of uniform convergence. Further we put $F = \{f: f \in G \text{ and } fZ = Z\}$ and we define the map φ of G onto Z^0 by the formula: $\varphi(g) = gZ$. It is easy to check that the triple G, F, φ satisfies all the conditions of Theorem 1, whence $Z^0 = \varphi(G)$ is Borel.

Finally let us observe that, in fact, the Lemma of [3] yields the following general proposition.

PROPOSITION. *If a continuous map φ from a metric, separable and complete space X into a metric space satisfies the condition*

(C′) $\varphi^{-1}\varphi(U)$ *is open for every open* $U \subseteq X$,

then $\varphi(X)$ is an absolutely Borel set.

This proposition yields Theorem 1 of this paper since (C) implies (C′).

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Some theorems of set theory and their topological consequences

by

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It is generally known that numerous theorems of topology have a set-theoretical content, and some even prove to be theorems of set theory formulated in a different language. In such a situation it is often possible to formulate the corresponding theorem of set theory and prove it, and finally — by applying some substitution — pass to a topological theorem. This usually permits a simplification of the proof and a better understanding of the content of the theorem.

The object of the present paper is to give such proofs of several theorems of general topology concerning Cartesian products, among them the well-known theorems of Bockstein and Marczewski (and some of their generalizations), and the Hewitt-Marczewski-Pondiczery theorem on dense subsets of Cartesian products (1). All topological theorems of the present paper are grouped in the second part and are deduced from three theorems of set theory which are presented in the first part and which, as we think, may be interesting in themselves.

1. Three theorems of set theory. In the proof of Theorem 1 we shall use theorem I(ii) of [4] (a simple proof of this theorem is given in [10]), which permits one to estimate the power of a family of sets with the help of the powers of its elements and the powers of its quasi-disjoint subfamilies. Recall that a family of sets $\mathfrak{A}$ is called *quasi-disjoint* if $A_1 \cap A_2 = \mathbf{I}(\mathfrak{A})$ for distinct $A_1, A_2 \in \mathfrak{A}$ (2). The theorem of Erdös-Rado mentioned above tells us that if a family $\mathfrak{A}$ is composed of sets of power at most $\mathfrak{n}$ and the power of every quasi-disjoint subfamily $\mathfrak{A}_0 \subset \mathfrak{A}$ does not exceed $\mathfrak{m} \geqslant \aleph_0$, then $\overline{\overline{\mathfrak{A}}} \leqslant \mathfrak{m}^{\mathfrak{n}}$. (3)

(1) Recently K. A. Ross and A. H. Stone gave in [12] a very simple and elegant proof of the theorem of Bockstein and certain of its generalizations. It appears (see [3]) that by a similar method one can also obtain some results of the present paper.

(2) By the symbol $\mathbf{I}(\mathfrak{A})$ we mean the intersection of all sets belonging to $\mathfrak{A}$. The union of all sets belonging to $\mathfrak{A}$ is denoted by $\mathbf{S}(\mathfrak{A})$.

(3) Small Gothic letters denote finite and infinite cardinal numbers. The symbol $\overline{\overline{A}}$ denotes the power of the set A.

THEOREM 1. *If families $\mathfrak{A}$ and $\mathfrak{B}$ are composed of sets of power not exceeding $\mathfrak{n}$ and $\mathfrak{m}$ respectively, where $\mathfrak{n} \leqslant \mathfrak{m} \geqslant \aleph_0$ and*

(1) $$A \cap B \neq 0 \quad \text{for all} \quad A \in \mathfrak{A},\ B \in \mathfrak{B},$$

then there exists a set N of power not greater than $\mathfrak{m}^{\mathfrak{n}}$ such that

(2) $$A \cap B \cap N \neq 0 \quad \text{for all} \quad A \in \mathfrak{A},\ B \in \mathfrak{B}.$$

Proof. Let us suppose that the family $\mathfrak{A}$ is well ordered by the relation $\leqslant$. Let

$$\mathfrak{B}^* = \{T \subset \mathbf{S}(\mathfrak{A}):\ T \cap B \neq 0 \text{ for all } B \in \mathfrak{B}\}.$$

For all $A_0 \in \mathfrak{A}$ the family

$$\mathfrak{A}(A_0) = \{A \in \mathfrak{A}:\ A \cap A_0 \in \mathfrak{B}^*\}$$

is non-empty, since it contains A_0. Denote by A_0^* the first set (with respect to the ordering $\leqslant$) belonging to it and let

$$T_{A_0} = A_0 \cap A_0^*;$$

clearly $T_A \in \mathfrak{B}^*$ for all $A \in \mathfrak{A}$.

Let us now consider sets $A_1, A_2 \in \mathfrak{A}$ such that $T_{A_1} \cap T_{A_2} \in \mathfrak{B}^*$. Then we have

$$T_{A_1} \cap T_{A_2} = A_1 \cap A_1^* \cap A_2 \cap A_2^* = (A_2^* \cap A_1) \cap (A_1^* \cap A_2),$$

from which it follows, in view of the fact that $\mathfrak{B}^*$ contains with each set all subsets of $\mathbf{S}(\mathfrak{A})$ which include it, that $A_1^* \leqslant A_2^*$ and $A_2^* \leqslant A_1^*$. Hence

(3) $$\text{if } T_{A_1} \cap T_{A_2} \in \mathfrak{B}^*, \quad \text{then} \quad A_1^* = A_2^*.$$

Let $\mathfrak{T} = \{T_A\}_{A \in \mathfrak{A}}$ and let $\mathfrak{T}_0$ be an arbitrary quasi-disjoint subfamily of $\mathfrak{T}$. We consider first the case

(i) $$\mathbf{I}(\mathfrak{T}_0) \in \mathfrak{B}^*.$$

Since $T_{A_1} \cap T_{A_2} = \mathbf{I}(\mathfrak{T}_0)$ for distinct $T_{A_1}, T_{A_2} \in \mathfrak{T}_0$, on the basis of (3) and the definition of the sets T_A, there exists an $A_0 \in \mathfrak{A}$ such that

$$T_A = A \cap A^* = A \cap A_0 \quad \text{for all} \quad T_A \in \mathfrak{T}_0.$$

From this it follows that $\mathbf{S}(\mathfrak{T}_0) \subset A_0$ and — in view of the quasi-disjointness of the family $\mathfrak{T}_0$ and the condition $\overline{\overline{A_0}} \leqslant \mathfrak{n} \leqslant \mathfrak{m}$ — that $\overline{\overline{\mathfrak{T}_0}} \leqslant \mathfrak{m}$.

If (i) does not hold, then

(ii) $\mathbf{I}(\mathfrak{T}_0) \notin \mathfrak{B}^*$, i.e. there exists a $B_0 \in \mathfrak{B}$ such that $\mathbf{I}(\mathfrak{T}_0) \cap B_0 = 0$.

Since $T_A \cap B_0 \neq 0$ for all $T_A \in \mathfrak{T}_0$, we have

$$(T_A \setminus \mathbf{I}(\mathfrak{T}_0)) \cap B_0 \neq 0 \quad \text{for} \quad T_A \in \mathfrak{T}_0,$$

and hence in view of the quasi disjointness of the family $\mathfrak{T}_0$ and the condition $\overline{\overline{B_0}} \leqslant \mathfrak{m}$ — we conclude that $\overline{\overline{\mathfrak{T}_0}} \leqslant \mathfrak{m}$.

Hence the power of a quasi-disjoint subfamily $\mathfrak{T}_0 \subset \mathfrak{T}$ does not exceed $\mathfrak{m}$ and from the theorem of Erdös-Rado we have $\overline{\overline{\mathfrak{T}}} \leqslant \mathfrak{m}^{\mathfrak{n}}$, from which it follows that the set $N = \mathbf{S}(\mathfrak{T})$ satisfies the conclusion of the theorem.

COROLLARY. *If families $\mathfrak{A}$ and $\mathfrak{B}$ are composed respectively of finite sets and sets of power at most $\mathfrak{m} \geqslant \aleph_0$ and satisfy condition* (1), *then there exists a set N of power at most $\mathfrak{m}$ which satisfies condition* (2).

Proof. Observe that $\mathfrak{A} = \bigcup_{i=1}^{\infty} \mathfrak{A}_i$, where

$$\mathfrak{A}_i = \{A \in \mathfrak{A}:\ A \text{ has exactly } i \text{ elements}\}.$$

Using Theorem 1 we obtain a set N_i of power $\leqslant \mathfrak{m}^i = \mathfrak{m}$ such that

$$A \cap B \cap N_i \neq 0 \quad \text{for} \quad A \in \mathfrak{A}_i \text{ and } B \in \mathfrak{B}.$$

It is easy to check that the set $N = \bigcup_{i=1}^{\infty} N_i$ satisfies the required condition.

In [2] a theorem is proved establishing a special case of the above corollary. By modifying slightly the proof of that theorem one gives a direct proof of the above corollary. A variant of Theorem 1 with $\mathfrak{n} = \mathfrak{m} = \aleph_\alpha$ and $\aleph_{\alpha+1}$ in place of $\mathfrak{m}^{\mathfrak{n}}$ is obtained, by using the continuum hypothesis, in [9].

THEOREM 2. *If the families of sets $\{A_t\}_{t \in T}$ and $\{B_t\}_{t \in T}$ are composed of sets of power not exceeding respectively $\mathfrak{n}$ and $\mathfrak{m}$, where $\mathfrak{n} \leqslant \mathfrak{m} \geqslant \aleph_0$ and*

(4) $$A_t \cap B_{t'} \neq 0 \quad \text{and} \quad A_t \cap B_t = 0 \quad \text{for all} \quad t, t' \in T,\ t \neq t',$$

then $\overline{\overline{T}} \leqslant \mathfrak{m}^{\mathfrak{n}}$.

Proof. The families $\mathfrak{A} = \{A_t \cup \{t\}\}_{t \in T}$ and $\mathfrak{B} = \{B_t \cup \{t\}\}_{t \in T}$ satisfy the hypotheses of Theorem 1, where in place of $\mathfrak{n}$ we must take $\mathfrak{n}+1$. For the set N satisfying the conclusion of that theorem and an arbitrary $t \in T$ we have

$$(A_t \cup \{t\}) \cap (B_t \cup \{t\}) = \{t\} \subset N$$

and hence $T \subset N$ and $\overline{\overline{T}} \leqslant \overline{\overline{N}} \leqslant \mathfrak{m}^{\mathfrak{n}+1} = \mathfrak{m}^{\mathfrak{n}}$.

The corollary below follows from Theorem 2 in the same way as the previous Corollary follows from Theorem 1; it can also be deduced from the Corollary to Theorem 1.

COROLLARY. *If the families of sets $\{A_t\}_{t \in T}$ and $\{B_t\}_{t \in T}$ are composed respectively of finite sets and sets of power at most $\mathfrak{m} \geqslant \aleph_0$ and satisfy condition* (4), *then the power of the set T does not exceed $\mathfrak{m}$.*

Remark. Theorem 2 can be derived directly from the theorem of Erdös-Rado. Let $T_0 \subset T$ be a non-empty set of indices such that the family $\{A_t\}_{t \in T_0}$ is quasi-disjoint. Choose $t_0 \in T_0$ and notice that $B_{t_0} \cap \bigcap_{t \in T_0} A_t = 0$. For every $t \in T_0 \setminus \{t_0\}$ we have $(A_t \setminus \bigcap_{t \in T_0} A_t) \cap B_{t_0} \neq 0$, and hence, in view of the quasi-disjointness of the family $\{A_t\}_{t \in T_0}$ and the condition $\overline{\overline{B}}_{t_0} \leqslant \mathfrak{m}$, we conclude that $\overline{\overline{T}}_0 \leqslant \mathfrak{m}+1 = \mathfrak{m}$, since $A_t \neq A_{t'}$ for $t \neq t'$. It follows that $\overline{\overline{T}} \leqslant \mathfrak{m}^{\mathfrak{n}}$.

In the proof of Theorem 3 we shall use theorem II of [5], which states that an arbitrary set A of power $\mathfrak{m} \geqslant \aleph_0$ includes $2^{\mathfrak{m}}$ independent subsets, i.e. that there exists a family $\mathfrak{S}$ of subsets of the set A such that $\overline{\overline{\mathfrak{S}}} = 2^{\mathfrak{m}}$ and

$$S_1^{i_1} \cap S_2^{i_2} \cap \ldots \cap S_k^{i_k} \neq 0\,, \quad \text{where} \quad S^0 = S\,,\ S^1 = A \setminus S\,,$$

for an arbitrary finite sequence $S_1, S_2, \ldots, S_k$ of distinct sets from $\mathfrak{S}$ and an arbitrary finite sequence $i_1, i_2, \ldots, i_k$ of zeros and ones.

THEOREM 3. *For every set A of power $\mathfrak{m} \geqslant \aleph_0$ there exists a family $\mathfrak{F}$ of functions mapping the set A into itself such that $\overline{\overline{\mathfrak{F}}} = 2^{\mathfrak{m}}$ and for an arbitrary finite sequence $f_1, f_2, \ldots, f_k$ of distinct functions from $\mathfrak{F}$ and an arbitrary finite sequence $a_1, a_2, \ldots, a_k$ of elements of the set A there exists an $a \in A$ such that*

$$f_i(a) = a_i \quad \text{for} \quad i = 1, 2, \ldots, k\,.$$

Proof. Let $\mathfrak{A}$ be the family of all sequences $(a_1, a_2, \ldots, a_k; F_1, F_2, \ldots, F_k)$ where k is an arbitrary natural number, $a_1, a_2, \ldots, a_k$ are elements of A, and $F_1, F_2, \ldots, F_k$ are finite subsets of A. Since $\overline{\overline{\mathfrak{A}}} = \mathfrak{m}$, there exists a one-to-one map h of the set A onto the family $\mathfrak{A}$.

Let $\mathfrak{S}$ be a family of power $2^{\mathfrak{m}}$ of independent subsets of A and a_0 an arbitrary element of A. For every $S \in \mathfrak{S}$ we define a function f_S mapping the set A into itself. Accordingly we consider for every $a \in A$ the sequence $h(a) = (a_1, a_2, \ldots, a_k; F_1, F_2, \ldots, F_k)$ and set

$$(5) \qquad f_S(a) = \begin{cases} a_i & \text{if there exists } i \leqslant k, \text{ such that } F_i \subset S \text{ and } F_j \not\subset S \text{ for } j \neq i\,, \\ a_0 & \text{otherwise.} \end{cases}$$

We shall show that the family $\mathfrak{F} = \{f_S\}_{S \in \mathfrak{S}}$ satisfies the conclusion of the theorem. Let $f_{S_1}, f_{S_2}, \ldots, f_{S_k}$ be an arbitrary finite sequence of functions, where $S_1, S_2, \ldots, S_k$ are distinct sets from $\mathfrak{S}$ and $a_1, a_2, \ldots, a_k$ are elements of A. From the independence of the sets of the family $\mathfrak{S}$ it follows that there exist elements a_{ij} such that

$$a_{ij} \in S_i \setminus S_j \quad \text{for} \quad i, j = 1, 2, \ldots, k \text{ and } i \neq j\,.$$

Let $F_i = \{a_{i1}, a_{i2}, \ldots, a_{ii-1}, a_{ii+1}, \ldots, a_{ik}\}$; then

$$F_i \subset S_i \quad \text{and} \quad F_j \not\subset S_i \quad \text{for} \quad j \neq i\,,$$

and hence, in agreement with (5), we conclude that for

$$a = h^{-1}\big((a_1, a_2, \ldots, a_k; F_1, F_2, \ldots, F_k)\big)$$

we have

$$f_{S_i}(a) = a_i \quad \text{for} \quad i = 1, 2, \ldots, k\,.$$

Since the functions f_S and $f_{S'}$ are distinct for $S \neq S'$, it follows that $\overline{\overline{\mathfrak{F}}} = \overline{\overline{\mathfrak{S}}} = 2^{\mathfrak{m}}$.

Remark 1. The property of a family $\mathfrak{F}$ satisfying the conclusion of Theorem 3 may be expressed differently by saying that, for an arbitrary sequence of distinct functions $f_1, f_2, \ldots, f_k$ chosen from the family $\mathfrak{F}$, the function attaching to a point $a \in A$ the point $(f_1(a), f_2(a), \ldots, f_k(a))$ belonging to the kth Cartesian power A^k of A, is a map of A *onto* A^k.

Remark 2. Theorem 3 may also be formulated as follows:

For every set A of power $\mathfrak{m} \geqslant \aleph_0$ there exists a family of subsets of A, $\{A_{s,t}\}_{s \in S, t \in T}$, where $\overline{\overline{S}} = \mathfrak{m}$ and $\overline{\overline{T}} = 2^{\mathfrak{m}}$, such that

$$(6) \qquad A_{s,t} \cap A_{s',t} = 0 \quad \textit{for} \quad s \neq s'\,,\ t \in T\,,$$

$$(7) \qquad \bigcup_{s \in S} A_{s,t} = A \quad \textit{for every} \quad t \in T\,,$$

(8) *for every function $s\colon T \to S$ the family $\{A_{s(t),t}\}_{t \in T}$ is composed of independent (and distinct) subsets of A.*

In fact, assuming that Theorem 3 holds, we set $S = A$, $T = \mathfrak{F}$, and $A_{a,f} = f^{-1}(a)$. It is easy to check that conditions (6), (7), and (8) will then be satisfied. If, on the other hand, there exists a family $\{A_{s,t}\}_{s \in S, t \in T}$ of subsets of A satisfying these conditions, then setting $S = A$ and putting for $t \in T$ and $a \in A$

$$f_t(a) = s \quad \text{where} \quad a \in A_{s,t}$$

(on the basis of (6) and (7) there exists exactly one s satisfying this condition) we obtain a family $\mathfrak{F} = \{f_t\}_{t \in T}$ satisfying the conclusion of Theorem 3.

Remark 3. It is not difficult to check that the above proof of Theorem 3 allows us to obtain the following stronger formulation:

For infinite cardinal numbers $\mathfrak{n}, \mathfrak{m}$ such that $\mathfrak{n} \leqslant \mathfrak{m}$ and an arbitrary set A of power $\sum_{\mathfrak{p}<\mathfrak{n}} \mathfrak{m}^{\mathfrak{p}}$ there exists a family $\mathfrak{F}$ of functions mapping A into itself such that $\overline{\overline{\mathfrak{F}}} = 2^{\mathfrak{m}}$ and for any index set S, where $\overline{\overline{S}} < \mathfrak{n}$, any family $\{f_s\}_{s \in S}$ composed of distinct functions from $\mathfrak{F}$ and any family $\{a_s\}_{s \in S}$ of elements of A, there exists $a \in A$ such that $f_s(a) = a_s$ for all $s \in S$.

Analogously we can also strengthen the formulation of Theorem 3 given in remarks 1 and 2.

Theorem 3 establishes a generalization of theorem I of Hausdorff [5] and, as follows from remark 2, also of the theorem about the existence of independent sets. Strengthening Theorem 3 (in the formulation given in remark 2) by the method indicated in remark 3 we obtain a generalization of lemma 3.16 of Tarski [13].

2. Topological consequences. We shall denote the Cartesian product (with the Tychonoff topology) of the family $\{X_s\}_{s\in S}$ of topological spaces by the symbol $\underset{s\in S}{P} X_s$. For every $S_0 \subset S$ the projection $p_{S_0}\colon \underset{s\in S}{P} X_s \to \underset{s\in S_0}{P} X_s$ is defined and continuous. In particular for each $s_0 \in S$ the continuous transformation $p_{s_0}\colon \underset{s\in S}{P} X_s \to X_{s_0}$ is defined, the projection on the s_0th axis of the product $\underset{s\in S}{P} X_s$. Subsets of the product $\underset{s\in S}{P} X_s$ of the form

$$K = \underset{s\in S}{P} K_s\,, \quad \text{where} \quad K_s \subset X_s\,,$$

we shall call *cubes*; the set K_s will be called the sth *face* of the cube K and the set

$$D(K) = \{s\colon K_s \neq X_s\}$$

will be called the set of its distinguished indexes. A criterion for the disjointness of cubes is given by the following easily provable

LEMMA 1. *For non-empty cubes $K = \underset{s\in S}{P} K_s$, $L = \underset{s\in S}{P} L_s$ of the Cartesian product $\underset{s\in S}{P} X_s$ the following equivalence holds:*

$(K \cap L = 0) \equiv$ (*there exists an $s_0 \in D(K) \cap D(L)$ such that $K_{s_0} \cap L_{s_0} = 0$*).

Let us consider now the class $\mathbf{R} = \{\mathfrak{R}_s\}_{s\in S}$, where $\mathfrak{R}_s$ is a family of subsets of the space X_s, and set

$$A(K) = \{K_s\colon s \in D(K)\}\,,$$
$$B(K, \mathbf{R}) = \bigcup_{s\in D(K)} \{R \in \mathfrak{R}_s\colon R \cap K_s = 0\}\,,$$

where, as above, K is a cube $\underset{s\in S}{P} K_s$.

Lemma 1 immediately implies

LEMMA 2. *Suppose we are given a family $\{X_s\}_{s\in S}$ of disjoint topological spaces and the class $\mathbf{R} = \{\mathfrak{R}_s\}_{s\in S}$, where $\mathfrak{R}_s$ is a family of subsets of X_s. For non-empty cubes $K = \underset{s\in S}{P} K_s$ and $L = \underset{s\in S}{P} L_s$ of the Cartesian product $\underset{s\in S}{P} X_s$, where $K_s \in \mathfrak{R}_s$ for $s \in D(K)$, the following equivalence holds:*

$$(K \cap L = 0) \equiv \big(A(K) \cap B(L, \mathbf{R}) \neq 0\big)\,.$$

A subset of a topological space X which is the intersection of $\mathfrak{m} \geqslant \aleph_0$ open sets is called a $G_\delta^{\mathfrak{m}}$ set, and the union of an arbitrary number of $G_\delta^{\mathfrak{m}}$ sets is called a $G_{\delta\Sigma}^{\mathfrak{m}}$ set. Instead of $G_\delta^{\aleph_0}$ and $G_{\delta\Sigma}^{\aleph_0}$ we shall write G_δ and $G_{\delta\Sigma}$.

For each $s \in S$ let $\mathfrak{B}_s$ be a base for X_s. A cube $K = \underset{s\in S}{P} K_s$, where $\overline{\overline{D(K)}} < \aleph_0$ and $K_s \in \mathfrak{B}_s$ for $s \in D(K)$, will be called a *basic cube with respect to the class* $\mathbf{B} = \{\mathfrak{B}_s\}_{s\in S}$. Clearly every open set in $\underset{s\in S}{P} X_s$ is the union of basic cubes. A cube $K = \underset{s\in S}{P} K_s$, where $\overline{\overline{D(K)}} \leqslant \mathfrak{m}$ and K_s is for every $s \in S$ the intersection of at most $\mathfrak{m}$ elements of the base $\mathfrak{B}_s$, will be called an $\mathfrak{m}$-*cube* [4] *with respect to the class* $\mathbf{B} = \{\mathfrak{B}_s\}_{s\in S}$. Every $\mathfrak{m}$-cube is a $G_\delta^{\mathfrak{m}}$ set in $\underset{s\in S}{P} X_s$.

LEMMA 3. *Every $G_{\delta\Sigma}^{\mathfrak{m}}$ set in a Cartesian product $\underset{s\in S}{P} X_s$ is the union of $\mathfrak{m}$-cubes with respect to an arbitrary class $\mathbf{B} = \{\mathfrak{B}_s\}_{s\in S}$.*

Proof. It suffices to consider $G_\delta^{\mathfrak{m}}$ sets. Hence let

$$M = \bigcap_{t\in T} H_t\,, \quad \text{where} \quad \overline{\overline{T}} \leqslant \mathfrak{m} \text{ and } H_t \text{ is open in } \underset{s\in S}{P} X_s\,,$$

be an arbitrary $G_\delta^{\mathfrak{m}}$ set. For every $p \in M$ and $t \in T$ let $K(p, t)$ be a basic cube with respect to the class $\mathbf{B}$ satisfying the condition

$$p \in K(p, t) \subset H_t\,.$$

We then have

$$p \in \bigcap_{t\in T} K(p, t) \subset M \quad \text{and} \quad M = \bigcup_{p\in M} \bigcap_{t\in T} K(p, t)\,.$$

The lemma follows from the fact that the intersection of at most $\mathfrak{m}$ basic cubes is an $\mathfrak{m}$-cube.

THEOREM 4. *Let $\{X_s\}_{s\in S}$ be an arbitrary family of topological spaces such that $w(X_s) \leqslant \mathfrak{w}$ for $s \in S$* [5] *and let $U, V \subset \underset{s\in S}{P} X_s$ be $G_{\delta\Sigma}^{\mathfrak{n}}$ and $G_{\delta\Sigma}^{\mathfrak{m}}$ sets respectively where $\mathfrak{n} \leqslant \mathfrak{m}$. If $U \cap V = 0$, then there exists a set $S_0 \subset S$ such that $\overline{\overline{S}}_0 \leqslant (\mathfrak{m}\cdot\mathfrak{w})^{\mathfrak{n}}$ and $p_{S_0}(U) \cap p_{S_0}(V) = 0$.*

Proof. Without loss of generality we may assume that $X_s \cap X_{s'} = 0$ for $s \neq s'$. Choose for each $s \in S$ a base $\mathfrak{B}_s$ of the space X_s such that $\overline{\overline{\mathfrak{B}}}_s \leqslant \mathfrak{w}$. From Lemma 3 we have

$$U = \bigcup_{t\in T} K_t \quad \text{and} \quad V = \bigcup_{r\in R} L_r\,,$$

where K_t and L_r are respectively non-empty $\mathfrak{n}$ and $\mathfrak{m}$-cubes with respect to the class $\mathbf{B} = \{\mathfrak{B}_s\}_{s\in S}$. In view of the disjointness of sets U and V we have

$$K_t \cap L_r = 0 \quad \text{for} \quad t \in T\,,\ r \in R\,. \tag{9}$$

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[4] The intersection of the empty subfamily of $\mathfrak{B}_s$ is the whole space X_s.

[5] The symbol $w(X)$ denotes the *weight* of the space X, i.e. the power of the least numerous base of the space.

We denote by $\mathfrak{R}_s$ the family of all subsets of X_s which are intersections of not more than $\mathfrak{n}$ elements of the base $\mathfrak{B}_s$ and put $\mathbf{R} = \{\mathfrak{R}_s\}_{s\in S}$. In particular the sth faces of the cubes $\{K_t\}_{t\in T}$ belong to $\mathfrak{R}_s$. From condition (9) and Lemma 2 it follows that the families

$$\mathfrak{A} = \{A(K_t)\}_{t\in T} \quad \text{and} \quad \mathfrak{B} = \{B(L_r, \mathbf{R})\}_{r\in R}$$

satisfy condition (1) of Theorem 1. Since $\overline{\overline{A(K_t)}} \leqslant \mathfrak{n}$ for all $t \in T$ and $\overline{\overline{B(L_r, \mathbf{R})}} \leqslant \mathfrak{m}\cdot\mathfrak{w}^{\mathfrak{n}}$ for every $r \in R$, it follows from Theorem 1 that there exists a set N of power at most $(\mathfrak{m}\cdot\mathfrak{w}^{\mathfrak{n}})^{\mathfrak{n}} = (\mathfrak{m}\cdot\mathfrak{w})^{\mathfrak{n}}$ such that

$$A(K_t) \cap B(L_r, \mathbf{R}) \cap N \neq 0 \quad \text{for} \quad t \in T \text{ and } r \in R\,.$$

Using Lemma 2 it is not difficult to check that the set

$$S_0 = \{s \in S\colon \text{the } s\text{th face of some cube of the family } \{K_t\}_{t\in T} \text{ belongs to } N\}$$

satisfies the conclusion of the theorem.

COROLLARY. *For every pair U, V of disjoint $G_{\delta\Sigma}$ sets in the Cartesian product $\underset{s\in S}{\boldsymbol{P}} X_s$ of a family of topological spaces $\{X_s\}_{s\in S}$, such that $w(X_s) \leqslant 2^{\aleph_0}$ for $s \in S$, there exists a set $S_0 \subset S$ such that $\overline{\overline{S_0}} \leqslant 2^{\aleph_0}$ and $p_{S_0}(U) \cap p_{S_0}(V) = 0$.*

Remark. The following example shows that the power of the set S_0 of the last corollary cannot be reduced.

Let I denote the closed interval $[0,1]$ and for each $t \in I$ let D_t be a copy of the two-point discrete space $\{0,1\}$. Let p denote the projection of the product $I \times \underset{t\in I}{\boldsymbol{P}} D_t$ on the I-axis and p_t the projection of this product on the D_t-axis. For every $t \in I$ consider the $\aleph_0$-cubes

$$K_t = p^{-1}(t) \cap p_t^{-1}(0) \quad \text{and} \quad L_t = p^{-1}(t) \cap p_t^{-1}(1)\,.$$

It is not difficult to check that the $G_{\delta\Sigma}$ sets

$$U = \bigcup_{t\in I} K_t \quad \text{and} \quad V = \bigcup_{t\in I} L_t$$

are disjoint but their projections on $\underset{t\in I}{\boldsymbol{P}} D_t$ and $I \times \underset{t\in I_0}{\boldsymbol{P}} D_t$, where $I_0 \not\supseteq I$, have a non-empty intersection.

The spaces considered in the above example have countable bases, whence our corollary cannot be strengthened even in this case. We have not succeeded, however, in finding an example of two disjoint G_δ sets in the Cartesian product of the family $\{X_s\}_{s\in S}$ of spaces with a countable base whose projections on every product $\underset{s\in S_0}{\boldsymbol{P}} X_s$, where $S_0 \subset S$ and $\overline{\overline{S_0}} \leqslant \aleph_0$, have a non empty intersection (compare Corollary 1 to Theorem 5 below). That such a situation is possible in a Cartesian product of spaces of weight $2^{\aleph_0}$ is shown by the example considered above in which I must be regarded as a discrete space of power $2^{\aleph_0}$; the sets U and V in this case will be open.

THEOREM 5. *Let $\{X_s\}_{s\in S}$ be a family of topological spaces such that $w(X_s) \leqslant \mathfrak{w}$ for $s \in S$ and let U, $V \subset \underset{s\in S}{\boldsymbol{P}} X_s$ be an open set and $G_{\delta\Sigma}^{\mathfrak{m}}$ set respectively. If $U \cap V = 0$, then there exists a set $S_0 \subset S$ such that $\overline{\overline{S_0}} \leqslant \mathfrak{m}\cdot\mathfrak{w}$ and $p_{S_0}(U) \cap p_{S_0}(V) = 0$.*

Proof. We proceed as in the proof of Theorem 4 except that the corollary to Theorem 1 is used in place of the Theorem itself.

COROLLARY 1. *For any two disjoint sets U, V, respectively open and $G_{\delta\Sigma}$ in the Cartesian product $\underset{s\in S}{\boldsymbol{P}} X_s$ of spaces with countable bases, there exists a set $S_0 \subset S$ such that $\overline{\overline{S_0}} \leqslant \aleph_0$ and $p_{S_0}(U) \cap p_{S_0}(V) = 0$.*

COROLLARY 2. *For any two disjoint open sets U and V in the Cartesian product $\underset{s\in S}{\boldsymbol{P}} X_s$ of spaces with countable bases there exists a set $S_0 \subset S$ such that $\overline{\overline{S_0}} \leqslant \aleph_0$ and $p_{S_0}(U) \cap p_{S_0}(V) = 0$.*

COROLLARY 3. *Any closed G_δ set in the Cartesian product $\underset{s\in S}{\boldsymbol{P}} X_s$ of spaces with countable bases is of the form $p_{S_0}^{-1}(F_0)$, where $S_0 \subset S$, $\overline{\overline{S_0}} \leqslant \aleph_0$ and F_0 is a closed G_δ set in the product $\underset{s\in S_0}{\boldsymbol{P}} X_s$.*

COROLLARY 4. *Any open F_σ set in the Cartesian product $\underset{s\in S}{\boldsymbol{P}} X_s$ of spaces with countable bases is of the form $p_{S_0}^{-1}(U_0)$, where $S_0 \subset S$, $\overline{\overline{S_0}} \leqslant \aleph_0$ and V_0 is an open F_σ set in the product $\underset{s\in S_0}{\boldsymbol{P}} X_s$* (6).

Notice that the proof of Theorem 5 requires only the corollary to Theorem 1 and hence may be carried out without the use of the Erdös-Rado theorem, modifying the proof of the theorem in [2]. Corollary 2 to Theorem 5 is proved (in a somewhat different formulation) in [1].

Let us turn now to the application of Theorem 2.

THEOREM 6. *Any family $\{K_t\}_{t\in T}$ of non-empty and pairwise disjoint $G_\delta^{\mathfrak{n}}$ sets in the Cartesian product $\underset{s\in S}{\boldsymbol{P}} X_s$ of topological spaces such that $w(X_s) \leqslant \mathfrak{w} \geqslant \aleph_0$ for $s \in S$ has power at most $\mathfrak{w}^{\mathfrak{n}}$.*

Proof. Without loss of generality we may assume that $X_s \cap X_{s'} = 0$ for $s \neq s'$. Choose for every $s \in S$ a base $\mathfrak{B}_s$ of X_s such that $\overline{\overline{\mathfrak{B}_s}} \leqslant \mathfrak{w}$. On the basis of Lemma 3 to Theorem 4, we can suppose that the family $\{K_t\}_{t\in T}$ is composed of $\mathfrak{n}$-cubes with respect to the class $\mathbf{B} = \{\mathfrak{B}_s\}_{s\in S}$. Denote by $\mathfrak{R}_s$ the family of all subsets of X_s which are intersections of not more than $\mathfrak{n}$ elements of the base $\mathfrak{B}_s$ and set $\mathbf{R} = \{\mathfrak{R}_s\}_{s\in S}$. From Lemma 2 to Theorem 4 it follows that the families

$$\{A(K_t)\}_{t\in T} \quad \text{and} \quad \{B(K_t, \mathbf{R})\}_{t\in T}$$

(6) For further consequences of Theorem 5, see Theorems 4 and 6 in [3].

satisfy the conditions of Theorem 2 if for $\mathfrak{m}$ we take $\mathfrak{n}\cdot\mathfrak{w}^{\mathfrak{n}} = \mathfrak{w}^{\mathfrak{n}}$ and hence $\overline{\overline{T}} \leqslant (\mathfrak{w}^{\mathfrak{n}})^{\mathfrak{n}} = \mathfrak{w}^{\mathfrak{n}}$.

COROLLARY. *Any family of non-empty and pairwise disjoint G_δ sets in the Cartesian product $\underset{s\in S}{\boldsymbol{P}} X_s$ of topological spaces, such that $w(X_s) \leqslant 2^{\aleph_0}$ for $s \in S$, has power at most $2^{\aleph_0}$.*

THEOREM 7. *Any family of non-empty and pairwise disjoint open sets in the Cartesian product $\underset{s\in S}{\boldsymbol{P}} X_s$ of a family of topological spaces $\{X_s\}_{s\in S}$ such that $w(X_s) \leqslant \mathfrak{w} \geqslant \aleph_0$ for $s \in S$, has power at most $\mathfrak{w}$.*

Proof. We proceed as in the proof of Theorem 6 except that the corollary to Theorem 2 is used in place of the Theorem itself.

COROLLARY. *Any family of non-empty and pairwise disjoint open sets in the Cartesian product $\underset{s\in S}{\boldsymbol{P}} X_s$ of spaces with countable bases is countable.*

Notice that the proof of Theorem 7 requires only the corollary to Theorem 2 (which follows from the corollary to Theorem 1) and hence may be carried out without the use of the Erdös-Rado theorem. Theorem 1.2 of [10] is similar in character to our Theorem 6, but it appears to us that Theorem 6 does not follows from it. The corollary to Theorem 7 was first proved in [7].

From the final theorem of the first part we derive:

THEOREM 8. *The Cartesian product of not more than $2^{\mathfrak{m}}$ topological spaces which contain dense subsets of power $\leqslant \mathfrak{m} \geqslant \aleph_0$ contains a dense subset of power $\leqslant \mathfrak{m}$.*

Proof. Such a product contains a dense subset which is a continuous image of the product of $2^{\mathfrak{m}}$ copies of the discrete space of power $\mathfrak{m}$, in fact a cube, the faces of which have power $\leqslant \mathfrak{m}$ and are dense in the respective spaces. We consider a set A of power $\mathfrak{m}$ and a family $\mathfrak{F}$ of functions mapping A into itself which satisfies the conclusion of Theorem 3. It suffices to show that the product $\underset{f\in\mathfrak{F}}{\boldsymbol{P}} X_f$, where for all $f \in \mathfrak{F}$, $X_f = A$ is the discrete space of power $\mathfrak{m}$, contains a dense subset of power $\mathfrak{m}$.

Let the function $\varphi\colon A \to \underset{f\in\mathfrak{F}}{\boldsymbol{P}} X_f$ attach to a point $a \in A$ the point $\{f(a)\}_{f\in\mathfrak{F}}$ of the product $\underset{f\in\mathfrak{F}}{\boldsymbol{P}} X_f$. The power of the set $\varphi(A) \subset \underset{f\in\mathfrak{F}}{\boldsymbol{P}} X_f$ does not exceed $\mathfrak{m}$. To prove that the set $\varphi(A)$ is dense in $\underset{f\in\mathfrak{F}}{\boldsymbol{P}} X_f$ it suffices to check that for any finite sequence $f_1, f_2, \ldots, f_k$ of distinct functions from $\mathfrak{F}$ and any finite sequence $a_1, a_2, \ldots, a_k$ of elements of A there exists an $a \in A$ such that $p_{f_i}(\varphi(a)) = a_i$, i.e. that $f_i(a) = a_i$, for $i = 1, 2, \ldots, k$. The existence of such a point a is ensured by Theorem 3.

Theorem 9 was proved in [11] and [6] and, for the case $\mathfrak{m} = \aleph_0$, in [8]. It is not difficult to check that the Cartesian product of more than $2^{\mathfrak{m}}$ spaces of more than one point does not have a dense subset of power $\mathfrak{m}$ (see [11] and [8]).

Finally we note that from the theorems of the first part we can also derive theorems analogous to those above for $\mathfrak{p}$-box topologies in the product $\underset{s\in S}{\boldsymbol{P}} X_s$, i.e. for topologies defined by a base composed of sets of the form $\bigcap_{s\in S_0} p_s^{-1}(U_s)$, where U_s is an open set in X_s and $\overline{\overline{S}}_0 < \mathfrak{p}$. We shall not formulate these theorems here since they are less interesting, but the reader, if he wishes, will be able to do so without the least difficulty.

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