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A Liouville algebra of non-entire functions

by

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1. Introduction. The results of [2] do not answer this question: Does there exist a Liouville F-algebra A of functions on the complex plane which is strictly larger than the algebra $\mathrm{Hol}(C)$ of entire functions? The purpose of this paper is to exhibit such an algebra.

A Liouville F-algebra is a metrizable complete locally multiplicatively convex topological algebra in which the only elements with bounded spectra are the constants. Singly-generated (polynomials in a single element are dense) Liouville F-algebras of continuous functions on the plane are always topologically and algebraically isomorphic to $\mathrm{Hol}(C)$ with the compact-open topology (cf. Theorem 3.5, [2]). The example which we construct is singly-generated. This leads to the conclusion that certain discontinuous functions on the plane are in this algebra. We do not show the existence of a Liouville F-algebra (not singly-generated!) of *continuous* functions on the complex plane which is strictly larger than $\mathrm{Hol}(C)$. (It is an open question as to whether or not such a Liouville F-algebra exists.)

2. The construction. We begin by describing a sequence of subsets M_m of the plane which satisfy:

(i) $M_{m+1} \supset M_m$ for all m,

(ii) each M_m is compact and non-separating,

(iii) $\bigcup_{m=1}^{\infty} M_m = C$, the complex numbers,

(iv) for each k, $\bigcap_{m=k}^{\infty} bd\, M_m = \{(0,y)\colon 0 \leqslant y \leqslant k\}$.

To this end, for each pair of positive integers m, n let

$$a_{m,n} = \left(\frac{2n+1}{2n(n+1)},\, m\right)$$

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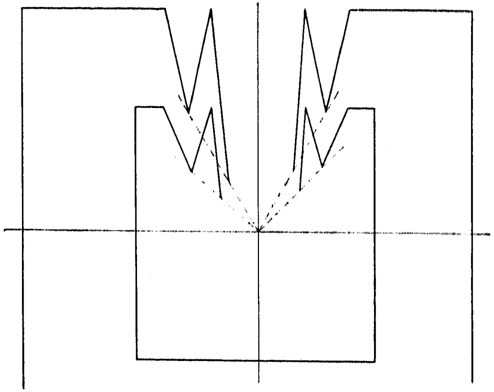
if n is odd and

$$a_{m,n} = \left(\frac{1}{n}, \frac{m}{n}\right)$$

if n is even. For each m, let B_m denote the broken line segments formed by joining $a_{m,n}$ to $a_{m,n+1}$ for every n, and let B'_m denote the reflection of B_m about the y-axis. Let M_m be the bounded closed subset of the plane bounded by: $B_m, B'_m, \{(x, m): 1 \leqslant x \leqslant m$ or $-m \leqslant x \leqslant -1\}$, $\{(0, y): 0 \leqslant y \leqslant m\}$, $\{(\pm m, y): -m \leqslant y \leqslant m\}$ and $\{(x, -m): -m \leqslant x \leqslant m\}$. It is evident that the sequence of subsets M_m so constructed satisfies properties (i)-(iv) (see figure below).



Now, for each m, let A_m be the uniform closure on M_n of the polynomials in z. By a well-known theorem of Mergelyan, each A_m is the commutative Banach algebra of all continuous functions on M_m which are analytic on the interior of M_m. Let $\pi^i_j: A_i \to A_j$ be the algebra isomorphism given by $\pi^i_j(f_i) = f_i \mid M_j$, $f \in A_i$, for $i \geqslant j$; $i, j = 1, 2, \ldots$ ($f \mid M_j$ denotes the restriction of f in A_i to the set M_j). The collection $\{A_i; \pi^i_j\}$ determines a dense inverse limit system; that is,

(i) $\pi^i_k = \pi^i_j \circ \pi^j_k$, $i \geqslant j \geqslant k$; $i, j, k = 1, 2, \ldots,$

(ii) $\pi^i_j(A_i)$ is dense in A_j, $i \geqslant j$; $i, j = 1, 2, \ldots$

Let A be the inverse limit of this system. Write π_i for the canonical projection of A into A_i, $i = 1, 2, \ldots$ In the terminology of Arens [1], A is a strongly dense inverse limit of the Banach algebras A_m. Thus A is a singly-generated F-algebra. The topology of A is determined by the norms $\|\cdot\|_m$ on A given by

$$\|f\|_m = \|\pi_m(f)\|_\infty \quad \text{for} \quad f \in A.$$

In order to realize A as an algebra of functions on the plane, we use the usual Gelfand representation of A on its closed maximal ideal space (or space of non-zero continuous complex-valued multiplicative functionals) M. Each maximal ideal of A_m is given by point-evaluation at a point of M_m, and the closed maximal ideals of A can likewise be identified with point-evaluations ϱ_z as follows:

$$\varrho_z(f) = (\pi_k f)(z) \quad \text{where} \quad z \in M_k \text{ and } f \in A.$$

Consequently, M can be identified set-wise with C and f with the function, also denoted by f, given by

$$f(z) = \varrho_z(f) \quad \text{for} \quad z \in C.$$

This provides an isomorphic representation of A as an algebra of functions on C. Thus, A is isomorphic to the algebra of all complex-valued functions f such that $f \mid M_m \in A_m$ for all m.

$M = C$ is topologized with the weak* topology determined by A, the weakest topology rendering all functions in A continuous. Since each M_m is compact, the relative weak* topology on M_m agrees with the usual Euclidean topology on M_n as a subset of the complex plane.

It will be convenient to consider another topology on $M = C$, which we call the δ-topology of C. A set $U \subset C$ is δ-open if and only if $U \subset M_m$ is relatively weak* open for each $m = 1, 2, \ldots$ Trivially, the δ-topology is stronger than the weak* topology.

3. Main results. The following lemmas will be established:

LEMMA 1. *The δ-topology on C is strictly stronger than the Euclidean topology on C.*

LEMMA 2. *There exist functions in A which are discontinuous when C is topologized with the Euclidean topology.*

LEMMA 3. *Every non-constant function in A is unbounded.*

From these lemma and our preliminary remarks we have

THEOREM. *A is a singly generated Liouville F-algebra strictly larger than the algebra* $\mathrm{Hol}(C)$ *of entire functions on the complex plane.*

4. Proofs of the lemmas. Proof of lemma 1 [1]. Let z_0 be any point on the positive y-axis and let S be an open Euclidean neighborhood of z_0. Choose a sequence $\{z_m\}$ such that

(i) $z_m \to z_0$ as $m \to \infty$,

(ii) $z_m \in M_m - M_{m-1}$ for each $m = 1, 2, \ldots$

From our construction, this choice is clearly possible. Let $S^* = S - \{z_m: m = 1, 2, \ldots\}$. Then $S^* \cap M_m = (S - \{z_k: k = 1, 2, \ldots, m\}) \cap M_m$ is

[1] We wish to thank Professor L. B. Treybig for this concise version of the proof.

relatively weak* open in each M_m. Hence z_m does not converge to z_0 in the δ-topology. Therefore, the δ-topology on C is strictly stronger than the Euclidean topology on C.

Proof of lemma 2. Suppose every function in A is continuous when C is given the Euclidean topology. Then, since basic weak* neighborhoods are of the form:

$$V_{z_0} = \{z \in C\colon |f_i(z) - f_i(z_0)| < \varepsilon,\ \varepsilon > 0,\ f_i \in A,\ 1 \leqslant i \leqslant n\},$$

every weak* neighborhood of z_0 is a Euclidean neighborhood. Conversely, every Euclidean neighborhood is a weak* neighborhood, since the identity function $(z \to z)$ is in A. Thus the weak* topology and Euclidean topology agree on C. It is well-known that the topology of a locally compact space is determined by the family of compact sets in the sense that a set is open if and only if its intersection with every compact set is relatively open. Even better, C is hemi-compact in the weak* topology: the sequence of weak* compact sets M_m has the property that any compact set is contained in some M_m. See Michael [3], p. 22. These facts combine to establish the equivalence of the weak* and δ-topologies of C, thereby producing a contradiction of Lemma 1.

Proof of lemma 3. Suppose there exists a bounded non-constant function f in A. Under this assumption we show f must be continuous at each point of the positive y-axis. But by construction f is analytic everywhere off the positive y-axis. So f is analytic and bounded in the entire plane and therefore constant.

Choose a closed sphere S centered at the origin of radius r. Let S_1 and S_2 denote, respectively, the intersection of S with the closed right and left half planes. $f \mid S_k$ is bounded and analytic on the interior of S_k, $k = 1, 2$. Consequently, $f \mid \operatorname{int}(S_k)$ has almost everywhere boundary values $F_k(z)$. Now $f \mid M_m$ is continuous and every point on the positive y-axis can be approached from within M_m for any fixed m, so $F_1(iy) = F_2(iy)$ for almost every y such that $0 \leqslant y < r$. Furthermore, $f \mid bd(S_k)$ is continuous with the possible exception of the point ir and agrees almost everywhere with F_k. Therefore $f \mid \operatorname{int} S_k$ can be recovered from $f \mid Bd\, S_k$. Finally, the fact that $f \mid \{iy\colon 0 \leqslant y < r\}$ is continuous implies the continuity of f at all points iy with $0 \leqslant y < r$. (More precisely, the argument can be carried through first by mapping S_k conformally onto the unit disc.)

5. Concluding remarks. Our theorem strengthens the results of [2] by showing that, in the absence of condition Φ of that paper, Liouville algebras strictly larger than $\mathrm{Hol}(C)$ can be constructed. It would be interesting to have explicitly constructed a function in A which is not entire.

References

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