(n = 1, 2, \ldots)$. Then the collection of all sequences $\{p_i^{(a)}\}$ has the required properties.

and a normalized basis $\{z_j\}_{j=p_{n-1}+1}^{p_n}$ of E_n, where $p_k = \frac{1}{2}(k+1)k$ $(k = 0, 1, 2, \ldots)$ such that we have

$$\Big\|\sum_{i=p_{n-1}+1}^{p_n} a_i z_i\Big\| \leqslant \sqrt{\sum_{i=p_{n-1}+1}^{p_n} |a_i|^2} \leqslant 2\Big\|\sum_{i=p_{n-1}+1}^{p_n} a_i z_i\Big\| \tag{35}$$

for any scalars $a_{p_{n-1}+1}, a_{p_{n-1}+2}, \ldots, a_{p_n}$ $(n = 1, 2, \ldots)$.

We claim that the sequence

$$\{z_j\} = \bigcup_{n=1}^{\infty} \{z_j\}_{j=p_{n-1}+1}^{p_n}$$

is an unconditional basic sequence. In fact, let $\{a_n\}$, $\{\lambda_n\}$ be two sequences of scalars, with $|\lambda_n| \leqslant 1$ $(n = 1, 2, \ldots)$. Let

$$y_n = \sum_{i=p_{n-1}+1}^{p_n} \lambda_i a_i z_i \qquad (n = 1, 2, \ldots).$$

Then, by (34), $\{y_n\}$ is a block basic sequence with respect to $\{x_n\}$, whence, by a remark of [7], $\{y_n\}$ is an unconditional basic sequence. Hence, by our hypothesis, the basic sequence $\{y_n/\|y_n\|\}$ is equivalent to $\{x_n\}$, and thus there exist two constants $A, B > 0$ such that we have

$$\begin{aligned} A\Big\|\sum_{n=1}^{l} y_n\Big\| &= A\Big\|\sum_{n=1}^{l} \|y_n\| \frac{y_n}{\|y_n\|}\Big\| \\ &\leqslant \Big\|\sum_{n=1}^{l} \|y_n\| x_n\Big\| \leqslant B\Big\|\sum_{n=1}^{l} y_n\Big\| \qquad (l = 1, 2, \ldots). \end{aligned} \tag{36}$$

On the other hand, by (35) and $|\lambda_i| \leqslant 1$ $(i = 1, 2, \ldots)$ we have

$$\begin{aligned} \|y_n\| &\leqslant \sqrt{\sum_{i=p_{n-1}+1}^{p_n} |\lambda_i a_i|^2} \\ &\leqslant \sqrt{\sum_{i=p_{n-1}+1}^{p_n} |a_i|^2} \leqslant 2\Big\|\sum_{i=p_{n-1}+1}^{p_n} a_i z_i\Big\| \qquad (n = 1, 2, \ldots). \end{aligned}$$

Hence, since $\{x_n\}$ is an unconditional basis, there exists a constant $C \geqslant 1$ such that

$$\begin{aligned} \Big\|\sum_{i=1}^{p_l} \lambda_i a_i z_i\Big\| &= \Big\|\sum_{n=1}^{l} y_n\Big\| \leqslant \frac{1}{A}\Big\|\sum_{n=1}^{l} \|y_n\| x_n\Big\| \\ &\leqslant \frac{2C}{A}\Big\|\sum_{n=1}^{l}\Big\|\sum_{i=p_{n-1}+1}^{p_n} a_i z_i\Big\| x_i\Big\| \qquad (l = 1, 2, \ldots). \end{aligned}$$

Thus, by (36) (applied for $\lambda_i = 1$, $i = 1, 2, \ldots$), we have

$$\Big\|\sum_{i=1}^{p_l}\lambda_i a_i z_i\Big\| \leqslant \frac{2BC}{A}\Big\|\sum_{n=1}^{l}\sum_{i=p_{n-1}+1}^{p_n} a_i z_i\Big\| = \frac{2BC}{A}\Big\|\sum_{i=1}^{p_l} a_i z_i\Big\| \quad (l = 1, 2, \ldots),$$

which proves that $\{z_j\}$ is an unconditional basic sequence.

Since by our hypothesis all normalized unconditional basic sequences in E are equivalent, it follows that the basic sequence $\{z_j\}$ is symmetric, whence there exists a constant $K \geqslant 1$ such that

$$\Big\|\sum_{i=1}^{m} a_i z_{q_i}\Big\| \leqslant K \Big\|\sum_{i=1}^{m} a_i z_{r_i}\Big\| \tag{37}$$

for any couple of increasing sequences of positive integers $\{q_i\}$, $\{r_i\}$ and any scalars $a_1, a_2, \ldots, a_m$ $(m = 1, 2, \ldots)$. This, together with (35) and $p_n - p_{n-1} = n$ $(n = 1, 2, \ldots)$ gives

$$\Big\|\sum_{i=1}^{n} a_i z_i\Big\| \leqslant K\Big\|\sum_{i=1}^{n} a_i z_{p_{n-1}+i}\Big\| \leqslant K\sqrt{\sum_{i=1}^{n}|a_i|^2} \leqslant 2K\Big\|\sum_{i=1}^{n} a_i z_{p_{n-1}+i}\Big\| \leqslant 2K^2\Big\|\sum_{i=1}^{n} a_i z_i\Big\|$$

for any scalars $a_1, a_2, \ldots, a_n$ $(n = 1, 2, \ldots)$, and thus the basic sequence $\{z_n\}$ is equivalent to the unit vector basis of l^2. Since by our hypothesis $\{x_n\}$ is equivalent to $\{z_n\}$, it follows that $\{x_n\}$ is equivalent to the unit vector basis of l^2, which completes the proof.

Problem 2. Let E be an infinite-dimensional Banach space with an unconditional basis, in which all normalized unconditional basic sequences are c-equivalent. Is E isomorphic to l^2?

We recall that two basic sequences $\{y_n\}$, $\{z_n\}$ are said to be *c-equivalent* [26] if there exists a permutation σ of $N = \{1, 2, 3, \ldots\}$ such that the basic sequences $\{y_n\}$, $\{z_{\sigma(n)}\}$ are equivalent.

Problem 3. Let E be an infinite-dimensional Banach space, non-isomorphic to l^2 and having an unconditional basis. Do there exist in E any two non-equivalent normalized unconditional bases?

In [26] it has been proved that the answer is affirmative for $E = l^p$ and $E = L^p$, with $1 < p \neq 2$, and it has been remarked that the answer is not known for $E = c_0$ and $E = l$. From the proof of proposition 4 (d) above it follows that an affirmative answer for $E = c_0$ and $E = l$ would imply an affirmative answer for all non-reflexive Banach spaces having an unconditional basis.

4.4. The following extension of some definitions of Bari [6] seems to be useful:

Definition 6. We shall say that a basic sequence $\{z_n\}$ in a Banach space E is *Besselian* if

$$\sum_{i=1}^{\infty} a_i z_i \text{ is convergent} \Rightarrow \sum_{i=1}^{\infty} |a_i|^2 < +\infty;$$

we shall say that the basic sequence $\{z_n\}$ is *Hilbertian* if

$$\sum_{i=1}^{\infty} |a_i|^2 < +\infty \Rightarrow \sum_{i=1}^{\infty} a_i z_i \text{ is convergent},$$

Remark 3. *In every infinite-dimensional Banach space E there exist two normalized basic sequences $\{y_n\}$, $\{z_n\}$ such that $\{y_n\}$ is non-Besselian and $\{z_n\}$ is non-Hilbertian.*

In fact, this can be proved by a method similar to that used by V. I. Gurariĭ in [19].

Problem 4. Does there exist in every infinite-dimensional Banach space with a basis, a normalized non-Besselian basis? Does there exist in every such space a normalized non-Hilbertian basis?

4.5. Problem 5. Let $(E, \{x_j\})$ be an infinite-dimensional symmetric space which admits a constant $C \geqslant 1$ such that for any 2^n-dimensional subspace E_{2^n} of E spanned by 2^n elements $\{x_{j_k}\}_{k=1}^{2^n} \subset \{x_j\}$ $(n = 1, 2, \ldots)$ the symmetric constant of the corresponding Haar system $\{y_{j_k}\}_{k=1}^{2^n}$ in $(E_{2^n}, \{x_{j_k}\}_{k=1}^{2^n})$ is $\leqslant C$. Is E then isomorphic to l^2?

We call the *symmetric constant* of a symmetric basic sequence $\{z_j\}$ in a Banach space E the least constant $K \geqslant 1$ for which (37) is satisfied.

4.6. Dynin and Mitiagin [11], [25] have proved that in an F-space (i. e. a complete metrizable locally convex space) which is nuclear all bases (and hence all basic sequences) are unconditional.

Problem 6. Let E be an F-space in which all basic sequences are unconditional. Is E nuclear?

An affirmative answer to the following problem would constitute a natural extension to F-spaces of the second assertion of our theorem:

Problem 7. Let E be an F-space with a basis, in which all bases are unconditional. Is E nuclear?

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Reçu par la Rédaction le 28. 12. 1963