Coincidences of real-valued maps from the n-torus *

by

P. Bacon (Knoxville, Tenn.)

1. Introduction. Let E_n denote euclidean n-space and suppose that f: $E_1 \to E_1$ is a continuous periodic function of period 1. In the number interval $[0, 1]$ there is a point p where f attains its maximum. Suppose that λ is a real number and let g denote the function from E_1 such that $gx = f(x+\lambda) - fx$. Then $gp \leqslant 0$ and $g(p-\lambda) \geqslant 0$. By the intermediate value theorem there is a point x^* such that $gx^* = 0$; $f(x^*+\lambda) = fx^*$. If we take $\lambda = \frac{1}{2}$ and identify points in the domain of f with coordinates congruent mod l, our result can be restated in the following form:

(1.1) *If f is a real-valued mapping from a circle C, there is a pair (x^*, y^*) of diametrically opposite points of C such that $fx^* = fy^*$.*

A generalization of (1.1) to higher dimensional spheres is commonly known as the Borsuk-Ulam theorem. If f is a continuous function from the n-sphere S^n into E_n, there is a pair (x^*, y^*) of diametrically opposite points of S^n such that $fx^* = fy^*$ ([1], p. 178, Satz II (1)). A different generalization of (1.1), applying to topological products of circles, has been devised by W. Schmidt ([4], p. 86, Satz 1). The 2-dimensional case of Schmidt's theorem runs as follows:

Suppose that each of f_1 and f_2 is a real-valued mapping from E_2 and for each number pair (x_1, x_2)

$$f_1(x_1, x_2) = f_1(x_1+1, x_2) = f_1(x_1, x_2+1/2),$$
$$f_2(x_1, x_2) = f_2(x_1, x_2+1) = f_2(x_1+1/2, x_2).$$

Then there is a number pair (x_1^*, x_2^*) such that $f_1(x_1^*, x_2^*) = f_1(x_1^*+1/2, x_2^*)$ and $f_2(x_1^*, x_2^*) = f_2(x_1^*, x_2^*+1/2)$. Thus there is a square in E_2, with sides of length 1/2, on the vertices of which each of f_1 and f_2 is constant.

We proved (1.1) by particularizing the parameter λ that appears in a more general theorem. This suggests a direction in which Schmidt's

* This paper constitutes my doctor's thesis, done at the University of Tennessee under the direction of W. S. Mahavier, to whom I am indebted for aid in removing errors and obscurities from an earlier version.

(1) The numbers which appear in brackets in this paper correspond to the numbers in the bibliography of this paper.

theorem might be extended. This paper is devoted to a proof of such an extension. A special case of the theorem proved here ((7.9) below) reads:

(1.2) *If p is a prime and each of f_1 and f_2 is a real-valued mapping from E_2 such that for any number pair (x_1, x_2)*

$$f_1(x_1, x_2) = f_1(x_1+1, x_2) = f_1(x_1, x_2+1/p)$$

and

$$f_2(x_1, x_2) = f_2(x_1, x_2+1) = f_2(x_1+1/p, x_2),$$

then there is a number pair (x_1^*, x_2^*) such that $f_1(x_1^*, x_2^*) = f_1(x_1^*+1/p, x_2^*)$ and $f_2(x_1^*, x_2^*) = f_2(x_1^*, x_2^*+1/p)$. Thus there is a square in E_2 with sides of length $1/p$ on the vertices of which each of f_1 and f_2 is constant.

The compact space obtained from E_n by identifying points with coordinates congruent (mod 1) will be called the *n-torus* T^n. T^1 is a simple closed curve; T^2 is an ordinary torus. Our principal theorem can be formulated as a statement about coincidences of real-valued maps from T^n and can be deduced from a theorem about the incidence relations in certain finite closed covers of T^n. The latter is deducible by a continuity argument from a theorem about intersections of subcomplexes of a complex K whose polyhedron is homeomorphic to T^n. We construct a homology theory on K and a homomorphism ν_K from the homology groups on K into a ring Z_p. To know that the intersection of two subcomplexes of K is nonempty, it suffices to show that their intersection carries a nonzero element of a homology group; to know that an element of a homology group is nonzero, it suffices to show that its image under ν_K is nonzero. We take advantage of the fact that T^n is a topological product by expressing K as a product of complexes $H_1, \ldots, H_n$, each with a polyhedron homeomorphic to the circle T^1, by defining a product $c_1, \ldots, c_n \to c_1 \times \ldots \times c_n$ from the chains on $H_1. \ldots, H_n$ into the chains on K, and by showing that if z_i is a cycle in the homology theory for H_i, $i \in \{1, \ldots, n\}$, then

$$(\nu_{H_1} z_1)(\nu_{H_2} z_2) \ldots (\nu_{H_n} z_n) = \nu_K(z_1 \times z_2 \times \ldots \times z_n),$$

where juxtaposition indicates the ring product in Z_p. To know that $\nu_K(z_1 \times \ldots \times z_n) \neq 0$ it suffices to prove that $\nu_{H_i} z_i = 1$, $i \in \{1, \ldots, n\}$. Such z_i exist.

The general method, then, is the same as that used by Schmidt; the proof that follows may be regarded as a refinement and completion of Schmidt's argument ([4], pp. 88-91).

2. Some definitions. Definitions of technical terms not defined in this paper may be found in [2] or [3].

If n is a nonnegative integer, the statement that S is an *n-simplex* means that there is a set $\{A^0, \ldots, A^n\}$ of $n+1$ objects such that S is the

set of all functions a from $\{A^0, \ldots, A^n\}$ into the positive real numbers such that $\sum_{i=0}^{n} aA^i = 1$. Each of $A^0, \ldots, A^n$ is called a *vertex* of S. Each member of S is called a *point* of S.

A *complex* K is a finite collection of one or more simplexes such that, if S is a simplex of K with vertex set V and L is a nonempty subset of V then, the simplex with vertex set L is in K.

If H is a finite collection of one or more nonempty sets, the *nerve* of H is defined to be the simplex collection to which S belongs if each vertex of S is in H and, if S has more than one vertex, the vertices of S have an element in common. The nerve of a finite collection of sets is a complex.

If n is a nonnegative integer and S is an n-simplex with vertex set $\{A^0, \ldots, A^n\}$, an *orientation* of S is a function f into $\{-1, 1\}$ from the set of simple orders for $\{A^0, \ldots, A^n\}$ such that, if $(A^{i_0}, \ldots, A^{i_n})$ is an odd permutation of $(A^0, \ldots, A^n)$, then $f(A^{i_0}, \ldots, A^{i_n}) = -f(A^0, \ldots, A^n)$. A simplex has two orientations, one the negative of the other. The orientation of S whose value at $(A^0, \ldots, A^n)$ is 1 will be denoted by $\langle A^0, \ldots, A^n \rangle$. If w is a simplicial map which is 1-1 on $\{A^0, \ldots, A^n\}$, $\langle wA^0, \ldots, wA^n \rangle$ will be denoted by $w\langle A^0, \ldots, A^n \rangle$.

If n is a nonnegative integer, X is a complex, and G is an abelian group, then a *G-valued n-chain on X* is a function c from the orientations of n-simplexes of X into G such that, if E is an orientation of an n-simplex of X, $c(-E) = -cE$. The G-valued n-chains on X form, under functional addition, an abelian group which will be denoted by $C_n(X, G)$.

Suppose that each of X and Y is a complex and $f\colon X \to Y$ is a simplicial map. Let $\check{f}$ be the function from $C_n(X, G)$ into $C_n(Y, G)$ such that, if $c \in C_n(X, G)$ and F is an orientation of an n-simplex of Y, $(\check{f}c)F = \sum_{E \in Q} cE$, where Q is the set of orientations of simplexes of X to which E belongs iff $fE = F$. $\check{f}$ is a *chain map*, i.e., is a homomorphism that commutes with the boundary operator ∂. Hereafter $\check{f}$ will be denoted by f. A similar convention holds for any other letter of the alphabet.

(2.1) *Notice that, if w is a 1-1 simplicial map from X onto X and c is an n-chain on X, then $(wc)E = c(w^{-1}E)$ for each orientation E of an n-simplex of X.*

Suppose that $\langle A^0, \ldots, A^n \rangle$ is an orientation of an n-simplex S of X, $g \in G$ and c is an n-chain on X such that

(a) $c\langle A^0, \ldots, A^n \rangle = g$ and

(b) if E is an orientation of a n-simplex of X different from S then $cE = 0$.

If $G = Z$, the additive group of the integers, and $g = 1$ then c is called an *elementary chain* and is denoted by $A^0 \dots A^n$. Whether $G = Z$ or not, c is denoted by $gA^0 \dots A^n$. Every member of $C_n(X, G)$ may be represented in the form $\sum_i g_i A_i^0 \dots A_i^n$.

If n is an integer and c is a chain, then nc is uniquely defined by the equations

$$0c = 0 \quad \text{and} \quad (n+1)c = nc + c.$$

Suppose that $g \in G$, $\{A_i^0 \dots A_i^n\}_{i=1}^r$ is a finite collection of elementary chains and each of $n_1, \dots, n_r$ is an integer. Then $\sum_{i=1}^r n_i g A_i^0 \dots A_i^n$ will sometimes be denoted by $g \sum_{i=1}^r n_i A_i^0 \dots A_i^n$.

The *support* of an n-chain c is the set of n-simplexes on whose orientations c does not assume the value 0.

3. The cartesian product of chains. Definitions for the *simplicial product* $K_1 \Delta K_2$ of complexes K_1 and K_2, an *order* for a complex and the *cartesian product* of ordered complexes are given in [2], pp. 66-67. (If $\leqslant$ is an order for a complex, the pair $(K, \leqslant)$ will be called an *ordered complex*.)

Remark. Any complex can be ordered by assigning a simple order to the vertices of the complex and then deleting from that order each vertex pair not connected by a 1-simplex. On the other hand there is a complex for which there is an order that cannot be imbedded in a simple order.

If each of G, H and J is an abelian group, a binary composition φ: $(G, H) \to J$ is called a *multiplication*, if

$$\varphi(g, h_1 + h_2) = \varphi(g, h_1) + \varphi(g, h_2)$$

and

$$\varphi(g_1 + g_2, h) = \varphi(g_1, h) + \varphi(g_2, h)$$

whenever $g, g_1, g_2 \in G$ and $h, h_1, h_2 \in H$.

(3.1) *Suppose that each of G, H and J is abelian group, φ: $(G, H) \to J$ is a multiplication, $g \in G$ and $h \in H$. Then $\varphi(g, 0) = \varphi(0, h) = 0$ and $\varphi(-g, h) = -\varphi(g, h) = \varphi(g, -h)$.* (Proof omitted.)

Suppose that X is a complex, $A^0 \dots A^a$ is an a-dimensional elementary chain on X, $B^0 \dots B^b$ is a b-dimensional elementary chain on X, and $\{A^0, \dots, A^a, B^0, \dots, B^b\}$ is the vertex set of an $(a+b+1)$-simplex of X. Then the elementary chain $A^0 \dots A^a B^0 \dots B^b$ is called the *join* of $A^0 \dots A^a$ and $B^0 \dots B^b$ and is denoted by $A^0 \dots A^a \circ B^b \dots B^b$. If each of $\{A_i^0 \dots A_i^a\}_{i=1}^r$ and $\{B_j^0 \dots B_j^b\}_{j=1}^r$ is a finite collection of elementary chains, each of $n_1, \dots, n_r, m_1, \dots, m_s$ is an integer and $A_i^0 \dots A_i^a \circ B_j^0 \dots B_j^b$ is defined when $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$, then $\sum_{i=1}^r n_i \sum_{j=1}^s m_j (A_i^0 \dots A_i^a \circ B_j^0 \dots B_j^b)$ will sometimes be denoted by $\left(\sum_{i=1}^r n_i A_i^0 \dots A_i^a\right) \circ \left(\sum_{j=1}^s m_j B_j^0 \dots B_j^b\right)$.

If each of $(K_1, \leqslant)$ and $K_2, \leqslant)$ is an ordered complex, there is a semi-linear homeomorphism from $|K_1 \times K_2|$ onto the topological product of $|K_1|$ and $|K_2|$ ([2], p. 68). We wish to define a "cartesian" product of chains on K_1 and K_2 into those on $K_1 \times K_2$ which is the algebraic counterpart of the cartesian product of complexes and the topological product of spaces. For $i \in \{1, 2\}$, suppose that S_i is a d_i-dimensional simplex in the complex K_i, H_i is the subcomplex of K_i consisting of S_i and all its faces and c_i is an elementary d_i-chain with support S_i. The cartesian chain product of c_1 and c_2 should be a chain whose dimension is $d_1 + d_2$, whose support is the set of (d_1+d_2)-simplexes in $H_1 \times H_2$ and whose boundary is 0 on any (d_1+d_2-1)-simplex of $H_1 \times H_2$ which is a common face of two (d_1+d_2)-simplexes in $H_1 \times H_2$. We now give a definition having these properties.

Suppose that each of $(K_1, \leqslant_1)$ and $K_2, \leqslant_2)$ is an ordered complex. If A is a vertex of K_1 and B is a vertex of K_2, the *cartesian chain product* of the elementary 0-chains A and B is defined to be the elementary 0-chain (A, B). For products of elementary chains the definition proceeds by induction on the dimensions. Suppose $A^0 \dots A^a$ is an elementary a-chain of K_1, $A^0 \leqslant_1 \dots \leqslant_1 A^a$, $B^0 \dots B^b$ is an elementary b-chain of K_2 and $B^0 \leqslant_2 \dots \leqslant_2 B^b$. If $a > 0$, $A^0 \dots A^a \times B^0$ means $(A^0 \dots A^{a-1} \times B^0) \circ (A^a, B^0)$. (Thus $A^0 \dots A^a \times B^0 = (A^0, B^0)(A^1, B^0) \dots (A^a, B^0)$.) If $b > 0$, $A^0 \times B^0 \dots B^b$ means $(A^0 \times B^0 \dots B^{b-1}) \circ (A^0, B^b)$. If $a, b > 0$, $A^0 \dots A^a \times B^0 \dots B^b$ means

$$[(A^0 \dots A^a \times B^0 \dots B^{b-1}) + (-1)^b (A^0 \dots A^{a-1} \times B^0 \dots B^b)] \circ (A^a, B^b).$$

(Since the join composition is applicable, these are genuinely definitions.) Suppose that each of G_1, G_2 and G_3 is an abelian group and φ: $(G_1, G_2) \to G_3$ is a multiplication. Then we define

$$\times: \left(C_a(X_1, G_1), C_b(X_2, G_2)\right) \to C_{a+b}(X_1 \times X_2, G_3)$$

by the equation

$$\left(\sum_i g_1^i A_i^0 \dots A_i^a\right) \times \left(\sum_j g_2^j B_j^0 \dots B_j^b\right) = \sum_{i,j} \varphi(g_1^i, g_2^j)(A_i^0 \dots A_i^a \times B_j^0 \dots B_j^b).$$

The definition of cartesian chain product given here, unlike the one given in [3], p. 138, is suitable for use in Čech homology theory.

The proofs of (3.2), (3.3), and (3.6) below are lengthy but routine and are omitted.

(3.2) *The cartesian chain product is a multiplication.*

(3.3) *Suppose that each of* $(K_1, \leqslant)$ and $(K_2, \leqslant)$ is an ordered complex, c_1 is an a-chain on K_1 and c_2 is a b-chain on K_2.

$$\partial(c_1\times c_2) = \begin{cases} \partial c_1 \times c_2 & \text{if} \quad a>0 \text{ and } b=0, \\ c_1\times \partial c_2 & \text{if} \quad a=0 \text{ and } b>0, \\ \partial c_1\times c_2 + (-1)^a c_1 \times \partial c_2 & \text{if} \quad a>0 \text{ and } b>0. \end{cases}$$

(3.4) COROLLARY. *If* ($a=0$ *or* $\partial c_1 = 0$) *and* ($b=0$ *or* $dc_1=0$), *then* ($a+b=0$ *or* $\partial(c_1\times c_2)=0$).

Suppose that each of $(K, \leqslant)$ and $(K', \leqslant')$ is an ordered complex and f is a simplicial map from K into K'. f will be said to be *order preserving* if $A^0 \leqslant A^1$ implies $fA^0 \leqslant' fA^1$ whenever A^0, A^1 are vertices of a 1-simplex of K.

(3.5) *If* $(K', \leqslant')$ *is an ordered complex and* f *is a simplicial map from the complex* K *into* K', *then there is an order* $\leqslant$ *for* K *such that* f *is order preserving.* (Such an assertion would have been false if order for a complex had been defined as a partial order.)

Construction. For each vertex v of K' let $\leqslant_v$ be an order on the subcomplex of K whose points are $f^{-1}v$. Define the relation $\leqslant$ as follows. Suppose that each of A and B is a vertex of K. If $fA=fB$, $A\leqslant B$ iff $A\leqslant_{fA} B$. If $fA\neq fB$, $A\leqslant B$ iff A and B are vertices of a simplex of K and $fA\leqslant' fB$.

(3.6) *Suppose that each of* $(K_1, \leqslant)$, $(K_1', \leqslant)$, $(K_2, \leqslant)$ *and* $(K_2', \leqslant)$ *is an ordered complex,* f_i *is an order preserving simplicial map of* K_i *into* K_i', $i\in\{1,2\}$, G_i *is an abelian group,* $i\in\{1,2,3\}$, *and* $\varphi\colon (G_1, G_2)\to G_3$ *is a multiplication. Let the chain multiplications*

$$\times\colon \big(C_a(K_1, G_1), C_b(K_2, G_2)\big)\to C_{a+b}(K_1\times K_2, G_3),$$
$$\times'\colon \big(C_a(K_1', G_1), C_b(K_2', G_2)\big)\to C_{a+b}(K_1'\times' K_2', G_3)$$

be defined using φ *as the underlying group multiplication.* (Note that each of $\times$ and $\times'$ is used in two ways here, to indicate a product of complexes and to indicate a product of chains.) *Let* (f_1, f_2) *denote the vertex map of* $K_1\times K_2$ *into* $K_1'\times' K_2'$ *such that* $(f_1, f_2)\,(A, B) = (f_1A, f_2B)$ *whenever* A *is a vertex of* K_1 *and* B *is a vertex of* K_2. *Then* (f_1, f_2) *defines a simplicial map which will also be denoted by* (f_1, f_2) *and* $(f_1, f_2)(c_1\times c_2) = f_1c_1\times' f_2c_2$ *for chains* c_1 *and* c_2 *on* K_1 *and* K_2, *respectively.*

4. Homology groups for actions. An *action* is an ordered triple $(W, X, *)$ such that W is a group under multiplication, X is a topological space and $*$ is a function from W such that

(1) if $w\in W$, then w^* is a homeomorphism of X onto X
and
(2) if $w, z\in W$ and $x\in X$, then $(wz)^*x = w^*(z^*x)$.

Note that $1^* = 1$ and $w^{*-1} = w^{-1*}$.

Throughout the remainder of this paper the conventions listed in the paragraphs (4.1) and (4.2) below will be observed except when explicitly suspended.

(4.1) If W is a group and X is a space, there will be under consideration at most one function $*$ such that $(W, X, *)$ is an action. In particular if Y is a subspace of X and $(W, Y, *)$ is an action, $w^* = w^*|Y$ for each w in W. Accordingly an action $(W, X, *)$ will be denoted by (W, X), $\{w^*\colon w\in W\}$ will be denoted by W and, for each w in W, w^* will be denoted by w.

(4.2) If (W, X) is an action, W is finite, X is a bicompact Hausdorff space and for each two different members w_1 and w_2 of W and each point x of X, $w_1x\neq w_2x$.

An example of an action satisfying our conditions is provided by (W, S^n) where S^n is the n-dimensional sphere, W is a group whose only elements are 1 and T, 1 is the identity map on S^n and T maps each point of S^m onto its antipode.

An action (W, X) is *simplicial* if X is a complex and each map in W is simplicial. We require in addition that, if v is a vertex of X and $w\in W$, $w\neq 1$, then no simplex has both v and wv as faces. Consequently, $\mathrm{St}v$ and $\mathrm{St}wv$ are disjoint and (4.2) holds. If S is a simplex and $w\in W$, $w\neq 1$, then $S\neq wS$. Any complex X may be regarded as a simplicial action $(\{1\}, X)$, where $\{1\}$ is the degenerate group.

(4.3) *Suppose that* (W, X) *is a simplicial action,* F *is a subcomplex of* X *such that* $\bigcup_{w\in W} wF = X$ *and* n *is a nonnegative integer. There is a subset* Y *of the set of* n-*simplexes of* F *such that* $\bigcup_{w\in W} wY$ *is the set* X_n *of* n-*simplexes of* X *and* $w_1Y\cap w_2Y=\emptyset$ *for* w_1, w_2 *in* W, $w_1\neq w_2$.

Construction. If each of H and K is in X_n, write $H\sim K$ if there is a w in W such that $H=wK$. Corresponding to the equivalence relation $\sim$ there is a partition of X_n into equivalence classes, each of which intersects F. Let Y be a subset of the n-simplexes of F consisting of just one simplex from each equivalence class.

An *ordered simplicial action* is a triple $(W, X, \leqslant)$ such that (W, X) is a simplicial action, $(X, \leqslant)$ is an ordered complex and each w in W is order preserving.

(4.4) *If* (W, X) *is a simplicial action there is an order* $\leqslant$ *for* X *such that* $(W, X, \leqslant)$ *is an ordered simplicial action.*

Proof. Let X/W denote the simplicial complex obtained from X by identifying each point of the polyhedron $|X|$ of X with all its images under W. Assign an order $\leqslant'$ to X/W. By (3.5) there is an order $\leqslant$ for X such that the natural map of X into X/W is order preserving. $\leqslant$ has the desired properties.

If each of $(W_1, X_1, *)$ and $(W_2, X_2, *)$ is an action, their *product* $(W_1, X_1)\times(W_2, X_2)$ is the action $(W_1\times W_2, X_1\times X_2, *)$, where $W_1\times W_2$ is the direct product, $X_1\times X_2$ is the topological product and $*$ is defined by

$$(w_1, w_2)^*(x_1, x_2) = (w_1^* x_1, w_2^* x_2).$$

It can be shown that if each of (W_1, X_1) and (W_2, X_2) is an action satisfying (4.2) then their product is an action satisfying (4.2).

If each of $(W_1, X_1, *)$ and $(W_2, X_2, *)$ is a simplicial action their *simplicial product* $(W_1, X_1)\Delta(W_2, X_2)$ is the simplicial action $(W_1\times W_2, X_1\Delta X_2, *)$, where $W_1\times W_2$ is the direct product, $X_1\Delta X_2$ is the simplicial product and, for $w_1\in W_1$ and $w_2\in W_2$,

$$(w_1, w_2)^*\colon X_1\Delta X_2\to X_1\Delta X_2$$

is the unique simplicial map such that

$$(w_1, w_2)^*(A, B) = (w_1^* A, w_1^* B)$$

whenever A is a vertex of X_1 and B is a vertex of X_2.

If each of $(W_1, X_1, \leqslant_1)$ and $(W_2, X_2, \leqslant_2)$ is an ordered simplicial action, their *cartesian product* is the simplicial action $(W_1\times W_2, X_1\times X_2)$, where $X_1\times X_2$ is the subcomplex of $X_1\Delta X_2$ associated with the orders $\leqslant_1$ and $\leqslant_2$ and the simplicial maps in $W_1\times W_2$ are the restrictions to $X_1\times X_2$ of the corresponding simplicial maps for $(W_1, X_1)\Delta(W_2, X_2)$.

The elements $(w_1, 1)$ and $(1, w_2)$ of the direct product $W_1\times W_2$ of groups W_1 and W_2 will sometimes be denoted by w_1 and w_2, respectively.

Suppose that (W, X) is a simplicial action, G is an abelian group and $R\colon C_q(X, G)\to C_q(X, G)$ is a chain map. A q-chain c on X is called an (R, q)-*chain* if there is a chain d such that $c = Rd$. The group of G-*valued* (R, q)-chains is denoted by $C_q(X, R, G)$. Thus $C_q(X, R, G) = RC_q(X, G)$. If $q>0$, the boundary operator ∂ is a homomorphism from $C_q(X, R, G)$ into $C_{q-1}(X, R, G)$. Define

$$\begin{aligned}
Z_0(X, R, G) &= C_0(X, R, G),\\
Z_q(X, R, G) &= \{c\colon c\in C_q(X, R, G), \partial c = 0\}, \qquad q>0,\\
B_q(X, R, G) &= \partial C_{q+1}(X, R, G),\\
H_q(X, R, G) &= Z_q(X, R, G)/B_q(X, R, G).
\end{aligned}$$

If each of (W, X) and (W, Y) is a simplicial action and $f\colon X\to Y$ is a simplicial map, f is said to be *equivariant* if $fwp = wfp$ for each w in W and each point p in X.

(4.5) *Suppose that each of (W, X) and (W, Y) is a simplicial action, G is an abelian group, for each w in W, n_w is an integer, and R is used to denote both the chain map from $C_q(X, G)$ into $C_q(X, G)$ such that*

$$Rc = \sum_{w\in W} n_w wc$$

and the chain map from $C_q(Y, G)$ into $C_q(Y, G)$ defined by the same formula. Then, if $f\colon X\to Y$ is an equivariant simplicial map, the chain map f maps $C_q(X, R, G)$ into $C_q(Y, R, G)$, $Z_q(X, R, G)$ into $Z_q(Y, R, G)$ and $B_q(X, R, G)$ into $B_q(Y, R, G)$ and induces a homomorphism

$$f_*\colon H_q(X, R, G)\to H_q(Y, R, G).$$

Proof. Since f is equivariant, it commutes with R. Suppose that $c\in C_q(X, R, G)$. There is a d such that $c = Rd$. $fc = fRd = Rfd\in C_q(Y, R, G)$. Suppose $q>0$ and $z\in Z_q(X, R, G)$. Since $\partial z = 0$, $\partial fz = 0$. $fz\in Z_q(Y, R, G)$. Suppose $z\in B_q(X, R, G)$. There is a u in $C_{q+1}(X, R, G)$ such that $z = \partial u$. Since $fu\in C_{q+1}(Y, R, G)$, $\partial fu\in B_q(Y, R, G)$. $fz = f\partial u = \partial fu\in B_q(Y, R, G)$.

The noun "cover" will be used in its usual sense: a collection of sets is a cover of any subset of the union of its members. The noun "covering" will be used in a more restricted sense. A *covering* of an action (W, X) is a finite open cover λ of X such that

(a) if $U\in\lambda$ and $w\in W$, $wU\in\lambda$,

and

(b) if $U\in\lambda$, $w_1, w_2\in W$ and $w_1\neq w_2$, then $w_1U\cap w_2U = \emptyset$.

If λ is a covering of (W, X), its nerve will be denoted by X_λ. (W, X_λ) is a simplicial action, where the simplicial maps in W are defined by the vertex maps $w\colon \lambda\to\lambda$, $w\in W$.

(4.6) *If α is an open cover of X, there is a covering of (W, X) that refines α.*

Proof. For each p in X there is an open set U_p containing p whose closure $\mathrm{Cl}\,U_p$ does not intersect the finite point set $\{wp\colon w\in W,\ w\neq 1\}$ and is a subset of a member of α. Let V_p denote $U_p - \bigcup_{w, w\neq 1} w\,\mathrm{Cl}\,U_p$. V_p is an open set containing p that is a subset of a member of α and does not intersect wV_p, if $w\in W$ and $w\neq 1$. Let S_p denote $\bigcap_{w\in W} w^{-1}V_{wp}$. S_p is an open set containing p such that

(a) $w_1S_p\cap w_2S_p = \emptyset$ if $w_1, w_2\in W$ and $w_1\neq w_2$

and

(b) wS_p is a subset of a member of α, if $w\in W$.

Since X is bicompact, some finite subcollection

$$\Big\{\bigcup_{w\in W}\{wS_{p_i}\colon i\in\{1,\dots,n\}\Big\} \quad\text{of}\quad \Big\{\bigcup_{w\in W} wS_p\colon p\in X\Big\}$$

covers X. $\{wS_{p_i}\colon w\in W,\ i\in\{1,\dots,n\}\}$ is a covering of (W, X) that refines α.

Suppose that each of α and β is a finite cover of the space X, β refines α and $\pi\colon \beta\to\alpha$ is a function such that, for all U in β, $U\subseteq\pi U$. The unique extension of π to a simplicial map from X_β into X_α is called a *projection* and will be denoted by π.

(4.7) *If each of λ and μ is a covering of the action (W, X) and μ refines λ, there is an equivariant projection π from X_μ into X_λ.*

Construction. By an argument similar to the one given for (4.3) there is a subset Y of μ such that $\mu = \bigcup_{w \in W} wY$ and $w_1 Y \cap w_2 Y = \emptyset$ if $w_1, w_2 \in W$ and $w_1 \neq w_2$. For each member V of μ there is just one pair (w, U) such that $w \in W$, $U \in Y$ and $V = wU$. For each U in Y, let πU be a member of λ that contains U. Extend π to all of μ by the rule.

$$\pi w U = w \pi U, \qquad U \in Y.$$

(4.8) *Suppose that each of λ and μ is a covering of the action (W, X), μ refines λ, R is a chain map of the form $\sum_{w \in W} n_w w$, each of π_0 and π_1 is an equivariant projection of X_μ into X_λ and π_{i*} is the homomorphism of $H_q(X_\mu, R, G)$ into $H_q(X_\lambda, R, G)$ induced by π_i, $i \in \{0, 1\}$. Then $\pi_{0*} = \pi_{1*}$.*

Proof. We construct a chain homotopy in the sense of [2], p. 129, or [3], p. 154. Let $(I, \leqslant_1)$ be an ordered 1-complex consisting of a 1-simplex and its vertices P^0 and P^1 and the relations $P^0 \leqslant_1 P^0$, $P^0 \leqslant_1 P^1$ and $P^1 \leqslant_1 P^1$. By (4.4) there is an order $\leqslant_2$ for X_μ such that $(W, X_\mu, \leqslant_2)$ is an ordered simplicial action. Let $I \times X_\mu$ be the product complex defined by the orders $\leqslant_1$ and $\leqslant_2$. $(W, I \times X_\mu)$ is a simplicial action where, for w in W, $w: I \times X_\mu \to I \times X_\mu$ is defined to be $(1, w)$. Let θ be the unique simplicial map from $I \times X_\mu$ into X_λ such that $\theta(P^i, U) = \pi_i U$, for $U \in \mu$, $i \in \{0, 1\}$. Let the cartesian chain product

$$\times: \big(C_a(I, Z), C_b(X_\mu, G)\big) \to C_{a+b}(I \times X_\mu, G)$$

be defined using the multiplication $\varphi: (Z, G) \to G$ such that $\varphi(1, g) = g$, $g \in G$. (Z is the additive group of the integers.) Let D denote the chain homomorphism from $C_q(X_\mu, G)$ into $C_{q+1}(X_\lambda, G)$ such that $Dc = \theta(P^0P^1 \times c)$. By (3.6) and the equivariance of θ, $wDc = w\theta(P^0P^1 \times c) = \theta w(P^0P^1 \times c) = \theta(1, w)(P^0P^1 \times c) = \theta(P^0P^1 \times wc) = Dwc$ for all chains c on X_μ and all w in W. Hence D commutes with R. It can be shown by induction of the dimension of c that $\theta(P^i \times Rc) = \pi_i Rc$, $i \in \{0, 1\}$. A calculation using (3.3) and the formulas $\partial\theta = \theta\partial$, $RD = DR$ and $\theta(P^i \times Rc) = \pi_i Rc$ shows that $\pi_1 Rc - \pi_0 RC = \partial RDc \in B_q(X_\lambda, R, G)$ whenever $Rc \in Z_q(X_\mu, R, G)$.

In view of (4.6) a covering of (W, X) exists and the coverings form a system directed by refinement. By (4.7) and (4.8) the homomorphisms

$$\pi_{\mu\lambda*}: H_q(X_\mu, R, G) \to H_q(X_\lambda, R, G)$$

induced by the equivariant projections $\pi_{\mu\lambda}: X_\mu \to X_\lambda$ form an inverse homomorphism system, whose limit, the q-dimensional *Čech-Smith homology group*, will be denoted by $\mathbf{H}_q(X, R, G)$.

For the purposes of this paper it is not necessary to know whether the homology group of a complex is isomorphic to the corresponding Čech-Smith homology group of its polyhedron.

5. *L*-systems. To say that (A, B, C) is an *L-triple* for (X, G) means that X is a complex, G is an abelian group, each of A, B and C is a chain map from the groups $C_q(X, G)$ into the groups $C_q(X, G)$ and, if c is a chain such that $Cc = 0$, there is a chain d such that $Bc = Ad$.

Examples. If X is a complex, $T: X \to X$ is a fixed-point free simplicial involution and Z_2 is the group of order 2, then $(1+T, 1, 1+T)$ is an L-triple for (X, Z_2) which plays a fundamental role in [6], [7], and [8]. If σ and δ are defined as in [5], p. 355, each of $(\sigma, 1, \delta)$ and $(\delta, 1, \sigma)$ is an L-triple. See also (6.3) below.

If X is a complex and G is an abelian group, we define a homomorphism In: $C_0(X, G) \to G$ by the rule $\mathrm{In}(\sum_i g_i V_i) = \sum_i g_i$ where the V_i's are the vertices of X and the g_i's are in G. The properties of In used in the sequel are:

(a) In is a homomorphism,

(b) $\mathrm{In}\,\partial c = 0$ if $c \in C_1(X, G)$,

(c) $\mathrm{In} fc = \mathrm{In}\, c$, if Y is a complex and $f: X \to Y$ is a simplicial map,

(d) $\mathrm{In}(c_1 \times c_2) = \varphi(\mathrm{In}\, c_1, \mathrm{In}\, c_2)$ if each of G_1, G_2 and G_3 is an abelian group, $\varphi: (G_1, G_2) \to G_3$ is a multiplication, each of $(X_1, \leqslant)$ and $(X_2, \leqslant)$ is an ordered complex, $\times$ is the cartesian chain product using φ as the underlying multiplication, and $c_i \in C_0(X_i, G_i)$, $i \in \{1, 2\}$.

To say that α is an *L-system* of depth n for (X, G) means that X is a complex, G is an abelian group, n is a nonnegative integer and α is a sequence

$$(A_0, S_1, A_1, \ldots, S_n, A_n)$$

of chain maps from $C_q(X, G)$ into $C_q(X, G)$ such that

(a) if $c \in C_0(X, G)$ and $A_0 c = 0$ then $\mathrm{In}\, c = 0$, and

(b) if $0 < q \leqslant n$ then (A_{q-1}, S_q, A_q) is an L-triple for (X, G).

(5.1) *Suppose that (A_0) is an L-system of depth 0 for (X, G). Let ν_0 be the relation to which (z, g) belongs iff there is a chain c such that $z = A_0 c \in Z_0(X, A_0, G)$ and $g = \mathrm{In}\, c$. Then ν_0 is a homomorphism from $Z_0(X, A_0, G)$ into G and $\nu_0 B_0(X, A_0, G) = 0$.*

Proof. (a) *If $z \in Z_0(X, A_0, G)$, there is a g such that $(z, g) \in \nu_0$.* Suppose that $z \in Z_0(X, A_0, G)$. There is a c such that $z = A_0 c$. $(z, \mathrm{In}\, c) \in \nu_0$.

(b) *ν_0 is a function.*

Suppose $(z, g_1), (z, g_2) \in \nu_0$. For $k \in \{1, 2\}$, there is a c_k such that $z = A_0 c_k$ and $g_k = \operatorname{In} c_k$. $A_0(c_1 - c_2) = A_0 c_1 - A_0 c_2 = z - z = 0$. $\operatorname{In}(c_1 - c_2) = 0$. $\operatorname{In} c_1 = \operatorname{In} c_2$. $g_1 = g_2$.

(c) *ν_0 is a homomorphism.*

Suppose $A_0 c, A_0 d \in Z_0(X, A, G)$. $\nu_0(A_0 c + A_0 d) = \nu_0(A_0(c+d)) = \operatorname{In}(c+d) = \operatorname{In} c + \operatorname{In} d = \nu_0(A_0 c) + \nu_0(A_0 d)$.

(d) $\nu_0 B_0(X, A_0, G) = 0$.

Suppose $z \in B_0(X, A_0, G)$. There is a u in $C_1(X, A_0, G)$ such that $z = \partial u$. There is a c in $C_1(X, G)$ such that $u = A_0 c$. $\nu_0 z = \nu_0 \partial u = \nu_0 \partial A_0 c = \nu_0 A_0 \partial c = \operatorname{In} \partial c = 0$.

(5.2) *Suppose that q is a positive integer, (A_{q-1}, S_q, A_q) is an L-triple for (X, G), ν_{q-1} is a homomorphism from $Z_{q-1}(X, A_{q-1}, G)$ into G such that $\nu_{q-1} B_{q-1}(X, A_{q-1}, G) = 0$ and ν_q is the relation to which (z, g) belongs iff there is a c in $C_q(X, G)$ such that $z = A_q c \in Z_q(X, A_q, G)$ and $g = \nu_{q-1} S_q \partial c$. Then ν_q is a homomorphism from $Z_q(X, A_q, G)$ into G and $\nu_q B_q(X, A_q, G) = 0$.*

Proof. (a) *If $z \in Z_q(X, A_q, G)$ there is a g such that $(z, g) \in \nu_q$.*

Suppose $z \in Z_q(X, A_q, G)$. There is a c such that $z = A_q c$. $A_q \partial c = \partial A_q c = \partial z = 0$. Since (A_{q-1}, S_q, A_q) is an L-triple, $S_q \partial c \in C_{q-1}(X, A_{q-1}, G)$. Since $\partial S_q \partial c = \partial \partial S_q c = 0$, $S_q \partial c \in Z_{q-1}(X, A_{q-1}, G)$. $\nu_{q-1} S_q \partial c$ is defined. $(z, \nu_{q-1} S_q \partial c) \in \nu_q$.

(b) *ν_q is a function.*

Suppose $(z, g_1), (z, g_2) \in \nu_q$. For $k \in \{1, 2\}$ there is a c_k such that $z = A_q c_k$ and $g_k = \nu_{q-1} S_q \partial c_k$. $A_q c(_1 - c_2) = A_q c_1 - A_q c_2 = z - z = 0$. Since (A_{q-1}, S_q, A_q) is an L-triple, $S_q(c_1 - c_2) \in C_q(X, A_{q-1}, G)$. $\partial S_q(c_1 - c_2) \in B_{q-1}(X, A_{q-1}, G)$. $g_1 - g_2 = \nu_{q-1} S_q \partial c_1 - \nu_{q-1} S_q \partial c_2 = \nu_{q-1} \partial S_q(c_1 - c_2) = 0$. $g_1 = g_2$.

(c) *ν_q is a homomorphism.*

Suppose $A_q c, A_q d \in Z_q(X, A_q, G)$.

$$\begin{aligned} \nu_q(A_q c + A_q d) &= \nu_q A_q(c+d) = \nu_{q-1} S_q \partial(c+d) = \nu_{q-1} S_q \partial c + \nu_{q-1} S_q \partial d \\ &= \nu_q A_q c + \nu_q A_q d. \end{aligned}$$

(d) $\nu_q B_q(X, A_q, G) = 0$.

Suppose $z \in B_q(X, A_q, G)$. There is a u in $C_{q+1}(X, A_q, G)$ such that $z = \partial u$. There is a c in $C_{q+1}(X, G)$ such that $u = A_q c$. $\nu_q z = \nu_q \partial u = \nu_q \partial A_q c = \nu_q A_q \partial c = \nu_{q-1} S_q \partial \partial c = 0$.

In the light of (5.1) and (5.2) we make the following definitions.

If $\alpha = (A_0, S_1, A_1, \ldots, S_n, A_n)$ is an L-system of depth n for (X, G), to say that ν is α's homomorphism means that ν is the functiton defined recursively by the rule

$$\nu A_q c = \begin{cases} \operatorname{In} c & \text{if} \quad q = 0, \\ \nu S_q \partial c & \text{if} \quad 1 < q \leqslant n \end{cases}$$

whenever $A_q c \in Z_q(X, A_q, G)$.

Since $\nu z_1 = \nu z_2$ whenever z_1 and z_2 are in the same element of $H_q(X, A_q, G)$, we can also define a homomorphism

$$\nu\colon H_q(X, A_q, G) \to G, \qquad q \in \{0, \ldots, n\}$$

by the rule $\nu[z] = \nu z$. (Here $[z]$ denotes the homology class containing z.)

Remark. Up to this point the discussion of L-triples and their associated homomorphisms can be interpreted even in the abstract context of chain complexes ([2], p. 124). But in concrete applications an L-system α will be defined just for simplicial actions (W, X) and the terms of α will be of the form $\sum_{w \in W} N_w w$, where the N_w's are integers.

(5.3) *Suppose that each of (W, X) and (W, Y) is a simplicial action, $\alpha = (R_0, S_1, R_1, \ldots, S_n, R_n)$ is an L-system of depth n for each of (X, G) and (Y, G), each term of α is of the form $\sum_{w \in W} N_w$ where each N_w is an integer, $f\colon X \to Y$ is an equivariant simplicial map and ν is α's homomorphism. Then for each q in $\{0, \ldots, n\}$ and each z in $Z_q(X, R_q, G)$, $\nu z = \nu f z$ (and, for $\zeta \in H_q(X, R_q, G)$, $\nu\zeta = \nu f_* \zeta$).*

Proof. Induction on q. Suppose $R_0 c \in Z_0(X, R_0, G)$. $\nu f R_0 c = \nu R_0 f c = \operatorname{In} f c = \operatorname{In} c = \nu R_0 c$. Suppose $0 < q \leqslant n$ and $R_q c \in Z_q(X, R_q, G)$.

$$\nu f R_q c = \nu R_q f c = \nu S_q \partial f c = \nu f(S_q \partial c) = \nu(S_q \partial c) = \nu R_q c.$$

Remark. It can be shown that if G is an abelian group, each of (W, X) and (W, Y) is a simplicial action, each of α, β and γ is a function from W into the integers and

$$\Big(\sum_{w \in W} \alpha_w w, \sum_{w \in W} \beta_w w, \sum_{w \in W} \gamma_w w\Big)$$

is an L-triple for (X, G) then it is also an L-triple for (Y, G). This fact does not contribute to the proof of our principal theorem and so is not proved here. I mention it only in order to render more palatable the following curious definition.

If (W, X) is an action or a simplicial action and, for every simplicial action (W, Y), $\alpha = (R_0, S_1, R_1, \ldots, S_n, R_n)$ is an α-system of depth n for (Y, G) and each term of α is of the form $\sum_{w \in W} N_w w$, where each N_w is an integer, then we shall say that α is an *L-system of depth n for (W, X, G)*.

Suppose that each of (W_1, X_1) and (W_2, X_2) is an ordered simplicial action, each of G_1, G_2 and G_3 is an abelian group,

$$A\colon C_q(X_1, G_1) \to C_q(X_1, G_1)$$

is a chain map of the form $\sum_{w \in W} N_w w$ and

$$B\colon C_q(X_2, G_2) \to C_q(X_2, G_2)$$

is a chain map of the form $\sum_{w\in W_2} M_w w$, where each of the N_w's and M_w's is an integer. Then A will also be used to denote the chain map

$$\sum_{w\in W_1} N_w(w,1)\colon\ C_q(X_1\times X_2,\,G_3)\to C_q(X_1\times X_2,\,G_3)$$

and B will be used to denote the chain map

$$\sum_{w\in W_2} M_w(1,w)\colon\ C_q(X_1\times X_2,\,G_3)\to C_q(X_1\times X_2,\,G_3)\,.$$

Note that the composite map

$$AB\colon\ C_q(X_1\times X_2,\,G_3)\to C_q(X_1\times X_2,\,G_3)$$

(B followed by A) can be expressed in the form $\sum_{w\in(W_1\times W_2)} J_w w$, where each of the J_w's is an integer.

(5.4) *Suppose that each of* (W_1, X_1) *and* (W_2, X_2) *is an ordered simplicial action,* $a_1=(A_0, S_1, A_1, \ldots, S_a, A_a)$ *is an L-system of depth a for* (W_1, X_1, G_1), $a_2=(B_0, Q_1, B_1, \ldots, Q_b, B_b)$ *is an L-system of depth b for* (W_2, X_2, G_2),

$$a_3=(A_0B_0,\,A_0Q_1,\,A_0B_1,\,\ldots,\,A_0Q_b,\,A_0B_b,\,S_1B_b,\,A_1B_b,\,\ldots,\,S_aB_b,\,A_aB_b)$$

is an L-system of depth $a+b$ for $(W_1\times W_2, X_1\times X_2, G_3)$, $\varphi\colon (G_1, G_2)\to G_3$ *is a multiplication and*

$$\times\colon\ \big(C_i(X_1, G_1),\,C_j(X_2, G_2)\big)\to C_{i+j}(X_1\times X_2,\,G_3)$$

is the cartesian chain product defined using φ as the underlying multiplication. Let ν_k denote a_k's homomorphism, $k\in\{1,2,3\}$. *If* $i\in\{0,\ldots,a\}$, $j\in\{0,\ldots,b\}$, $j=b$ *if* $i>0$, $z_1\in Z_i(X_1, A_i, G_1)$ *and* $z_2\in Z_j(X_2, A_j, G_2)$, *then* $z_1\times z_2\in Z_{i+j}(X_1\times X_2, A_iB_j, G_3)$ *and* $\nu_3(z_1\times z_2)=\varphi(\nu_1 z_1, \nu_2 z_2)$.

Proof. Suppose $A_ic\in Z_i(X_1, A_i, G_1)$ and $B_jd\in Z_j(X_2, B_j, G_2)$. If $i+j>0$, $\partial(A_ic\times B_jd)=0$ by (3.4). Suppose

$$A_i=\sum_{w\in W_1}\alpha_w w \quad\text{and}\quad B_j=\sum_{w\in W_2}\beta_w w\,.$$

With the aid of (3.2) and (3.6) we calculate:

$$\begin{aligned}
A_iB_j(c\times d) &= \sum_{w_1\in W_1}\alpha_{w_1}(w_1,1)\sum_{w_2\in W_2}\beta_{w_2}(1,w_2)(c\times d)\\
&= \Big(\sum_{w_1\in W_1}\sum_{w_2\in W_2}\alpha_{w_1}\beta_{w_2}\Big)(w_1,w_2)(c\times d)\\
&= \Big(\sum_{w_1\in W_1}\sum_{w_2\in W_2}\alpha_{w_1}\beta_{w_2}\Big)(w_1c\times w_2d)\\
&= \Big(\sum_{w_1\in W}\alpha_{w_1}w_1c\Big)\times\Big(\sum_{w_2\in W_2}\beta_{w_2}w_2d\Big)\\
&= A_ic\times B_jd\,.
\end{aligned}$$

Thus $A_ic\times B_jd\in Z_{i+j}(X_1\times X_2, A_iB_j, G_3)$. To show that $\nu_3(A_ic\times B_jd)=\varphi(\nu_1A_ic, \nu_2B_jd)$ we use induction on i.

(I) Suppose $i=0$. Induction on j.

(I.I) Suppose $j=0$.

$$\begin{aligned}
\nu_3(A_0c\times B_0d) &= \nu_3A_0B_0(c\times d) && \text{(3.2) and (3.6)}\\
&= \mathrm{In}(c\times d) && \text{definition of } \nu_3\\
&= \varphi(\mathrm{In}\,c,\,\mathrm{In}\,d)\\
&= \varphi(\nu_1A_0c,\,\nu_2B_0d) && \text{definitions of } \nu_1, \nu_2.
\end{aligned}$$

(I.II) Suppose $0<j\leqslant b$. The proof consists of a calculation similar to, but simpler than, the one given for case (II) below.

(II) Suppose $0<i\leqslant a$. $j=b$. Suppose $b>0$.

$$\begin{aligned}
\nu_3(A_ic\times B_bd) &= \nu_3A_iB_b(c\times d) && \text{(3.2) and (3.6),}\\
&= \nu_3S_iB_b\partial(c\times d) && \text{definition of } \nu_3,\\
&= \nu_3\partial S_iB_b(c\times d) && S_iB_b \text{ is a chain map,}\\
&= \nu_3\partial(S_ic\times B_bd) && \text{(3.2) and (3.6)}\\
&= \nu_3\big(\partial S_ic\times B_bd+(-1)^iS_ic\times\partial B_bd\big) && \text{(3.3)}\\
&= \nu_3(\partial S_ic\times B_bd) && \text{(3.1) and (3.2)}\\
&= \varphi(\nu_1\partial S_ic,\,\nu_2B_bd) && \text{inductive hypothesis}\\
&= \varphi(\nu_1A_ic,\,\nu_2B_bd). && \text{definition of } \nu_1.
\end{aligned}$$

In case $b=0$ the calculation is similar but simpler.

Suppose that (W, X) is an action and $a=(R_0, S_1, R_1, \ldots, S_n, R_n)$ is an L-system of depth n for (W, X, G). By (5.3) all of the coordinates of an element of $\boldsymbol{H}_q(X, R_q, G)$, $q\in\{0,\ldots,n\}$, have the same value under a's homomorphism ν. A homomorphism

$$\nu\colon\ \boldsymbol{H}_q(X, R_q, G)\to G$$

is defined by taking the value of an element of $\boldsymbol{H}_q(X, R_q, G)$ under ν to be the common value of its coordinates under ν.

If (W, X) is a simplicial action {action} and $a=(A_0, S_1, A_1, \ldots, S_n, A_n)$ is an L-system of depth n for (W, X, G), (W, X) will be said to be *a-admissible* if there is an element ζ of $H_n(X, A_n, G)$ $\{\boldsymbol{H}_n(X, A_n, G)\}$ such that $\nu\zeta\neq 0$, where ν is a's homomorphism.

Suppose that K is a complex, $\mathrm{Sd}\,K$ is its first barycentric subdivision and G is an abelian group. Define a homomorphism

$$\mathrm{Sd}\colon\ C_q(K, G)\to C_q(\mathrm{Sd}\,K, G)$$

as follows. If $c \in C_0(K, G)$, let $\mathrm{Sd}\,c$ be ic where i is the inclusion map of the vertices of K into those of $\mathrm{Sd}\,K$. Suppose that $q > 0$ and Sd has been defined on $C_{q-1}(K, G)$ and $C_{q-1}(K, Z)$. If $c = A^0 \dots A^q$ is an elementary q-chain in $C_q(K, Z)$ and B is the barycenter of the simplex with vertices $A^0, \dots, A^q$, define $\mathrm{Sd}\,c$ to be $B \circ \mathrm{Sd}\,\partial c$. If $d = \sum_i g_i A_i^0 \dots A_i^q \in C_q(K, G)$ or $C_q(K, Z)$, define $\mathrm{Sd}\,d$ to be $\sum_i g_i \mathrm{Sd}\,A_i^0 \dots A_i^q$.

(5.5) Sd *is a chain map* ([2], p. 177) *such that, if* π *is a projection of* $\mathrm{Sd}\,K$ *into* K, $\pi\,\mathrm{Sd} = 1$ ([2], Remark, p. 178).

(5.6) *Suppose that* (W, X) *is a simplicial action,* G *is an abelian group,* $c \in C_q(X, G)$ *and* $w \in W$. *Then* $w\,\mathrm{Sd}\,c = \mathrm{Sd}\,wc$.

Proof. Induction on q. Obvious if $q = 0$. Suppose that $q > 0$ and c is elementary. Let B be the barycenter of the support of c.

$$\begin{aligned} w\,\mathrm{Sd}\,c &= w(B \circ \mathrm{Sd}\,\partial c) && \text{definition of Sd} \\ &= (wB) \circ (w\,\mathrm{Sd}\,\partial c) \\ &= (wB) \circ (\mathrm{Sd}\,w\partial c) && \text{inductive hypothesis} \\ &= (wB) \circ (\mathrm{Sd}\,\partial wc) \\ &= \mathrm{Sd}\,wc. && \text{definition of Sd} \end{aligned}$$

Since each of w and Sd is a homomorphism, the result extends to arbitrary c.

(5.7) *If* (W, X) *is a simplicial action,* a *is an* L-*system for* (W, X, G) *and* (W, X) *is* a-*admissible, then* $(W, |X|)$ *is* a-*admissible.*

Proof. Suppose $a = (R_0, S_1, R_1, \dots, S_n, R_n)$, $\zeta_0 \in H_n(X, R_n, G)$ and $\nu\zeta_0 \neq 0$. For each positive integer j let π_j denote an equivariant projection from the jth barycentric subdivision $(\mathrm{Sd})^j X$ of X into $(\mathrm{Sd})^{j-1}X$. By (5.5) and (5.6), π_j maps $Z_n\big((\mathrm{Sd})^n X, R_n, G\big)$ *onto* $Z_n\big((\mathrm{Sd})^{j-1} X, R_n, G\big)$. Consequently,

$$\pi_{j*}\colon H_n\big((\mathrm{Sd})^j X, R_n, G\big) \to H_n\big((\mathrm{Sd})^{j-x} X, R_n, G\big)$$

is an onto function. There is a sequence $\{\zeta_j\}_{j=0}^{\infty}$ such that, for each positive integer j, $\zeta_j \in H_n\big((\mathrm{Sd})^j X, R_n, G\big)$ and $\pi_{j*}\zeta = \zeta_{j-1}$. For each covering λ of $(W, |X|)$ there is a non-negative integer $m(\lambda)$ such that the stars of the vertices of $(\mathrm{Sd})^{m(\lambda)}X$ form a covering of $(W, |X|)$ that refines λ. In particular, let $m\big((\mathrm{Sd})^j X\big) = j$ for each nonnegative integer j. For each covering λ of $(W, |X|)$ let F_λ denote the image of $\zeta_{m(\lambda)}$ under an equivariant projection from $H_n\big((\mathrm{Sd})^{m(\lambda)} X, R_n, G\big)$ into $H_n(X_\lambda, R_n, G)$. F is a member of $\mathbf{H}_n(X, R_n, G)$ that has ζ_0 as a representative. $\nu F = \nu\zeta_0 \neq 0$.

6. The admissibility of the n-torus.

(6.1) *Suppose that* (W, X) *is a simplicial action,* G *is an abelian group and* c *a chain in* $C_0(X, G)$ *such that* $\sum_{w \in W} wc = 0$. *Then* $\mathrm{In}\,c = 0$.

Proof. By an argument similar to the one used for (4.3) there is a subset Y of the set V of 0-simplexes of X such that $\sum_{w \in W} wY = V$ and $w_1 Y \cap w_2 Y = \emptyset$ whenever $w_1, w_2 \in W$ and $w_1 \neq w_2$. With the aid of (2.1) we have:

$$\mathrm{In}\,c = \sum_{v \in V} c\langle v\rangle = \sum_{v \in Y} \sum_{w \in W} c\langle wv\rangle = \sum_{v \in Y} \sum_{w \in W} w^{-1}c\langle v\rangle = \sum_{v \in Y} 0\langle v\rangle = 0.$$

The statements in (6.2) below will be hypotheses in (6.3) and (6.4).

(6.2) (W, X) is a simplicial action,

p is an integer >1, m is a positive integer, $e = mp$,

W is the (internal) direct product of a subgroup H and a cyclic subgroup W_e of order e with generator T,

n is a nonnegative integer.

Y' is a subset of the set X_n of n-simplexes of X such that

(i) $$hY' = Y' \quad \text{for each} \quad h \text{ in } H,$$

(ii) $$\bigcup_{a=0}^{e-1} T^a Y' = X_n,$$

(iii) $$T^i Y' \cap T^j Y' \neq \emptyset \quad \text{if} \quad i \not\equiv j \pmod{e},$$

N_h is an integer for each h in H,

M denotes the chain map $\sum_{h \in H} N_h h\colon C_q(X, Z_p) \to C_q(X, Z_p)$,

Z_p is the cyclic group of order p,

A denotes the chain map $\sum_{j=0}^{e-1} T^j M\colon C_q(X, Z_p) \to C_q(X, Z_p)$,

S denotes the chain map $\sum_{j=0}^{e-1} jT^j M\colon C_q(X, Z_p) \to C_q(X, Z_p)$.

(6.3) *Suppose* (6.2), $c \in C_n(X, Z_p)$ *and* $Ac = 0$. *Let* d *be the unique chain such that, if* E *is an orientation of an* n-*simplex in* Y',

$$d(T^a E) = \begin{cases} \Big(\sum_{j=0}^{e-1} jT^j c\Big) E & \text{if} \quad a = 0, \\ 0 & \text{if} \quad a \in \{1, \dots, e-1\}. \end{cases}$$

Then $Sc = Ad$. (*Thus,* (A, S, A) *is an* L-*triple for* (X, Z_p).)

Proof. Suppose that E is an orientation of an n-simplex in Y' and $b \in \{0, \ldots, e-1\}$. Then

$$\Big(\sum_{a=0}^{e-1} T^a Md\Big)(T^b E)$$

(a) $$= \Big(\sum_{a=0}^{e-1} T^a \sum_{u \in H} N_u ud\Big)(T^b E)$$

$$= \sum_{a=0}^{e-1} \sum_{u \in H} N_u[(T^a ud)(T^b E)]$$

(b) $$= \sum_{a=0}^{e-1} \sum_{u \in H} N_u[d(u^{-1} T^{b-a} E)]$$

(c) $$= \sum_{u \in H} N_u\Big[\sum_{a=0}^{e-1} d(T^{b-a} u^{-1} E)\Big]$$

(d) $$= \sum_{u \in H} N_u[d(u^{-1}E)]$$

(e) $$= \sum_{u \in H} N_u\Big[\Big(\sum_{j=1}^{e-1} jT^j c\Big)(u^{-1}E)\Big]$$

(f) $$= \sum_{u \in H} N_u\Big[u\Big(\sum_{j=0}^{e-1} T^j c\Big)E\Big]$$

(g) $$= \Big(\sum_{j=0}^{e-1} jT^j Mc\Big)E$$

(h) $$= \Big(\sum_{j=b}^{e-1+b} jT^j Mc\Big)E$$

(i) $$= \Big(\sum_{j=b}^{e-1+b} jT^j Mc\Big)E + b\Big(\sum_{j=b}^{e-1+b} T^j Mc\Big)E$$

$$= \Big(\sum_{j=b}^{e-1+b} (j+b)T^j Mc\Big)E$$

$$= \Big(\sum_{j=0}^{e-1} jT^{j-b} Mc\Big)E$$

(j) $$= \Big(\sum_{j=0}^{e-1} jT^j Mc\Big)(T^b E).$$

Equations (b), (f) and (j) are justified by (2.1); (a) and (g), by the definition of M; (c) and (g), by the fact that T commutes with each member of H; (d) and (e), by the definition of d; (h), by the fact that the order of Z_p divides the period of T; and (i), by the fact that $Ac = 0$.

(6.4) *Suppose* (6.2) *and* $b \in C_n(X, Z_p)$. *There is a chain* d *with support in* Y' *such that* $Ab = Ad$.

Proof. Let c denote the chain $(1-T)b$. Since $A(1-T)$ is the 0 operator, $Ac = 0$. Let d be defined as in the hypothesis of (6.3). The support of d is a subset of Y'. By (6.3), $Sc = Ad$. Since $S(1-T) = A$, we have $Ab = Ad$.

(6.5) *Suppose that:*

m is a positive integer, p is an integer >1, $e = pm$,

X is a 1-complex consisting of the $2e$ vertices $v_0, \ldots, v_{2e-1}$ and the $2e$ 1-simplexes $s_0, \ldots, s_{2e-1}$,

the 0-faces of s_i are v_i and v_{i-1}, if $i \in \{0, \ldots, 2e-1\}$ (addition in the subscripts is modulo $2e$),

k is an integer in $\{1, \ldots, e-1\}$ such that the greatest common divisor of k and e is 1,

T: $X \to X$ is the simplicial map such that

$$Tv_j = v_{j+2k}, \quad \text{if} \quad j \in \{0, \ldots, 2e-1\}.$$

Let W denote the group of homeomorphisms generated by T, A denote the chain map $\sum_{j=1}^{e-1} T^j$: $C_q(X, Z_p) \to C_q(X, Z_p)$, S denote the chain map $\sum_{j=1}^{e-1} jT^j$: $C_q(X, Z_p) \to C_q(X, Z_p)$, and a denote the triple (A, S, A). Then T is periodic of period e, (W, X) is a simplicial action, a is an L-system of depth 1 for (W, X, Z_p) and, if ν is a's homomorphism and the members of Z_p are denoted by $0, 1, \ldots, p-1$, there is a z in $Z_1(X, A, Z_p)$ such that $\nu z = 1$.

Proof. That (A, S, A) is an L-system follows from (6.1) and (6.3). Let c denote the 1-chain: $-\sum_{i=1}^{2k} v_{i-1}v_i$ and let d denote the 0-chain: $1v_0$. $A\partial c = 0$. Let Y' denote $\{v_0, v_1\}$. By (6.4), $S\partial c = Ad$. $\nu Ac = \nu S\partial c = \nu Ad = \operatorname{In} d = 1$.

(6.6) *Suppose that p is an integer >1, n is a positive integer and, for $i \in \{1, \ldots, n\}$,*

m_i is a positive integer,

$e_i = pm_i$,

k_i is an integer in $\{1, \ldots, e_i-1\}$ relatively prime to e_i,

T_i: $T^n \to T^n$ is the onto homeomorphism defined by

$$T_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_i + k_i/e_i, x_{i+1}, \ldots, x_n).$$

Let W denote the group of homeomorphisms generated by $\{T_1, \ldots, T_n\}$, A denote the chain map $\sum_{w \in x} w\colon C_q(X, Z_p) \to C_q(X, Z_p)$ and, for $i \in \{1, \ldots, n\}$ let H_i denote the subgroup of W generated by $\{T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n\}$ and S_i denote the chain map

$$\sum_{j=0}^{e_i-1} jT_i^j \sum_{h \in H_i} h\colon C_q(X, Z_p) \to C_q(X, Z_p).$$

Then $a = (A, S_1, A, \ldots, S_n, A)$ is an L-system of depth n for (W, T^m, Z_p) and (W, T^m) is a-admissible.

Proof. That (W, T^m) is an action satisfying (4.2) may be verified by the reader. That a is an L-system of depth n for (W, T^m, Z_p) is a consequence of (6.1) and (6.3).

Suppose $i \in \{1, \ldots, n\}$. The collection of points and segments in the real numbers

$$\{j/2e_i,\ (j/2e_i, (j+1)/2e_i)\colon j \in \{0, \ldots, 2e_i-1\}\}$$

defines a simplicial decomposition of T^1 which will be denoted by K_i. There is an order $\leqslant_i$ for K_i such that $j/2e_i \leqslant_i (j+1)/2e_i$ if $j \in \{0, \ldots, 2e_i-2\}$ and $(2e_i-1)/2e_i \leqslant_i 0$. Let $T_i\colon K_i \to K_i$ denote the simplicial map such that $T_i x = x + k_i/e_i$ and let W_i denote the group generated by T_i. $(W_i, K_i, \leqslant_i)$ is an ordered simplicial action. Let K denote $K_1 \times \ldots \times K_n$. $(W, K) = (W_1, K_1) \times \ldots \times (W_n, K_n)$ and $(W, T^m) = (W, |K|)$ ([2], p. 68). By (5.7), to prove the a-admissibility of (W, T^n) it will suffice to prove the a-admissibility of (W, K). An induction on n will show that there is a z in $Z_n(K, A, Z_p)$ such that $\nu z = 1$. In case $n = 1$, (6.5) disposes of the question. Suppose $n > 1$, By the inductive hypothesis there is a z in $Z_{n-1}\big(K_1 \times \ldots \times K_{n-1}, \sum_{w \in W_1 \times \ldots \times W_{n-1}} w, Z_p\big)$ such that $\nu z = 1$. By (6.5) there is a u in $Z_1\big(K_n, \sum_{w \in W_n} w, Z_p\big)$ such that $\nu u = 1$. A chain product

$$\times\colon \big(C_i(K_1 \times \ldots \times K_{n-1}, Z_p),\ C_j(K_n, X_p)\big) \to C_{i+j}(K, Z_p)$$

can be defined using the usual ring product $(Z_p, Z_p) \to Z_p$ as the underlying multiplication. Then by (5.4) $\nu(z \times u) = (\nu z)(\nu u) = 1 \cdot 1 = 1$.

7. The principal theorem. Throughout this section we will use the fact, that if X is a set with subsets A and B and $T\colon X \to X$ is 1-1 and onto, then $T(A \cup B) = TA \cup TB$ and $T(A \cap B) = TA \cap TB$.

(7.1a) {(7.1b)} *Suppose that*

p is an integer >1, Z_p is the cyclic group of order p,

(W, Z) is (a) *a simplicial action* {(b) *an action*},

m is a positive integer, $e = pm$,

W is the (internal) direct product of a subgroup H and a cyclic subgroup W_e of order e with generator T,

F is a (a) *subcomplex* {(b) *closed subset*} *of X such that*

$$\text{(i)} \qquad X = \bigcup_{j=0}^{p-1} T^j F,$$

$$\text{(ii)} \qquad hF = F = T^p F, \qquad h \in H,$$

K denotes $\bigcup_{j=1}^{p-1} \bigcup_{i=j+1}^{p} (T^j F \cap T^i F)$,

$a = (A_0, S_1, A_1, \ldots, S_n, A_n)$ is an L-system of depth n $(n > 0)$ for both (W, X, Z_p) and (W, K, Z_p),

ν is a's homomorphism,

there is a function N from H into the integers such that

$$A_n = A_{n-1} = \sum_{j=1}^{e-1} T^j \sum_{h \in H} N_h h, \qquad S_n = \sum_{j=1}^{p-1} jT^j \sum_{x=0}^{m-1} T^{px} \sum_{h \in H} N_h h.$$

Then (a) *if Δ is the relation to which (ζ, ω) belongs iff there is a chain c such that*

the support of c is a subset of F,

ζ is a homology class in $H_n(X, A_n, Z_p)$ that contains $A_n c$,

ω is a homology class in $H_{n-1}(K, A_n, Z_p)$ that contains the restriction of $S_n \partial c$ to K,

then Δ is a homomorphism from $H_n(X, A_n, Z_p)$ into $H_{n-1}(K, A_{n-1}, Z_p)$ such that $\nu\zeta = \nu\Delta\zeta$ for each ζ in $H_n(X, A_n, Z_p)$;

{(b) *there is a homomorphism Δ from $\mathbf{H}_n(X, A_n, Z_p)$ into $\mathbf{H}_{n-1}(K, A_{n-1}, Z_p)$ such that $\nu\zeta = \nu\Delta\zeta$ for each ζ in $H_n(X, A_n, Z_p)$*}.

Proof. Notice first that K is invariant under W, so that (W, K) is (a) a simplicial action {(b) an action}. Let A denote $A_n = A_{n-1}$ and S denote S_n. We now restrict our attention to (7.1a).

(A) If q is a nonnegative integer, $c \in C_q(X, Z_p)$, $Ac = 0$ and the support of c is a subset of F, then the support of Sc is a subset of K and $(Sc)|K \in C_q(K, A, Z_p)$.

Let b denote $\sum_{x=0}^{m-1} T^{px} \sum_{h \in H} N_h hc$. Since the support of c is a subset of F and F is invariant under both H and T^p, the support of b is a subset of F. Since $Sc = \sum_{j=0}^{p-1} jT^j b$, the support J of Sc is a subset of $\bigcup_{j=1}^{p-1} T^j F$. Since (A, S, A) is an L-triple for (W, X, Z_p) (see (6.3)), and $Ac = 0$, there is a chain d such that $Sc = Ad$. Since $Ad = TAd$, $J \subseteq TJ$. Hence, for any integer i, $J \subseteq T^i J \subseteq T^i \bigcup_{j=1}^{p-1} T^j F$. Therefore $J \subseteq \bigcap_{i=0}^{p-1} (T^i \bigcup_{j=1}^{p-1} T^j F) \subseteq K$.

Since K is W-invariant, we have $(Ad)|K = A(d|K)$. Thus $(Ad)|K \in C_q(K, A, Z_p)$.

(B) If $\zeta \in H_n(X, A, Z_p)$, then there is an ω such that $(\zeta, \omega) \in \Delta$.

By (4.3) there is a subset Y of the n-simplexes of F such that $\bigcup_{w \in W} wY$ is the set of n-simplexes of X and $w_1 Y \cap w_2 Y = \emptyset$ if $w_1 \neq w$ and $w_1, w_2 \in W$. Let Y' denote $\bigcup_{h \in H} hY$. Suppose $\zeta \in H_n(X, A, Z_p)$. Let z be in ζ. By (6.4) there is a chain c with support in Y' such that $z = Ac$. Since F is closed and $Y' \subseteq F$, the support of ∂c is a subset of F. Since $z \in Z_n(X, A, Z_p)$, $A\partial c = \partial z = 0$. By (A) the support of $S\partial c$ is a subset of K and $(S\partial c)|K \in C_{n-1}(K, A, Z_p)$. Since $\partial(S\partial c) = 0$ and the support of $S\partial c$ is a subset of K, $\partial\big((S\partial c)|K\big) = 0$. Thus $(S\partial c)|K \in Z_{n-1}(K, A, Z_p)$. Let ω be the homology class in $H_{n-1}(K, A, Z_p)$ that contains $(S\partial c)|K$. $(\zeta, \omega) \in \Delta$.

(C) Δ is a function.

Suppose that $(\zeta, \omega_1), (\zeta, \omega_2) \in \Delta$. By (B), for $k \in \{1, 2\}$, there is an n-chain c_k with support in F such that $Ac_k \in \zeta$ and ω_k is the member of $H_{n-1}(K, A, Z_p)$ that contains $(S\partial c_k)|K$. Since $Ac_1, Ac_2 \in \zeta$, there is a chain Ad in $C_{n+1}(X, A, Z_p)$ such that $Ac_2 - Ac_1 = \partial Ad$. By (6.4) there is an $(n+1)$-chain u with support in F such that $Ad = Au$. The support of $c_2 - c_1 - \partial u$ is a subset of F and $A(c_2 - c_1 - \partial u) = 0$. By (A), K contains the support of $S(c_2 - c_1 - \partial u)$ and $\big(S(c_2 - c_1 - \partial u)\big)|K \in C_n(K, A, Z_p)$.

$$\begin{aligned}(S\partial c_2)|K - (S\partial c_1)|K &= (\partial S c_2)|K - (\partial S c_1)|K - (\partial S \partial u)|K = \big(\partial S(c_2 - c_1 - \partial u)\big)|K\\ &= \partial\Big(\big(S(c_2 - c_1 - \partial u)\big)\big|K\Big) \in B_{n-1}(K, A, Z_p).\quad \omega_1 = \omega_2 .\end{aligned}$$

(D) A routine calculation shows that Δ is a homomorphism.

(E) If $\zeta \in H_n(X, A, Z_p)$, then $\nu\Delta\zeta = \nu\zeta$.

Adopt the following convention: if z is a cycle, $[z]$ denotes the homology class containing z. Suppose $\zeta \in H_n(X, A, Z_p)$ and let c be an n-chain with support in F such that $Ac \in \zeta$. Since the inclusion map $i\colon K \to X$ is equivariant, (5.3) justifies the starred equation: $\nu\Delta\zeta = \nu\Delta[Ac] = \nu[(S\partial c)|K] = \nu\big((S\partial c)|K\big) =_* \nu i\big((S\partial c)|K\big) = \nu S\partial c = \nu Ac = \nu[Ac] = \nu\zeta$.

We now turn to the proof of (7.1b). For each covering λ of (W, X) let X_λ denote the nerve of λ and let $F_\lambda\{K_\lambda\}$ denote the subcomplex of X_λ to which a simplex belongs iff the common part of its vertices intersects $F\{K\}$. Let D denote the collection of coverings of (W, X) to which λ belongs iff

$$K_\lambda = \bigcap_{j=1}^{p-1} \bigcap_{i=j+1}^{p} (T^j F_\lambda \cap T^i F_\lambda) .$$

It can be shown that D is cofinal in the collection of all coverings of (W, X). For each λ in D define

$$\Delta\colon H_n(X_\lambda, A, Z_p) \to H_{n-1}(K_\lambda, A, Z_p)$$

as in (7.1a). Suppose that each of λ and μ is in D, μ refines λ, $\pi\colon X_\mu \to X_\lambda$ is an equivariant projection and $Ac \in Z_n(X_\mu, A, Z_p)$, where the support of c is a subset of F_μ. Then the support of πc is a subset of F_λ and $\pi Ac \in Z_n(X_\lambda, A, Z_p)$ (see (4.5)). Let π' denote the restriction of π to K_μ.

$$\begin{aligned}\Delta_\lambda \pi_*[Ac] &= \Delta_\lambda[\pi Ac] = \Delta_\lambda[A\pi c] = [(S\partial\pi c)|K_\lambda] = [(\pi S\partial c)|K_\lambda]\\ &= \big[\pi'\big((S\partial c)|K_\mu\big)\big] = \pi'_*[(S\partial c)|K_\mu] = \pi'_*\Delta_\mu[Ac] .\end{aligned}$$

Thus $\Delta_\lambda \pi_* = \pi'_* \Delta_\mu$. It is now clear that

$$\Delta\colon \boldsymbol{H}_n(X, A, Z_p) \to \boldsymbol{H}_{n-1}(K, A, Z_p)$$

can be defined thus: if $\zeta \in \boldsymbol{H}_n(X, A, Z_p)$, $\lambda \in D$ and ζ_λ is the member of $H_n(X_\lambda, A, Z_p)$ that is the λ-coordinate of ζ then $\Delta\zeta$ is the member of $\boldsymbol{H}_{n-1}(K, A, Z_p)$ whose μ-coordinate is $\Delta_\lambda\zeta_\lambda$, where $\mu = \{U \cap K\colon U \in \lambda\}$. Suppose $\zeta \in \boldsymbol{H}(X, A, Z_p)$, $\lambda \in D$, ζ_λ is the λ-coordinate of ζ, $Ac \in \zeta_\lambda$ and the support of c is a subset of F_λ. $\nu\zeta = \nu\zeta_\lambda = \nu Ac = \nu(Ac|K_\lambda) = \nu\big((S\partial c)|K_\lambda\big) = \nu[(S\partial c)|K_\lambda] = \nu\Delta\zeta_\lambda = \nu\Delta\zeta$.

The statements in (7.2) below will be hypotheses for some definitions and theorems.

(7.2) W is a finite group, p is an integer greater than 1, n is a positive integer and for each i in $\{1, \ldots, n\}$

m_i is a positive integer,

$e_i = pm_i$,

W is the internal direct product of a subgroup H_i and a cyclic subgroup W_i of order e_i with generator T_i.

(7.3) Definition. Suppose (7.2) and that (W, X) is an action, not necessarily bicompact Hausdorff. We define sentences $A_i(X)$, $i \in \{0, \ldots, n\}$ as follows:

$A_0(X)$. $X \neq \emptyset$.

$A_i(X)$, $0 < i \leqslant n$. If F is a closed subset of X such that

(a) $\quad F = hF, \quad h \in H_i,$

(b) $\quad T_i^p F = F,$

(c) $\quad \bigcup_{j=0}^{p-1} T_i^j F = X,$

then

$$A_{i-1}\Big(\bigcup_{j=1}^{p-1} \bigcup_{k=j+1}^{p} (T_i^j F \cap T_i^k F)\Big) .$$

(7.4) *Suppose* (7.2), *that* (W, X) *is anacti on and let* $\alpha = (A, S_1, A, \ldots$ $\ldots, S_n, A)$ *be the L-system of depth n $(n > 0)$ for (W, X, Z_p) such that* $A = \sum_{w \in W} w$ *and*

$$S_i = \sum_{j=1}^{p-1} j T_i^j \sum_{x=0}^{m_i - 1} T_i^{px} \sum_{h \in H_i} h, \qquad i \in \{1, \ldots, n\} .$$

If (W, X) *is α-admissible, then* $A_n(X)$.

Proof. Suppose F is a closed subset of X such that $f = hF$ for all h in H_n, $T_n^pF = F$ and $\bigcup_{j=0}^{p-1} T_n^jF = X$. Let K denote $\bigcup_{j=1}^{p-1}\bigcup_{k=j+1}^{p}(T_n^j \cap T_n^k)$ and let ν be α's homomorphism. Since (W, X) is α-admissible, there is a ζ in $\boldsymbol{H}_n(X, A, Z_p)$ such that $\nu\zeta \neq 0$. By (7.1b), $\nu\Delta\zeta \neq 0$, where $\Delta\zeta \in \boldsymbol{H}_{n-1}(K, A, Z_p)$. The remainder of the argument is an induction on n. If $n = 1$, $\nu\Delta\zeta \neq 0 \Rightarrow \Delta\zeta \neq 0 \Rightarrow K \neq \emptyset \Rightarrow A_1(x)$. Suppose that $n > 1$ and let β denote the L-system $(A, S_1, \ldots, S_{n-1}, A)$ of depth $n-1$ for (W, K, Z_p). Since $\nu\Delta\zeta \neq 0$, (W, K) is β-admissible. By the inductive hypothesis $A_{n-1}(K)$. $A_n(X)$.

(7.5) Corollary. *If $n > 0$ and (W, T^n) is the action specified in the hypothesis of* (6.5), *then $A_n(T^m)$.*

(7.6) *Suppose that*

(a) *p is a prime number,*

(b) *u is a positive integer and $e = p^u$,*

(c) *$m = e/p$,*

(d) *the group W is the internal direct product of a subgroup H and a cyclic subgroup W' of order e with generator T,*

(e) *(W, X) is an action, not necessarily bicompact Hausdorff,*

(f) *f is a continuous function from X into the real numbers such that, if $x \in X$ and $h \in H$, $fx = fhx$.*

Then there is a closed subset F of X such that

(h) *$F = T^pF$,*

(i) *$F = hF$ for each h in H,*

(j) $\bigcup_{j=0}^{p-1} T^jF = X$,

(k) *if* $x \in \bigcup_{j=1}^{p-1}\bigcup_{k=j+1}^{p}(T^jF \cap T^kF)$,

then there is an integer a such that $fT^ax = fT^{a+1}x$.

Proof. For each i in the integers Z define $B(i, 1)$ to be $\{x\colon x \in X, fT^ix \leqslant fT^{i+1}x\}$ and $B(i, -1)$ to be $\{x\colon x \in X, fT^ix \geqslant fT^{i+1}x\}$. Notice that

(m) $B(i, -1) \cup B(i, 1) = X$, $\quad i \in Z$,

(n) $\bigcap_{i=1}^{e} B(i, -1) = \bigcap_{i=1}^{e} B(i, 1) \subseteq B(j, \delta)$, $j \in Z$, $\delta \in \{1, -1\}$,

(o) $Tb(j, \delta) = B(j-1, \delta)$, $j \in Z$, $\delta \in \{-1, 1\}$,

(p) *if* $h \in H$, $hB(j, \delta) = B(j, \delta)$, $j \in Z$, $\delta \in \{-1, 1\}$.

Let C be the set of functions from the integers Z onto $\{-1, 1\}$ such that, if $c \in C$ and $n \in Z$, $cn = c(n+e)$. C has $2^e - 2$ members. As a consequence of (m) and (n) we have

(q) $X = \bigcup_{d \in C}\bigcap_{i=1}^{e} B(i, di)$.

Let t denote the function from C into C such that, if $d \in C$,

$$(td)i = d(i+1), \qquad i \in Z.$$

If $d_1, d_2 \in C$ write $d_1 \sim d_2$ if there is an integer i such that $d_1 = t^id_2$. Associated with the equivalence relation $\sim$ is a partition of C into equivalence classes. Let M be a subset of C containing just one member of each equivalence class and let N denote $\bigcup_{x=0}^{m-1} t^{px}M$.

Suppose $d \in C$, $a \in Z$ and $d = t^ad$. Since a is a period of d and e is a period of d, the greatest common divisor (a, e) of a and e is a period of d. (a, e) divides e and is >1. Since $e = p^u$, (a, e) is a power of p. p divides a. Thus, for any d in C and a in Z, $d = t^ad$ implies that p divides a. This enables us to show that

(r) $\{t^0N, \ldots, t^{p-1}N\}$ *is a partitioning of C into p disjoint sets.*

Let β be the function from $\{tN^0, \ldots, t^{p-1}N\}$ such that, if $i \in \{0, \ldots, p-1\}$,

$$\beta(t^iN) = \bigcup_{d \in t^iN}\bigcap_{n=1}^{e} B(j, dj).$$

By using (o) above it can be shown that

(s) $TB(t^iN) = \beta t(t^iN)$, $\quad i \in \{0, \ldots, p-1\}$.

Let F denote βN. By (q), (r), and (s),

$$\bigcup_{i=0}^{p-1} T^iF = \bigcup_{i=0}^{p-1} T^i\beta N = \bigcup_{i=0}^{p-1} \beta t^iN = \bigcup_{d \in C}\bigcap_{j=1}^{e} B(j, dj) = X,$$

which proves (j). Suppose $x \in \bigcup_{i=1}^{p-1}\bigcup_{j=i+1}^{p}(T^iF \cap T^jF)$. There are two integers i and k in $\{0, \ldots, p-1\}$ such that

$$x \in (T^i\beta N) \cap (T^k\beta N) = (\beta t^iN) \cap (\beta^k tN)$$

$$= \Big[\bigcup_{d \in t^iN}\bigcap_{j=1}^{e} B(j, dj)\Big] \cap \Big[\bigcup_{c \in T^kN}\bigcap_{j=1}^{e} B(j, dj)\Big].$$

There is a d in T^iN such that $x \in \bigcap_{j=1}^{e} B(j, dj)$ and there is a c in T^kN such that $x \in \bigcap_{j=1}^{e} B(j, cj)$. Since t^iN and t^kN are disjoint, $c \neq d$. There is an a in $\{1, \ldots, e\}$ such that $da \neq ca$, in fact $da = -ca$. $x \in B(a, da) \cap B(a, ca) = B(a, -1) \cap B(a, 1)$. Hence $fT^ax \leqslant fT^{a+1}x$ and $fT^ax \geqslant f^{a+1}Tx$, which proves (k); (h) and (i) follow from the definition of F and from (p).

(7.7) DEFINITION. Suppose (7.2) and that (W, X) is an action, not necessarily bicompact Hausdorff. We define sentences $Q_k(X)$, $k \in \{1, \ldots, n\}$, as follows:

$Q_k(X)$. If each of $f_1, \ldots, f_k$ is a continuous function from X into the real numbers and

$$f_j T_i x = f_j x, \quad x \in X,\ i \in \{1, \ldots, n\},\ j \in \{1, \ldots, k\},$$

then there is an x^* in X such that $f_i x^* = f_i T_i x^*$ for $i \in \{1, \ldots, k\}$.

(7.8) *Suppose* (7.2), *p is prime, u_i is a positive integer and $e_i = p^{u_i}$, $i \in \{1, \ldots, n\}$, and (W, X) is an action, not necessarily bicompact Hausdorff. For each i in $\{1, \ldots, n\}$, $A_i(X)$ implies $Q_i(X)$.*

Proof. Induction on i. Suppose $i = 1$. By (7.6) there is a closed subset F of X such that $F = T_1^p F$, $F = hF$ for each h in H_1, $\bigcup_{j=0}^{p-1} T_1^j F = X$ and if $x \in K = \bigcup_{j=1}^{p-1} \bigcup_{k=j+1}^{p} (T_1^j F \cap T_1^k F)$ then there is an integer a such that $f_1 T_1^a x = f_1 T_1^{a+1} x$. Since $A_1(X)$, $K \neq \emptyset$. Let x be a point of K, let a be an integer such that $f_1 T x^a = f_1 T_1^{a+1} x$ and let x^* be $T^a x$.

Suppose $1 < i \leqslant n$. By (7.6) there is a closed subset F of X such that $F = T_i^p F$, $F = hF$ for each h in H_i, $\bigcup_{j=0}^{p-1} T_i^j F = X$ and if $x \in K = \bigcup_{j=1}^{p-1} \bigcup_{k=j+1}^{p} (T_i^j F \cap T_i^k F)$ then there is an integer a such that $f_i T_i^a x = f_i T_i^{a+1} x$. Since $A_i(X)$, $A_{i-1}(K)$ and, by the inductive hypothesis, $Q_{i-1}(K)$. Let x' be a point of K such that $f_j x' = f_j T_j x'$ for $j \in \{1, \ldots, i-1\}$. By (7.6) there is an integer a such that $f_i T_i^a x' = f_i T_i^{a+1} x'$. Let x^* denote $T_i^a x'$.

From (7.5) and (7.8) we have our principal theorem:

(7.9) *Suppose that n is a positive integer, p is a prime and, for each is in $\{1, \ldots, n\}$, u_i is a positive integer and k_i is a positive integer not divisible by p. Let T_i, $i \in \{1, \ldots, n\}$, denote the function from T^n into T^n such that*

$$T_i(x_1, \ldots, x_n) = \big(x_1, \ldots, x_{i-1}, x_i + (k_i/p^{u_i}), x_{i+1}, \ldots, x_n\big).$$

If each of $f_1, \ldots, f_n$ is a continuous function from T^n into the real numbers and

$$f_j T_i x = f_j x, \quad x \in T^n,\ i, j \in \{1, \ldots, n\},\ j \neq i,$$

then there is an x^* in T^n such that $f_i x^* = f_i T_i x^*$ for i in $\{1, \ldots, n\}$.

Remark. Taking $p = 2$ and $u_1 = \ldots = u_n = 1$ yields Schmidt's Satz 1 ([4], p. 86).

References

[1] K. Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. 20 (1933), pp. 177-190.

[2] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton Math. Series, No. 15, Princeton, N. J. 1952.

[3] S. Lefschetz, *Algebraic topology*, Amer. Math. Soc. Coll. Public., Vol. 27, New York City, 1942.

[4] W. Schmidt, *Stetige Funktionen auf dem Torus*, J. reine angew. Math. 207 (1961), pp. 86-95.

[5] P. A. Smith, *Fixed points of periodic transformations; Algebraic topology*, Amer. Math. Soc. Coll. Publ., Vol. 27, Appendix B, New York City, 1942.

[6] C. T. Yang, *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobo and Dyson* I, Ann. of Math. 60 (1954), pp. 262-282.

[7] — *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobo and Dyson* II, ibidem 62 (1955), pp. 271-283.

[8] — *Continuous functions from spheres to euclidean spaces*, ibidem 62 (1955), pp. 284-292.

Reçu par la Rédaction le 4. 6. 1964