and whether a paracompact semi-metric space is a Nagata space. An answer to the following question should help considerably to solve those problems.

QUESTION. *What is a necessary condition on an open mapping f from a metric space onto a T_2-space Y for Y to be normal (or paracompact)?*

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A closure and complement result for nested topologies

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It is well known that from a given set one can in a topological space construct at most 14 different sets by repeatedly using the operations of complementation and closure. The main purpose of this note is to establish a similar result for any finite number of nested topologies. Given any finite sequence of topologies each member of which is finer than its predecessor, from a fixed set one can construct only a finite number of different sets by repeatedly using the operations of complementation and of closure with respect to any topology of the sequence. It will be shown how this number is determined, and an upper bound will be given to it.

This will be accomplished by means of methods developed in modal logic. It is well known that the system of modal logic which is called S4 is interpretable as the closure algebra: If M and N are understood as the closure and the interior operator, respectively, and if $\sim$, &, and $\vee$ are understood in their usual Boolean sense, then a function formed by their means from set variables is identically the whole space in all topological spaces if and only if the same function is provable in S4 when the variables are interpreted as propositional variables, when $\sim$, &, and $\vee$ have their normal propositional senses, and when M and N are interpreted as the symbols for possibility and necessity, respectively [1]. This connection is extended to the case of a finite sequence of finer and finer topologies by considering a sequence of modal operators $M_0, N_0, M_1, N_1, \ldots, M_{n-i}, N_{n-i}$ where each pair M_i, N_i $(i = 0, 1, \ldots, n-1)$ is subject to the laws of S4 and where we have as an additional assumption the axiom schema

(1) $$M_i f \supset M_{i-1} f$$

for each $i = 1, \ldots, n-1$ (or, equivalently,

(2) $$N_{i-1} f \supset N_i f$$

for each $i = 1, \ldots, n-1$), where f is an arbitrary formula.

[1] For a lucid summary of many interesting results concerning the relation of modal logics to topology, see H. Rasiowa [6].

A simple semantical method of dealing with such modal logics as S4 has been outlined by the author [2]. (This method is easily seen to be equivalent with certain methods of Kripke's [3].) A formula is provable in the usual systems of S4 if and only if it is valid in this semantical approach [4]. A formula is valid if and only if its unit set is not satisfiable. A set of formulae is satisfiable if and only if it is a part of a member of some *model system*. A model system Ω is a set of *model sets* on which a transitive dyadic relation, called the *relation of alternativeness*, is defined. This transititive relation must satisfy the following conditions (f an arbitrary formula):

(C.M*) If $Mf \in \mu \in \Omega$, then there is in Ω at least one alternative ν to μ such that $f \in \nu$.

(C.N+) If $Nf \in \mu \in \Omega$ and if $\nu \in \Omega$ is an alternative to μ, then $f \in \nu$.

A model set may be defined for our present purposes as a set of formulae μ satisfying the following conditions (f and g arbitrary formulae):

(C.$\sim$) If $f \in \mu$, then not $\sim f \in \mu$.

(C.&) If $(f \& g) \in \mu$, then $f \in \mu$ and $g \in \mu$.

(C.$\vee$) If $(f \vee g) \in \mu$, then $f \in \mu$ or $g \in \mu$ (or both).

(C.N) If $Nf \in \mu$, then $f \in \mu$.

No further conditions are here imposed on model sets or model systems. If quantifiers are present, similar conditions are formulated for them. It has been assumed that all our formulae are in the negational normal form, i.e. that all their negation-signs have been pushed as deep into the formulae as they go until they all stand in front of propositional variables.

From these conditions (with the exception of (C.$\sim$)) one can obtain rules for constructing better and better approximations towards a model system by starting from a given set λ whose satisfiability we are investigating. These rules will be called (A.M*), (A.N+), ..., (A.N), respectively, and their formulation is left to the reader. If all the ways of trying to build a model system which would show that λ is satisfiable lead to a violation of (C.$\sim$) after a finite number of steps, then λ is not satisfiable. The construction which gave rise to this conclusion may be considered as a disproof of λ. Conversely, it may be shown that if λ is not satisfiable, it may be disproved in this way. This is a kind of completeness result for our system of rules [5].

(2) See my papers [1], [2].

(3) See S. A. Kripke [3], [4], [5].

(4) This follows from Kripke's completeness results together with the obvious equivalence of his semantical conditions with mine.

Every disproof of the kind just mentioned may also be considered as a *reductio ad absurdum* proof which starts from the assumption that there exists a member μ of a model system Ω which includes λ. It is seen at once from our rules that they have the following *subformula property*: Each formula introduced by them is a subformula (a proper subformula) of an earlier formula.

We can deal with the case in which we have a finite sequence of modal operators $M_0, N_0, M_1, N_1, \ldots, M_{n-1}, N_{n-1}$ for which (1) holds as follows: Instead of one alternativeness relation, we now use n different alternativeness relations, called 0-alternativeness, 1-alternativeness, etc. We assume that each $(i+1)$-alternative to a set is always also an i-alternative to it ($i = 0, 1, \ldots, n-2$). Furthermore, we relativize the conditions (C.M*) and (C.N+) (as well as the corresponding rules) as follows ($i = 0, 1, \ldots, n-1$):

(C.M*) If $M_i f \in \mu \in \Omega$, then there is in Ω at least one i-alternative ν to μ such that $f \in \nu$.

(C.N+) If $N_i f \in \mu \in \Omega$ and if $\nu \in \Omega$ is an i-alternative to μ, then $f \in \nu$.

The condition (C.N) holds for each subscript of N.

This is our apparatus, of which only a small part will be needed in what follows. By means of it, the problem we are investigating may be formulated very simply. The question is: How many irreducible formulae can we form from a given propositional variable p by the sole means of the operations $\sim$, M_i, and N_i ($i = 0, 1, \ldots, n-1$)? By an irreducible formula we here mean a formula which is not provably equivalent to a formula of the same or a smaller number of operators. Because of the connection between M and N it suffices to ask: How many irreducible formulae can we form from p and $\sim p$ by means of the operations M_i and N_i? Let us call this desired number $K = K(n)$.

First of all, we can reduce this question to the question concerning p only. In order to see this, let α be an arbitrary string of the operators M_i, N_i, and let $\overline{\alpha}$ be the result of interchanging the letters M and N in α. (This notation will be used consistently in the sequel.) In order to prove the equivalence of αp and $\beta \sim p$ (where β is another sequence of the operators M_i, N_i) we would have to show that αp and $\overline{\beta} p$ are not satisfiable together. However, this cannot be shown by means of our rules. Neither of the formulae in question contains a negation-sign, and

(5) This follows from Kripke's results concerning the equivalence of semantical tableaux to models, together with the obvious connection between our rules and Kripke's rules for his tableaux.

from the subformula property of our rules it therefore follows that they cannot give rise to a violation of (C.$\sim$). From the completeness result it follows that αp and $\bar{\beta}p$ are satisfiable together.

There are obviously just as many irreducible formulae of the form αp as there are irreducible formulae of the form $\alpha\sim p$.

Our first main observation is the following:

Formulae of the form

$$N_iN_jf \equiv N_if \quad \text{or} \quad N_iN_jf \equiv N_jf \tag{3}$$

are valid according to whether $j \geqslant i$ or $i \geqslant j$, respectively.

Proof (a). Assume that $j \geqslant i$. In order to show that the first equivalence (3) is valid we have to show that the two implications $N_iN_jf \supset N_if$ and $N_if \supset N_iN_jf$ are valid. (i) In order to show the first of the two we have to try to imbed the formulae N_iN_jf and $M_i\sim f$ into one and the same member μ of a model system Ω. The attempt may proceed as follows:

(3.11) $N_iN_jf \in \mu \in \Omega$
(3.12) $M_i\sim f \in \mu$ } counter-assumption,

(3.13) $\sim f \in \nu$ from (3.12) by (A.M*); here ν is an i-alternative to μ in Ω,

(3.14) $N_jf \in \nu$ from (3.11) by (A.N+),

(3.15) $f \in \nu$ from (3.14) by (A.N).

Here (3.14) and (3.15) violate (C.$\sim$), showing the desired validity.

(ii) In order to prove the validity of the converse implication we have to try to see whether there is a model system Ω and a model set μ such that

(3.21) $N_if \in \mu \in \Omega$
(3.22) $M_iM_j\sim f \in \mu$ } counter-assumption.

The attempt may proceed as follows:

(3.23) $M_j\sim f \in \nu$ from (3.22) by (A.M*); here $\nu \in \Omega$ is an i-alternative to μ.

(3.24) $\sim f \in \pi$ from (3.23) by (A.M*); here $\pi \in \Omega$ is a j-alternative to ν; hence (because $j \geqslant i$) an i-alternative to ν, hence (because of transitivity) an i-alternative to μ.

(3.25) $f \in \pi$ from (3.21) by (A.N+).

Here (3.24) and (3.25) violate (C.$\sim$), showing the desired validity.

(b) The validity of the second equivalence of (3) is proved (from the appropriate assumptions) in a closely similar fashion.

By duality we have the following analogous result: Formulae of the form

$$M_iM_jf \equiv M_if \quad \text{or} \quad M_iM_jf \equiv M_jf \tag{4}$$

are valid according to whether $j \geqslant i$ or $i \geqslant j$, respectively.

Equivalences (3) and (4) show us that there is no need to consider strings which contain two adjacent N's or two adjacent M's. If they do, a formula containing the string in question is provably equivalent to a shorter formula, and hence does not add to the number of irreducible formulae of the form we are considering.

Hence it suffices to consider, in addition to p, formulae which are either of the form

$$\ldots M_{a_3}N_{b_2}M_{a_2}N_{b_1}M_{a_1}p \tag{5}$$

or of the form

$$\ldots N_{a_3}M_{b_2}N_{a_2}M_{b_1}N_{a_1}\sim p\,. \tag{6}$$

Now a formula of form (5) cannot be provably equivalent to one of form (6). For in order to show their equivalence one would have to show that a pair of formulae of form

$$\{\alpha M_a p,\ \beta M_b \sim p\} \tag{7}$$

is not satisfiable. By examining our rules one can see, however, that this cannot be accomplished by their means. Consider, for the purpose, the set ν which is supposed to violate (C.$\sim$). The violating pair of formulae must obviously be $\{p, \sim p\}$. Here p can have been imported to ν only by means of an application of (A.M*); and the same applies to $\sim p$. However, they cannot both have been imported to ν by (A.M*). Hence a disproof of (7) by means of our rules is impossible, and by completeness (7) is therefore satisfiable.

Hence it suffices to consider formulae of form (5) only. If the number of non-equivalent formulae of this form is k, the total number K we are looking for is $K = 2\cdot(1+2k)$.

What conditions must the indices of (5) satisfy in order for (5) to be irreducible? An answer to this question is obtained from the following result:

Formulae of the form

$$N_bM_{a+i}N_{b+j}M_af \equiv N_bM_af \tag{8}$$

are valid for arbitrary indices a, b, and for all $i \geqslant 0$, $j \geqslant 0$.

Proof. In order to show that equivalence (8) is valid, we have to show that the corresponding pair of implications are valid.

(a) In order to show this for the implication from the left to the right, we may start from the following counter-assumption (Ω a model system):

(8.11) $$N_b M_{a+i} N_{b+j} M_a f \in \mu \in \Omega .$$

(8.12) $$M_b N_a \sim f \in \mu .$$

The argument may proceed as follows:

(8.13) $N_a \sim f \in \nu$ from (8.12) by (A.M*); $\nu \in \Omega$ is a b-alternative to μ.

(8.14) $M_{a+i} N_{b+j} M_a f \in \nu$ from (8.11) by (A.N+).

(8.15) $N_{b+j} M_a f \in \pi$ from (8.14) by (A.M*); $\pi \in \Omega$ is an $(a+i)$-alternative to ν; hence an a-alternative to ν.

(8.16) $M_a f \in \pi$ from (8.15) by (A.N).

(8.17) $f \in \varrho$ from (8.16) by (A.M*); $\varrho \in \Omega$ is an a-alternative to π; hence (by transitivity) an a-alternative to ν.

(8.18) $\sim f \in \varrho$ from (8.13) by (A.N+).

Here (8.17) and (8.18) violate (C.$\sim$), proving the validity of the implication.

(b) The validity of the converse implication may be shown as follows:

(8.21) $N_b M_a f \in \mu \in \Omega$ }
(8.22) $M_b N_{a+i} M_{b+j} N_a \sim f \in \mu$ } counter-assumption,

(8.23) $N_{a+i} M_{b+j} N_a \sim f \in \nu$ from (8.22) by (A.M*); $\nu \in \Omega$ is a b-alternative to μ.

(8.24) $M_{b+j} N_a \sim f \in \nu$ from (8.23) by (A.N).

(8.25) $N_a \sim f \in \pi$ from (8.24) by (A.M*); $\pi \in \Omega$ is a $(b+j)$-alternative to ν; hence a b-alternative to ν; hence (by transitivity) a b-alternative to μ.

(8.26) $M_a f \in \pi$ from (8.21) by (A.N+).

Here (8.25) and (8.26) violate (C.$\sim$).

By duality, formulae of the form

(9) $$M_b N_{a+i} M_{b+j} N_a f \equiv M_b N_a f$$

are valid whenever $i \geqslant 0$, $j \geqslant 0$.

Hence in order for (5) to be irreducible its indices must satisfy the following conditions:

(10) $$\begin{array}{l} \text{either} \quad a_1 > a_2 \quad \text{or} \quad b_2 > b_1 ; \\ \text{either} \quad b_1 > b_2 \quad \text{or} \quad a_3 > a_2 ; \\ \text{either} \quad a_2 > a_3 \quad \text{or} \quad b_3 > b_2 . \\ \dots\dots\dots\dots\dots\dots \end{array}$$

Here all the inequalities must be strict. Clearly, if at any line we choose the second of the alternatives in (10), we must at all the subsequent lines also choose the second alternative. Hence the general form of the conditions which the indices of (5) must satisfy may be written in one of the following forms (in both cases with a suitable i):

(11) (a) $$a_1 > a_2 > a_3 > \dots > a_i ; \qquad a_{i+1} < a_{i+2} < a_{i+3} < \dots$$
$$b_1 > b_2 > b_3 > \dots > b_i ; \qquad b_i < b_{i+1} < b_{i+2} < \dots$$

(b) $$a_1 > a_2 > a_3 > \dots > a_i ; \qquad a_i < a_{i+1} < a_{i+2} < \dots$$
$$b_1 > b_2 > b_3 > \dots > b_{i-1} ; \qquad b_i < b_{i+1} < b_{i+2} < \dots$$

Since the indices can only assume a finite number of values $0, 1, \dots, \dots, n-1$, the longest irreducible formula of form (5) is clearly obtained by making the two sequences of (11) as long as possible. It is easily seen that this is obtained by choosing the alternative (a) and by putting $i = n-1$. The longest irreducible formula is thus the following:

(12) $$M_{n-1} N_{n-1} M_{n-2} N_{n-2} \dots M_2 N_2 M_1 N_1 M_0 N_0 M_0 N_1 M_1 \dots \dots N_{n-2} M_{n-2} N_{n-1} M_{n-1} p .$$

The number of operators it contains is $(4n-1)$. As an upper bound for k we therefore obtain $k \leqslant (n+n^2+\dots+n^{4n-1}) =$ (if $n>1$) $(n^{4n}-1)/(n-1)$ and as an upper bound to K therefore $K \leqslant 2\big(1+2(n^{4n}-n)/(n-1)\big)$. This upper bound is obviously much too large in all cases $n>1$. For $n=1$, we obtain as the upper bound 14, which is known to be the exact number in question.

On the topological interpretation (12) is the most complicated set which can be formed from p by using repeatedly the closure and interior operations with respect to any of the n nested topologies and which does not identically reduce to a simpler set of the same kind.

The number $k = k(n)$ does not appear to obey any very simple law as a function of n. An interesting further fact concerning this number can easily be established, however. It is the fact that among all the different formulae of form (5) satisfying (10) there are no provably equivalent ones. Requirements (10) constitute, in short, not only a necessary but also a sufficient condition for (5) to be irreducible.

In order to show this, consider first a pair of formulae of form (5), say (5) itself and

$$\ldots N_{d_2} M_{c_2} N_{d_1} M_{c_1} p \,. \tag{13}$$

In order to prove the equivalence of (5) with (13) we would have to demonstrate that (5) is not satisfiable together with the following formula:

$$\ldots M_{d_2} N_{c_2} M_{d_1} N_{c_1} \sim p \,. \tag{14}$$

What would such a demonstration (by means of our rules) look like? The following schematic outline shows how the last few stages of the argument must run:

$$\begin{array}{ccccc}
\mu_{i+1} & \mu_i & \ldots & \mu_1 & \mu_0 \\
\hline
\longrightarrow & M_{a_i} N_{b_{i-1}} \ldots p \in \mu_i \rightarrow & \ldots \rightarrow & M_{a_1} p \in \mu_1 \rightarrow & p \in \mu_0 \\
\rightarrow M_{d_1} N_{c_1} \sim p \in \mu_{i+1} \longrightarrow & N_{c_1} \sim p \in \mu_i & \longrightarrow & \longrightarrow & \sim p \in \mu_0 \\
(d_2\text{-alt.} & (d_1\text{-alt.} & \ldots & (a_2\text{-alt.} & (a_1\text{-alt.} \\
\text{to } \mu_{i+2}) & \text{to } \mu_{i+1}) & & \text{to } \mu_2) & \text{to } \mu_1)\,.
\end{array}$$

Most of this outline is self-explanatory. In order to be able to infer $\sim p \in \mu_1$ from $N_{c_1} \sim p \in \mu_1$ we must have $c_1 \leqslant a_1, c_1 \leqslant a_2, \ldots, c_1 \leqslant a_i$. In general, (5) implies (13) if and only if there are integers $i, j, k, \ldots \geqslant 1$ such that the following conditions are satisfied:

(15) (a) $a_1 \geqslant c_1,\ a_2 \geqslant c_1,\ \ldots,\ a_i \geqslant c_1\,;$

(b) $d_1 \geqslant b_i,\ d_2 \geqslant b_i,\ \ldots,\ d_j \geqslant b_i\,;$

(c) $a_{i+1} \geqslant c_{j+1},\ a_{i+2} \geqslant c_{j+1},\ \ldots,\ a_{i+k} \geqslant c_{j+1}\,;$

. . . .

Conversely, (5) is implied by (13) if and only if there are integers $x, y, z, \ldots$ such that the analogous conditions are satisfied. These analogous conditions will be called (16).

In addition, we know that the indices of (5) satisfy our conditions (10). The indices of (13) must satisfy analogous conditions which will be referred to as conditions (17).

Can conditions (10), (15)-(17) be all satisfied together? In order to find the answer, let us first assume that $i > 1$, $x > 1$. Then by the first conditions of (15) (a) and of (16) (a) we have $a_1 = c_1$. By the second inequality of (15) (a) we have $a_2 \geqslant a_1$; hence in (10) the second alternative is always satisfied. Thus by (15) (b) and (10) $d_1 \geqslant b_i > b_1$. But by symmetry we also have $b_1 \geqslant d_x > d_1$, which yields a contradiction.

Assume, then, that $i > 1$, $x = 1$. Then we have as before $a_1 = c_1$, $d_1 \geqslant b_i > b_1$. By (16) (b) we have $b_1 \geqslant d_x = d_1$, which contradicts the inequality just obtained.

Thus there only remains the possibility that $i = x = 1$. Then by (15) (b) and (16) (b) $b_1 = d_1$. By repeatedly using the same argument we can show that $i = j = k = \ldots = x = y = z = \ldots = 1$, $a_r = c_r$, $b_r = d_r$ $(r = 1, 2, \ldots)$. But this means that (5) and (13) are identical, which was to be proved. This is the only case in which their equivalence can be proved by our rules, and by the completeness of these rules it is the only case in which (5) and (13) are provably equivalent.

Although we have not found any simple expression which would yield the number of the different irreducible formulae of form (5), we have found sufficient and necessary conditions for such formulae to be irreducible.

It is of some interest to see what our results amount to when $n = 2$. The only possible values of the indices of (5) are then 0 and 1. We obtain a convenient shorthand way referring to a formula of form (5) by thinking of the string of its indices as a dyadic number. The formula (i, j) will then be that formula (5) which has i operators the string of whose subscripts constitutes the dyadic number j. Then by our conditions the irreducible formulae of form (5) trivially include the formulae (1,0)-(1,1), (2,0)-(2,3), (3,0)-(3,7). Of the formulae with four operators the following are irreducible: (4,1), (4,3), (4,8), (4,9), (4,11), (4,12), (4,13); while the following equivalences are valid (4,0) = (4,2) = (4,4) = (4,6) = (2,0); (4,5) = (4,7) = (2,1); (4,10) = (4,14) = (2,2); (4,15) = (2,3). Of the formulae with five operators the following are irreducible: (5,3), (5,17), (5,19), (5,24), (5,25), (5,27); while of those which do not already reduce because of their last four operators we can say the following: (4,1) gives rise to (5,1) = (3,1) and (5,17); (4,3) gives rise to (5,3) and (5,19); (4,8) gives rise to (5,8) = (3,0) and (5,24); (4,9) gives rise to (5,9) = (3,1) and (5,25); (4,11) gives rise to (5,11) = (3,3) and (5,27); (4,12) gives rise to (5,12) = (3,0) and (5,28) = (3,4); (4,13) gives rise to (5,13) = (3,1) and (5,29) = (3,5). Of the formulae with six operators the only irreducible ones are (6,35), (6,49), and (6,51) which arise from (5,3), (5,17), and (5,19), respectively. As implied by our results, there is only one irreducible formula (5) with seven operators; it is (7,99) or

$$M_1 N_1 M_0 N_0 M_0 N_1 M_1 p \,.$$

It arises from (6,35). No irreducible formulae (5) exist with more than seven operators.

All told, we therefore have $k(2) = (2+4+8+7+6+3+1) = 31$ irreducible formulae of form (5) for $n = 2$. From what we have found, it follows that no two of them are provably equivalent. Hence we have $K = K(2) = 2\cdot(1+2\cdot 31) = 126$.

The proofs of our crucial equivalences (3) and (8) turned essentially on the fact that the topologies we are dealing with are nested, i.e. that

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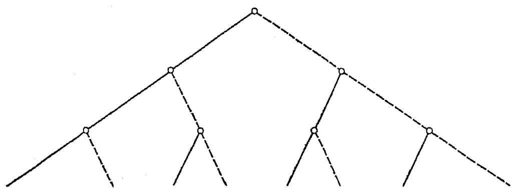
each of the n modal operators M we are dealing with is stronger than its predecessors. This suggests that no finitude result is possible without this assumption of nesting (linear order). In particular, since the proof of (3) already turns on this assumption, it may be expected that without the assumption of nesting we could already have an infinity of irreducible formulae of the form ... $N_{a_2}N_{a_1}p$. This expectation turns out to be justified. It is well known that of each partly ordered set we can obtain a topology by taking for the closure of each set S the set of all the elements e for which $e \leqslant s$ for at least one $s \in S$. The following infinite double tree will then serve as an example which shows the justifiability of our expectation:



Two partly ordering relations are defined on it whose covering relations are indicated by solid and dotted lines, respectively. From the unit set of the highest element one can obviously form an infinity of different sets by repeatedly using the two closure operations.

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Spaces in which sequences suffice*

by

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0. Introduction. Venkataraman [5] poses the following problem.

0.1. PROBLEM. Characterize "the class of topological spaces which can be specified completely by the knowledge of their convergent sequences".

It is a well known and useful fact that every first-countable space falls into this class. Indeed, this is so by virtue of either of two properties of first-countable spaces:

(a) A point lies in the closure of a set iff there is a sequence in the set converging to the point.

(b) A set is open iff every sequence converging to a point in the set is, itself, eventually in the set.

But these properties are not equivalent (see Example 2.2 below) and each is of independent interest (see Arhangel'skiĭ [1], Dudley [3], Franklin and Sorgenfrey [4], Hukuhara and Sibuyo [6], Kelley and Namioka [8], Mazur [10]). Hence problem 0.1 becomes by mitosis the two problems (0.1 (a) and 0.1 (b)) of characterizing the class of spaces satisfying (a) and the class satisfying (b).

The first of these (0.1 (a)) has two known solutions. Kowalsky [9] has given a characterization in terms of the neighborhoods of a point as follows: *A space satisfies* (a) *iff the filter of neighborhoods of each of its points is a union of Fréchet filters.* Since little is known of unions of Fréchet filters, this solution is not completely satisfactory.

A more penetrating solution is given by Arhangel'skii who calls spaces satisfying (a) *Fréchet spaces*. In [1] he asserts, without proof, that *among Hausdorff spaces, Fréchet spaces, and only these, are pseudo-open images of metric spaces*: (Pseudo-open maps form a class between the open maps and the quotient maps. See section 2 below for the definition.) An analogous result, due to Ponomarev [3], characterizes first-countable

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