On the convergence of orthogonal series of polynomials

by

H. RATAJSKI (Poznań)

1. Introduction. Young [5] proved a test for the convergence of trigonometric series formulated by means of generalized variation. The aim of this paper is to formulate and to prove this test in the case of a possibly large class of polynomial orthogonal series (1). First we give the definition of generalized Φ-variation of a function, as introduced by Wiener [4] and generalized in various directions by Young [5], and Musielak and Orlicz [3].

Let $\Phi(u)$ be a continuous function defined for $u \geqslant 0$, strictly increasing, $\Phi(0) = 0$, $\Phi(u) \to \infty$ as $u \to \infty$, and let $f(x)$ be a real-valued function defined in the interval $[a, b]$. Young defined *Φ-variation* of the function $f(x)$ by the formula

$$V_\Phi[f, [a, b]] = \sup_{\Pi} \sum_{i=0}^{k-1} \Phi(|f(x_{i+1}) - f(x_i)|),$$

where Π runs over all partitions $a = x_0 < x_1 < \ldots < x_k = b$ of the interval $[a, b]$. He modified the classical theorem on limits of Stieltjes integrals, and applying this modification he proved the following test for the convergence of trigonometric Fourier series ([5], p. 610):

If a function $f(x)$ continuous in $[-\pi, \pi]$ has a bounded Φ-variation in this interval, where

$$\lim_{u \to 0+} e^{u^{-a}} \Phi(u) = 1 \textit{ with an } a < \tfrac{1}{2},$$

then the trigonometric Fourier series of $f(x)$ is convergent to $f(x)$ at every point $x \in (-\pi, \pi)$.

Remark. The theorem is quoted here in a slightly weaker form than the original formulation by Young, namely we assume $f(x)$ to be continuous in the whole interval. However, this case seems to be the most interesting one; moreover, limiting ourselves to continuous functions, we need not mention special properties of functions of bounded generalized variation.

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(1) This problem was raised by W. Orlicz.

Let $\varrho(x)$ be a positive integrable function in $[a, b]$. We shall deal with a system of polynomials $\{p_n(x)\}$ orthonormal in $[a, b]$ with respect to $\varrho(x)$ as a weight-function, i. e. such that

$$\int_a^b p_n(x)p_k(x)\varrho(x)dx = \begin{cases} 1 & \text{for} \quad n = k, \\ 0 & \text{for} \quad n \neq k. \end{cases}$$

Here $p_n(x)$ denotes a polynomial of degree n, with a positive coefficient of x^n. If $S_n(x)$ denotes the n-th partial sum of the Fourier series of a function $f(x)$ with respect to the system $\{p_n(x)\}$, it is well known that

$$S_n(x) = \int_a^b f(t)K_n(t, x)\varrho(t)dt, \tag{1.1}$$

where

$$K_n(t, x) = \sum_{k=0}^{n} p_k(t)p_k(x) \tag{1.2}$$

is called the *kernel* of the integral (1.1).

In the problems of convergence of Fourier series, the following summation formula of Christoffel and Darboux is of importance:

$$\sum_{k=0}^{n} p_k(t)p_k(x) = \frac{a_n}{a_{n+1}} \frac{p_n(x)p_{n+1}(t)-p_n(t)p_{n+1}(x)}{t-x}; \tag{1.3}$$

here a_n, a_{n+1} are the positive coefficients of x^n, x^{n+1} in polynomials $p_n(x)$, $p_{n+1}(x)$, respectively. It is known that $a_n/a_{n+1} \leqslant \max(|a|, |b|)$ (see [1], p. 33). In the sequel $\Psi(u)$ will be a continuous function defined for $u \geqslant 0$, strictly increasing, $\Psi(0) = 0$, $\Psi(u) \to \infty$ as $u \to \infty$, and such that

$$\Psi(u_1)+\Psi(u_2) \leqslant \Psi(u_1+u_2) \tag{1.4}$$

for arbitrary $u_1, u_2 \geqslant 0$. Moreover, we shall write

$$g_n(t, x) = \int_a^t K_n(u, x)\varrho(u)du \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{1.5}$$

In our further considerations we shall limit ourselves to the interval $[-1, 1]$, since the general case is reduced to this one by the substitution $t = -1+2(x-a)/(b-a)$.

2. We now formulate Young's test for convergence in the case of orthonormal polynomial series.

THEOREM. *Let $\{p_n(x)\}$ be a system of polynomials orthonormal in $[-1, 1]$ with respect to the weight-function $\varrho(x)$, and let $f(x)$ be defined in $[-1, 1]$. We suppose that* (²)

(2.1) *there exist constants $c_1 > 0$, $c_2 \geqslant 0$, $c_3 > 0$, $c_4 \geqslant 0$ such that*

$$0 < \varrho(x) \leqslant \frac{c_1}{(1-x^2)^{c_2}}, \quad |p_n(x)| \leqslant \frac{c_3}{(1-x^2)^{c_4}}$$

for all $x \in (-1, 1)$ and $n = 0, 1, 2, \ldots$,

(2.2) *$f(x)$ is continuous in $[-1, 1]$,*

(2.3) *there exists a constant $a < \frac{1}{2}$ such that $f(x)$ is of bounded Φ-variation in $[-1, 1]$, where*

$$\Phi(u) \sim \exp(-u^{-a}) \quad \textit{as} \quad u \to 0+. \tag{2.4}$$

Then

$$f(x) = \sum_{n=0}^{\infty} d_n p_n(x) \quad \textit{for every} \quad x \in (-1, 1),$$

where

$$d_n = \int_{-1}^{1} f(t)p_n(t)\varrho(t)dt.$$

Remark. It is easy to give an example of a function which is of infinite Φ-variation for all $\Phi(u) = u^p$, $p > 1$, but satisfies conditions (2.3) and (2.4).

3. Proof. We shall prove the Theorem basing ourselves on the following lemmas.

LEMMA 3.1. *If the system $\{p_n(x)\}$ satisfies condition* (2.1) *and $[a, \beta] \subset (-1, 1)$, then*

$$\int_{\xi_1}^{\xi_2} p_n(x)p_m(x)\varrho(x)dx = O\left(\frac{1}{|n-m|}\right)$$

uniformly with respect to ξ_1, ξ_2, where $a \leqslant \xi_1 \leqslant \xi_2 \leqslant \beta$.

The proof of this lemma may be found in [2], p. 91-92.

LEMMA 3.2 ([5], p. 602, Theorem 5.5). *Let $f(t)$, $g(t)$ be of bounded generalized variation of the type Φ_1, Ψ_1 respectively, and let f, g have no common points of discontinuity. Moreover, let φ_1 resp. ψ_1 be the functions inverse to Φ_1 resp. Ψ_1. We suppose that*

$$\sum \varphi_1\left(\frac{1}{n}\right)\psi_1\left(\frac{1}{n}\right) < \infty.$$

Then the Riemann-Stieltjes integral $\int_a^b g(t)df(t)$ exists.

LEMMA 3.3. *Let the following conditions be satisfied:*

(3.1) *$G_n(t)$ $(n = 1, 2, \ldots)$ have uniformly bounded Ψ-variation, where Ψ is a convex function, and $F(t)$ is of bounded Φ-variation in $[a, b]$.*

(²) Condition (2.1) is satisfied, for instance, in the case of normalized Jacobi polynomials $\{p_n^{(\alpha,\beta)}(x)\}$ where $\varrho(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha > -1$, $\beta > -1$ (see [2], p. 89, footnote (²)).

(3.2) $G_n(t)$ $(n = 1, 2, \ldots)$ *and* $F(t)$ *are continuous in* $[a, b]$, $|G_n(a)| \leqslant M$ *for* $n = 1, 2, \ldots$

(3.3) $\sum_{n=1}^{\infty} \varphi(1/n)\psi(1/n) < \infty$, *where* φ, ψ *are the functions inverse to* Φ, Ψ, *respectively.*

(3.4) $G_n(t) \to G(t)$ *as* $n \to \infty$ *for* $t \in [a, \tau_0)$ *and* $t \in (\tau_0, b)$ *where* τ_0 *is a given point in* (a, b).

Then

$$\lim_{n\to\infty} \int_a^b G_n(t)\,dF(t) = \int_a^b G(t)\,dF(t).$$

The proof of this lemma is obtained by a slight modification of the proof of Theorem (6.2), p. 606, [5].

LEMMA 3.4. *Let* $\Psi(u)$ *be a continuous, strictly increasing function, defined for* $u \geqslant 0$, $\Psi(0) = 0$, $\Psi(u) \to \infty$ *as* $u \to \infty$, *satisfying* (1.4). *Given any* $\delta > 0$, *we then have*

$$V_\Psi[g_n(t,x), -1 \leqslant t \leqslant 1] \leqslant C(\delta)\left[1 + \sum_{m=1}^{\infty} \Psi\left(\frac{k(\delta)}{m}\right)\right] \text{ for } -1+\delta \leqslant x \leqslant 1-\delta$$

where $g_n(t,x)$ *are defined by* (1.5), *and* $C(\delta)$, $k(\delta)$ *are constants dependent only on* δ.

Proof of Lemma 3.4. Let $x \in [-1+\delta, 1-\delta]$, where $\delta > 0$. Then $[x-1/n, x+1/n] \subset [-1+\frac{1}{2}\delta, 1-\frac{1}{2}\delta]$ for sufficiently large n. The points $-1+\delta/2$, $-1+\delta$, $x-1/n$, $x+1/n$, $1-\delta$, $1-\delta/2$ divide the interval $[-1, 1]$ into seven subintervals. On the other hand, let us divide the interval $[-1, 1]$ by means of the points

$$a_m = x + \frac{m}{n}$$

where m takes such integer values that $a_m \in [-1, 1]$.

We now give some auxiliary estimations:

(i) *If either* $(\alpha, \beta) \subset [-1, -1+\delta/2]$ *or* $(\alpha, \beta) \subset [1-\delta/2, 1]$ *then*

$$\int_\alpha^\beta |K_n(t,x)|\,\varrho(t)\,dt \leqslant \gamma_1(\delta).$$

Applying the Christoffel-Darboux summation formula for polynomials $p_n(x)$, we obtain

$$\int_\alpha^\beta |K_n(t,x)|\,\varrho(t)\,dt$$
$$< \frac{a_n}{a_{n+1}}|p_n(x)| \int_{-1}^{-1+\delta/2} \frac{|p_{n+1}(t)|}{|t-x|}\varrho(t)\,dt + \frac{a_n}{a_{n+1}}|p_{n+1}(x)| \int_{-1}^{-1+\delta/2} \frac{|p_n(t)|}{|t-x|}\varrho(t)\,dt.$$

Since $x \in [-1+\delta, 1-\delta]$, there exists by (2.1) a constant $h_1 = h_1(\delta)$ such that $|p_n(x)| < h_1(\delta)$ for every index n. Changing the interval of integration to the whole $[-1, 1]$ and applying $|t-x| \geqslant \delta/2$ and Schwarz's inequality

$$\int_{-1}^{1} |p_n(t)|\,\varrho(t)\,dt = \int_{-1}^{1} |p_n(t)|\sqrt{\varrho(t)}\sqrt{\varrho(t)}\,dt \leqslant \Big\{\int_{-1}^{1} p_n^2(t)\,\varrho(t)\,dt \int_{-1}^{1} \varrho(t)\,dt\Big\}^{1/2},$$

we find that the integral

$$\int_\alpha^\beta |K_n(t,x)|\,\varrho(t)\,dt$$

is bounded by a constant γ_1 dependent on δ. The second part of (i) is obtained in a similar way.

(ii) *If* $(\alpha, \beta) \subset [x-1/n, x+1/n]$, *then*

$$\int_\alpha^\beta |K_n(t,x)|\,\varrho(t)\,dt < \gamma_2(\delta).$$

Indeed, we have $-1+\delta/2 < x-1/n \leqslant t \leqslant x+1/n < 1-\delta/2$ for sufficiently large n. Hence, by (2.1),

$$\int_\alpha^\beta |K_n(t,x)|\,\varrho(t)\,dt \leqslant \int_{x-1/n}^{x+1/n} \sum_{k=0}^{n} |p_k(t)|\,|p_k(x)|\,\varrho(t)\,dt \leqslant h_2(\delta)(n+1)\frac{2}{n} \leqslant \gamma_2(\delta).$$

(iii) *Let* $(\alpha, \beta) \subset [a_m, a_{m+1}]$, *where* m *is such an integer that* $m \neq -1$, $m \neq 0$ *and* $a_m \in [-1+\frac{1}{2}\delta, 1-\frac{1}{2}\delta]$. *Then*

$$\int_\alpha^\beta |K_n(t,x)|\,\varrho(t)\,dt \leqslant \gamma_3(\delta)/|m|.$$

Indeed, since $-1+\delta/2 \leqslant x+m/n < t < x+(m+1)/n \leqslant 1-\delta/2$, the last inequality follows by applying the summation formula for polynomials $p_n(x)$ and the inequalities

$$|t-x| \geqslant \begin{cases} m/n & \text{for } m > 0, \\ |m+1|/n & \text{for } m < 0. \end{cases}$$

In the case of $m > 0$ we have

$$\int_\alpha^\beta |K_n(t,x)|\,\varrho(t)\,dt$$
$$\leqslant \frac{a_n}{a_{n+1}}|p_n(x)| \int_{x+m/n}^{x+(m+1)/n} \frac{|p_{n+1}(t)|\,\varrho(t)}{|t-x|}dt + \frac{a_n}{a_{n+1}}|p_{n+1}(x)| \int_{x+m/n}^{x+(m+1)/n} \frac{|p_n(t)|\,\varrho(t)}{|t-x|}dt$$
$$\leqslant h_3(\delta)\,\frac{n}{m}\cdot\frac{1}{n},$$

and the proof is finished.

In the case of $m<0$ analogous estimations hold.

(iv) *If either* $(\alpha,\beta)\subset[a_m, 1-\delta/2]$ *for a fixed* $m>0$, *or* $(\alpha,\beta)\subset\subset[-1+\delta/2, a_m]$ *for a fixed* $m<0$, *then*

$$\Big|\int_\alpha^\beta K_n(t,x)\varrho(t)\,dt\Big|\leqslant\gamma_4(\delta)\frac{1}{|m|}.$$

Indeed, by (1.3) we have

$$\int_\alpha^\beta K_n(t,x)\varrho(t)\,dt=\frac{a_n}{a_{n+1}}p_n(x)\int_\alpha^\beta\frac{p_{n+1}(t)}{t-x}\varrho(t)\,dt-\frac{a_n}{a_{n+1}}p_{n+1}(x)\int_\alpha^\beta\frac{p_n(t)}{t-x}\varrho(t)\,dt.$$

Supposing $m>0$, let us apply the mean-value theorem to the first of the integrals. Taking into account the inequality $m/n\leqslant|\alpha-x|$, we get

$$\Big|\int_\alpha^\beta\frac{p_{n+1}(t)}{t-x}\varrho(t)\,dt\Big|=\frac{1}{|\alpha-x|}\Big|\int_\alpha^{\beta'}p_{n+1}(t)\varrho(t)\,dt\Big|\leqslant\frac{n}{m}\Big|\int_\alpha^{\beta'}p_{n+1}(t)\varrho(t)\,dt\Big|,$$

where $\alpha<\beta'<\beta$. By lemma 3.1,

$$\int_\alpha^{\beta'}p_{n+1}(t)\varrho(t)\,dt=O\Big(\frac{1}{n+1}\Big).$$

The second integral is estimated analogously. Hence

$$\Big|\int_\alpha^\beta K_n(t,x)\varrho(t)\,dt\Big|\leqslant\frac{\gamma_4(\delta)}{m}.$$

In the case of $m<0$, the argument is analogous to the above one.

We now turn to the proof of lemma 3.4. We take a partition $-1=t_0<t_1<\ldots<t_N=1$ of the interval $[-1,1]$ and we consider the sum

$$\sigma=\sum_{r=1}^N\Psi(|g_n(t_r,x)-g_n(t_{r-1};x)|)=\sum_{r=1}^N\Psi\Big(\Big|\int_{t_{r-1}}^{t_r}K_n(t,x)\varrho(t)\,dt\Big|\Big).\tag{3.5}$$

Let

$$I_1=\Big(-1,-1+\frac{\delta}{2}\Big),\quad I_2=\Big(-1+\frac{\delta}{2},x-\frac{1}{n}\Big),\quad I_3=\Big(x-\frac{1}{n},x+\frac{1}{n}\Big),$$
$$I_4=\Big(x+\frac{1}{n},1-\frac{\delta}{2}\Big),\quad I_5=\Big(1-\frac{\delta}{2},1\Big).$$

We group the intervals (t_{r-1},t_r) in three classes, namely: (t_{r-1},t_r) belongs

1. to the first class if $(t_{r-1},t_r)\subset I_1\cup I_3\cup I_5$,
2. to the second one if $(t_{r-1},t_r)\subset I_2\cup I_4$,
3. to the third one if (t_{r-1},t_r) contains at least one of the points $-1+\delta/2$, $x-1/n$, $x+1n$, $1-\delta/2$.

Denoting by σ_1,σ_2, and σ_3 the sums in (3.5) extended over intervals (t_{r-1},t_r) belonging to the first, the second and the third class, respectively, we now prove all sums $\sigma_1,\sigma_2,\sigma_3$ to be bounded; this will give the boundedness of the sum (3.5).

As regards σ_1, applying (1.4) and the estimations (i), (ii), we obtain

$$\sigma_1\leqslant\Psi\Big(\int_{-1}^{-1+\delta/2}|K_n(t,x)|\varrho(t)\,dt\Big)+\Psi\Big(\int_{x-1/n}^{x+1/n}|K_n(t,x)|\varrho(t)\,dt\Big)$$
$$+\Psi\Big(\int_{1-\delta/2}^{1}|K_n(t,x)|\varrho(t)\,dt\Big)\leqslant\gamma_5(\delta).$$

In order to estimate the sum σ_2 we divide the indices r from the second class again in two subclasses, denoting by r' such r that $(t_{r-1},t_r)\subset(a_m,a_{m+1})$ and by r'' such r that $t_{r-1}<a_m<t_r$ for some m. Denoting the respective sums by σ_2' and σ_2'', we have $\sigma_2=\sigma_2'+\sigma_2''$, and by (1.4) and the estimation (iii) we get

$$\sigma_2'=\sum_{r'}\Psi\Big(\Big|\int_{t_{r'-1}}^{t_{r'}}K_n(t,x)\varrho(t)\,dt\Big|\Big)\leqslant\sum_{r'}\Psi\Big(\int_{t_{r'-1}}^{t_{r'}}|K_n(t,x)|\varrho(t)\,dt\Big)$$
$$\leqslant\sum_{\substack{m\neq0\\m\neq-1}}\Psi\Big(\int_{a_m}^{a_{m+1}}|K_n(t,x)|\varrho(t)\,dt\Big)\leqslant2\sum_{m=1}^\infty\Psi\Big(\frac{\gamma_3(\delta)}{m}\Big).$$

In order to estimate the sum σ_2'', let us note that to every m there exists at most one value r'' such that

$$t_{r''-1}<a_m<t_{r''}.\tag{3.6}$$

We limit ourselves only to such intervals $(t_{r''-1},t_{r''})$ which are contained in I_4, since the sum of intervals contained in I_2 is estimated analogously. Denoting the first interval of the type $(t_{r''-1},t_{r''})$ on the right-hand side of the point $x+1/n$ by (τ_1,τ_2), we obtain by the estimation (iv)

$$\Big|\int_{\tau_1}^{\tau_2}K_n(t,x)\varrho(t)\,dt\Big|\leqslant\frac{\gamma_4(\delta)}{1}.$$

Now writing (τ_3,τ_4) for the next interval of the type $(t_{r''-1},t_{r''})$ on the right-hand side of (τ_1,τ_2) and applying (3.6), we obtain $a_2\leqslant\tau_3$. Hence, by (iv),

$$\Big|\int_{\tau_3}^{\tau_4}K_n(t,x)\varrho(t)\,dt\Big|\leqslant\frac{\gamma_4(\delta)}{2}.$$

Proceeding further in the same way, we get the inequality

$$\sigma_2''\leqslant\sum_{m\neq0}\Psi\Big(\frac{\gamma_4(\delta)}{|m|}\Big)\leqslant2\sum_{m=1}^\infty\Psi\Big(\frac{\gamma_4(\delta)}{m}\Big).$$

We now estimate σ_3; it is easily seen that σ_3 contains at most four terms. Let (α, β) denote the interval containing, say, the point $-1+\delta/2$. We write the integral $\int_\alpha^\beta K_n \varrho dt$ in the form

$$\int_\alpha^\beta K_n(t, x)\varrho(t)\,dt = \int_\alpha^{-1+\delta/2} K_n(t, x)\varrho(t)\,dt + \int_{-1+\delta/2}^\beta K_n(t, x)\varrho(t)\,dt.$$

Hence

$$\Big|\int_\alpha^\beta K_n(t, x)\varrho(t)\,dt\Big| \leqslant \int_\alpha^{-1+\delta/2} \big|K_n(t, x)\big|\varrho(t)\,dt + \Big|\int_{-1+\delta/2}^\beta K_n(t, x)\varrho(t)\,dt\Big|.$$

The first integral is bounded by a constant independent of x, n in virtue of (i). By (iv), the second integral is estimated by a constant dependent only on δ in case where $\beta \leqslant a_{-1}$. If $\beta > a_{-1}$, we must estimate the integrals of the form

$$\int_{a_{-1}}^{a_1} K_n(t, x)\varrho(t)\,dt, \qquad \int_{a_1}^{\beta} K_n(t, x)\varrho(t)\,dt.$$

But the second integral is estimated by (iv) if $\beta \leqslant 1-\delta/2$, and by (iv) and (i) if $\beta > 1-\delta/2$. Hence

$$\Big|\int_\alpha^\beta K_n(t, x)\varrho(t)\,dt\Big| < \gamma_6(\delta)$$

and this shows σ_3 to be bounded by a constant dependent on δ. Thus lemma 3.4. is proved completely.

4. We now proceed to the proof of the Theorem. Formula (1.1) and the definition of the functions $g_n(t, x)$ give

$$S_n(x) = \int_{-1}^{1} f(t)\,d_t[g_n(t, x)]. \tag{4.1}$$

Integrating (4.1) by parts, we obtain

$$S_n(x) = f(1) - \int_{-1}^{1} g_n(t, x)\,df(t). \tag{4.2}$$

In particular, let the function Ψ be defined for small $u > 0$ by formula $\Psi(u) = u/|\ln u|^{1+\varepsilon}$, where $\varepsilon > 0$, and let it be defined for other $u > 0$ arbitrarily but in such a way that $\Psi(u)$ satisfies condition (1.4) and is convex. We now apply lemma 3.3; we obtain

$$g_n = G_n, \quad \Phi(u) \sim e^{-u^{-\alpha}} \quad f = F, \quad \varphi(u) \sim |\ln u|^{-1/\alpha} \quad (u \to 0+);$$

$$\Psi(u) \sim \frac{u}{|\ln u|^{1+\varepsilon}}, \quad \Psi(u) \sim u|\ln u|^{1+\varepsilon}, \quad \tau_0 = x,$$

$$G(t) \stackrel{\text{def}}{=} G(t, x) = \begin{cases} 0 & \text{for} \quad -1 \leqslant t < x, \\ 1 & \text{for} \quad x < t \leqslant 1. \end{cases}$$

It is easily seen that the assumptions of lemma 3.3 are satisfied; property (3.4) follows from the generalized Riemann-Lebesgue theorem. Hence, by (4.2), we obtain

$$S_n(x) \to f(1) - \int_{-1}^{1} G(t, x)\,df(t) = f(1) - \int_x^1 df(t) = f(x),$$

as $n \to \infty$, and the proof of the Theorem is thus completed.

References

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