On operators preserving a conjugate space

by

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Let X be a linear space over real or complex scalars. Let A be a linear operator transforming X into itself. By the *nullity* α_A of the operator A we mean the dimension of the space $\{x \in X: Ax = 0\}$. By the *deficiency* β_A we mean the dimension of the quotient space X/AX. If numbers α_A, β_A are both finite, we say that the operator A possesses a *finite d-characteristic* (Kato [7], Gochberg and Krein [6], Przeworska-Rolewicz and Rolewicz [10]).

In the classical theorems, for example in the Fredholm theory of integral equations (Fredholm [2]-[5]), instead of β_A a characteristic number β_A^Ξ has been considered. To define this number, we consider simultaneously with the space X a total(¹) space Ξ of linear functionals, which will further be called a *conjugate space*. Now β_A^Ξ is the dimension of the space

$$\{\xi \in \Xi: \xi Ax = 0 \text{ for all } x \in X\}.$$

Let $\alpha_A, \beta_A < +\infty$. If $\beta_A^\Xi = \beta_A$, we say that A is a *Φ_Ξ-operator*.

Obviously for each functional $\xi \in \Xi$ we can consider a functional $\eta x = \xi Ax$. We shall write $\eta = A'\xi$ and the operator A' will be called a *conjugate operator* to the operator A. We have $\alpha_{A'} = \beta_A^\Xi$. We do not always have $A'\Xi \subset \Xi$, but if A' possesses this property, we shall say that A *preserves* the conjugate space Ξ. The set of all linear operators preserving Ξ will be denoted by $\mathscr{L}(X, \Xi)$; it constitutes an algebra.

If $\Xi = X'$ is the space of all linear functionals, then each linear operator preserves Ξ. If X is a linear topological locally convex space and $\Xi = X^+$ the space of all linear continuous functionals, then every continuous operator preserves Ξ. On the other hand, in these cases we have $\beta_A^\Xi = \beta_A$. But there are also operators preserving a conjugate space which are not Φ_Ξ-operators (see [11]).

(¹) A space Ξ of linear functionals is called *total* if $\xi x = 0$ for all $\xi \in \Xi$ implies $x = 0$.

In this note we give some theorems which permit us to prove when the operators A preserving the conjugate space Ξ are Φ_Ξ-operators. In this note we use the method of *regularization* considered in papers [8], [9] and [10].

Given an algebra of linear operators $\mathcal{X}$, let $\mathcal{I}$ be a two-sided ideal contained in $\mathcal{X}$. We say that an operator $A \in \mathcal{X}$ possesses a *left-sided* (*right-sided*) *regularizer* R_A to the ideal $\mathcal{I}$ if $R_A A - I \in \mathcal{I}$ (resp. $AR_A - I \in \mathcal{I}$). We say that a regularizer is *simple* if it is simultaneously left-sided and right-sided.

We shall employ the following results from paper [10]:

(A) If the ideal $\mathcal{I}$ is such that for each $T \in \mathcal{I}$, $I+T$ is a Φ_Ξ-operator, then each operator A which possesses a simple regularizer to the ideal $\mathcal{I}$ is a Φ_Ξ-operator ([10], proposition 5.8).

(B) By $\mathcal{K}(X, \Xi)$ we denote the set (the two-sided ideal) of all finite dimensional operators contained in $\mathcal{L}(X, \Xi)$. If $K \in \mathcal{K}(X, \Xi)$, then $I+K$ is a Φ_Ξ-operator ([10], proposition 4.2).

Let X and Ξ be Banach spaces. We denote the respective norms by $\| \|_X$, $\| \|_\Xi$. In the space $\mathcal{L}(X, \Xi)$ define the norm

$$\|A\|^* = \max\{\|A\|_X, \|A'\|_\Xi\}.$$

If the topology in Ξ is equivalent to the norm-topology of functionals, then obviously the norm $\| \|^*$ is equivalent to the usual norm $\| \|_X$ of operator $X \to X$.

By $\bar{\mathcal{K}}(X, \Xi)$ we denote the closure of $\mathcal{K}(X, \Xi)$ in the norm $\| \|^*$. Obviously $\bar{\mathcal{K}}(X, \Xi)$ is a two-sided ideal.

THEOREM 1. *If an operator $A \in \mathcal{L}(X, \Xi)$ possesses a left-sided (right-sided) regularizer to the ideal $\bar{\mathcal{K}}(X, \Xi)$, then it possesses a left-sided (right-sided) regularizer to the ideal $\mathcal{K}(X, \Xi)$.*

Proof. Let us assume that there exists a left-side regularizer R_A to the ideal $\bar{\mathcal{K}}(X, \Xi)$, i. e. such an operator R_A that

$$R_A A = I+T, \quad \text{where} \quad T \in \bar{\mathcal{K}}(X, \Xi).$$

Take an operator $K \in \mathcal{K}(X, \Xi)$ such that

$$\|T-K\|^* < 1$$

and write $B = T-K$. It is easy to check, on the basis of the completeness of X and Ξ, that the operator $I+B$ is invertible and $(I+B)^{-1} \in \mathcal{L}(X, \Xi)$. Let

$$R_A^0 = (I+B)^{-1}R_A;$$

then

$$R_A^0 A = (I+B)^{-1}R_A A = (I+B)^{-1}(I+T) = (I+B)^{-1}(I+B+K) = I+(I+B)^{-1}K.$$

But $(I+B)^{-1}K \in \mathcal{K}(X, \Xi)$; therefore R_A^0 is a left-sided regularizer of the operator A to the ideal $\mathcal{K}(X, \Xi)$.

The proof in the case of a right-sided regularizer is identical. From theorem 1 and result (A) follows

COROLLARY 1. *If an operator $A \in \mathcal{L}(X, \Xi)$ possesses a simple regularizer to the ideal $\bar{\mathcal{K}}(X, \Xi)$ in the particular case of $A = I+T$, where $T \in \bar{\mathcal{K}}(X, \Xi)$, then A is a Φ_Ξ-operator.*

Remark. We do not know whether it is possible to replace in Corollary 1 the assumption that $T \in \bar{\mathcal{K}}(X, \Xi)$ by the assumption that $T \in \mathcal{T}(X, \Xi)$, where $\mathcal{T}(X, \Xi)$ is an ideal of compact operators contained in $\mathcal{L}(X, \Xi)$.

Consider the following application of theorem 1.

Example 1. Let L be a regular arc with a finite length on a plane. Let $X = C(L)$ be the space of all continuous real or complex-valued functions defined on L. Let $\Xi = C(L)$ be a space of functionals ξ of the type

$$\xi x = \int_L x(t)\xi(t)dt,$$

where $\xi(t)$ is a continuous real or respectively complex-valued function defined on L. Let T be an integral operator

$$Tx = \int_L K(s,t)x(t)d\omega(t),$$

where $\omega(t)$ is a linear Hausdorff measure, $K(s,t) = K_0(s,t)k(|s-t|)$, where $K_0(s,t)$ is a continuous function and $k(u)$ is a non-negative and summable function of one real variable continuous for $u \neq 0$. Then $T \in \bar{\mathcal{K}}(X, \Xi)$.

Indeed, let

$$k_m(u) = \begin{cases} k(u) & \text{if} \quad k(u) < m, \\ m & \text{if} \quad k(u) \geqslant m. \end{cases}$$

$k_m(u)$ is obviously a continuous function. The arc L is regular and therefore

$$e_m = \sup_{s \in L} \int_L k(|s-t|) - k_m(|s-t|)\,d\omega(t)$$

tends to 0 when $m \to \infty$. But if we write

$$T_m x = \int_L K_0(s,t)k_m(|s-t|)x(t)d\omega(t),$$

then $\|(T_m - T)x\|_x \leqslant Me_m\|x\|_x$, where $M = \sup_{s,t \in L}|K_0(s,t)|$. The kernels of the operators T_m are continuous functions. Basing ourselves on Weierstrass theorem, we can approximate each kernel uniformly by polynomials. Hence each operator T_m can be approximated in the norm topology by operators belonging to $\mathscr{K}(X, \Xi)$. Therefore T is approximable by operators belonging to $\mathscr{K}(X, \Xi)$ in the topology induced by the norm $\| \|_X$. But the topology in Ξ is the norm topology in the conjugate space, whence $T \in \bar{\mathscr{K}}(X, \Xi)$.

Applying corollary 1 we find that $I+T$ is a Φ_C-operator. In the particular case of $ku = 1/|u|^a$, $0 < a < 1$, we obtain a well known theorem for weakly singular equations, without using the classical method of iteration.

Obviously the condition that $\omega(t)$ is a linear Hausdorff measure can be replaced by the condition that t is a complex-valued measure continuous with respect to t; in particular it is true when we consider a complex plane and integration is considered as integration on a complex arc.

THEOREM 2. *Assume that X is a linear space and Ξ is a conjugate space, X_0 and Ξ_0 are subspaces of X and Ξ respectively. Let $A \in \mathscr{L}(X_0, \Xi_0)$. Let there be such a simple regularizer R_A that the operators $T = I - R_A A$ and $T_1 = I - R_A A$ can be extended to operators $\overline{T}$, $\overline{T}_1$ belonging to $\mathscr{L}(X, \Xi)$ and $I+\overline{T}$, $I+\overline{T}_1$ are Φ_Ξ-operators. Then A is a Φ_{Ξ_0}-operator.*

Proof. Operators $I+T$, $I+T_1$ can be considered on the whole space X. According to the assumption these operators are Φ_Ξ-operators. By theorem 4.1 of [10] these operators considered on X_0 are Φ_{Ξ_0}-operators. i. e. operators AR_A and $R_A A$ are Φ_{Ξ_0}-operators. Therefore proposition 5.8 of [10] implies that A is a Φ_{Ξ_0}-operator.

We consider the following application of theorem 2.

Example 2. Let L be a regular closed Jordan curve. Let $X_0 = H^\mu$ be a space of all functions $x(t)$ defined on L and satisfying a Hölder inequality with an exponent μ, i. e. $|x(t)-x(t')| \leqslant c|t-t'|^\mu$, $0 < \mu < 1$. Let $\Xi_0 = H^{\mu/2}$ be a space of functionals ξ of the type

$$\xi x = \int_L \xi(t) x(t) dt,$$

where $\xi(t) \in H^{\mu/2}$.

Let us consider an operator

$$Ax = A_0(t)x(t) + \frac{1}{\pi i}\int_L \frac{K(s,t)}{s-t} x(s)\, ds,$$

where the integral is considered as an integral in the sense of the Cauchy principal value, $A_0(t) \in H^{\mu/2}$, and $K(s,t)$ satisfies a Hölder inequality with an exponent μ, i. e.

$$|K(s',t') - K(s,t)| \leqslant c[|t-t'|^\mu + |s-s'|^\mu].$$

Let $A^2(t) - K^2(t,t) \neq 0$ for all $t \in L$. Then the operator A is a $\Phi_{H^{\mu/2}}$ operator. Indeed, let

$$R_A x = [A_0^2(t) - K^2(t,t)]^{-1} A_0(t) x(t) - \frac{1}{\pi i}\int_L \frac{K(s,t)}{s-t} x(s)\, ds.$$

Then by classical considerations (see also [8]) we obtain

$$R_A A = I + T, \quad AR_A = I + T_1,$$

where T, T_1 are weakly singular operators transforming C into $H^{\mu/2}$. By example 1 we infer that $I+T$, $I+T_1$ are Φ_Ξ-operators. Therefore theorem 2 shows that A is a $\Phi_{H^{\mu/2}}$-operator.

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