11. Lemma. *If every set can be linearly ordered, then* FS_n^* *holds.*

The truth of 11 was first observed by Kuratowski. The proof is not difficult and may be found in Sierpiński [6], p. 412.

12. Lemma. *The conjunction of* FS_1 *and* FS_0^* *is equivalent to* FS_0.

Proof. Obviously FS_0 implies both FS_1 and FS_0^*. On the other hand, FS_1 says that given a set X of non-empty sets there is a function f such that for $Y \in X$, $f(Y) \in \mathfrak{P}^*(Y)$, while FS_0^* says there is a choice function g on $\mathfrak{P}^*(\bigcup X)$. Since $\mathfrak{P}^*(Y) \subset \mathfrak{P}^*(\bigcup X)$ for each $Y \in X$, it follows that $g(f)$ is the desired choice function.

We are now in a position to prove

13. Theorem. *If* $\mathfrak{S}$ *is consistent, then the axioms of* $\mathfrak{S}$ *do not imply* FS_n *for any* $n \in X$.

Proof. Since for each $n \in I$, FS_n implies FS_1, it is sufficient to prove that FS_1 is independent of the axioms of $\mathfrak{S}$. In view of Lemmas 12 and 13 we see that FS_1, together with the supposition that every set can be linearly ordered, implies the axiom of choice. Since Mostowski [4] has shown that the axioms of $\mathfrak{S}$, together with the principle of linear ordering do not imply the axiom of choice, it is clear that they cannot imply FS_1. It follows that the axioms of $\mathfrak{S}$ alone certainly cannot imply FS_1.

It is clear from the above independence results that FS_1 is independent of the conjunction of the axioms of $\mathfrak{S}$ and FS_0^*, and conversely that FS_0^* is independent of the conjunction of the axioms of $\mathfrak{S}$ and FS_1. In view of this independence 12 gives a nice decomposition of the axiom of choice into independent, heuristically complementary statements. The feeling that the statements are complementary is strengthened by the fact that if FS_0^* is replaced by $\langle I \rangle$ the new conjunction does not imply the axiom of choice.

In view of 9 it would be interesting to know if there is a prime $p \in \mathfrak{I}$ such that $FS_{\mathfrak{I}}$ implies FS_P where $P = \mathfrak{I} \setminus \{p\}$.

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An interpolation theorem for denumerably long formulas *

by

E. G. K. Lopez-Escobar (Cambridge, Mass.)

0. Introduction. $\mathbf{L}_{\alpha\beta}$ is the first-order language obtained by modifying the usual formation rules for the finite first-order formulas so that conjunctions and disjunction of less that α formulas and quantifications over fewer than β individual variables are allowed (thus in particular, $\mathbf{L}_{\omega\omega}$ is the usual (finite) first-order language) [1]. We say that the interpolation theorem is true for $\mathbf{L}_{\alpha\beta}$ just in case that for all formulas φ, Φ of $\mathbf{L}_{\alpha\beta}$ if the implication $\varphi \to \Phi$ is valid, then there exists a formula π of $\mathbf{L}_{\alpha\beta}$ such that (1) $\varphi \to \pi$ and $\pi \to \Phi$ are valid, (2) if a variable occurs free in π, then it occurs free both in φ and in Φ, and (3) if a relational symbol occurs (occurs positively, occurs negatively) in π, then it occurs (occurs positively, occurs negatively) both in φ and in Φ. The interpolation theorem is known to be true for $\mathbf{L}_{\omega\omega}$ (see Craig [2] and Lyndon [10]) and, whenever θ is an inaccessible cardinal, for $\mathbf{L}_{\theta\theta}$ (see Maehara-Takeuti [11])**. In this paper we show that the interpolation theorem is true for $\mathbf{L}_{\omega_1\omega}$.

The interpolation theorem is obtained as a consequence of the completeness of a Gentzen type formalization for $\mathbf{L}_{\omega_1\omega}$. The essential property of our rules of inference, in addition to the usual subformula property associated with Gentzen type systems, is that the number of variables that occur free in the premise(s) and do not occur free in the conclusion (i.e. are quantified out) is always finite [2].

* The results in this paper form part of the results contained in a dissertation submitted in partial satisfaction of the requirements for the Ph. D. degree at the University of California, Berkeley, prepared under the direction of Professor Dana S. Scott, to whom the author wishes to express his sincere thanks for much valuable advice.

[1] These kinds of infinitary languages (i.e. infinitary languages $\mathbf{L}_{\alpha\beta}$ where α need not be equal to β) have been studied by C. R. Karp [4].

** Added in proof. J. Malitz has shown that the interpolation theorem for $\mathbf{L}_{\theta\theta}$ is false, cf. Notices of Amer. Math. Soc. 12(1965), p. 379.

[2] E. Engeler [3] has obtained a formalization for $\mathbf{L}_{\omega_1\omega}$ which has the subformula property, however his formalization does not have the property mentioned above and thus it is unsuitable for deriving the interpolation theorem (the author did not learn

After giving some immediate applications of the interpolation theorem for $\mathbf{L}_{\omega_1\omega}$ we show that unlike the case of $\mathbf{L}_{\omega\omega}$, the interpolation theorem is not true for sets of formulas of $\mathbf{L}_{\omega_1\omega}$ (i.e. there exists disjoint $\mathbf{PC}_\Delta(\mathbf{L}_{\omega_1\omega})$ classes which can not be separated by a class which is closed under $\mathbf{L}_{\omega_1\omega}$ elementary equivalence (cf. Theorem 6.3)).

1. Preliminaries. We will distinguish between classes and sets, namely, a set is a class which is a member of some other class. We assume the axiom of choice and any of its equivalent formulations. For the most part standard set-theoretical notation will be used. Thus, for example, ϵ is the membership relation and $\subseteq$, 0, $\cup$, $\bigcup$, $\cap$ and $\bigcap$ are respectively the relation of inclusion, the empty set and the operations of forming unions (intersections) of two and arbitrary many classes. $X \sim Y$ is the set-theoretical difference of the classes X and Y and if Z is a set then $\mathbf{P}(Z)$ is then the set of all subsets of Z. If $t(x)$ is a term containing x as a free variable, then a symbolic expression of the form $\{t(x)\colon \Phi\}$ where Φ is to be replaced by any formula containing x as free variable denotes the class of all $t(x)$ for which x satisfies the given formula. (x, y) is the ordered pair with x as its first term and y as its second term. A function is a set of ordered pairs satisfying the usual conditions and if f is a function then $\mathbf{D}f$ and $\mathbf{C}f$ are respectively the domain and the range of f. If f is a function and $x \in \mathbf{D}f$, then $f(x)$ or sometimes f_x is the value of f at x; thus $f = \{(x, f(x))\colon x \in \mathbf{D}f\}$. A^B is the class of all functions whose domain is B and whose range is inclued in A. $f \upharpoonright X$, the restriction of f to X, is the class $\{(x, f(x))\colon x \in \mathbf{D}f \cap X\} \cup \{(x, x)\colon x \in X \sim \mathbf{D}f\}$. Note that the domain of $f \upharpoonright X$ is always X.

Ordinals are assumed to have been defined in such a way that an ordinal is the set of all smaller ordinals and we shall use the letters: $\mu, \eta, \zeta, \varrho, \delta$ to represent arbitrary ordinals. The formulas $\mu < \eta$, $\eta > \mu$ and $\mu \in \eta$ are equivalent and will be used interchangeably. $\mu + \eta$ is the ordinal addition of μ and η. If a function f has an ordinal μ as its domain, then f is called a *μ-sequence* (or simply: a *sequence*) and $\langle f_\eta \rangle_{\eta<\mu} = f$. The letters: m, n, p, q will be used to represent natural numbers (i.e. finite ordinals) and $\langle a_0, \ldots, a_{n-1} \rangle$ is the n-sequence $\langle a_i \rangle_{i<n}$. If $s = \langle s_\eta \rangle_{\eta<\varrho}$ and $t = \langle t_\mu \rangle_{\mu<\delta}$ are sequences, then the concatenation of s and t, in symbols: $s * t$, is the $\varrho+\delta$-sequence $s \cup \{(\varrho+\mu, t_\mu)\colon \mu < \delta\}$; $\langle a, t_\mu \rangle_{\mu<\delta} = \langle a \rangle * \langle t_\mu \rangle_{\mu<\delta}$.

By a *cardinal* it is understood an initial ordinal, i.e. an ordinal whose power exceeds the power of each smaller ordinal. ω is the smallest infinite cardinal and ω_1 is the least cardinal greater than ω (i.e. the smallest uncountable ordinal). $|A|$ is the cardinality of the set A.

of Engeler's axiomatization till after obtaining the interpolation theorem). For another axiomatization of $\mathbf{L}_{\omega_1\omega}$, see Karp [4].

A sequence ν such that the range of ν is a set of natural numbers will be called a *similarity type*. We shall use the letter ν to represent similarity types and then ϱ_ν (or simply: ϱ) to represent the domain of ν. By a *relational system of type* ν (or simply: a *relational system*) we understand a $\varrho_\nu+1$-sequence $\mathfrak{A} = \langle A, R_\mu \rangle_{\mu<\varrho_\nu}$ such that $A \neq 0$ (and then A is called the *universe* of $\mathfrak{A}$) and such that $R_\mu \in A^{\nu_\mu}$ for all $\mu < \varrho_\nu$. Relational systems will in general be represented by capital German letters and it will be always understood that the corresponding capital Roman letter denotes its universe. If $\mathfrak{A} = \langle A, R_\mu \rangle_{\mu<\varrho}$ and $\delta \leqslant \varrho$, then $\mathfrak{A} \upharpoonright \delta$ is the relational system $\langle A, R_\mu \rangle_{\mu<\delta}$ and is called the *δ-reduct of* $\mathfrak{A}$.

If T is a set of finite sequences of natural numbers such that exactly one element of T, called the *root of* T, has no immediate predecessor (where an element $s \in T$ is an immediate predecessor of an element $t \in T$ just in case that for some $n < \omega$, $t = s * \langle n \rangle$) then T is a *pseudo-tree*. If T is a pseudo-tree, then a *branch of* T is a maximal subset of T satisfying the condition that no two (distinct) elements have the same immediate predecessor; if $s \in T$ and there is no $n < \omega$ such that $s * \langle n \rangle \in T$ then s is an *uppermost element of* T. By a *tree* we understand a pseudo-tree T such that all branches of T are finite; clearly if T is a tree then every branch of T has an uppermost element.

To every element s of a tree T we can associate an ordinal $\mathrm{dt}_T(s)$, called the *T-depth of* s, as follows:

(i) if s is an uppermost element of T, then $\mathrm{dt}_T(s) = 0$,

(ii) if s is not an uppermost element of T, then

$$\mathrm{dt}_T(s) = \bigcup \{\mathrm{dt}_T(t)\colon s \text{ is an immediate predecessor of } t\} + 1\,.$$

If T is a tree, then the *length of* T is the T-depth of the root of T.

2. The language $\mathbf{L}_{\omega_1\omega}$. The language $\mathbf{L}_{\omega_1\omega}$ can briefly be described as follows. The symbols of $\mathbf{L}_{\omega_1\omega}$ include a set $\mathbf{V}$ of ω_1 distinct individual variables and we shall use the letters (with or without subscripts): v, w, x, y, and z to denote the elements of $\mathbf{V}$ (i.e. individual variables). The equality symbol '$\equiv$' and the relational symbols: $P_0, \ldots, P_\eta, \ldots$ are included amongst the symbols of $\mathbf{L}_{\omega_1\omega}$; the number and type of relational symbols is determined by the similarity type [3]. Hence the atomic formulas of $\mathbf{L}_{\omega_1\omega}$ are of the form $v \equiv w$ or $P_\eta v_0 \ldots v_{\nu_\eta - 1}$. $\mathbf{L}_{\omega_1\omega}$ is the least class W such that:

(i) every atomic formula is in W,

(ii) if φ is in W, then its negation $\neg\varphi$ is in W,

[3] For the purposes of this paper the equality symbol is not considered to be a relational symbol.

(iii) if $X = \{\varphi_i \colon i \in I\} \subseteq W$ and $|I| < \omega_1$, then the conjunction of X, in symbols: $\prod X$ (also written $\prod_{i\in I} \varphi_i$), is also in W,

(iv) if φ is in W, then its universal generalization $(\forall w)\varphi$ is also in W.

When writing formulas of $\mathbf{L}_{\omega_1\omega}$ we will sometimes use the existential quantifiers $(\mathrm{E}v)\varphi$, (countable) infinite disjunctions $\sum_{i\in I}\varphi_i$ (or $\sum X$) and the usual finite sentential connectives: $\rightarrow, \longleftrightarrow, \wedge, \vee$; we shall assume that they have been defined in the usual way in terms of $\neg, \prod, (\forall v)$. Note that in (iii) it is not assumed that $X \neq 0$ and thus the empty conjunction $\prod 0$ is a formula of $\mathbf{L}_{\omega_1\omega}$. We let $\mathbf{T} = \prod 0$ and call $\mathbf{T}$ the *truth symbol.*

We assume it to be known what it means for an ω_1-sequence s of elements from the set A to satisfy the formula $\varphi \in \mathrm{L}_{\omega_1\omega}$ in the relational system $\mathfrak{A}$, in symbols: $(\mathfrak{A}, s) \models \varphi$ (see [4]). In the case that φ is satisfied by all ω_1-sequences from A we say that φ *is true in* $\mathfrak{A}$ (or *valid in* $\mathfrak{A}$) and we express this condition by: $\mathfrak{A} \models \varphi$. If φ is valid in all relational systems (of the suitable similarity type) then φ *is valid*, in symbols: $\models \varphi$. It is clear that $\mathbf{T}$ is a valid formula.

DEFINITION 2.0. *If $\varphi \in \mathbf{L}_{\omega_1\omega}$ then the set of variables occurring free in φ, in symbols*: $\mathrm{FV}(\varphi)$, *is defined as follows:*

(i) *if* $\varphi = v \equiv w$, *then* $\mathrm{FV}(\varphi) = \{v, w\}$,

(ii) *if* $\varphi = P_\eta v_0 \ldots v_{\nu_\eta - 1}$, *then* $\mathrm{FV}(\varphi) = \{v_i \colon i < \nu_\eta\}$,

(iii) *if* $\varphi = \neg\Phi$, *then* $\mathrm{FV}(\varphi) = \mathrm{FV}(\Phi)$,

(iv) *if* $\varphi = \prod_{i\in I} \Phi_i$ *then* $\mathrm{FV}(\varphi) = \bigcup_{i\in I} \mathrm{FV}(\Phi_i)$,

(v) *if* $\varphi = (\forall v)\Phi$, *then* $\mathrm{FV}(\varphi) = \mathrm{FV}(\Phi) \sim \{v\}$.

In a similar way we can define the set of variables occurring bound in φ, in symbols: $\mathrm{BD}(\varphi)$, and then $\mathrm{VS}(\varphi) = \mathrm{FV}(\varphi) \cup \mathrm{BD}(\varphi)$ is the set of variables occuring in φ.

DEFINITION 2.1. *If $\varphi \in \mathbf{L}_{\omega_1\omega}$ then the set of subformulas of φ, in symbols*: $\mathrm{SF}(\varphi)$, *is the least set W containing φ and such that whenever it contains*

(i) $\neg\Phi$, *it contains* Φ,

(ii) $(\forall v)\Phi$, *it contains* Φ,

(iii) $\prod X$, *it includes* X.

DEFINITION 2.2. *If $\varphi, \Phi \in \mathbf{L}_{\omega_1\omega}$ then "Φ is an immediate member of φ", in symbols: $\Phi \lesssim \varphi$, just in case that either*

(i) $\varphi = \neg\Phi$ *or*

(ii) *for some* v, $\varphi = (\forall v)\Phi$ *or*

(iii) *for some* X, $\varphi = \prod X$ *and* $\Phi \in X$.

The following lemma is immediate from the above definitions.

LEMMA 1.3. *If $\Phi, \varphi \in \mathbf{L}_{\omega_1\omega}$ and $\Phi \in \mathrm{SF}(\varphi)$, then*

(1) $$|\mathrm{FV}(\Phi) \sim \mathrm{FV}(\varphi)| < \omega,$$

(2) *for some $n < \omega$ there exist an $n+1$-sequence $\langle\pi_i\rangle_{i\leqslant n}$ of formulas of $\mathbf{L}_{\omega_1\omega}$ such that:*

(i) $$\pi_0 = \Phi \quad \textit{and} \quad \pi_n = \varphi,$$

(ii) $$\textit{for all } i < n, \quad \pi_i \lesssim \pi_{i+1}.$$

DEFINITION 2.4. *A relational symbol P_η (equality symbol $\equiv$) has a positive (negative) occurrence in a formula $\varphi \in \mathbf{L}_{\omega_1\omega}$ just in case that there exists a finite sequence $\langle\pi_i\rangle_{i\leqslant n}$ of formulas such that:*

(i) $\pi_n = \varphi$ *and* π_0 *is of the form* $P_\eta v_0 \ldots v_{\nu_\eta-1}$ $[v \equiv w]$,

(ii) *for all* $i < n$, $\pi_i \lesssim \pi_{i+1}$,

(iii) $|\{i \colon i < n \;\&\; \pi_{i+1} = \neg\pi\}|$ *is even (odd).*

DEFINITION 2.5. *A formula $\varphi \in \mathbf{L}_{\omega_1\omega}$ is a "positive formula" just in case that neither the equality symbol nor any relational symbol has a negative occurrence in φ.*

It us clear that a relational (or the equality) symbol has a negative (positive) occurrence in φ if and only if it has a positive (negative) occurrence in $\neg\varphi$.

A *replacement function* is a function f such that $\mathbf{D}f = \mathbf{V}$ and $\mathbf{\Pi}f \subseteq \mathbf{V}$.

DEFINITION 2.6. *If f is a replacement function, then* Rp_f *and* Sb_f *are defined as follows:*

(i) $\mathrm{Rp}_f(v \equiv w) = \mathrm{Sb}_f(v \equiv w) = f(v) \equiv f(w)$,

(ii) $\mathrm{Rp}_f(P_\eta v_0 \ldots v_{\nu_\eta-1}) = \mathrm{Sb}_f(P_\eta v_0 \ldots v_{\nu_\eta-1}) = P_\eta f(v_0) \ldots f(v_{\nu_\eta-1})$,

(iii) $\mathrm{Rp}_f(\prod_{i\in I}\varphi_i) = \prod_{i\in I}\mathrm{Rp}_f(\varphi_i)$ *and* $\mathrm{Sb}_f(\prod_{i\in I}\varphi_i) = \prod_{i\in I}\mathrm{Sb}_f(\varphi_i)$,

(iv) $\mathrm{Rp}_f(\neg\varphi) = \neg\mathrm{Rp}_f(\varphi)$ *and* $\mathrm{Sb}_f(\neg\varphi) = \neg\mathrm{Sb}_f(\varphi)$,

(v) $\mathrm{Rp}_f\big((\forall v)\varphi\big) = \big(\forall f(v)\big)\mathrm{Rp}_f(\varphi)$ *and* $\mathrm{Sb}_f\big((\forall v)\varphi\big) = (\forall v)\mathrm{Sb}_g(\varphi)$ *where* $g = \big(f \upharpoonright (\mathbf{V} \sim \{v\})\big) \upharpoonright \mathbf{V}$.

Intuitively $\mathrm{Rp}_f(\varphi)$ $\big(\mathrm{Sb}_f(\varphi)\big)$ is the formula obtained by simultaneously replacing all occurrences (all free occurrences) of the individual variable v in φ by the variable $f(v)$.

If f is a function such that $\mathbf{D}f \subseteq \mathbf{V}$ and $f\,\mathbf{\Pi} \subseteq \mathbf{V}$, then we let $\mathrm{Rp}_f(\varphi) = \mathrm{Rp}_h(\varphi)$ and $\mathrm{Sb}_f(\varphi) = \mathrm{Sb}_h(\varphi)$ where $h = f \upharpoonright \mathbf{V}$. We will usually only be concerned with substituting finitely many free variables and thus we introduce the following more suggestive notation: Suppose that $u \in \mathbf{V}^n$ is such that $u_i \neq u_j$ whenever $i < j < n$, $\varphi \in \mathbf{L}_{\omega_1\omega}$, $u' \in \mathbf{V}^n$, $f = \{(u_i, u_i') \colon i < n\}$, then we let

$$\varphi[u_0/u_0', \ldots, u_{n-1}/u_{n-1}'] = \mathrm{Sb}_f(\varphi).$$

DEFINITION 2.7. *If $\varphi \in \mathbf{L}_{\omega_1\omega}$, then "$w$ is free for v in φ" just in case that either (1) $v = w$ or (2) for all $n < \omega$ and all $(n+1)$-sequences $\langle\pi_i\rangle_{i\leqslant n}$ of subformulas of φ such that (i) $\pi_n = \varphi$, (ii) for all $i < n$, $\pi_i \lesssim \pi_{i+1}$, and (iii) for some Φ, $\pi_0 = (\forall w)\Phi$ and $v \in \mathrm{FV}(\Phi)$; then for some $i < n$, $\pi_{i+1} = (\forall v)\pi_i$.*

DEFINITION 2.8. *If* $\varphi \in \mathbf{L}_{\omega_1\omega}$ *and* A *is a set of individual variables and* $\Phi = \varphi[u_0/u_0', \ldots, u_{n-1}/u_{n-1}']$, *then* Φ *is a "proper* A*-substitution instance of* φ" (*or simply: a proper substitution of* φ) *if and only if* $\{u_i'\colon i < n\} \subseteq A$ *and for all* $i < n$, u_i' *is free for* u_i *in* φ.

DEFINITION 2.9. Φ *is a normal form of* φ *just in case that no variable occurs both free and bound in* Φ *and* $\varphi = \mathrm{Rp}_f(\Phi)$ *where* f *is a replacement function such that* (i) $f(v) = v$ *for all* $v \in \mathrm{FV}(\Phi)$ *and* (ii) *for every subformula of* Φ *of the form* $(\forall v)\pi$, $f(v) \notin \mathrm{FV}(\pi) \sim \{v\}$.

Intuitively, if Φ is a normal form of φ, then Φ can be obtained from φ by "renaming" the variables that occur bound in φ in such a way that no variable occurs both free and bound in Φ (the purpose of clause (ii) is to make sure that if Φ is a normal form of φ, then $\Phi \longleftrightarrow \varphi$ is valid). It is clear that to every formula φ there corresponds at least one formula Φ such that Φ is a normal form of φ.

As mentioned in the introduction we will consider a Gentzen type formalization of $\mathbf{L}_{\omega_1\omega}$. Hence the following definition.

DEFINITION 2.10. *If* X *and* Y *are countable sets of formulas of* $\mathrm{L}_{\omega_1\omega}$ *then we let* $X \to Y = (X, Y)$ *and it is called a "*$\mathbf{L}_{\omega_1\omega}$*-sequent"* (*or: a sequent*); X *is the "antecedent" and* Y *is the "succedent" of the sequent* $X \to Y$.

If $S = X \to Y$ is a sequent, then the set of variables occurring free in S, in symbols: $\mathrm{FV}(S)$, is the set $\bigcup_{\varphi\in X} \mathrm{FV}(\varphi) \cup \bigcup_{\varphi\in Y} \mathrm{FV}(\varphi)$. The set of variables occurring bound (occurring) in S, in symbols: $\mathrm{BD}(S)$ $(\mathrm{VS}(S))$, are similarly defined. If r is either the equality symbol or a relational symbol and X is a set of formulas, then r has a positive (negative) occurrence in X just in case that r has a positive (negative) occurrence in some formula of X; if $X \to Y = S$ is a sequent, then r has a positive (negative) occurrence in S if and only if r has a positive (negative) occurrence in $\{\neg\varphi\colon \varphi \in X\} \cup Y$.

DEFINITION 2.11. $X' \to Y'$ *is a "normal form of* $X \to Y$" *if and only if* $\prod_{\varphi\in X' \cup Y'} \varphi$ *is a normal form of* $\prod_{\varphi \in X \cup Y} \varphi$.

If s is an ω_1-sequence of elements from the universe of the relational system $\mathfrak{A}$, then s satisfies the sequent $X \to Y$ in $\mathfrak{A}$ if and only if

$$(\mathfrak{A}, s) \models \prod_{\varphi\in X} \varphi \to \sum_{\varphi \in Y} \varphi\,.$$

The notions which are defined in terms of satisfaction are extended in the natural way to sequents. Thus, e.g., we have that a formula $\varphi \in \mathbf{L}_{\omega_1\omega}$ is a valid formula if and only if the sequent $0 \to \{\varphi\}$ is a valid sequent.

3. Proof theory for $\mathbf{L}_{\omega_1\omega}$. In the axioms and rules of inference given below it is assumed that:

(a) φ *and* Φ *are formulas of* $\mathbf{L}_{\omega_1\omega}$,

(b) M, M', N *and* N' *are countable sets of formulas of* $\mathbf{L}_{\omega_1\omega}$.

STRUCTURAL RULES OF INFERENCE

1. PRINCIPAL STRUCTURAL RULE OF INFERENCE.

$$\frac{M \to N}{M \cup M' \to N' \cup N}\,.$$

2. RULE OF INFERENCE FOR RENAMING BOUND VARIABLES. (4) If $M' \to N'$ is a normal form of $M \to N$, then the following is a rule of inference:

$$\frac{M' \to N'}{M \to N}\,.$$

AXIOMS

1. LOGICAL AXIOMS.

(A.0) $$M \cup \{\varphi\} \to N \cup \{\varphi\}\,,$$

(A.1) $$M \to N \cup \{\mathbf{T}\}\,.$$

2. AXIOM FOR EQUALITY. Whenever v is an individual variable the following is an axiom:

$$M \to N \cup \{v \equiv v\}\,.$$

RULES OF INFERENCE FOR THE LOGICAL SYMBOLS

1. RULES OF INFERENCE FOR NEGATION.

$(\to \neg)$ $$\frac{M \cup \{\varphi\} \to N}{M \to N \cup \{\neg\varphi\}}$$

$(\neg \to)$ $$\frac{M \to N \cup \{\varphi\}}{M \cup \{\neg\varphi\} \to N}$$

2. RULES OF INFERENCE FOR CONJUNCTION. If $|I| < \omega_1$ and $\{\varphi_i\colon i \in I\} \subseteq \mathbf{L}_{\omega_1\omega}$, then the following two rules are rules of inference:

$(\prod \to)$ $$\frac{M \cup \{\varphi_i\colon i \in I\} \to N}{M \cup \{\prod_{i\in I} \varphi_i\} \to N}$$

$(\to \prod)$ $$\frac{M \to N \cup \{\varphi_i\}\ :\ \text{for all } i \in I}{M \to N \cup \{\prod_{i\in I} \varphi_i\}}$$

3. UNRESTRICTED RULE OF INFERENCE FOR THE QUANTIFIER. If x and y are free for v in φ, then the following is a rule of inference:

$(\forall \to)$ $$\frac{M \cup \{\varphi[v/x]\} \to N}{M \cup \{(\forall y)\varphi[v/y]\} \to N}$$

(4) This rule of inference is needed because we have omitted the rule of inference usually known as a *cut*, i.e. from $M \to N \cup \{\varphi\}$ and $M' \cup \{\varphi\} \to N'$ to obtain $M \cup M' \to N \cup N'$, and yet we allow a variable to occur both free and bound in a formula; c.f. Kleene [7].

4. RESTRICTED RULE OF INFERENCE FOR THE QUANTIFIER. If x and y are free for v in φ and, unless x is v, x does not occur free in φ, then the following is a rule of inference provided that x (called the *eigenvariable of the inference*) does not occur free in the conclusion:

$$(\to\forall)\qquad \frac{M\to N\cup\{\varphi[v/x]\}}{M\to N\cup\{(\forall y)\varphi[v/y]\}}$$

RULES OF INFERENCE FOR EQUALITY [5]

If x and y are free for v in φ and in Φ, then the following two rules are rules of inference:

$$(\equiv_1)\qquad \frac{M\cup\{\varphi[v/x]\}\to N\cup\{\Phi[v/x]\}}{M\cup\{x\equiv y,\varphi[v/y]\}\to N\cup\{\Phi[v/y]\}}$$

$$(\equiv_2)\qquad \frac{M\cup\{\varphi[v/x]\}\to N\cup\{\Phi[v/x]\}}{M\cup\{y\equiv x,\varphi[v/y]\}\to N\cup\{\Phi[v/y]\}}\,.$$

DEFINITION 3.0. *If* $\mathbf{A}$ *is a set of sequents,* S *is a sequent and* P *a function such that:*

(i) $\mathbf{D}P$ *is a tree,*

(ii) *if* s *is an uppermost element of* $\mathbf{D}P$, *then* $P(s)\in\mathbf{A}$,

(iii) *if* $s\in\mathbf{D}P$ *and* s *is not an uppermost element of* $\mathbf{D}P$, *then there exists a rule of inference* R *such* $P(s)$ *is the conclusion of an application of* R *with* $\{P(s^*\langle n\rangle)\colon n<\omega\ \&\ s^*\langle n\rangle\in\mathbf{D}P\}$ *as the set of premises (and then we shall say that "*R *has been applied in* P*"),*

(iv) $P(r)=S$, *where* r *is the root of* $\mathbf{D}P$,

then "P *is an* $\mathbf{A}$*-derivation of* S*"*

DEFINITION 3.1. *A sequent* S *is "provable", in symbols:* $\vdash S$, *if and only if there exists an* $\mathbf{Ax}$*-derivation of* S *where* $\mathbf{Ax}$ *is the set of axioms of* $\mathbf{L}_{\omega_1\omega}$.

Next we show that certain sequents which express some elementary properties about equality are provable in our system. Hence

DEFINITION 3.2. *If* Γ *is a set of equations (i.e. atomic formulas of the form* $x\equiv y$*) and* v *and* w *are individual variables, then "*v *is congruent to* w *modulo* Γ*", in symbols:* $v\asymp w[\Gamma]$, *if and only if for some* $n<\omega$ *there exists a* $n+1$*-sequence* $\langle y\rangle_{i\leqslant n}$ *of individual variables such that* (1) $y_0=v$, (2) $y_n=w$, *and* (3) *for all* $i<n$, *either* $y_i=y_{i+1}$ *or else* $\Gamma\cap\{y_i\equiv y_{i+1},\ y_{i+1}\equiv y_i\}\neq 0$.

[5] The author is indebted to Maehara-Takeuti [12] for this form of the rules of inference for equality.

The following properties of $v\asymp w[\Gamma]$ are easily verified:

LEMMA 3.3.

(.1) *If* $v\asymp w[\Gamma]$, *then there exists a finite set* $\Gamma'\subseteq\Gamma$ *such that* $v\asymp w[\Gamma']$,

(.2) *if* $v\asymp w[\Gamma]$ *and* $w\asymp z[\Gamma]$, *then* $v\asymp z[\Gamma]$,

(.3) *if* $v\asymp w[\Gamma]$ *then* $w\asymp v[\Gamma]$,

(.4) $v\asymp v[0]$.

LEMMA 3.4. *If for all* $i<n$, $x_i\asymp y_i[\Gamma]$, *then*

(.1) $\vdash\Gamma\to\{x_i\equiv y_i\}$, *whenever* $i<n$,

(.2) *if* φ *is an atomic formula, then*
$\vdash\Gamma\cup\{\varphi[v_0/x_0,\ldots,v_{n-1}/x_{n-1}]\}\to\{\varphi[v_0/y_0,\ldots,v_{n-1}/y_{n-1}]\}$.

Proof. It follows from the axioms and rules of inference for equality (noting that if $x\equiv y\in\Gamma$, then $\Gamma\cup\{x\equiv y\}=\Gamma$).

DEFINITION 3.5. *A sequent* $X\to Y$ *is a "fundamental sequent" just in case that either* (a) $X\to Y$ *is an axiom or* (b) *there exists a finite set of equations* $\Gamma\subseteq X$ *and a finite number of individual variables* x_i, y_i $(i<n)$ *such that* $x_i\asymp y_i[\Gamma]$ *for all* $i<n$ *and either* (i) $x_0\equiv y_0\in Y$ *or* (ii) *for some atomic formula* φ,

$$\varphi[v_0/x_0,\ldots,v_{n-1}/x_{n-1}]\in X\quad \textit{and}\quad \varphi[v_0/y_0,\ldots,v_{n-1}/y_{n-1}]\in Y\,.$$

Using lemma 3.4 we immediately obtain (by an application of the principal structural rule of inference) that

LEMMA 3.6. *If* S *is a fundamental sequent, then* S *is provable.*

DEFINITION 3.7. *If* P *is a function and* S *a sequent such that* P *is an* $\mathbf{F}$*-derivation of* S *in which the rules of inference for equality are not applied and where* $\mathbf{F}$ *is the set of fundamental sequents, then "*P *is a proof of* S*".*

By $\Vdash S$ we understand that there exists a proof of S. The following theorem can easily be proven by induction on the lengths of the domain of the proof and of the domain of the derivation respectively:

THEOREM 3.8.

(.1) *If* $\Vdash S$, *then* $\vdash S$.

(.2) *If* $\vdash S$, *then* $\models S$.

We will show in theorem 3.16 that the three notions: $\vdash S$, $\Vdash S$ and $\models S$ are equivalent. The equivalence of $\vdash S$ with $\Vdash S$ tells us that if we enlarge the set if axioms of $\mathbf{L}_{\omega_1\omega}$ to include all the fundamental sequents, then we can omit the rules of inference for equality.

To show that if $\models S$ then $\Vdash S$ we proceed as follows. With every sequent S we associate a function TA_S (called the *tableau of* S) such that (1) the domain of TA_S is a pseudo-tree, (2) the range of TA_S is a set of sequents, (3) $TA_S(0) = S$, and (4) if S is valid, then from TA_S we can obtain a proof of S, while if S is not valid then from TA_S we can obtain a (countable) relational system in which S is not valid. However, instead of defining TA_S directly we first define an auxiliary function Ta_S from which TA_S is easily defined. We shall give an informal definition for Ta_S (i.e. we define simultaneously the domain and the functional values) because the content of Ta_S more clearly expressed. The gist of the definition of Ta_S is as follows:

(a) the domain of Ta_S is to be a pseudo-tree,

(b) if $s \in \mathbf{D}Ta_S$; then $Ta_S(s)$ is to be a 2-sequence such that $Ta_S(s)_0$ and $Ta_S(s)_1$ are finite sequences of sequences (of type $\leqslant \omega$) of substitution instances of subformulas occuring in a (fixed) normal form of S ($TA_S(s)$ will then be defined by eliminating the structure of $Ta_S(s)_0$ and $Ta_S(s)_1$, i.e. by making them into sets of formulas),

(c) whenever $Ta_S(s)$ is defined, then $\{Ta_S(s^*\langle n\rangle): n < \omega \;\&\; s^*\langle n\rangle \in \mathbf{D}Ta_S\}$ is defined in such a way that there exists a $\{TA_S(s^*\langle n\rangle): n < \omega \;\&\; s^*\langle n\rangle \in \mathbf{D}Ta_S\}$ — derivation of $TA_S(s)$ in which the rules of inference for equality are not applied, and this in turn is done by breaking down the formulas occurring in the first $|s|$ places of the sequences in $Ta_S(s)_0$ and $Ta_S(s)_1$.

DEFINITION 3.9. If $S = X \rightarrow Y$ is a sequent, then Ta_S is defined as follows:

PART 1 OF DEFINITION 3.9. First we choose a normal form $S' = X' \rightarrow Y'$ of S. Then let W be a set of individual variables such that $|W| = \omega$ and $W \cap VS(S') = 0$. Then let $Z = W \cup FV(S')$ and finally let $\langle a_i\rangle_{i<\omega}$ be an enumeration of Z (by an *enumeration* we understand an enumeration without repetitions).

PART 2 OF DEFINITION 3.9. Let $S_0 = \langle S_0(i)\rangle_{i<\mu\leqslant\omega}$ and $T_0 = \langle T_0(i)\rangle_{i<\mu'\leqslant\omega}$ be enumerations of X' and Y' respectively. We "start" the definition of Ta (we omit the subscript S because there is no risk of confusion) by letting $\langle 0\rangle \in \mathbf{D}Ta$ and

$$Ta(\langle 0\rangle) = \langle S_0, T_0\rangle .$$

PART 3 OF DEFINITION 3.9. Suppose that $Ta(\langle n_0, \ldots, n_{d-1}\rangle)$ has already been defined ($d < \omega$) and that $Ta(\langle n_0, \ldots, n_{d-1}\rangle) = \langle S, T\rangle$ where S and T are finite sequences of sequences of type $\leqslant \omega$ of proper Z-substitution instances of subformulas of formulas occurring in S'. Then we define

(1) $$A = \bigcup\{\{S_i(j): j \in d \cap \mathbf{D}S_i\}: i \in \mathbf{D}S\},$$

(2) $$B = \bigcup\{\{T_i(j): j \in d \cap \mathbf{D}T_i\}: i \in \mathbf{D}T\} \sim \{\mathrm{T}\},$$

(3) $$\vec{n} = \langle n_0, \ldots, n_{d-1}\rangle .$$

Case 1. $d = 6h$. Let $\langle \neg\varphi_i\rangle_{i<m}$ be an enumeration of the negation formulas occurring in A (note that A is finite). Then $\vec{n}^*\langle j\rangle \in \mathbf{D}Ta$ if and only if $j = 0$ and

$$Ta(\vec{n}^*\langle 0\rangle) = \langle S, T^*\langle\langle\varphi_i\rangle_{i<m}\rangle\rangle .$$

Case 2. $d = 6h+1$. Corresponding case for the negation formulas in B.

Case 3. $d = 6h+2$. Let $\langle\prod_{i\in I_p}\varphi_i\rangle_{p<m}$ be an enumeration of the conjunctions occurring in A. Then let M be an m-sequence such that for each $p < m$, $M(p)$ is an enumeration of $\{\varphi_i: i \in I_p\}$ into either a finite or an ω-sequence.

Then $\vec{n}^*\langle j\rangle \in \mathbf{D}Ta$ if and only if $j = 0$ and

$$Ta(\vec{n}^*\langle 0\rangle) = \langle S^*M, T\rangle .$$

Case 4. $d = 6h+3$. Let $\langle\prod_{i\in I_p}\varphi_i\rangle_{p<m}$ be an enumeration of the conjunctions occurring in B (note then that for all $p < m$, $I_p \neq 0$). Then let $\langle\langle\pi_{jp}\rangle_{p<m}\rangle_{j<\mu\leqslant\omega}$ be an enumeration of all possible m-sequences of formulas such that for all $p < m$ and $j < \mu$, $\pi_{jp} \in \{\varphi_i: i \in I_p\}$. Then $\vec{n}^*\langle j\rangle \in \mathbf{D}Ta$ if and only if $j < \mu$ and

$$Ta(\vec{n}^*\langle j\rangle) = \langle S, T^*\langle\langle\pi_{jp}\rangle_{p<m}\rangle\rangle ,$$

whenever $j < \mu$.

Case 5. $d = 6h+4$. Let $\langle(\forall w_p)\varphi_p\rangle_{p<m}$ be an enumeration of the universally quantified formulas occurring in A. Then let

$$C = \langle\varphi_0[w_0/a_i]\rangle^*_{i<d}\langle\varphi_1[w_1/a_i]\rangle^*_{i<d}\ldots\langle\varphi_{m-1}[w_{m-1}/a_i]\rangle_{i<d} .$$

Then $\vec{n}^*\langle j\rangle \in \mathbf{D}Ta$ if and only if $j = 0$ and

$$Ta(\vec{n}^*\langle 0\rangle) = \langle S^*\langle C\rangle, T\rangle .$$

Case 6. $d = 6h+5$. Let $\langle(\forall w_p)\varphi_p\rangle_{p<m}$ be an enumeration of the universally quantified formulas in B. Then let $b_0, \ldots, b_{m-1}$ be the first m individual variables from the set Z (see part 1 of the definition) which occur neither in the range of S_i, $i \in \mathbf{D}S$, nor of T_i, $i \in \mathbf{D}T$. Then $\vec{n}^*\langle j\rangle \in \mathbf{D}Ta$ if and only if $j = 0$ and

$$Ta(\vec{n}^*\langle 0\rangle) = \langle S, T^*\langle\langle\varphi_p[w_p/b_p]\rangle_{p<m}\rangle\rangle .$$

Note that because S' is a normal form of S (i.e. no variable occurs both free and bound in S') and because of the choice of the set Z all the substitutions in cases 5 and 6 are proper Z-substitutions. Thus in all cases $Ta(\vec{n}^*\langle j\rangle)$ (when defined) is again of the required form. Thus the definition of Ta is complete.

DEFINITION 3.10. *The tableau of a sequent S, in symbols: TA_S, is the function defined as follows:*

(i) $\mathbf{D}\mathrm{TA}_S = \mathbf{D}\mathrm{Ta}_S \cup \{0\}$,

(ii) $\mathrm{TA}_S(0) = S$,

(iii) *if* $s \in \mathbf{D}\mathrm{Ta}_S$, *then*

$$\mathrm{TA}_S(s) = \bigcup\{\mathfrak{C}f\colon f \in \mathfrak{C}(\mathrm{Ta}_S(s)_0)\} \to \bigcup\{\mathfrak{C}f\colon f \in \mathfrak{C}(\mathrm{Ta}_S(s)_1)\}\,.$$

The definition of TA_S was chosen so that the following lemma holds.

LEMMA 3.11.

(.1) $\mathbf{D}\mathrm{TA}_S$ *is a pseudo-tree.*

(.2) $\mathrm{TA}_S(0) = S$.

(.3) S *is obtained from* $\mathrm{TA}_S(\langle 0\rangle)$ *by an application of the rule of inference for renaming the bound variables.*

(.4) *If* $s \in \mathbf{D}\mathrm{TA}_S \sim \{0\}$, *then there exists a* $\{\mathrm{TA}_S(s^*\langle n\rangle)\colon n < \omega \;\&\; s^*\langle n\rangle \in \mathbf{D}\mathrm{TA}_S\}$*-derivation of* $\mathrm{TA}_S(s)$ *in which neither the rule of inference for renaming the bound variables nor a rule of inference for equality are applied.*

If F is a function whose domain is a pseudo-tree and whose range is a set of sequents, then (a) by a *branch of* F we understand F restricted to a branch of the domain of F, (b) a sequent S *occurs in* F (or the *occurrence of* S *in* F *is at* s) just in case that S occurs in the range of F (or $F(s) = S$), and (c) an *occurrence of* S' *is above an occurrence of* S *in* F just in case that $F(s) = S$ and $F(s^*s') = S'$ for some $s' \neq 0$.

LEMMA 3.12. *If in each branch of* TA_S *there occurs a fundamental sequent, then there is a proof of* S *(and hence* S *is provable).*

Proof. By 3.11, 3.8 and 3.0.

LEMMA 3.13. *If* B *is a branch of* TA_S, $Z = \{a_i\colon i < \omega\}$ *is the set defined in part* 1 *of definition* 3.9 *(i.e. of* Ta_S*)*, $\mathcal{A}$ *is the union of the antecedents of the sequents occurring in* B *and* $\mathcal{S}$ *is the union of the succedents occurring in* B, *then*:

(.1) *if* $\neg\varphi \in \mathcal{A}$, *then* $\varphi \in \mathcal{S}$,

(.2) *if* $\neg\varphi \in \mathcal{S}$, *then* $\varphi \in \mathcal{A}$,

(.3) *if* $\prod_{i\in I}\varphi_i \in \mathcal{A}$, *then for all* $i \in I$, $\varphi_i \in \mathcal{A}$,

(.4) *if* $\prod_{i\in I}\varphi_i \in \mathcal{S}$ *and* $I \neq 0$, *then for some* $i \in I$, $\varphi_i \in \mathcal{S}$,

(.5) *if* $(\forall w)\varphi \in \mathcal{A}$, *then for all* $i < \omega$, $\varphi[w/a_i] \in \mathcal{A}$,

(.6) *if* $(\forall w)\varphi \in \mathcal{S}$, *then for some* $i < \omega$, $\varphi[w/a_i] \in \mathcal{S}$.

If in addition B is a branch in which there occurs no fundamental sequent, then we have:

LEMMA 3.14. *If* B *is a branch of* TA_S *in which there does not occur a fundamental sequent*, $Z = \{a_i\colon i < \omega\}$, $\mathcal{A}$ *and* $\mathcal{S}$ *are defined as in lemma* 3.13, *and* Γ *is the set of equations occurring in* $\mathcal{A}$, *then*:

(.1)-(.6) *as in lemma* 3.13,

(.7) $\mathbf{T} \notin \mathcal{S}$,

(.8) *if* $v \asymp w[\Gamma]$, *then* $v \equiv w \notin \mathcal{S}$,

(.9) *if* φ *is an atomic formula*, $n < \omega$, $x_i \asymp y_i[\Gamma]$ *for all* $i < n$ *and* $\varphi[v_0/x_0, \ldots, v_{n-1}/x_{n-1}] \in \mathcal{A}$, *then* $\varphi[v_0/y_0, \ldots, v_{n-1}/y_{n-1}] \notin \mathcal{S}$,

(.10) $\mathcal{A} \cap \mathcal{S} = 0$.

Proof. (.7) follows immediately from the condition that in B there does not occur a fundamental sequent. Suppose next that $v \asymp w[\Gamma]$. Then there exists a finite subset Γ' of Γ such that $v \asymp w[\Gamma']$. But then because Γ' is finite, Γ' must be included in the antecedent of some sequent S_1 occurring in B. Then, if contrary to (.8), $v \equiv w \in \mathcal{S}$ we would have that $v \equiv w$ occurs in the succedent of some sequent S_2 occurring in B; but then any sequent which occurred above S_1 and S_2 in B would be a fundamental sequent. Thus (.8). (.9) is proven in an analogous way (note that essential use is made of the fact that the atomic formulas are finite). From (.8) and (.9) it follows that if $\varphi \in \mathcal{A} \cap \mathcal{S}$, then φ is not an atomic formula; using (.1)-(.7) we see then that φ cannot be a compound formula. In other words $\mathcal{A} \cap \mathcal{S} = 0$.

LEMMA 3.15. *If in some branch of* TA_S *there does not occur a fundamental sequent, then* S *is not valid.*

Proof. Let B a branch of TA_S in which there does not occur a fundamental sequent. Let then $Z = \{a_i, i < \omega\}$, $\mathcal{A}$, $\mathcal{S}$ and Γ be defined as in lemma 3.14. Then for all $x \in Z$, let $[x] = \{y\colon y \in Z \;\&\; x \asymp y[\Gamma]\}$ and then let $N = \{[x]\colon x \in Z\}$. If P_η is an n-ary relational symbol occurring in S, then we define

$$R_\eta = \{\langle [x_0], \ldots, [x_{n-1}]\rangle\colon P_\eta x_0 \ldots x_{n-1} \in \mathcal{A}\}\,,$$

while if P_η does not occur in S, then we let $R_\eta = 0$. It is then immediate to verify that S is not valid in the relational system $\langle N, R_\eta\rangle_{\eta<\varrho}$.

Combining the above lemmas we finally obtain:

THEOREM 3.16. *If* S *is a sequent, then the following three conditions are equivalent*:

(i) $\models S$,

(ii) $\vdash S$,

(iii) $\Vdash S$.

It follows from theorem 3.16 that the system of axioms and rules of inferences given for $\mathbf{L}_{\omega_1\omega}$ is complete. It also follows from the proof of theorem 3.16 that if S a valid sequent in which the equality symbol does not occur, then there exists an $\mathbf{Ax}'$-derivation of S in which no rule of inference for equality is applied and where $\mathbf{Ax}'$ is the set of logical axioms.

4. The interpolation theorem for $\mathbf{L}_{\omega_1\omega}$.

THEOREM 4.0. *If $X \to Y$ is a valid sequent, then to every pair of partitions (A_1, A_2) and (B_1, B_2) of X and Y respectively there corresponds a formula φ, called an "interpolating formula for $(A_1 \to B_1, A_2 \to B_2)$", such that*

(.1) $A_1 \to B_1 \cup \{\varphi\}$ *and* $A_2 \cup \{\varphi\} \to B_2$ *are provable,*

(.2) $\mathrm{FV}(\varphi) \subseteq \mathrm{FV}(A_1 \to B_1) \cap \mathrm{FV}(A_2 \to B_2)$,

(.3) *if the equality symbol occurs in φ then it occurs in $X \to Y$,*

(.4) *if a relational symbol has a positive (negative) occurrence in φ, then it has a positive (negative) occurrence in $A_2 \to B_2$ and a negative (positive) occurrence in $A_1 \to B_1$.*

Proof. Assume that $X \to Y$ is a valid sequent. Then by theorem 3.16 there exists a proof P of $X \to Y$. Let μ be the length of $\mathbf{D}P$.

Part I: $\mu = 0$, Then $X \to Y$ is a fundamental sequent and for fundamental sequents the theorem is easily verified.

Part II: $\mu > 0$. Let r be the root of $\mathbf{D}P$ (hence $P(r) = X \to Y$) and then let $C = \{r^*\langle n\rangle : n < \omega \;\&\; r^*\langle n\rangle \in \mathbf{D}P\}$. For each $c \in C$, let $P_c = P{\restriction}\{s : s \in \mathbf{D}P \;\&\; c \subseteq s\}$; then P_c is a proof of the of the sequent $P(c)$ such that the length of $\mathbf{D}P_c$ is strictly smaller than μ. Thus we assume (induction hypothesis) that the theorem is true for all the sequents $P(c)$ where $c \in C$, and then we shall show that the theorem is also true for $P(r)$ (i.e. $X \to Y$). Because of the definition of a proof (cf. definition 3.7) it follows that $X \to Y$ is the conclusion of a rule of inference, other than a rule of inference for equality, with $\{P(c) : c \in C\}$ as the set of premises. We shall consider only the case when the rule applied is $\forall \to$, the remaining cases being similar. In this case then $C = \{c\}$ and $P(c)$ must be of the form

$$A_1' \cup A_2' \cup \{\Phi[v/w]\} \to B_1 \cup B_2$$

where either

(a) $$A_1' = A_1 \quad \text{and} \quad A_2' \cup \{(\forall z)\Phi[v/z]\} = A_2$$

or

(b) $$A_1' \cup \{(\forall z)\Phi[v/z]\} = A_1 \quad \text{and} \quad A_2' = A_2 .$$

Suppose (a) (the case for (b) is analogous). Then by the induction hypothesis there exists an interpolating formula π for $(A_1 \cup B_1, A_2' \cup \{\Phi[v/w]\} \to B_2)$. Hence

(i) $$\vdash A_1 \to B_1 \cup \{\pi\}$$

(ii) $$\vdash A_2' \cup \{\Phi[v/w]\} \cup \{\pi\} \to B_2 .$$

Hence from (ii) and $\forall \to$, we obtain:

$$\vdash A_2' \cup \{(\forall z)\Phi[v/z]\} \cup \{\pi\} \to B_2 ,$$

that is:

(iii) $$\vdash A_2 \cup \{\pi\} \to B_2 .$$

Because π is an interpolating formula for $(A_1 \to B_1, A_2' \cup \{\Phi[v/w]\} \to B_2)$ it follows that $\mathrm{FV}(\pi) \subseteq (\mathrm{FV}(A_1 \to B_1) \cap \mathrm{FV}(A_2 \to B_2)) \cup \{w\}$ and that π, $(\forall w)\pi$ and $(\mathrm{E}w)\pi$ (i.e. $\neg(\forall w)\neg\pi$) satisfy conditions (.3) and (.4) of the theorem.

Case 1. $w \in \mathrm{FV}(A_1 \to B_1) \cap \mathrm{FV}(A_2 \to B_2)$.

Then we may take π as an interpolating formula for $(A_1 \to B_1, A_2 \to B_2)$.

Case 2. $w \in \mathrm{FV}(A_2 \to B_2)$ and $w \notin \mathrm{FV}(A_1 \to B_1)$.

Then from (i) and (iii) we obtain

(iv) $$\vdash A_1 \to B_1 \cup \{(\forall w)\pi\} ,$$

(v) $$\vdash A_2 \cup \{(\forall w)\pi\} \to B_2 .$$

That is, $(\forall w)\pi$ is a suitable interpolation formula for $(A_1 \to B_1, A_2 \to B_2)$.

Case 3. $w \in \mathrm{FV}(A_1 \to B_1)$ and $w \notin \mathrm{FV}(A_2 \to B_2)$.

In this case $(\mathrm{E}w)\pi$ is a suitable interpolation formula for $(A_1 \to B_1, A_2 \to B_2)$.

Case 4. $w \notin \mathrm{FV}(A_1 \to B_1) \cup \mathrm{FV}(A_2 \to B_2)$.

In this case either $(\forall w)\pi$ or $(\mathrm{E}w)\pi$ would be a suitable interpolation formula for $(A_1 \to B_1, A_2 \to B_2)$.

THEOREM 4.1. *If $\varphi \to \Phi$ is a valid formula of $\mathbf{L}_{\omega_1\omega}$, then there exists a formula π, called an "interpolating formula for $\varphi \to \Phi$" such that:*

(.1) *$\varphi \to \pi$ and $\pi \to \Phi$ are valid formulas of $\mathbf{L}_{\omega_1\omega}$,*

(.2) $\mathrm{FV}(\pi) \subseteq \mathrm{FV}(\varphi) \cap \mathrm{FV}(\Phi)$,

(.3) *if the equality symbol occurs in π, then it occurs in $\varphi \to \Phi$,*

(.4) *if a relational symbol has a positive (negative) occurrence in π, then it has a positive (negative) occurrence both in φ and in Φ.*

Proof. Apply theorem 4.0 to the sequent $\{\varphi\} \to \{\Phi\}$.

5. Applications of the interpolation theorem for $\mathbf{L}_{\omega_1\omega}$.

Just as in the case of $\mathbf{L}_{\omega\omega}$ an immediate consequence of the interpolation theorem is the definability theorem, i.e. that explicit definability coincides with implicit definability (cf. Beth [1]).

THEOREM 5.0. *If φ, Φ are formulas of $\mathbf{L}_{\omega_1\omega}$ such that Φ is obtained from φ by replacing, for every $\mu < \eta$, the n_μ-ary relational symbol P_μ by the n_μ-ary relational symbol Q_μ not already occurring in φ (and such that $Q_\mu \neq Q_{\mu'}$ whenever $\mu \neq \mu'$) and if*

$$\models (\varphi \wedge \Phi \to (\forall v_0)\ldots(\forall v_{n-1})(P_0 v_0 \ldots v_{n_0-1} \leftrightarrow Q_0 v_0 \ldots v_{n_0-1}))$$

then there exists a formula π such that

(i) $\mathrm{FV}(\pi) \subseteq \{v_i : i < n_0\}$,

(ii) *for all* $\mu < \eta$, P_μ *does not occur in* π,

(iii) *if a relational symbol occurs in* π, *then it occurs in* φ,

(iv) $\models \big(\varphi \to (\nabla v_0)\dots(\nabla v_{n_0-1})(\pi \longleftrightarrow P_0 v_0 \dots v_{n_0-1})\big)$.

The proof of theorem 5.0 is essentially the same proof as in the case of $\mathbf{L}_{\omega\omega}$ and is thus omitted.

A sentence $\varphi \in \mathbf{L}_{\omega_1\omega}$ is *preserved under homomorphism* just in case that: for all structures $\mathfrak{A}$, if $\mathfrak{A} \models \varphi$, then $\mathfrak{B} \models \varphi$ whenever $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$. If we repeat (mutatis mutandis) the proof in Lyndon [10] we obtain:

THEOREM 5.1 (Homomorphism theorem). *If* φ *is a sentence of* $\mathbf{L}_{\omega_1\omega}$ *then* φ *is preserved under homomorphisms if and only if* φ *is equivalent to a positive sentence.*

As a final application of the interpolation theorem for $\mathbf{L}_{\omega_1\omega}$ we give theorem 5.3 below. However we must first give some definitions.

Suppose that f is a function whose domain is the power set of ω^2 and whose range is included in the power set ω^3; then f is an *elementary function* just in case that there exists a formula $\varphi \in \mathbf{L}_{\omega_1\omega}$ (which will then be called a *description of* f) such that (i) $\mathrm{FV}(\varphi) = \{v_i\colon\ i < 3\}$, (ii) the only relational symbol occurring in φ is a binary relation symbol and (iii) for all $R \subseteq \omega^2$,

$$f(R) = \{\langle n_i\rangle_{i<3}\colon\ (\langle\omega, R\rangle, \langle n_i\rangle_{i<3}) \models \varphi\}\,. \ (^6)$$

Introducing the natural topology in the power sets of ω^2 and ω^3 it is clear what is meant by "f is a Borel function". If σ is a permutation of ω and $R \subseteq \omega^i$, then we let $\sigma(R) = \{\langle\sigma(n_j)\rangle_{j<i}\colon\ \langle n_j\rangle_{j<i} \in R\}$. If f is a function with domain the power set of ω^2 and range consisting of subsets of ω^3, then f is an *invariant function* just in case that for all permutations σ of ω and all $R \subseteq \omega^2$, $f\big(\sigma(R)\big) = \sigma\big(f(R)\big)$. A set $X \subseteq \mathbf{P}(\omega^2)\times\mathbf{P}(\omega^3)$ is an *elementary set* just in case that there exists a sentence Φ of $\mathbf{L}_{\omega_1\omega}$ such that the only relational symbols occurring in Φ are a binary and ternary relational symbol and such that $X = \{\langle R, S\rangle\colon\ \langle\omega, R, S\rangle \models \Phi\}$. The notions of an *invariant set* and a *Borel set* are analogously defined.

Before the author had shown the interpolation theorem for $\mathbf{L}_{\omega_1\omega}$, D. S. Scott had shown that: *if the interpolation theorem is true for* $\mathbf{L}_{\omega_1\omega}$ *then a set is invariant and Borel if and only if it is an elementary set* (⁷) (and in fact D. S. Scott's result was a motivation for studying the interpolation theorem for $\mathbf{L}_{\omega_1\omega}$). If we slightly modify the proof of Scott's (unpublished) result, in particular if we use the definability theorem instead of the interpolation theorem, we obtain that the elementary functions are exactly the invariant Borel functions. First we show the following lemma

LEMMA 5.2. *To every Borel set* $X \subseteq \mathbf{P}(\omega^2)\times\mathbf{P}(\omega^3)$ *there corresponds a sentence* $\varphi \in \mathbf{L}_{\omega_1\omega}$ *such that* (i) *the relational symbols occurring in* φ *are: two binary relation symbols, a ternary relation symbol, and for each* $n < \omega$, *a unary relation symbol* P_n *and* (ii)

$$X = \{\langle R, S\rangle\colon\ \langle\omega, <, \{n\}, R, S\rangle_{n<\omega} \models \varphi\}\,.$$

Proof. Let Φ be the sentence of $\mathbf{L}_{\omega_1\omega}$ which characterizes (up to isomorphism) the relational system $\langle\omega, <, \{n\}\rangle_{n<\omega}$. Note that the elements of the subbase for the topology for $\mathbf{P}(\omega^2)\times\mathbf{P}(\omega^3)$ are of form either $B_{ij} = \{\langle R, S\rangle\colon\ \langle i, j\rangle \in R\}$, or $B^{ijk} = \{\langle R, S\rangle\colon\ \langle i, j, k\rangle \in S\}$ or their complements. Now it is clear that

$$B_{ij} = \{\langle R, S\rangle\colon\ \langle\omega, <, \{n\}, R, S\rangle_{n<\omega} \models (\mathrm{E}v_0)(\mathrm{E}v_1)(P_i v_0 \wedge P_j v_1 \wedge Q v_0 v_1) \wedge \Phi\}$$

and similarly for B^{ijk} or their complements. But the Borel sets are built up from these sets by means of countable (propositional) operations. Thus a simple induction on the rank of the Borel set completes the proof of the lemma.

THEOREM 5.3. *If* f *is a function whose domain is the power set of* ω^2 *and whose range is included in the power set of* ω^3, *then* f *is an elementary function if and only if* f *is an invariant Borel function.* (⁶)

Proof. Let Φ be the sentence of $\mathbf{L}_{\omega_1\omega}$ which characterizes the relational system $\langle\omega, <, \{n\}\rangle_{n<\omega}$. The proof is completed in 3 parts.

Part I. *If* f *is elementary then* f *is an invariant and Borel function.* Suppose that f is elementary. Then let φ be the description of f. It is clear, because of the way satisfaction is defined, that f is then invariant. To prove that f is Borel we use induction on the rank of the formula (noting that the quantifier also corresponds to a countable operation since we are considering relational systems whose domains are ω).

Part II. *If* f *is a Borel function, then the graph of* f (i.e. $\{\langle R, f(R)\rangle\colon R \subseteq \omega^2\}$) *is a Borel set.*

For a proof of the above, see Kuratowski [8], p. 291.

Part III. *If* f *is an invariant Borel function whose domain is the power set of* ω^2, *then* f *is an elementary function.*

By part II, the graph of f is a Borel set, and hence by lemma 5.2 there exists a sentence φ such that:

(a) $$\{\langle R, f(R)\rangle\colon\ R \subseteq \omega^2\} = \{\langle R, S\rangle\colon\ \langle\omega, <, \{n\}, R, S\rangle_{n<\omega} \models \varphi\}\,.$$

Then let ψ be the formula obtained from $\varphi \wedge \Phi$ by replacing the ternary relational symbol Q by a new ternary relation symbol Q', the relation

(⁶) The restriction to functions f such that $\mathbf{D}f = \mathbf{P}(\omega^2)$ and $\mathbf{G}f \subseteq \mathbf{P}(\omega^3)$ is clearly not essential, i.e. corresponding results hold for functions f such that $\mathbf{D}f = \mathbf{P}(\omega^{n_0}\times\dots\times\omega^{n_s})$ and $\mathbf{G}f \subseteq \mathbf{P}(\omega^{m_0}\times\dots\times\omega^{m_t})$.

(⁷) C. Ryll-Nardzewski has shown the converse.

symbol corresponding to $<$ by a new binary relation symbol and also replacing, for every $n<\omega$, the unary relational symbol P_n by a new unary relational symbol P'_n. Then because f is an invariant function and Φ characterizes (up to isomorphism) the relational system $\langle\omega, <, \{n\}\rangle_{n<\omega}$ we then obtain from (a) that

(b) $$\models \Psi \wedge \varphi \wedge \Phi \rightarrow (\forall v_0)(\forall v_1)(\forall v_2)(Qv_0v_1v_2 \longleftrightarrow Q'v_0v_1v_2)\,.$$

Thus applying the definability theorem (theorem 5.0) we obtain that there exists a formula π, such that $\mathrm{FV}(\pi)\subseteq\{v_i\colon i<3\}$, the only relational symbol occurring in π is a binary relational symbol and such that

(c) $$\models \Phi \wedge \varphi \rightarrow (\forall v_0)(\forall v_1)(\forall v_2)(\pi \longleftrightarrow Qv_0v_1v_2)\,.$$

But because the domain of f is the whole of the power set of ω^2 we also have that

(d) $$\models \Phi \rightarrow \big((\forall v_0)(\forall v_1)(\forall v_2)(\pi \longleftrightarrow Qv_0v_1v_2) \rightarrow \varphi\big)\,.$$

Because suppose that (d) were false, then there would exist $R_0\subseteq\omega^2$ and $S_0\subseteq\omega^3$ such that

(i) $$\langle\omega, <, \{n\}, R_0, S_0\rangle_{n<\omega} \models (\forall v_0)(\forall v_1)(\forall v_2)(\pi \longleftrightarrow Qv_0v_1v_2)\,,$$

(ii) $$\langle\omega, <, \{n\}, R_0, S_0\rangle_{n<\omega} \models \neg\varphi\,.$$

From (a) we then obtain that

(iii) $$\langle\omega, <, \{n\}, R_0, f(R_0)\rangle_{n<\omega} \models \varphi\,.$$

From (iii) and (c) we obtain that

(iv) $$\langle\omega, <, \{n\}, R_0, f(R_0)\rangle_{n<\omega} \models (\forall v_0)(\forall v_1)(\forall v_2)(\pi \longleftrightarrow Qv_0v_1v_2)\,.$$

From (iv) and (i), because Q does not occur in π, we obtain that $S_0=f(R_0)$, but then (iii) and (ii) would contradict each other. Thus combining (c) with (d) we have that

$$\models \Phi \rightarrow \big(\varphi \longleftrightarrow (\forall v_0)(\forall v_1)(\forall v_2)(\pi \longleftrightarrow Qv_0v_1v_2)\big)\,,$$

i.e.

$$f(R) = \{\langle n_i\rangle_{i<3}\colon (\langle\omega, R\rangle, \langle n_i\rangle_{i<3}) \models \pi\}\,.$$

6. Model-theoretical versions of the interpolation theorem for $\mathbf{L}_{\omega_1\omega}$.

DEFINITION 6.0. *If* $\mathbf{K}$ *is a class of relational systems of type* ν, *then* $\mathbf{K}$ *is an "elementary class" in symbols:* $\mathbf{K}\in\mathbf{EC}(\mathbf{L}_{\omega_1\omega})$ [*"elementary class in the wide sense" in symbols:* $\mathbf{K}\in\mathbf{EC}_\Delta(\mathbf{L}_{\omega_1\omega})$] *just in case that there exists a sentence* $\varphi\in\mathbf{L}_{\omega_1\omega}$ [*a set of sentences* $\Gamma\subseteq\mathbf{L}_{\omega_1\omega}$] *such that the relational symbols occurring in* φ [*in* Γ] *correspond to the similarity type* ν *and such that* $\mathbf{K}$ *is the class of all models of* φ [*of* Γ].

DEFINITION 6.1. *If* $\mathbf{K}$ *is a class of relational systems of type* ν, *then* $\mathbf{K}$ *is a "pseudo-elementary class", in symbols:* $\mathbf{K}\in\mathbf{PC}(\mathbf{L}_{\omega_1\omega})$ [*a "pseudo-elementary class in the wide sense" in symbols:* $\mathbf{K}\in\mathbf{PC}_\Delta(\mathbf{L}_{\omega_1\omega})$] *just in class there exists a similarity type* $\nu'\supseteq\nu$ *and a class* $\mathbf{M}$ *of relational systems of type* ν' *such that* $\mathbf{K}$ *is the class of all* ϱ_ν*-reducts of the members of* $\mathbf{M}$ *and such that* $\mathbf{M}\in\mathbf{EC}(\mathbf{L}_{\omega_1\omega})$ [$\mathbf{M}\in\mathbf{EC}_\Delta(\mathbf{L}_{\omega_1\omega})$].

The interpolation theorem for $\mathbf{L}_{\omega_1\omega}$ (theorem 4.1) can then be expressed in the form:

THEOREM 6.2. *If* $\mathbf{K}$, $\mathbf{K}'$ *are classes of relational systems of type* ν *such that* $\mathbf{K}\cap\mathbf{K}'=0$ *and such that* $\mathbf{K}\in\mathbf{PC}(\mathbf{L}_{\omega_1\omega})$, $\mathbf{K}'\in\mathbf{PC}(\mathbf{L}_{\omega_1\omega})$, *then, there exists a class* $\mathbf{N}$ *of relational systems of type* ν *such that* $\mathbf{K}\subseteq\mathbf{N}$, $\mathbf{N}\cap\mathbf{K}'=0$ *and* $\mathbf{N}\in\mathbf{EC}(\mathbf{L}_{\omega_1\omega})$.

In the case of $\mathbf{L}_{\omega\omega}$ the corresponding theorem is also true for $\mathbf{PC}_\Delta$-classes (cf. Keisler [6]). However in the case of $\mathbf{L}_{\omega_1\omega}$ it is not so, in fact we have the following:

THEOREM 6.3. *The classes* $\mathbf{K}_1=\{\mathfrak{A}\colon \mathfrak{A}$ is isomorphic to $\langle\omega_1, \varepsilon_{\omega_1}\rangle\}$, *and* $\mathbf{K}_2=\{\mathfrak{A}\colon \mathfrak{A}$ is a non-well-ordered linear ordering$\}$ *are such that* $\mathbf{K}_1\in\mathbf{PC}_\Delta(\mathbf{L}_{\omega_1\omega})$, $\mathbf{K}_2\in\mathbf{PC}(\mathbf{L}_{\omega_1\omega})$, $\mathbf{K}_1\cap\mathbf{K}_2=0$ *and such that there does not exists a class* $\mathbf{N}$ *which is closed under* $\mathbf{L}_{\omega_1\omega}$ *elementary equivalence and* $\mathbf{K}_1\subseteq\mathbf{N}$ *and* $\mathbf{K}_2\cap\mathbf{N}=0$. (8)

Proof. That $\mathbf{K}_2\in\mathbf{PC}(\mathbf{L}_{\omega_1\omega})$ and $\mathbf{K}_1\cap\mathbf{K}_2=0$ is immediate. To show that $\mathbf{K}_1\in\mathbf{PC}_\Delta(\mathbf{L}_{\omega_1\omega})$, we note first that $\langle\omega_1, \epsilon_{\omega_1}\rangle$ can be characterized up to isomorphism as a relational system $\langle A, R\rangle$ such that (i) R is a linear ordering of A, (ii) for every $\mu<\omega_1$, there exists an initial segment of $\mathfrak{A}$ isomorphic to $\langle\mu, \epsilon_\mu\rangle$, and (iii) for every $a\in A$, the set of R-predecessors of a is countable. Condition (iii) can be expressed in the following form: (iii′) there exists a ternary relation F on A such that for each $a\in A$, $\{(x, y)\colon \langle a, x, y\rangle\in F\}$ is a function whose domain is (isomorphic to) ω and whose range is the set of R-predecessors of a. Since every countable ordinal can be characterized by a single sentence of $\mathbf{L}_{\omega_1\omega}$ (cf. [9] and [13]), it follows then from (i), (ii) and (iii′) that $\mathbf{K}_1\in\mathbf{PC}_\Delta(\mathbf{L}_{\omega_1\omega})$. Suppose next that $\mathbf{N}$ is a class such that:

(*) $$\mathbf{K}_1\subseteq\mathbf{N} \quad\text{and}\quad \mathbf{K}_2\cap\mathbf{N}=0\,.$$

Then $\langle\omega_1, \varepsilon_{\omega_1}\rangle\in\mathbf{N}$ and all linear orderings in $\mathbf{N}$ must be well-orderings. Hence since ω_1 and $\omega_1+\omega_1\omega^*$ (considered as linear orderings and where ω^* is the order type of the negative integers) are $\mathbf{L}_{\omega_1\omega}$-elementarily equivalent (cf. [5] and [9]), it follows that if $\mathbf{N}$ satisfies the condition (*) then $\mathbf{N}$ cannot be closed under $\mathbf{L}_{\omega_1\omega}$-elementary equivalence (and in particular $\mathbf{N}$ cannot be an elementary class).

(8) Two relational systems $\mathfrak{A}$ and $\mathfrak{B}$ of the same similarity type ν are $\mathbf{L}_{\omega_1\omega}$-elementarily equivalent just in case that for every sentence $\varphi\in\mathbf{L}_{\omega_1\omega}$ such that the relational symbols occurring in φ correspond to the similarity type ν, $\mathfrak{A}\models\varphi$ if and only if $\mathfrak{B}\models\varphi$.

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On a Freedman's problem

by

C. Ryll-Nardzewski (Wrocław)

Given a compact and metrisable space X, let us consider the space 2^X of all closed subsets of X (cf. [1], p. 106) and an arbitrary fixed $Z \in 2^X$. Let $Z^0 \subset 2^X$ denote the set of all homeomorphs of Z contained in X. D. Freedman has conjectured that Z^0 is always a Borel set in 2^X and, in fact, for the case where X is the Cantor dyadic set this was proved by D. Scott ([2], pp. 126-128). Our aim now is to prove the general statement for an arbitrary X. This is based on a refinement of the method of [3] (1).

THEOREM 1. *If F is an arbitrary group of autohomeomorphisms of a separable topological space G admitting a complete metrization* (2) *and a continuous map φ from G into a metric space satisfies the condition*

(C) *the class of all level-sets of φ and the class of all F-orbits are identical, i.e.*

$$\big\{\{g\colon \varphi(g)=\varphi(g_0)\}\colon g_0 \in G\big\} = \big\{\{gf\colon f\in F\}\colon g\in G\big\},$$

then $\varphi(G)$ is an absolutely Borel set (i.e. every homeomorph of $\varphi(G)$ in any metric space is Borel).

Proof. The decomposition of G given by $F_g = \{gf\colon f\in F\}$ is open in the sense of [3] since

$$\{g\colon F_g \cap U \neq 0\} = \bigcup_{f\in F}\{g\colon gf \in U\}.$$

Let S be a Borel selector given by the Lemma (see [3], p. 129). The continuous mapping φ is one-to-one on S and $\varphi(S)=\varphi(G)$ (S is a selector), whence $\varphi(G)$ is an absolutely Borel set (cf. [1], p. 396).

THEOREM 2. *The set Z^0* (introduced at the begining) *is Borel.*

Proof. The set G of all homeomorphisms of Z into X is a G_δ set in the space X^Z of all continuous maps of Z into X with the topology

(1) [2] and [3] give information on other topics similar to those presented in this note.

(2) Let us recall that every G_δ set in a complete metric space always admits a complete metrization topologically equivalent to the original one ([1], p. 316), e.g. the set $N!$ of [2] is such a set in the space N^N of [2].