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Reçu par la Rédaction le 21. 12. 1964

ACTA ARITHMETICA
XI (1965)

A theorem on generalized Dedekind sums*

by

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1. Put

$$((x)) = \begin{cases} x-[x]-\frac{1}{2} & (x \neq \text{integer}), \\ 0 & (x \text{ integer}). \end{cases}$$

The Dedekind sum $s(h,k)$ is defined by

$$s(h,k) = \sum_{\mu \pmod k} \left(\left(\frac{h\mu}{k}\right)\right)\left(\left(\frac{\mu}{k}\right)\right), \tag{1.1}$$

where the summation is extended over a complete residue system (mod k). It is well known that $s(h,k)$ satisfies

$$12hk\{s(h,k)+s(k,h)\} = h^2-3hk+k^2+1, \tag{1.2}$$

where h and k are relatively prime.

Rademacher, at the 1963 Number Theory Institute in Boulder, proved the following generalization of (1.2). Define

$$s(k,h;x,y) = \sum_{\mu \pmod k} \left(\left(h\frac{\mu+y}{k}+x\right)\right)\left(\left(\frac{\mu+y}{k}\right)\right), \tag{1.3}$$

where x, y are arbitrary real numbers. Then

$$\begin{aligned} &s(h,k;x,y)+s(k,h;y,x) \\ &= -\frac{1}{4}\delta(x)\delta(y)+((x))((y))+\frac{1}{2}\left\{\frac{h}{k}\bar{B}_2(y)+\frac{1}{hk}\bar{B}_2(hy+kx)+\frac{k}{h}\bar{B}_2(x)\right\}, \end{aligned} \tag{1.4}$$

where $(h,k)=1$,

$$\delta(x) = \begin{cases} 1 & (x \text{ integral}), \\ 0 & (\text{otherwise}) \end{cases}$$

* Supported in part by NSF grant GP-1593.

and

$$\bar{B}_2(x) = B_2(x-[x]), \qquad B_2(x) = x^2-x+\tfrac{1}{6}.$$

When $x = y = 0$, (1.4) reduces to (1.2). Rademacher's proof of (1.4) appeared in [6].

Let $B_n(x)$ be the Bernoulli polynomial of degree n defined by

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$$

and let $\bar{B}_n(x)$ be the Bernoulli function defined by means of

$$\bar{B}_n(x) = B_n(x-[x]).$$

Apostol ([1], [2]) introduced the sum

$$s_p(h,k) = \sum_{\mu \,(\mathrm{mod}\, k)} \bar{B}_p\left(\frac{h\mu}{k}\right)\bar{B}_1\left(\frac{\mu}{k}\right) \tag{1.5}$$

and proved the reciprocity formula

$$(p+1)\{hk^p s_p(h,k)+kh^p s_p(k,h)\} = (hB+kB)^{p+1}+pB_{p+1}, \tag{1.6}$$

where $(h,k) = 1$, p odd, $p>1$ and $B_p = B_p(0)$. For another proof of (1.6) see [5].

Rademacher's definition of $s(h,k;x,y)$ suggests that we define

$$s_p(h,k;x,y) = \sum_{\mu \,(\mathrm{mod}\, k)} \bar{B}_p\left(h\frac{\mu+y}{k}+x\right)\bar{B}_1\left(\frac{\mu+y}{k}\right), \tag{1.7}$$

which reduces to $s_p(x,y)$ when $x = y = 0$. It is evident from (1.7) that

$$s_p(h,k;x+1,y) = s_p(h,k;x,y+1) = s_p(h,k;x,y), \tag{1.8}$$

so that there is no loss in generality in assuming that

$$0 \leqslant x < 1, \qquad 0 \leqslant y < 1. \tag{1.9}$$

The writer ([3]) has proved the following

THEOREM. *Let $(h,k) = 1$ and assume that x, y satisfy (1.9). Then*

$$(p+1)\{hk^p s_p(h,k;x,y)+kh^p s_p(k,h;y,x)\} = (hB+kB+hy+kx)^{p+1}+p\bar{B}_{p+1}(hy+kx) \tag{1.10}$$

for all $p \geqslant 0$.

It is understood that to evaluate $(Bh+Bk+hy+kx)^{p+1}$ we expand by the multinomial theorem and then replace B^n by B_n; alternatively we have

$$(Bh+Bk+hy+kx)^{p+1} = \sum_{r=0}^{p+1}\binom{p+1}{r}h^r k^{p-r+1}B_r(y)B_{p-r+1}(x).$$

This suggests the following equivalent formulation of (1.10) in which (1.9) is no longer assumed:

$$(p+1)\{hk^p s_p(k,h;x,y)+kh^p s_p(k,h;y,x)\} = \sum_{r=0}^{p+1}\binom{p+1}{r}h^r k^{p-r+1}\bar{B}_r(y)\bar{B}_{p-r+1}(x)+p\bar{B}_{p+1}(hy+kx). \tag{1.11}$$

The object of the present paper is to give a new and simpler proof of (1.10).

2. We recall that $\bar{B}_n(x)$ satisfies the multiplication theorem:

$$\bar{B}_n(kx) = k^{1-n}\sum_{\mu \,(\mathrm{mod}\, k)} \bar{B}_n\left(x+\frac{\mu}{k}\right). \tag{2.1}$$

Applying (2.1) to (1.7) we get

$$s_p(h,k;x,y) = h^{p-1}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{h-1} \bar{B}_1\left(\frac{\mu+y}{k}\right)\bar{B}_p\left(\frac{\mu}{k}+\frac{\nu}{h}+\frac{y}{k}+\frac{x}{h}\right),$$

so that

$$\begin{aligned} hk^p s_p(h,k;x,y)&+kh^p s_p(k,h;x,y) \\ &= (hk)^p\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{h-1}\left\{\bar{B}_1\left(\frac{\mu+y}{k}\right)+\bar{B}_1\left(\frac{\nu+x}{h}\right)\right\}\bar{B}_p\left(\frac{\mu}{k}+\frac{\nu}{h}+\frac{y}{k}+\frac{x}{h}\right). \end{aligned} \tag{2.2}$$

We shall now assume that x, y satisfy

$$0 \leqslant x < 1, \qquad 0 \leqslant y < 1. \tag{2.3}$$

As we have seen above there is no loss in generality in making this assumption. Put

$$\sigma = \frac{\mu}{k}+\frac{\nu}{h}+\frac{y}{k}+\frac{x}{h}, \tag{2.4}$$

so that

$$0 \leqslant \sigma < 2 \qquad (0 \leqslant \mu < k,\ 0 \leqslant \nu < h). \tag{2.5}$$

Since

$$\bar{B}_1(x) = x-\tfrac{1}{2} \qquad (0 \leqslant x < 1),$$

it follows that, for x, y satisfying (2.3),

$$\bar{B}_1(x)+\bar{B}_1(y) = \bar{B}_1(x+y)+\tfrac{1}{2}f(x+y), \tag{2.6}$$

where

$$f(x) = \begin{cases} -1 & (0 \leqslant x < 1), \\ +1 & (1 \leqslant x < 2). \end{cases} \tag{2.7}$$

For brevity put

$$S_p = hk^p s_p(h,k;x,y) + kh^p s_p(k,h;y,x). \tag{2.8}$$

Then (2.2) becomes

$$(hk)^{-p} S_p = \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{h-1} \{\bar{B}_1(\sigma) + \tfrac{1}{2} f(\sigma)\} \bar{B}_p(\sigma).$$

In view of (2.7) this becomes

$$(hk)^{-p} S_p = \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{h-1} \bar{B}_1(\sigma)\bar{B}_p(\sigma) + \tfrac{1}{2} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{h-1} \bar{B}_p(\sigma) - \sum_{\substack{\mu=0 \\ \sigma<1}}^{k-1} \sum_{\nu=0}^{h-1} \bar{B}_p(\sigma). \tag{2.9}$$

Now by (2.1) we have

$$\sum_{\mu=0}^{k-1} \sum_{\nu=0}^{h-1} \bar{B}_p(\sigma) = (hk)^{1-p} \bar{B}_p(hy+kx),$$

so that

$$(hk)^{-p} S_p = T_p - U_p + \tfrac{1}{2}(hk)^{1-p} \bar{B}_p(hy+kx), \tag{2.10}$$

where

$$T_p = \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{h-1} \bar{B}_1(\sigma)\bar{B}_p(\sigma), \tag{2.11}$$

$$U_p = \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{h-1} \bar{B}_p(\sigma) = \sum_{\substack{\mu=0 \\ \sigma<1}}^{k-1} \sum_{\nu=0}^{h-1} B_p(\sigma). \tag{2.12}$$

3. It will be convenient to put

$$z = hy + kx. \tag{3.1}$$

Since h and k are relatively prime, it is evident from (2.11) that

$$T_p = \sum_{\lambda=0}^{hk-1} \bar{B}_1\left(\frac{\lambda}{hk} + \frac{z}{hk}\right) \bar{B}_p\left(\frac{\lambda}{hk} + \frac{z}{hk}\right). \tag{3.2}$$

If we put

$$z = [z] + \zeta, \qquad 0 \leqslant \zeta < 1, \tag{3.3}$$

so that ζ is the fractional part of z, then (3.2) reduces to

$$T_p = \sum_{\lambda=0}^{hk-1} B_1\left(\frac{\lambda}{hk} + \frac{\zeta}{hk}\right) B_p\left(\frac{\lambda}{hk} + \frac{\zeta}{hk}\right). \tag{3.4}$$

Now we have, for $p \geqslant 1$,

$$B_1(x)B_p(x) = B_{p+1}(x) + \frac{1}{p+1} \sum_{0<2r\leqslant p} \binom{p+1}{2r} B_{2r} B_{p-2r+1}(x) + \frac{1}{p+1} B_{p+1}. \tag{3.5}$$

Indeed (3.5) is a special case of a formula for $B_m(x)B_n(x)$ (see e. g. [4]).

Put $x = (\mu+\zeta)/k$ in (3.5) and sum over μ. We get

$$\sum_{\mu=0}^{k-1} B_1\left(\frac{\mu+\zeta}{k}\right) B_p\left(\frac{\mu+\zeta}{k}\right)$$

$$= k^{-p} B_{p+1}(\zeta) + \frac{1}{p+1} \sum_{0<2r\leqslant p} \binom{p+1}{2r} B_{2r} B_{p-2r+1}(\zeta) k^{2r-p} + \frac{k}{p+1} B_{p+1}$$

$$= \frac{p}{p+1} k^{-p} B_{p+1}(\zeta) + \frac{k-p}{p+1} \sum_{r=0}^{p+1} \binom{p+1}{r} B_r B_{p-r+1}(\zeta) k^r + \tfrac{1}{2} k^{1-p} B_p(\zeta)$$

$$= \frac{p}{p+1} k^{-p} B_{p+1}(\zeta) + \frac{k-p}{p+1} (Bk + B + \zeta)^{p+1} + \tfrac{1}{2} k^{1-p} B_p(\zeta).$$

Therefore by (3.4)

$$T_p = \frac{p}{p+1} (hk)^{-p} B_{p+1}(\zeta) + \frac{(hk)^{-p}}{p+1} (Bhk + B + \zeta)^{p+1} + \tfrac{1}{2}(hk)^{1-p} B_p(\zeta). \tag{3.6}$$

As for U_p, we have

$$U_p = \sum_{\substack{\mu=0 \\ \sigma<1}}^{k-1} \sum_{\nu=0}^{h-1} B_p\left(\frac{\mu}{k} + \frac{\nu}{h} + \frac{z}{hk}\right) \tag{3.7}$$

$$= \sum_{r=0}^{p} \binom{p}{r} B_{p-r}\left(\frac{\mu}{k}\right) \sum_{\substack{\mu=0 \\ \sigma<1}}^{k-1} \sum_{\nu=0}^{h-1} \left(\frac{\nu}{h} + \frac{z}{hk}\right)^r.$$

Now

$$\sum_{\substack{\mu=0 \\ \sigma<1}}^{k-1} \sum_{\nu=0}^{h-1} \left(\frac{\nu}{h} + \frac{z}{hk}\right)^r = \sum_{\frac{\mu}{k} + \frac{z}{hk} < 1} \sum_{\nu<N_\mu} \left(\frac{\nu}{h} + \frac{z}{hk}\right)^r, \tag{3.8}$$

where

$$N_\mu = h - \frac{h\mu}{k} - \frac{z}{k}. \tag{3.9}$$

We may assume without loss of generality that $k < h$, so that

$$\frac{z}{h} = y + \frac{kx}{h} < 2.$$

Hence in the outer summation on the right of (3.8) we have

$$\mu \leqslant k-1-\left[\frac{z}{h}\right]. \tag{3.10}$$

Let us assume first, in evaluating the inner sum on the right of (3.8), that $h\mu+z$ is not an integral multiple of k. Then the sum is equal to

$$\frac{h^{-r}}{r+1}\left\{B_{r+1}\left(\left[h-\frac{h\mu}{k}-\frac{z}{k}\right]+1+\frac{z}{k}\right)-B_{r+1}\left(\frac{z}{k}\right)\right\}$$
$$=\frac{h^{-r}}{r+1}\left\{B_{r+1}\left(h+\frac{z}{k}-\left[\frac{h\mu}{k}+\frac{z}{h}\right]\right)-B_{r+1}\left(\frac{z}{k}\right)\right\}.$$

Thus by (3.7)

$$U_p=\frac{h^{-p}}{p+1}\sum_{\mu}\sum_{r=0}^{p+1}\binom{p+1}{r}h^{p-r+1}B_{p-r+1}\left(\frac{\mu}{k}\right) \tag{3.11}$$
$$=\left\{B_r\left(h+\frac{z}{k}-\left[\frac{h\mu}{k}+\frac{z}{k}\right]\right)-B_r\left(\frac{z}{k}\right)\right\}.$$

But

$$\sum_{\mu=0}^{k-1}\sum_{r=0}^{p+1}h^{p-r+1}B_{p-r+1}\left(\frac{\mu}{k}\right)B_r\left(h+\frac{z}{k}-\left[\frac{h\mu}{k}+\frac{z}{k}\right]\right)$$
$$=\sum_{\mu=0}^{k-1}\left\{\left(hB+\frac{h\mu}{k}\right)+\left(B+h+\frac{z}{k}-\left[\frac{h\mu}{k}+\frac{z}{k}\right]\right)\right\}^{p+1}$$
$$=\sum_{\mu=0}^{k-1}\left\{(hB+h)+\left(B+\frac{h\mu}{k}+\frac{z}{k}-\left[\frac{h\mu}{k}+\frac{z}{k}\right]\right)\right\}^{p+1}$$
$$=\sum_{r=0}^{p+1}\binom{p+1}{r}(hB+h)^{p-r+1}\sum_{\mu=0}^{k-1}B_r\left(\frac{h\mu}{k}+\frac{z}{k}-\left[\frac{h\mu}{k}+\frac{z}{k}\right]\right)$$
$$=\sum_{r=0}^{p+1}\binom{p+1}{r}(hB+h)^{p-r+1}\sum_{\mu=0}^{k-1}\bar{B}_r\left(\frac{h\mu}{k}+\frac{z}{k}\right)$$
$$=\sum_{r=0}^{p+1}\binom{p+1}{r}(hB+h)^{p-r+1}\bar{B}_r(z)k^{1-r}=k^{-p}\sum_{r=0}^{p+1}\binom{p+1}{r}(hkB+hk)^{p-r+1}B_r(\zeta)$$
$$=k^{-p}(hkB+hk+B+\zeta)^{p+1},$$

with ζ defined by (3.3).

On the other hand,

$$\sum_{\mu=0}^{k-1}\sum_{r=0}^{p+1}\binom{p+1}{r}h^{p-r+1}B_{p-r+1}\left(\frac{\mu}{k}\right)B_r\left(\frac{z}{k}\right)$$
$$=\sum_{r=0}^{p+1}\binom{p+1}{r}h^{p-r+1}k^{r-p}B_{p-r+1}B_r\left(\frac{z}{k}\right)$$
$$=k^{-p}\sum_{r=0}^{p+1}\binom{p+1}{r}(hB)^{p-r+1}(kB+z)^r=k^{-p}(hB+kB+z)^{p+1}.$$

Thus (3.11) yields

$$U_p=\frac{(hk)^{-p}}{p+1}(hkB+hk+B+\zeta)^{p+1}-\frac{(hk)^{-p}}{p+1}(hB+kB+z)^{p+1} \tag{3.12}$$

provided $z<h$. When $z\geqslant h$ we must delete the terms corresponding to $\mu=k-1$ in the right member of (3.11). Since

$$\left[\frac{h(k-1)}{k}+\frac{z}{k}\right]=h+\left[\frac{z-h}{k}\right]=h,$$

it is evident that (3.12) holds for all z.

We have

$$(hkB+hk+B+\zeta)^{p-1}=\big(hk(B+1)+(B+\zeta)\big)^{p+1}$$
$$=\sum_{r=0}^{p+1}\binom{p+1}{r}(hk)^r(B+1)^r(B+\zeta)^{p-r+1}.$$

Since $(B+1)^r=B^r$ $(r\neq 1)$, the above sum is equal to

$$\sum_{r=0}^{p+1}\binom{p+1}{r}(hkB)^r(B+\zeta)^{p-r+1}+(p+1)hk(B+\zeta)^p$$
$$=(hkB+N+\zeta)^{p+1}+(p+1)hk(B+\zeta)^p.$$

Therefore (3.12) becomes

(3.13)

$$U_p=\frac{(hk)^{-p}}{p+1}(hkB+B+\zeta)^{p+1}-\frac{(hk)^{-p}}{p+1}(hB+kB+z)^{p+1}+(hk)^{1-p}(B+\zeta)^p.$$

When $h\mu+z$ is an integral multiple of k, the above proof requires modification. We get in this case

$$\sum_{0\leqslant\nu<N_\mu}\left(\frac{\nu}{h}+\frac{z}{hk}\right)^r=h^{-r}\sum_{\nu=0}^{N_\mu-1}\left(\nu+\frac{z}{k}\right)^r=\frac{h^{-r}}{r+1}\left\{B_{r+1}\left(N_\mu+\frac{z}{k}\right)-B_{r+1}\left(\frac{z}{k}\right)\right\}$$
$$=\frac{h^{-r}}{r+1}\left\{B_{r+1}\left(h+\frac{z}{k}-\left[\frac{h\mu}{k}+\frac{z}{k}\right]\right)-B_{r+1}\left(\frac{z}{k}\right)\right\}.$$

The remainder of the proof goes through without change. Therefore (3.13) holds without exception.

We now substitute from (3.6) and (3.12) in (2.10) to get

$$\begin{aligned}(hk)^{-p}S_p &= \frac{p}{p+1}(hk)^{-p}B_{p+1}(\zeta)+\frac{(hk)^{-p}}{p+1}(Bhk+B+\zeta)^{p+1}+\tfrac{1}{2}(hk)^{1-p}B_p(\zeta)-\\ &\quad -\frac{(hk)^{-p}}{p-1}\{(hkB+B+\zeta)^{p+1}-(hB+kB+z)^{p+1}-\\ &\quad -(hk)^{1-p}(B+\zeta)^p+\tfrac{1}{2}(hk)^{1-p}\bar{B}_p(z)\\ &= \frac{p}{p+1}(hk)^{-p}B_{p+1}(\zeta)+\frac{(hk)^{-p}}{p+1}(hB+kB+hy+ky)^{p+1}.\end{aligned}$$

This evidently completes the proof of (1.10) when $p \geqslant 1$.

When $p = 0$, we have

$$S_0(h, k; x, y) = \sum_{\mu \pmod k} \bar{B}_1\left(\frac{\mu+y}{k}\right) = \bar{B}_1(y),$$

so that

$$S_0 = hs_0(h, k; x, y)+ks_0(k, h; y, x) = h\bar{B}_1(y)+k\bar{B}_1(x).$$

On the other hand,

$$(Bh+Bk+hy+kx)^1 = hB_1(y)+kB_1(x).$$

Since $0 \leqslant x < 1$, $0 \leqslant y < 1$, it is clear that (1.10) holds when $p = 0$.

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Reçu par la Rédaction le 23. 12. 1964

ACTA ARITHMETICA
XI (1965)

Errata zur Arbeit „Eine Bemerkung zur Fermatschen Vermutung"

(Acta Arithmetica 11 (1965), S. 129-131)

von

M. Eichler (Basel)

S. 129[8] statt „zu 1 teilerfremden Zahlen" lies „zu l teilerfremden Zahlen";

S. 131[4] statt „$r(r+1)+1 < l_{-1}$" lies „$r(r+1) < l_{-1}$".