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On Fourier transforms of distributions with one-sided bounded carriers

by

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Ehrenpreis [2], [3], Gelfand and Šilov [4] have defined Fourier transforms of all distributions as functionals on a space of entire functions. The space D′ of these functionals is provided with a topology to make the Fourier transformation a topological isomorphism. By a generalization of the Paley-Wiener theorem (see [4]), Fourier transforms of distributions with compact carriers are entire functions of exponential type, slowly increasing for real values of the arguments. Moreover, the convolution $T*S$ of an arbitrary distribution T and a distribution S with compact carrier is transformed into a product FG, which is defined in D′ in a natural way.

However, if T and S are distributions of one variable and both have, for example, left-sided bounded carriers, the convolution $T*S$ (in $\mathscr{D}'$) still exists. The problem now arrises to define the corresponding operation between elements of D′ and to characterize all those elements, which are Fourier transforms of distributions with left-sided bounded carriers.

In the case of square integrable functions, Fourier transforms of functions with left-sided bounded carriers are extendable analytically to the upper half-plane (see [9]). A similar result was proved (by Bogolubov and Parasiuk [1], Lions [5], Mikusiński [6]) for tempered distributions.

The purpose of this paper is to characterize Fourier transforms of distributions and infinitely differentiable functions with left-sided bounded carriers in the general case by means of a semi-regularity property, and to introduce in the corresponding sets D'_+ and D_+ a multiplication, which we prove corresponds to the convolution of the original distributions or functions.

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§ 1. Preliminary notions. We denote by C the plane of complex numbers and by R its real line. For any $u \in C$ we write $u = \xi + i\eta$, where ξ and η are the real and imaginary parts of u.

All functions under consideration are complex valued. A function f on R is *slowly increasing*, if $f(\xi) = O(|\xi|^k)$ as $|\xi| \to \infty$, for some k; f is *rapidly decreasing*, if $|\xi|^k f(\xi) = o(1)$ as $|\xi| \to \infty$, for every k. The *carrier* of a continuous function f on R is the closure of the set of points, where f is different from zero.

Throughout the paper we take as known the general notions and fundamental theorems of the theory of distributions [7]. We denote by $\mathscr{E}$ the space of all infinitely differentiable functions on R, by $\mathscr{D}$, $\mathscr{D}_-$ and $\mathscr{D}_+$ its subspaces formed of functions with compact, right-sided bounded and left-sided bounded carriers respectively. The topology in each case is that introduced in [7]. $\mathscr{D}$ is the strict inductive limit of (F)-spaces $\mathscr{D}_{(-l,l)}$, consisting of infinitely differentiable functions, whose carriers are contained in the interval $(-l, l)$. Similarly, $\mathscr{D}_-$ is the strict inductive limit of spaces $\mathscr{E}_{(-\infty,l)}$ of infinitely differentiable functions vanishing for $x < l$.

The corresponding strong dual spaces are $\mathscr{E}'$, $\mathscr{D}'$, etc.; $\mathscr{E}'$ is the space of distributions of compact carriers, $\mathscr{D}'$ the space of all distributions and $\mathscr{D}'_+$ the space of distributions with left-sided bounded carriers. The carrier of a distribution T is the complement of the largest open set Ω, such that $T(\varphi) = 0$ for every function $\varphi \in \mathscr{D}$ with carrier $\subset \Omega$. By $\mathscr{E}'_{(a,b)}$ we denote the space of distributions, whose carriers are contained in (a, b). We write e. g. $\mathscr{D}'_x$ or $\mathscr{E}'_x$, when considering spaces of distributions of the "variable" x.

The value of the functional T on φ is often denoted by $T \cdot \varphi$ instead of $T(\varphi)$. We also adopt the notation $\check{T}(\varphi) = T(\check{\varphi})$, where $\check{\varphi}(x) = \varphi(-x)$.

For any function $\varphi \in \mathscr{D}$, the Fourier transform $\Phi = \mathscr{F}\{\varphi\}$ is defined by

$$\Phi(u) = \int_{-\infty}^{\infty} \varphi(x) e^{2\pi i x u} dx.$$

$\Phi(u)$ is an entire function of exponential type, rapidly decreasing on R. Conversely, if $\Phi(u)$ has the latter properties, then it is the Fourier transform of a function $\varphi \in \mathscr{D}$, given by Fourier's inversion formula

$$\varphi(x) = \int_{-\infty}^{\infty} \Phi(\xi) e^{-2\pi i x \xi} d\xi;$$

we write briefly $\varphi = \mathscr{F}^{-1}\{\Phi\}$.

In order to define Fourier tranforms of all distributions we proceed in the following way. Let us denote by D the space of Fourier transforms of functions $\varphi \in \mathscr{D}$, provided with a topology making the Fourier transformation a topological isomorphism (see [2] and [3]). Then, for any distribution T, the Parseval equation

(1) $$F \cdot \Phi = T \cdot \check{\varphi},$$

where $\varphi \in \mathscr{D}$ and $\Phi = \mathscr{F}\{\varphi\}$, determines F as a continuous linear functional on D. We call F the *Fourier transform* of T and we write $F = \mathscr{F}\{T\}$.

Conversely, given the functional F, one can use equation (1) to define the distribution $T = \mathscr{F}^{-1}\{F\}$. Thus the strong dual D′ of D is the space of Fourier transforms of distributions.

A continuous function f on R, slowly increasing at infinity, determines an element $\hat{f} \in \mathscr{D}'$ by equation

$$\hat{f} \cdot \Phi = \int_{-\infty}^{\infty} f(\xi) \Phi(\xi) d\xi;$$

we identify $\hat{f}$ with f.

Fourier transforms of distributions with compact carriers are entire functions of exponential type, slowly increasing on R. The space of these functions, with the topology introduced in [3], will be denoted by E′.

We also use infinitely differentiable functions and distributions of two variables together with their Fourier transforms. The corresponding spaces are $\mathscr{D}_{x,y}$, $\mathscr{D}'_{x,y}$, $\mathrm{D}_{u,v}$ and $\mathrm{D}'_{u,v}$.

§ 2. The tensor product and semi-regularity. The *tensor product* $F_u \otimes G_v$ of elements $F \in \mathrm{D}'_u$ and $G \in \mathrm{D}'_v$ is an element of $\mathrm{D}'_{u,v}$ defined by equation

(2) $$F_u \otimes G_v \cdot \Phi(u)\Psi(v) = F(\Phi) G(\Psi),$$

where $\Phi \in \mathrm{D}_u$ and $\Psi \in \mathrm{D}_v$. Its existence and unicity can be proved similarly as for distributions (see [7], vol. I).

We denote by $F_{u+v} \otimes G_v$ the element of $D'_{u,v}$ obtained from the tensor product $F_u \otimes G_v$ by means of the substitution $u \to u+v$, $v \to v$. In other words,

(3) $$F_{u+v} \otimes G_v \cdot \Theta(u, v) = F_u \otimes G_v \cdot \Theta(u-v, v),$$

for every $\Theta \in \mathrm{D}_{u,v}$. The definition (3) is consistent, because the substitution $u \to u-v$, $v \to v$ is a continuous operation in $\mathrm{D}_{u,v}$.

Similarly, for $T \in \mathscr{D}_x$ and $S \in \mathscr{D}_y$, $T_x \otimes S_{y-x}$ denotes the distribution obtained from the tensor product $T_x \otimes S_y$ by the substitution $x \to x$, $y \to y-x$.

If F_u and G_v are Fourier transforms of the distributions T_x and S_y respectively, then

(4) $$F_u \otimes G_v = \mathscr{F}\{T_x \otimes S_y\},$$

where the Fourier transformation $\mathscr{F}$ is understood in two dimensions.

LEMMA 1. *If F_u and G_v are Fourier transforms of distributions T_x and S_y respectively, then*

(5) $$F_{u+v} \otimes G_v = \mathscr{F}\{T_x \otimes S_{y-x}\}.$$

Proof. Let $\vartheta(x,y)$ be any function of $\mathcal{D}_{x,y}$ and $\Theta(u,v)$ its Fourier transform. Then

$$\Theta(u-v,v) = \iint\limits_{-\infty}^{\infty} \vartheta(x,y)e^{2\pi i(ux-vx+vy)}dx\,dy$$

$$= \iint\limits_{-\infty}^{\infty} \vartheta(x,y+x)e^{2\pi i(ux+vy)}dx\,dy = \mathcal{F}\{\vartheta(x,y+x)\}.$$

Hence, in view of (3) and (4), we obtain

$$F_{u+v}\otimes G_v\cdot\Theta(u,v) = F_u\otimes G_v\cdot\Theta(u-v,v)$$

$$= T_x\otimes S_y\cdot\check{\vartheta}(x,y+x) = T_x\otimes S_{y-x}\cdot\check{\vartheta}(x,y),$$

and so the correspondence (5).

In what follows the elements of $\mathrm{D}'_{u,v}$ are called *kernels*. If K is a kernel and $\Psi\in\mathrm{D}_v$, then

$$I = K\cdot\Psi$$

is an element of D'_u defined by equation

$$I\cdot\Phi = K\cdot\Phi\Psi$$

where $\Phi\in\mathrm{D}_u$.

Definition. A kernel K is said to be *semi-regular* in u, if

$$\Psi\to K\cdot\Psi \tag{6}$$

is a continuous mapping of D_v into E'_u. If (6) transforms continuously D_v into D_u, then K is called *strongly semi-regular* in u [1].

In this paper we are concerned solely with kernels of the form $K_{u,v} = F_{u+v}\otimes G_v$.

§ 3. Fourier transforms of distributions and infinitely differentiable functions with left-sided bounded carriers. We denote by D'_+ and D_+ the subsets of D' formed of those elements, which are Fourier transforms of distributions and infinitely differentiable functions with left-sided bounded carriers respectively.

For the characterization we need the element $H\in\mathrm{D}'$ defined by equation

$$H\cdot\Phi = -\int\limits_{-\infty+ci}^{\infty+ci}\frac{\Phi(u)}{2\pi iu}\,du \tag{7}$$

[1] The notion of semi-regularity is similar to that introduced by L. Schwartz for distributions (see [7] and [8]).

where $\Phi\in\mathrm{D}$ and $c>0$. It is an easy matter to verify that H is the Fourier transform of Heaviside unit function

$$h(x) = \begin{cases} 0 & \text{for} \quad x<0, \\ 1 & \text{for} \quad x\geqslant 0, \end{cases}$$

and therefore $H\in\mathrm{D}'_+$.

THEOREM 1. *For any two elements $F, G\in\mathrm{D}'_+$, the kernel $F_{u+v}\otimes G_v$ is semi-regular in u; moreover, if $F\in\mathrm{D}_+$, $F_{u+v}\otimes G_v$ is strongly semi-regular in u.*

Conversely, if $F\in\mathrm{D}'$ and the kernel $F_{u+v}\otimes H_v$, with the element H defined by (7), is semi-regular or strongly semi-regular in u, then $F\in\mathrm{D}'_+$ or $F\in\mathrm{D}_+$ respectively.

Proof. We prove the part of theorem 1, which relates to the semi-regularity of the kernels in question; the proof of the remaining part is similar.

Let F and G be Fourier transforms of the distributions T and S respectively. Then, by virtue of lemma 1,

$$F_{u+v}\otimes G_v\cdot\Psi(v) = \mathcal{F}\{T_x\otimes S_{y-x}\cdot\check{\psi}(y)\}, \tag{8}$$

where $\Psi\in\mathrm{D}$ and $\psi = \mathcal{F}^{-1}\{\Psi\}$. This correspondence shows that the kernel $F_{u+v}\otimes G_v$ is semi-regular in u if and only if

$$\psi\to T_x\otimes S_{y-x}\cdot\psi(y) \tag{9}$$

is a continuous mapping of $\mathcal{D}_y$ into $\mathcal{E}'_x$.

The distribution on the right-hand side of (9) may be represented as a product of T with the infinitely differentiable function $\check{S}*\psi$. If now $T=S=0$ for $x<a<0$, then

$$\psi\to\check{S}*\psi \tag{10}$$

transforms continuously $\mathcal{D}_{(-l,l)}$ into $\mathcal{E}_{(-\infty,l-a)}$, and also

$$\vartheta\to\vartheta T \tag{11}$$

transforms continuously $\mathcal{E}_{(-\infty,l-a)}$ into $\mathcal{E}'_{(a,l-a)}$. Therefore the mapping (9), being compound of (10) and (11), transforms continuously $\mathcal{D}_{(-l,l)}$ into $\mathcal{E}'_{(a,l-a)}$ for every positive l. Hence it follows that (9) is a continuous linear mapping of $\mathcal{D}$ into $\mathcal{E}'$.

Conversely, let ψ be a function of $\mathcal{D}$ such that $\psi(x)=0$ for $x<a$ and $\int\limits_{-\infty}^{\infty}\psi(x)\,dx = 1$. Then

$$\tilde{\vartheta}(x) = (\check{h}*\psi)(x) = \int\limits_x^{\infty}\psi(y)\,dy$$

is a function of $\mathscr{D}_-$ and equals 1 for $x < a$. Consequently, for any distribution T, condition $\tilde{\vartheta}T \in \mathscr{E}'$ implies that $T \in \mathscr{D}'_+$, which is the desired result.

We use the notation

$$\Gamma(F, G, \Psi; u) = F_{u+v} \otimes G_v \cdot \Psi(v),$$

where $F, G \in \mathrm{D}'_+$ and $\Psi \in \mathrm{D}$. For fixed F, G and Ψ, $\Gamma(u) = \Gamma(F, G, \Psi; u)$ is a function of E'_u.

Remark. The following conditions are equivalent:

(C_1) $F = \mathscr{F}\{T\}$, where $T = 0$ for $x < a$.

(C_2) Given any $G \in \mathrm{D}'_+$ and $\Psi \in \mathrm{D}$, there exist constants k and M such that

$$|\Gamma(F, G, \Psi; \xi + i\eta)| \leqslant M(1 + |\xi|^k)e^{-2\pi a\eta} \tag{12}$$

for $\eta > 0$.

In fact, by a theorem on analytic continuation of tempered distributions (see [1], [5], and [6]), condition (12) in the upper half-plane $\eta > 0$ is necessary and sufficient for the function $\Gamma(u) = \Gamma(F, G, \Psi; u)$ of E'_u to be the Fourier transform of a distribution, which vanishes for $x < a$. The equivalence of (C_1) and (C_2) can now be obtained by use of (8).

§ 4. Multiplication in D'_+ and D_+. Let f and g be functions of $\mathscr{D}_+$. Then the convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x-y)\, dy \tag{13}$$

also belongs to $\mathscr{D}_+$.

The convolution of a distribution $T \in \mathscr{D}'_+$ and a function $\psi \in \mathscr{D}$, defined as

$$(T * \psi)(x) = T_y \cdot \psi(x-y),$$

represents a function of $\mathscr{D}_+$. Thus, if S is another distribution of $\mathscr{D}'_+$, we may write

$$(T * S) \cdot \psi = T \cdot (\check{S} * \psi). \tag{14}$$

We now introduce in D'_+ a multiplication and we prove that, under the Fourier transformation, it corresponds to the convolution (14). In particular, the product of elements of D_+ corresponds to the convolution (13).

According to the results of the preceding section, for any pair of elements $F, G \in \mathrm{D}'_+$ and any $u_0 \in C$,

$$\Gamma(F, G, \Psi; u_0) \tag{15}$$

determines a continuous linear functional on D. We find the corresponding distribution.

LEMMA 2. *The element of D'_+ determined by* (15) *is the Fourier transform of $(aT) * S$, where $T = \mathscr{F}^{-1}\{F\}$, $S = \mathscr{F}^{-1}\{G\}$ and $a(x) = e^{2\pi i x u_0}$.*

Proof. For $F, G \in \mathrm{D}'_+$ and $\Psi \in \mathrm{D}$,

$$\Gamma(u) = \Gamma(F, G, \Psi; u)$$

is a function of E', and so $V = \mathscr{F}^{-1}\{\Gamma\}$ is a distribution with compact carrier. Moreover,

$$\Gamma(u_0) = V \cdot a.$$

But from lemma 1 it follows that

$$V = \vartheta T,$$

where $\vartheta = \check{S} * \check{\psi}$ and $\psi = \mathscr{F}^{-1}\{\Psi\}$. Therefore $\Gamma(u_0) = \vartheta T \cdot a = aT \cdot \vartheta = aT \cdot (\check{S} * \check{\psi}) = [(aT) * S] \cdot \check{\psi}$. The lemma is thus proved.

Definition. Let F and G be elements of D'_+. We define the product $F \circ G$ as follows:

$$(F \circ G) \cdot \Psi = \Gamma(F, G, \Psi; 0) \tag{16}$$

for every $\Psi \in \mathrm{D}$.

From lemma 2 we obtain immediately

THEOREM 2. *For any two elements $F, G \in \mathrm{D}'_+$, their product $F \circ G$ is an element of D'_+ and*

$$F \circ G = \mathscr{F}\{T * S\}, \tag{17}$$

where $T = \mathscr{F}^{-1}\{F\}$ and $S = \mathscr{F}^{-1}\{G\}$.

If either F or G belongs to D_+, then so does the product $F \circ G$. In particular, the space D_+ is closed with respect to this operation.

From (17) it follows that the product (16) is commutative, associative and distributive with respect to addition. Moreover, it satisfies the formula

$$D(F \circ G) = DF \circ G + F \circ DG,$$

where D is the differential operator.

If the elements F and G of D'_+ are continuous slowly increasing functions on R, then the new product coincides with the ordinary product.

References

[1] Н. Н. Боголюбов и О. С. Парасюк, *Об аналитическом продолжении обобщенных функций*, Докл. Акад. Наук СССР 109 (1956), p. 717-719.

[2] L. Ehrenpreis, *Solution of some problems of division I*, Amer. Journal of Math. 76 (1954), p. 883-903.

[3] — *Analytic functions and the Fourier transform of distributions I*, Annals of Math. 63 (1956), p. 129-159.

[4] И. М. Гелфанд, и Г. Е. Шилов, *Переобразования Фурье быстро растущих функций и вопросы единственности решения задачи Коши*, Успехи Мат. Наук 8 (1953), p. 3-54.

[5] J. L. Lions, *Supports dans la transformation de Laplace*, Journal d'Analyse Math. 2 (1952-53), p. 369-380.

[6] J. Mikusiński, *Analytic functions of polynomial growth*, Studia Mathematica 22 (1962), p. 7-13.

[7] L. Schwartz, *Théorie des distributions I, II*, Paris 1950/51.

[8] — *Distributions semi-régulières et changements de coordonnées*, Journal de Math. Pures et Appl. 36 (1957), p. 109-127.

[9] E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Oxford 1937.

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Smoothness and differentiability in L_p

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1. A measurable function $f\colon I_0 \to R$, $I_0 = [0,1]$, R reals, will be called *L_p-symmetric*, *L_p-smooth*, if for each $x \in I_0^0$, $I_0^0 = (0,1)$,

$$\left\{\frac{1}{h}\int_0^h |\Delta^2 f(x,t)|^p\,dt\right\}^{1/p} = o(1),\, o(h), \quad \text{as} \quad h \to 0, \tag{1}$$

respectively, where $\Delta^2 f(x,t) = f(x+t)+f(x-t)-2f(x)$. Throughout this paper p will be $\geqslant 1$. The well-known notions of symmetry and smoothness given by

$$\Delta^2 f(x,t) = o(1),\, o(h), \quad \text{as} \quad h \to 0, \tag{2}$$

respectively, can be viewed as the $p = \infty$ versions of (1). The question arises whether certain of the results for (2) are also true for (1) with perhaps estimating some of the inequalities in the metric of L_p.

In particular, it is known that a measurable smooth function has a derivative on a set which is of the power of the continuum in each interval [4, 10]. In [2], A. P. Calderon and A. Zygmund introduced the notion of L_p-differentiability. We say that f has at x_0 a first *L_p-derivative* provided there is a linear polynomial a_0+a_1t such that

$$\left\{\frac{1}{2h}\int_{-h}^{h} |f(x_0+t)-a_0-a_1t|^p dt\right\}^{1/p} = o(h), \quad \text{as} \quad h \to 0. \tag{3}$$

The polynomial a_0+a_1t is unique, and we write $a_1 = f'_{L_p}(x_0)$. One of the results that we obtain shows that L_p-smoothness implies L_p-differentiability on a set which is of the power of the continuum in each interval. That this may be the case was noted by A. Zygmund as the author learned in a conversation with E. M. Stein. We will first prove that the theorem is true for continuous functions, and then we will show that a measurable L_p-smooth function is continuous on a dense open set. We will show that this is the best possible continuity property for an L_p-smooth function and that in the case $p = \infty$ a substantial improvement is possible;

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