On linear processes of approximation (III)

by

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1. The results of this paper differ from those of papers [4] and [5] in two respects. In both of our previous papers the test-conditions for approximation referred to the trigonometric system. In the present paper our conditions refer to more general systems of functions. The prerequisites for such basic systems are two simple properties, one of which we call Fejérian, the other Jacksonian resp. quasi-Jacksonian property. Beside the trigonometric system there are several other orthogonal systems which enjoy both of these properties, e. g. the eigenfunctions of a non-singular Sturm-Liouville problem, polynomials orthogonal over a finite interval with a weight function bounded from below, Hermite polynomials and the Franklin system (1). As to the other respect, we specify presently our approximation-operators to coefficient-transformations of the corresponding expansions. In order to include all of our applications, we shall formulate our theorem in a somewhat abstract form. The necessary preparations for it will be expounded in Chapter 2. The formulation itself along with the proof will constitute Chapter 3. Chapter 4 is concerned with applications to Jacksonian systems, Chapter 5 deals with Hermite polynomials.

2. Let X be a real linear vector space which forms a normal linear vector space with norm $\| \|_1$ and also with norm $\| \|_2$. $\| \|_1$ will refer to approximation properties, $\| \|_2$ to the modulus of continuity. The corresponding spaces will be called X_1 and X_2 respectively. Beside X_1 and X_2 we take into consideration another normed linear space Y and a bounded linear operator B which maps X_1 to Y. The norm in Y will be denoted by $\| \|_Y$. In X_2 there is defined an abelian semigroup U_h, $h \geqslant 0$, of linear transformations of X_2 (it is enough to have U_h defined for all $0 \leqslant h \leqslant 1$ and then set $U_h = U_1$ for $h \geqslant 1$). Thus

$$U_{h_1} U_{h_2} = U_{h_1+h_2}, \qquad U_0 \equiv 1.$$

(1) Apart from the Hermite polynomial system which is quasi-Jacksonian all the other systems listed above are Jacksonian.

Additionally, we assume for every h [2]

(2.1) $$\|U_h\| \leqslant 1.$$

Let for $f \epsilon X$, $\delta \geqslant 0$

$$\omega(\delta; f) \stackrel{\text{def}}{=} \sup_{0 \leqslant h \leqslant \delta} \|U_h(f) - f\|_2.$$

We assert

(2.2) $$\omega(\vartheta\delta; f) < 2\vartheta\omega(\delta; f) \quad \text{for} \quad \vartheta > 1,\ \delta > 0.$$

Indeed, we first have

$$\omega(2\delta; f) \leqslant 2\omega(\delta; f)$$

since, for $0 \leqslant h \leqslant \delta$,

$$\begin{aligned} \|U_{2h}(f) - f\|_2 &\leqslant \|U_{2h}(f) - U_h(f)\|_2 + \|U_h(f) - f\|_2 \\ &= \|U_h(U_h(f) - f)\|_2 + \|U_h(f) - f\|_2 \\ &= (\|U_h\|_2 + 1)\|U_h(f) - f\|_2 \leqslant 2\omega(\delta; f). \end{aligned}$$

Let now $2^\mu < \vartheta \leqslant 2^{\mu+1}$. Then by induction

$$(2^{\mu+1}\delta; f) < 2^{\mu+1}\omega(\delta; f)$$

and

$$\omega(\vartheta\delta; f) \leqslant \omega(2^{\mu+1}\delta; f) \leqslant 2^{\mu+1}\omega(\delta; f) < 2\vartheta\omega(\delta; f).$$

Further, noting that $\omega(\delta; f)$ is always bounded, we put

$$\Omega(\delta; f) \stackrel{\text{def}}{=} \delta \int_\delta^\infty \frac{\omega(x; f)}{x^2}\, dx.$$

$\Omega(\delta; f)$ is an increasing function (which can be easily proved by differentiating); we also have

$$\Omega(2\delta; f) = 2\delta \int_{2\delta}^\infty \frac{\omega(x; f)}{x^2}\, dx \leqslant 2\delta \int_\delta^\infty \frac{\omega(x; f)}{x^2}\, dx = 2\Omega(\delta; f)$$

so that, following our previous argument,

(2.3) $$\Omega(\vartheta\delta; f) < 2\vartheta\Omega(\delta; f) \quad \text{for} \quad \vartheta > 1,\ \delta > 0.$$

Suppose that we are given in X_1 a sequence of linearly independent elements $\{\varphi_n\}$, $n = 0, 1, 2, \ldots$, and a sequence of bounded linear functionals $a_k(f)$ with the property [3]

$$a_k(\varphi_n) = \begin{cases} 0 & \text{if} \quad k \neq n, \\ 1 & \text{if} \quad k = n. \end{cases}$$

[2] Here $\| \ \|$ stands for the operator-norm in X_2.

[3] For the existence of such functionals—under rather general conditions—see [7], p. 151, Theorem 9, and p. 172, Remark.

A linear combination

$$\sum_{k=0}^n a_k \varphi_k$$

with real $a_1, a_2, \ldots, a_n$ will be called a *φ-polynomial of n-th degree*; the set of φ-polynomials of degree n will be denoted by Φ_n. We define, further, $E_n(f)$ as the greatest lower bound of $\|f-p\|_1$ as p runs over Φ_n. We call $\{\varphi_n\}$ a *Jacksonian system* if, for every $f \epsilon X$, $n \geqslant 1$,

$$E_{n-1}(f) \leqslant c_1\left\{\omega\left(\frac{1}{n}; f\right) + \varrho\frac{\|f\|_1}{n}\right\},$$

where $\varrho \geqslant 0$; actually only $\varrho = 0$ or $\varrho = 1$ are of interest.

Similarly, $\{\varphi_n\}$ is said to be a *γ-quasi-Jacksonian system* if with $0 < \gamma < 1$

$$E_{n-1}(f) \leqslant c_2\left\{\omega\left(\frac{1}{n^\gamma}; f\right) + \varrho\frac{\|f\|_1}{n^\gamma}\right\}.$$

The expansion of an element $f \epsilon X$ in the system $\{\varphi_n\}$ is defined by

(2.4) $$f \sim \sum_k a_k(f)\varphi_k.$$

Having that, we define the *partial sum-operator*

$$s_n(f) = \sum_{k \leqslant n} a_k(f)\varphi_k.$$

As a consequence of boundedness of $a_k(f)$, this is a continuous linear transformation of X_1. We define similarly

(2.5) $$B_n(f) \stackrel{\text{def}}{=} \sum_{k \leqslant n} (1 - b_{k,n}) a_k(f)\varphi_k$$

and in particular the operators of $(C, 1)$-summation

$$F_n(f) \stackrel{\text{def}}{=} \sum_{k \leqslant n} \left(1 - \frac{k}{n}\right) a_k(f)\varphi_k.$$

The expansion (2.4) will be called of *B-Fejérian type* if

(2.6) $$\|B(F_n(f))\|_Y \leqslant c_3\|f\|_1, \quad f \epsilon X,\ n = 1, 2, \ldots$$

In the simple but rather important case of $Y = X_1$, $B \equiv I$, I-Fejérian type will be briefly called *Fejérian*.

Introducing the de la Vallée-Poussin means

$$v_n(f) \stackrel{\text{def}}{=} \frac{nF_n(f)-n_1F_{n_1}(f)}{n-n_1},$$

where $n_1 = [n/2]$, we have the following propositions which are well known in more special cases:

(2.7) $$v_n(f) = f \quad \text{whenever} \quad f \in \Phi_{n_1},$$

(2.8) $$\|B(v_n(f))\|_Y \leqslant c_4\|f\|, \quad f \in X,\ n = 1, 2, \ldots,$$

(2.9) $$\|B(f-v_n(f))\|_Y \leqslant c_5 E_{n_1}(f), \quad f \in X,\ n = 1, 2, \ldots$$

Relation (2.7) is an easy consequence of the relation

$$v_n(f) = \frac{\sum_{k=n_1}^{n-1} s_k(f)}{n-n_1}$$

and of the fact that $s_k(f) = f$, whenever $f \in \Phi_{n_1}$, $k \geqslant n_1$.

Relation (2.8) follows directly from (2.6).

As to (2.9), fixing an $\varepsilon > 0$ we have with some $p \in \Phi_{n_1}$

$$\|f-p\|_1 \leqslant E_{n_1}(f)+\varepsilon.$$

Then

$$\|B(v_n(f)-f)\|_Y = \|B(v_n(f-p)+p-f)\|_Y$$
$$\leqslant c_4\|f-p\|_1 + \|B\|\,\|f-p\|_1 \leqslant c_5\{E_{n_1}(f)+\varepsilon\}$$

and we let ε tend to 0.

In the sequel we shall use the following

LEMMA. *Let* (2.4) *be of Fejérian type. If* $\{\varphi_n\}$ *is a Jacksonian system, then*

$$\|B(F_n(f)-f)\|_Y \leqslant c_6\left(\Omega\left(\frac{1}{n};f\right)\right)+\varrho\|f\|_1\frac{\log n}{n}\Big), \quad f \in X,\ n = 1, 2, \ldots;$$

if $\{\varphi_n\}$ *is a* γ*-quasi-Jacksonian system, then*

$$\|B(F_n(f)-f)\|_Y \leqslant c_7\left(\omega\left(\frac{1}{n^\gamma};f\right)+\varrho\|f\|_1\frac{1}{n^\gamma}\right), \quad f \in X,\ n = 1, 2, \ldots$$

Proof. Let $n_0 = n$, $n_{k+1} = [n_k/2]$. Then

$$F_n(f)-f = \sum_{j=0}^{\infty}\frac{n_j-n_{j+1}}{n}\left(v_{n_j}(f)-f\right).$$

By (2.9) we get in case of a Jacksonian system

$$\|B(F_n(f)-f)\|_Y \leqslant \frac{c_5}{n}\sum_{j=0}^{\infty}(n_j-n_{j+1})E_{n_{j+1}}(f)$$
$$\leqslant \frac{c_8}{n}\Big\{E_0(f)+\sum_{j=0}^{\infty}\big(E_{n_{j+1}}(f)+E_{n_{j+1}-1}(f)+\ldots+E_{n_{j+2}+1}(f)\big)\Big\}$$
$$\leqslant \frac{c_8}{n}\sum_{k=0}^{n}E_k(f) \leqslant \frac{c_9'}{n}\sum_{k=1}^{n}\left\{\omega\left(\frac{1}{k};f\right)+\varrho\frac{\|f\|_1}{k}\right\}$$
$$\leqslant \frac{c_{10}}{n}\left\{\int_0^n \omega\left(\frac{1}{x};f\right)dx+\varrho\|f_1\|\log n\right\}$$
$$= \frac{c_{10}}{n}\left\{\int_{n^{-1}}^{1}\frac{\omega(y;f)}{y^2}\,dy+\varrho\|f\|_1\log n\right\}$$
$$= c_{10}\left\{\Omega\left(\frac{1}{n};f\right)+\varrho\|f_1\|\frac{\log n}{n}\right\}.$$

Similarly, for a γ-quasi-Jacksonian system, we obtain

$$\|B(F_n(f)-f)\|_Y \leqslant \frac{c_5}{n}\sum_{j=0}^{\infty}(n_j-n_{j+1})E_{n_{j+1}}(f)$$
$$\leqslant c_{11}\sum_{n_{j+1}\geqslant 1}2^{-j}\left\{\omega\left(\frac{1}{n_{j+1}^\gamma};f\right)+\varrho\|f\|_1\frac{1}{n_{j+1}^\gamma}\right\}.$$

Using the inequality [4], valid for $n_j \geqslant 1$,

(2.10) $$\frac{n}{2^{j+1}} \leqslant n_j \leqslant \frac{n}{2^j}$$

we get

$$\|B(F_n(f)-f)\|_Y \leqslant c_{12}\sum_{j=0}^{\infty}2^{-j}\left\{\omega\left(\frac{2^{(j+2)\gamma}}{n^\gamma};f\right)+\varrho\|f\|_1\frac{2^{\gamma j}}{n^\gamma}\right\}$$
$$\leqslant c_{13}\left\{\omega\left(\frac{1}{n^\gamma};f\right)\sum_{j=0}^{\infty}2^{-j(1-\gamma)}+\varrho\|f\|_1 n^{-\gamma}\right\}$$
$$= c_{14}\left\{\omega\left(\frac{1}{n^\gamma};f\right)+\varrho\|f\|_1 n^{-\gamma}\right\}.$$

[4] (2.10) follows from the fact that $2^{\mu-1} < n < 2^\mu$ implies $[2^{\mu-1}/2^j] < n_j$.

3. THEOREM. *Let (2.4) be a Fejérian expansion; we consider a sequence of coefficient-transformations (2.5) with the properties*

$$\|B(B_n(f))\|_Y \leqslant c_{15}\|f\|_1 \quad \text{for} \quad f\in X,\ n=1,2,\ldots, \tag{3.1}$$

$$b_{0,n}=0, \quad n=1,2,\ldots, \tag{3.2}$$

$$b_{1,n}=O\left(\frac{1}{n}\right), \quad n=1,2,\ldots, \tag{3.3}$$

$$\sum_{k=0}^{n-2}|\Delta^2 b_{k,n}|=O\left(\frac{1}{n}\right), \quad n=1,2,\ldots \tag{3.4}$$

These conditions imply that

a. *if $\{\varphi_n\}$ is a Jacksonian system, then for $f\in X$*

$$\|B(B_n(f)-f)\|_Y \leqslant c_{16}\left\{\Omega\left(\frac{1}{n};f\right)+\varrho\|f\|_1\frac{\log n}{n}\right\};$$

b. *if $\{\varphi_n\}$ is a γ-quasi-Jacksonian, then for $f\in X$*

$$\|B(B_n(f)-f)\|_Y \leqslant c_{17}\left\{\omega\left(\frac{1}{n^\gamma};f\right)+\varrho\|f\|_1 n^{-\gamma}\right\}.$$

Proof. Let $\psi(n)$ stand either for $\Omega(1/n;\, f)+\varrho\|f\|_1\frac{\log n}{n}$ or for $\omega(1/n^\gamma;\, f)+\varrho\|f\|_1 n^{-\gamma}$ in case a or b respectively. We write, as in our paper [4],

$$B(B_n(f)-f)=B(F_n(f)-f)+B(B_n(f-F_n))+B(B_n(F_n)-F_n(f)). \tag{3.5}$$

As to the first two terms, their Y-norms are $O(\psi(n))$ by our Lemma, (3.1) and the boundedness of B. For the last term we have

$$B_n(F_n)-F_n(f)=-\sum_{k\leqslant n} b_{k,n}\left(1-\frac{k}{n}\right)a_k(f)\varphi_k.$$

As in our paper [4], formula (4.1), we put $b_{n+1,n}=b_{n+2,n}=0$, $\Delta b_{k,n}=b_{k,n}-b_{k+1,n}$, $\Delta^2 b_{k,n}-2b_{k+1,n}+b_{k+2,n}$ $(k=0,1,\ldots,n)$, and obtain

$$B_n(F_n)-F_n(f)$$
$$=-\sum_{k=0}^{n}\left(1-\frac{k}{n}\right)\Delta^2 b_{k,n}(k+1)F_{k+1}(f)-\frac{2}{n}\sum_{k=0}^{n}\Delta b_{k+1,n}(k+1)F_{k+1}(f).$$

Noting (see [4], formula (4.2)) that

$$\sum_{k=0}^{n}\left(1-\frac{k}{n}\right)\Delta^2 b_{k,n}(k+1)+\frac{2}{n}\sum_{k=0}^{n}\Delta b_{k+1,n}(k+1)=b_{0,n}$$

and using (3.2), we come to

$$B_n(F_n)-F_n(f)$$
$$=-\sum_{k=0}^{n}\left(1-\frac{k}{n}\right)\Delta^2 b_{k,n}(k+1)\{F_{k+1}(f)-f\}-\frac{2}{n}\sum_{k=0}^{n}\Delta b_{k,n}(k+1)\{F_{k+1}(f)-f\}.$$

By Lemma, multiplying by B,

$$\|B(B_n(F_n)-F_n)\|_Y$$
$$\leqslant\sum_{k=0}^{n}\left(1-\frac{k}{n}\right)|\Delta^2 b_{k,n}|(k+1)\psi(k+1)+\frac{2}{n}\sum_{k=0}^{n}|\Delta b_{k,n}|(k+1)\psi(k+1).$$

We have by (2.2) and (2.3)

$$\psi(k+1)<c_{18}\frac{n}{k+1}\psi(n),$$

so that

$$\|B(B_n(F_n)-F_n)\|_Y\leqslant c_{18}n\psi(n)\left\{\sum_{k=0}^{n}\left(1-\frac{k}{n}\right)|\Delta^2 b_{k,n}|+\frac{2}{n}\sum_{k=0}^{n}|\Delta b_{k,n}|\right\}. \tag{3.6}$$

Since

$$\Delta b_{k,n}=b_{0,n}-b_{1,n}-\{\Delta^2 b_{0,n}+\Delta^2 b_{1,n}+\ldots+\Delta^2 b_{k-1,n}\}, \quad k=0,1,\ldots,n-1,$$
$$b_{k,n}=b_{0,n}-\{\Delta b_{0,n}+\Delta b_{1,n}+\ldots+\Delta b_{k-1,n}\}, \quad k=1,2,\ldots,n,$$

we get

$$|\Delta b_{k,n}|\leqslant c_{19}\frac{1}{n}, \quad k=1,2,\ldots,n-1, \tag{3.7}$$

and

$$|b_{k,n}|\leqslant c_{19}\frac{k}{n}, \quad k=1,2,\ldots,n. \tag{3.8}$$

These inequalities and (3.4) yield our statement.

4. We turn now to applications. In all of them elements of X are real or complex-valued functions of a real variable $t\in\Theta$. The operator U_h will be in most cases the operator of translation

$$T_h\{f(t)\}\overset{\text{def}}{=}f(t+h). \tag{4.1}$$

For $\Theta=(-\infty,+\infty)$ or in case of periodic functions, this definition makes sense. In the remaining cases which we treat, $\Theta=[a,b]$ is a finite interval and elements of X are continuous functions. Here we extend the function $f(t)$ by $f(b)$ for $t\geqslant b$ and by $f(a)$ for $t\leqslant a$, so that our definition

(4.1) applies again. A more general type of a semi-group satisfying (2.1) is defined in the following way: we have $\Theta = [a, b]$ and a strictly monotoneous function $g(x)$ defined on $[c, d]$ and transforming $[c, d]$ into $[a, b]$. Further, we have an operator $\Lambda(f)$ defined on X such that

$$\Lambda f(x) = f\big(g(y)\big), \quad \Lambda^{-1}\varphi(y) = \varphi\big(g^{-1}(x)\big).$$

Then we put

$$U_h(f) = T_h^{(g)}\{f(x)\} \stackrel{\text{def}}{=} \Lambda^{-1}T_h\{\Lambda f(x)\},$$

where T_h is the operator of translation defined above. (2.1) is satisfied if we take as X_2-norm the usual C-norm. An important particular case of this kind is $[a, b] = [-1, +1]$, $[c, d] = [0, \pi]$, $g(x) = \cos x$; this operator $T_h^{(g)}$ is denoted by T_h^*. $\{\varphi_n\}$ will be always an orthonormal system of functions and $a_k(f)$ will stand for the corresponding Fourier coefficients of f.

a) *The trigonometric system.* Let $X = X_1 = X_2 = L^p[-\pi, \pi]$ $(p \geqslant 1)$ or $C_{2\pi}$ and $\varphi_n = e^{int}$. As well known, this system is Fejérian, so that we can put $Y = X$, $B \equiv I$. We further set $U_h = T_h$. With this notation $\omega(\delta)$ is the ordinary modulus of continuity and, according to Jackson's theorem, the system is Jacksonian with $\varrho = 0$. Thus in this case the conditions of our theorem turn out to be identical with (2.4), (2.5), (2.6) of Theorem 1 in [4], the conclusion being possibly only slightly less precise. E. g. for $f \epsilon \operatorname{Lip}\alpha$, $0 < \alpha \leqslant 1$, we obtain exactly the same statement

$$B_n(f) = \begin{cases} f + O(n^{-\alpha}) & \text{if} \quad \alpha < 1, \\ f + O\left(\dfrac{\log n}{n}\right) & \text{if} \quad \alpha = 1. \end{cases}$$

b) *Polynomials orthogonal over a finite interval* $[a, b]$. Here $X = X_1 = X_2 = C[a, b]$ and $\varphi_n = p_n(t)$, where $p_n(t)$ are polynomials orthonormal with respect to a distribution $d\alpha(t)$ whose support is contained in $[a, b]$. $\alpha(t)$ is supposed to be absolutely continuous in $[a, b]$ and

$$0 < m \leqslant \operatorname*{vrai\,min}_{a \leqslant t \leqslant b} \alpha'(t) \leqslant \operatorname*{vrai\,max}_{a \leqslant t \leqslant b} \alpha'(t) \leqslant M < \infty.$$

Let $B = B_\delta$, $\delta > 0$, be the operator which restricts a given function defined on $[a, b]$ to the same function defined on $[a+\delta, b-\delta]$. $Y = Y_\delta$ is the space $C[a+\delta, b-\delta]$. As a consequence of investigations of the first of us (see [3], § 1) we can state that $\{p_n(t)\}$ form a B_δ-Fejérian system for each $\delta > 0$. We define $U_h = T_h^*$; with this choice of U_h, $\{p_n(t)\}$ is, as is well known, a Jacksonian system with $\varrho = 0$ [5]. Hence we have

[5] We might put as well $U_h = T_h$. In this case, however, our final statement would be slightly less precise.

COROLLARY 1. *Under the conditions imposed on $\alpha(t)$ and (3.1)-(3.4) we have*

$$\max_{a+\delta \leqslant t \leqslant b-\delta} |B_n(f) - f(t)| < c_{16}\Omega\left(\frac{1}{n}; f\right),$$

where $B_n(f; t) = B_n(f)$ is defined by (2.5).

c) *Sturm-Liouville expansions.* Let $X = X_1 = X_2 = Y = C[0, \pi]$, $B \equiv I$, $U_h = T_h$. Let $\{u_n(t)\}$ be the system of eigenfunctions of the Sturm-Liouville problem

$$u''(t) + \big(\lambda - q(t)\big)u(t) = 0, \quad q \epsilon C[0, \pi],$$
$$u'(0) - hu(0) = 0, \quad u'(\pi) + Hu(\tau) = 0,$$

ordered according to the increasing eigenvalues. We set $\varphi_n = u_n(t)$. As a consequence of the equiconvergence theorem of A. Haar, this system enjoys the Fejérian property. On the other hand, it is also Jacksonian with $\varrho = 1$ (see [6]), so that our theorem can be applied. What is more, in this case we can formulate also a sufficient condition for (3.1) in terms of $b_{k,n}$. We assume (3.2)-(3.4) and compare the

$$B_n(f; t) = \sum_{k \leqslant n} (1 - b_{k,n}) a_k(f) u_k(t)$$

with the corresponding trigonometric-Fourier expression

$$B_n^*(f; t) = \frac{1}{\sqrt{\pi}}\left(1 + \sqrt{2}\sum_{k=1}^{n} (1 - b_{k,n}) a_k^*(f) \cos kt\right).$$

By Haar's theorem and (3.7) we get, applying partial summation,

$$\|B_n^*(f; t) - B_n(f; t)\|_1 = O(\|f\|_1), \quad n = 1, 2, \ldots,$$

which means that condition (3.1) is in our case equivalent with the corresponding statement for the trigonometric system. In the latter case, however, we have a simple criterion of S. M. Nikolski stating that

$$\Delta^2 b_{k,n} \geqslant 0 \quad \text{or} \quad \Delta^2 b_{k,n} \leqslant 0, \quad k = 0, 1, \ldots, n-1, \tag{4.2}$$

and

$$\sum_{k=0}^{n} \frac{1 - b_k^{(n)}}{n - k + 1} = O(1) \tag{4.3}$$

imply (3.1) for B_n^* and consequently also for B_n. Owing to (4.2), condition (3.4) can be expressed in a much relaxed form:

$$\Delta b_{n-1,n} = O\left(\frac{1}{n}\right).$$

All in all, we have

COROLLARY 2. *Suppose* (4.2), (4.3) *and*

$$b_{0,n}=0,\quad b_{1,n}=O\left(\frac{1}{n}\right),\quad \Delta b_{n-1,n}=O\left(\frac{1}{n}\right).$$

Then

$$\max_{0\leqslant t\leqslant\pi}\left|\sum_{k=0}^{n}(1-b_{k,n})a_k(f)u_k(t)-f(t)\right|\leqslant c_{17}\left(\Omega\left(\frac{1}{n};f\right)+\max_{0\leqslant t\leqslant\pi}|f(t)|\frac{\log n}{n}\right).$$

d) *Franklin system.* Let $X=X_1=X_2=Y=C[0,1]$, $B=I$, $U_h=T_h$ and $\varphi_n=\chi_n(t)$, where $\{\chi_n(t)\}$ is the orthonormal Franklin system (see e. g. [1]). Using the results of [1], this system is both Jacksonian and Fejérian, so that our theorem applies. Nevertheless, in this case one can deduce in a trivial way a more precise result.

5. Let X be the linear space of those functions $f(x)$ defined on $(-\infty,+\infty)$, which are bounded and uniformly continuous on the whole real axis. Let

$$\|f\|_1=\sup_{-\infty<x<+\infty}|e^{-x^2/2}f(x)|,$$

$$\|f\|_2=\sup_{-\infty<x<+\infty}|f(x)|$$

and $U_h=T_h$.

Let further $\{\varphi_n(t)\}$ be the system of orthonormal Hermite-polynomials $\{2^{-n/2}(n!)^{-1/2}\pi^{-1/4}H_n(x)\}$. Refining a theorem of the first of us (see [3]), we will prove that $\{\varphi_n\}$ is Fejérian. We note also that it is $\frac{1}{2}$-quasi-Jacksonian; this follows from a more general theorem of M. M. Dzrbašian (see [2] p. 430-431, Theorem 7b).

We refer to the following inequality, proved in [3]:

$$\frac{1}{n}\sum_{\nu=0}^{n-1}|s_\nu(f;x)|\leqslant\{K_n(x)\int_{x-\delta_n}^{x+\delta_n}f^2(\xi)e^{-\xi^2}d\xi\}^{1/2}+ \tag{5.1}$$

$$+\frac{2}{n}\max_{\nu\leqslant n}\frac{\gamma_{\nu-1}}{\gamma_\nu}\left\{K_{n+1}(x)\left(\int_{-\infty}^{x-\delta_n}\frac{f^2(\xi)}{(\xi-x)^2}e^{-\xi^2}d\xi+\int_{x+\delta_n}^{\infty}\frac{f^2(\xi)}{(\xi-x)^2}e^{-\xi^2}d\xi\right)\right\},$$

where

$$K_n(x)=\sum_{k=0}^{n-1}\varphi_k^2(x)$$

and

$$\gamma_\nu=2^{\nu/2}(\nu!)^{-1/2}\pi^{-1/4}$$

is the coefficient at x_ν in $\varphi_\nu(x)$, so that

$$\max_{\nu\leqslant n}\frac{\gamma_{\nu-1}}{\gamma_\nu}=O(n^{1/2}). \tag{5.2}$$

As in [3], we insert $t=1-n^{-1}$ into the formula

$$\pi^{1/2}\sum_{\nu=0}^{\infty}\varphi_\nu^2(x)t^\nu=(1-t^2)^{-1/2}\exp\left(\frac{2tx^2}{1+t}\right)$$

and obtain

$$K_n(x)\leqslant(1-n^{-1})\sum_{\nu=0}^{\infty}\varphi_\nu^2(x)(1-n^{-1})^\nu=e^{x^2}O(n^{1/2}). \tag{5.3}$$

Putting in (5.1) $\delta_n=n^{-1/2}$ and using (5.2) and (5.3), we have finally

$$e^{-x^2/2}\frac{1}{n}\sum_{\nu=0}^{n-1}|s_\nu(f;x)|=O(\sup_\xi|e^{-\xi^2/2}f(\xi)|),$$

whence

$$\|F_n(f)\|_1=O(\|f\|_1),\ \text{q. e. d.}$$

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