Quasi-universal flows and semi-flows

by

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1. All spaces on which groups are to act will be compact and metric. (1) (2) As general references for language and background we cite [4], [5], and [6], Chapter 5. A *flow* on a space X is a transformation group (G, X) with G either isomorphic to the integers (a *discrete* flow) or the reals (a *continuous* flow). A discrete flow may be considered as generated by any one homeomorphism (the unit of G) of X onto itself. A *semi-flow* may be considered as arising from a mapping (3) μ of X into itself and consists of the semi-group of mappings formed by the identity and the various (positive) iterates of μ.

A flow (G, X_1) on a space X_1 is called *universal* if for every flow (G, X_2), for X_2 a copy of X_1, there exists a homeomorphism φ of a closed subset X' of X_1 onto X_2 such that for each $p \in X'$ and $g \in G$, $\varphi g(p) = g\varphi(p)$. The set X' will be invariant under G. In short, (G, X_1) is universal provided every flow (G, X_2), for X_2 homeomorphic to X_1, can be imbedded in (G, X_1). If (G, X_1) is universal and the flow is discrete, we call that homeomorphism of X_1 which is the unit of G a *universal homeomorphism* of X_1.

A flow (G, X) is called *quasi-universal* (4) provided that for any flow (G, Y), for Y any compact metric space, there exist a closed subset Z of X and a mapping φ of Z onto Y such that for each $z \in Z$ and $g \in G$,

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(1) As usual, to get theorems about locally compact metric spaces we may use one-point compactifications and require that all homeomorphisms carry the adjoined point onto itself.

(2) The main results of this paper follow from results and methods of [1] and two theorems of Baayen and de Groot communicated orally to me. Proofs are included herein. It is my understanding that they intend to publish a rather general theory of linearization of mappings which will include these theorems. The two theorems follow from methods developed earlier by de Groot [2] as used in [3].

(3) A mapping is a continuous transformation.

(4) or, more specifically, quasi-universal with respect to all flows (discrete or continuous as appropriate) on any compact metric space.

$\varphi g(z) = g\varphi(z)$. The set Z will be invariant under G. We say also that (G, Z) is a *subflow* of (G, X) and that (G, Y) is *raised* to (G, Z).

We similarly define universal and quasi-universal semi-flows. A mapping μ of X into itself is called *universal* and generates a *universal semi-flow* provided that for any mapping λ of X into itself, there exists a homeomorphism φ of a closed subset X' of X onto X such that for each $p \in X'$, $\varphi\mu(p) = \lambda\varphi(p)$.

A mapping μ of X into itself generates a *quasi-universal semi-flow* provided that for any mapping ϱ of a compact metric space Y into itself there exist a closed subset Z of X and a mapping φ of Z onto Y such that, for each $p \in Z$, $\varphi\mu(p) = \varrho\varphi(p)$. We may also say that ϱ is *raised* to μ (acting on Z).

In this paper we shall prove that there exists a quasi-universal discrete flow on the Cantor set C (5) (Theorem III), that there exist quasi-universal continuous flows on the solid torus in Euclidean 3-space and on a certain 1-dimensional continuum (Theorems VII and VI), and that there exists a quasi-universal semi-flow on C (Theorem X). In Theorems IV and V, some characterizations of a natural quasi-universal discrete flow on C are given.

2. Discrete flows. The following theorem is established in [1]; we shall not prove it here.

THEOREM I. *Any discrete flow on a compact metric space can be raised to a flow on a Cantor set.*

Let the Cantor set C be represented as the infinite product $\prod_{-\infty<i<\infty} C_i$ of copies of the Cantor set and let $p \in C$ be represented as $\{p_i\}$, $p_i \in C_i$. For each i, let φ_i be a homeomorphism of C_{i+1} onto C_i which, for convenience, could be taken as canonical if all the C_i's have the same representation. Let Φ be a homeomorphism of C onto C given by $\Phi(\{p_i\}) = \{p'_i\}$ where $p'_i = \varphi_i(p_{i+1})$, i.e. the image of $\{p_i\}$ under Φ is that point whose ith coordinate is $\varphi_i(p_{i+1})$ for each i.

THEOREM II (Baayen and de Groot). *The homeomorphism Φ is a universal homeomorphism of C onto itself.*

Proof. Let C' be a copy of the Cantor set and let η be a homeomorphism of C' onto itself. We wish to exhibit a homeomorphism θ of C' into C such that $\theta^{-1}\Phi\theta = \eta$, (in which case θ^{-1} is the definitional homeomorphism desired). Let θ_0 be a homeomorphism of C' onto C_0. For each $c \in C'$, we define $\theta(c) = \{p_i\}$, $p_i \in C_i$ where

$$p_0 = \theta_0(c)\,,$$
$$p_i = \begin{cases} \varphi_{i-1}^{-1}\cdots\varphi_1^{-1}\varphi_0^{-1}\theta_0\eta^i(c)\,, & i>0\,,\\ \varphi_i\cdots\varphi_{-1}\theta_0\eta^i(c)\,, & i<0\,.\end{cases}$$

It is easy to see that θ is 1-1 and continuous into C. As C' is compact so is $\theta(C')$ and thus θ is a homeomorphism.

We wish to observe that Φ carries $\theta(C')$ onto itself and that for each $c \in C'$, $\Phi\theta(c) = \theta\eta(c)$. Applying η and then θ means that $p_0 = \theta_0\eta(c)$.

$$p_i = \begin{cases} \varphi_{i-1}^{-1}\cdots\varphi_1^{-1}\varphi_0^{-1}\theta_0\eta^{i+1}(c)\,, & i>0\,,\\ \varphi_i\cdots\varphi_{-1}\theta_0\eta^{i+1}(c)\,, & i<0\,.\end{cases}$$

However first applying θ produces

$$p'_0 = \theta_0(c)\,.$$
$$p'_i = \begin{cases} \varphi_{i-1}^{-1}\cdots\varphi_1^{-1}\varphi_0^{-1}\theta_0\eta^i(c)\,, & i>0\,,\\ \varphi_i\cdots\varphi_{-1}\theta_0\eta^i(c)\,, & i<0\,.\end{cases}$$

Then applying Φ, noting that $p'_{i+1} \to p_i$, we get

$$p_0 = \varphi_0\varphi_0^{-1}\theta_0\eta(c)\,,$$
$$p_i = \begin{cases} (\varphi_i\varphi_i^{-1})\varphi_{i-1}^{-1}\cdots\varphi_1^{-1}\varphi_0^{-1}\theta_0\eta^{i+1}(c)\,, & i>0\,,\\ (\varphi_i)\varphi_{i+1}\cdots\varphi_{-1}\theta_0\eta^{i+1}(c)\,, & i<0\,,\end{cases}$$

as we wished to show. We also note that this last observation shows that Φ carries each point of $\theta(C')$ to a point of $\theta(C')$. A similar argument suffices for Φ^{-1}. Thus $\theta(C')$ is invariant under Φ, completing the proof.

As an immediate corollary of Theorems I and II we have

THEOREM III. *The flow generated by Φ is a quasi-universal discrete flow on C.*

Remark. Since Φ is a universal homeomorphism of C onto itself, it is immediate that any homeomorphism of C which "contains" a copy of Φ on a closed subset is also a universal homeomorphism. There are many such, including, for instance, one which is the identity on some open set in C. Thus it is not a priori clear that Φ is "the" universal homeomorphism which merits study. However, in the ensuing discussion of Φ, it will be shown (Theorems IV and V) that Φ is, in fact, a naturally arising homeomorphism from various points of view.

OTHER REPRESENTATIONS OF Φ. The Cantor set C can be thought of as the space of all functions f_a defined over the set I of all integers into the set $\{0, 1\}$. The shift σ_0 of symbolic dynamics (see [5], for instance)

(5) By a *Cantor set* is meant a space homeomorphic to a Cantor ternary set. A cantor set may be characterized as a zero-dimensional compact metric space with no isolated points. In this note, various representations of the Cantor set will be used most of which are standard and all of which are easy to verify as being the Cantor set.

is the homeomorphism of C onto itself that corresponds a function f_α to a function f_β where, for each i, $|i| < \infty$, $f_\alpha(i) = f_\beta(i-1)$. The shift σ_0 has exactly two fixed points, the two constant functions, and thus is not "the same" as Φ which has a Cantor set of fixed points. Φ can be thought of as defined like σ_0 but with a geometric Cantor Set replacing the pair $\{0, 1\}$.

In [1], an "infinite" shift σ is defined essentially as follows: Let $\{N_j\}$, $|j| < \infty$, be a collection of disjoint subsets of I with $\bigcup_{|j|<\infty} N_j = I$ and with each N_j unbounded positively and negatively. For each i, let i' denote the unique predecessor of i in the N_j to which i belongs. Then σ is that homeomorphism of C onto itself that corresponds f_α to f_β where for each i, $|i| < \infty$, $f_\alpha(i) = f_\beta(i')$. Thus σ is the homeomorphism obtained by simultaneously using "ordinary" shifts on each of countably many Cantor sets C_j whose product may be considered as C. Hence, σ is an "infinite shift". In a sense, Φ permutes coordinate Cantor sets while σ preserves them. However we have the following theorem.

THEOREM IV. *There is a representation of C such that Φ and σ can be interpreted as the same homeomorphism.*

Proof. Let C be regarded as the set of all functions g_α from the set of all ordered pairs (i, j) of integers to the set $\{0, 1\}$. Then σ may be regarded as the homeomorphism of C onto itself that corresponds a function g_α to a function g_β where for each (i, j), $|i|, |j| < \infty$, $g_\alpha(i, j) = g_\beta(i-1, j)$. But let $C_i = \{f_{i,\alpha} | f_{i,\alpha}$ is a function from $\{(i, j)\}$ for fixed i and all $|j| < \infty$, into $\{0, 1\}\}$ and let φ_i map C_{i-1} onto C_i canonically. Then if we regard C as $\prod_{|i|<\infty} C_i$, it follows that Φ is precisely the homeomorphism σ of C onto itself. Thus Theorem IV is proved.

For any n, $1 \leqslant n < \infty$, C may be represented as the space of all functions from the set of all ordered n-tuples of integers to the set $\{0, 1\}$. For any j, $1 \leqslant j \leqslant n$, one can define a "shift" $\sigma(j, n)$ with respect to the first j of the coordinates of the n-tuples (the "shift" being simultaneous in all of the first j coordinates and the homeomorphism being the identity on the other $(n-j)$-coordinates). The following theorem is easy to prove either by analogy with the original description of σ or with that of Φ.

THEOREM V. *If $n > 1$, every $\sigma(j, n)$, $1 \leqslant j \leqslant n$, is conjugate to σ (or Φ), i.e. there is a homeomorphism η of C onto itself such that $\sigma(j, n) = \eta^{-1}\sigma\eta$. If $n = 1$, then $\sigma(1, 1)$ is σ_0 by definition.*

Thus it is seen that, in the above sense, the shift σ_0 is a rather special homeomorphism while the infinite shift besides being a universal homeomorphism on C is, in the sense of Theorem V at least, *the* natural generalization of σ_0.

3. Continuous flows. While the results of this section follow easily from constructions of [1] and the results of Section 2, the final result (Theorem VII) that there exists a quasi-universal continuous flow on the solid torus in E^3 seems rather surprising.

For any discrete flow (I_g, Y) with g the unit of I_g, let $\mathrm{TC}(Y, g)$ denote the *twisted cylinder over Y with respect to g*, i.e. the image of the set which is Y cross the unit interval $[0, 1]$ under a map β which for each $y \in Y$ identifies the point $(y, 1)$ with the point $(g(y), 0)$ and is 1-1 otherwise. Let $(R_g, \mathrm{TC}(Y, g))$ represent the continuous flow on $\mathrm{TC}(Y, g)$ induced by (I_g, Y) and the $[0, 1]$ parametrization (with Y identified as the image under β of $Y \times \{0\}$). For Φ the homeomorphism of Section 2, (I_Φ, C) induces $(R_\Phi, \mathrm{TC}(C, \Phi))$ and as C is zero-dimensional, $\mathrm{TC}(C, \Phi)$ is 1-dimensional.

Let (R_g, X) be any continuous flow with g the unit of the reals R_g. Let (I_g, X) denote the discrete flow whose generating (unit) homeomorphism is g. Let α be the canonical (identity) imbedding of (I_g, X) in (R_g, X). Let, by Theorem I, (I_g, C) be a discrete flow on C for which there is a mapping ν of C onto X such that for each $g' \in I_g$, $g'\nu = \nu g'$, with g' though of as acting on the appropriate space. As ν maps C onto X, there exists an induced mapping Γ of $\mathrm{TC}(C, g)$ onto $\mathrm{TC}(X, g)$ such that Γ raises $(R_g, \mathrm{TC}(X, g))$ to $(R_g, \mathrm{TC}(C, g))$, i.e. Γ commutes with each $g' \in R_g$ thought of as a homeomorphism of the appropriate space.

Let λ denote the map of $\mathrm{TC}(X, g)$ onto X induced by the canonical imbeddings of (I_g, X) in (R_g, X) and in $(R_g, \mathrm{TC}(X, g))$ and such that (R_g, X) is thus raised to $(R_g, \mathrm{TC}(X, g))$ by means of λ.

Thus we have the following diagram:

$$\begin{tikzcd}
(R_g, \mathrm{TC}(C, g)) \arrow[r, "\Gamma"] & (R_g, \mathrm{TC}(X, g)) \arrow[r, "\lambda"] & (R_g, X) \\
(I_g, C) \arrow[u] \arrow[r, "\nu"] & (I_g, X) \arrow[u] \arrow[ur, "\alpha"] &
\end{tikzcd}$$

with the vertical arrows representing the cannonical imbeddings implied by the twisted cylinder construction. The map $\lambda\Gamma$ raises (R_g, X) to $(R_g, \mathrm{TC}(C, g))$.

Further we have by Theorem III that there is an imbedding of (I_g, C) in the flow (I_Φ, C) and thus an induced imbedding of $(R_\Phi, \mathrm{TC}(C, g))$ in the flow $(R_\Phi, \mathrm{TC}(C, \Phi))$. Hence (R_g, X) may be raised to a closed subflow of $(R_\Phi, \mathrm{TC}(C, \Phi))$. Thus we have established

THEOREM VI. *The flow $(R_\Phi, \mathrm{TC}(C, \Phi))$ is a quasi-universal continuous flow and $\mathrm{TC}(C, \Phi)$ is one-dimensional.*

By definition, any continuous flow in which $(R_\Phi, \mathrm{TC}(C, \Phi))$ is imbedded is also a quasi-universal flow. But in [7] it is shown that for any imbedding of C in the interioir of the 2-dimensional disc M_2, the flow (I_Φ, C) can be extended to a flow (I_{Φ^*}, M_2) with Φ^* the identity on the boundary of M_2. From the last condition it follows that the set $\mathrm{TC}(M_2, \Phi^*)$ is in fact a solid torus. Then we may assert that the continuous flow $(R_{\Phi^*}, \mathrm{TC}(M_2, \Phi^*))$ is a quasi-universal flow, for $(R_\Phi, \mathrm{TC}(C, \Phi))$ is imbedded in it. Thus we have

THEOREM VII. *There exists a quasi-universal flow on the solid torus in* E^3.

The exhibited flow has no fixed points and may be thought of as having circular orbits on the boundary of the torus. Further each orbit may be thought of as parametrized by the number of revolutions around the "hole" of the torus.

4. Semi-flows. In connection with Theorem IIA and IIB of [1] it was observed that with obvious modifications of the argument given in [1] for Theorem I of this paper, the following theorem may be established. ("Onto" maps were used in [1] but the suggested argument is just as valid for "into" maps.)

THEOREM VIII. *Any mapping of a compact metric space into itself may be raised to a mapping of C into C.*

There is also an easy modification (given below) of the proof of our Theorem II to establish.

THEOREM IX (Baayen and de Groot). *There exists a universal semi-flow on C.*

Proof. Let $C = \prod_{0 \leqslant i < \infty} C_i$ where, for each i, C_i is a Cantor set and let ψ_i be a homeomorphism of C_{i+1} onto C_i. Let Ψ be the mapping of C onto C such that, for $p = \{p_i\} \in C$, $p_i \in C_i$, $\Psi(p)$ is the point whose ith coordinate is $\psi_i(p_{i+1})$ for each $i \geqslant 0$. We shall show that Ψ is a universal mapping of C into C.

Let C' be a Cantor Set and let λ be an arbitrary mapping of C' into C'. We wish to describe a homeomorphism $\varkappa$ of C' into C such that, for each $c \in C'$, $\Phi\varkappa(c) = \varkappa\lambda(c)$.

Let $\varkappa_0$ be a homeomorphism of C' onto C_0. For each $c \in C'$, we define

$$\varkappa(c) = \{p_i\}, \quad p_i \in C_i,$$

where

$$p_0 = \varkappa_0(c) \quad \text{and} \quad p_i = \psi_{i-1}^{-1} \dots \psi_1^{-1}\psi_0^{-1}\varkappa_0\lambda^i(c) \quad \text{for} \quad i > 0,$$

It is easy (and routine) to see that $\varkappa$ is one-to-one and continuous. As C' is compact, $\varkappa(C')$ is compact and thus $\varkappa$ is a homeomorphism.

We wish to observe that Ψ carries $\varkappa(C')$ into itself and that for each $c \in C'$, $\Psi\varkappa(c) = \varkappa\lambda(c)$.

Applying λ and then $\varkappa$ means that

$$p_0 = \varkappa_0\lambda(c) \quad \text{and} \quad p_i = \psi_{i-1}^{-1} \dots \psi_1^{-1}\psi_0^{-1}\varkappa_0\lambda^{i+1}(c)$$

whereas applying $\varkappa$ produces

$$p_0' = \varkappa_0(c) \quad \text{and} \quad p_i' = \psi_{i-1}^{-1} \dots \psi_1^{-1}\psi_0^{-1}\varkappa_0\lambda^{i}(c)$$

and then applying Ψ produces

$$p_0 = (\psi_0\psi_0^{-1})\varkappa_0\lambda(c) \quad \text{and} \quad p_i = (\psi_i\psi_i^{-1})\psi_{i-1}^{-1} \dots \psi_1^{-1}\psi_0^{-1}\varkappa_0\lambda^{i+1}(c)$$

as we wished to show.

We also note that this last observation shows that Ψ carries each point of $\varkappa(C')$ to a point of $\varkappa(C')$.

As a corollary of Theorems VIII and IX we have

THEOREM X. *There exists a quasi-universal semi-flow on the Cantor set, i.e. a mapping Ψ of C onto C such that each mapping of any compact metric space into itself may be raised to Ψ restricted to a suitable closed sub-Cantor set of C.*

Question: Let C and C' be Cantor sets. Does there exist a homeomorphism ω of C onto itself such that for any homeomorphism α of C' onto itself there exists a map φ of C onto C' such that $\varphi\omega = \alpha\varphi$? Here we are asking that the closed subset of C (of quasi-universality) be C itself. There does exist such a strongly quasi-universal p-adic homeomorphism with respect to all p-adic homeomorphisms but the author does not know the answer to the general question. An affirmative solution to such question would imply, by the devices of Section 3, the existence of a continuous flow on a certain 1-dimensional compactum M such that all continuous flows on compacta can be raised to such flow itself (as distinct from closed subflows of such flow as in Theorem VI). As the solid torus can only map onto locally connected continua, there could be no similar general analogue for Theorem VII.

Added in proof: Ellard Nunnally has answered this question in the negative

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Linear-compact congruence topologies in *-lattices *

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1. Introduction. The notion of "linear-compactness" was first introduced by Lefschetz in topological linear spaces. This concept has been further extended to topological groups and modules by Leptin. This paper gives a formulation for linear-compactness in a class of topological lattices the C-lattices. Here the general properties of linear-compact C*-lattices are analysed and it is shown that the study of any Hausdorff linear-compact C*-lattice can, in some sense, be reduced to the study of certain discrete linear-compact lattices. We then proceed to establish that the centre of a discrete linear-compact C*-lattice is finite which enables us to prove that the centre of a linear-compact Hausdorff C*-lattice is compact. Next we investigate the structure of the compact complemented modular C*-lattices from which we deduce that any linear-compact Hausdorff C-Boolean algebra is the direct product of (two element) simple Boolean algebras. Hence the question naturally arises as to whether every linear-compact C*-lattice admits such a direct product decomposition into simple lattices. In this paper we shall answer this question in the affirmative for a certain class of C*-lattices viz., the generalized continuous geometries. We also define the concept of a PC*-lattice and show that a Hausdorff PC*-generalized continuous geometry is linear-compact if and only if its centre is compact. The paper ends with a brief discussion on some unsolved problems concerning the PC*-lattices.

2. Preliminaries and basic results. In our notations and terminology in lattice theory and topology we shall generally follow [2] and [5], respectively.

It is seen that in a lattice L, given any set $C = [\theta_i]$ $(i \in I)$ of congruences directed below in the lattice of congruences, the subsets $V_i = [(x, y)/x\theta_i y]$ $(i \in I)$ define a uniformity V on L. Further the lattice sum and product in L are uniformly continuous with respect to V. A complete study of these uniformities, termed "congruence uniformites"

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