On the theory of inductive families

by

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Introduction. This paper presents the complete and final version of the ideas primarily developed in [4] and [5], i. e. a certain method for handling the closed graph and the open mapping type of theorems in inductive limits of $(\mathscr{F})$-spaces. It is essentially based on the results of [11] and should be read with the assistance of [11] or [9]. The results presented here have been announced in [10]. Though no applications are given here, the paper is meant to give a background for the theory which was partially outlined in [6]-[8]. It is only natural that in the course of time and in connection with increasing number of applications the form of [4] and [5] necessarily changed. After several attempts had been made to improve the version of the theory, the most elegant and convincing way was accepted and it is thoroughly carried out in this paper.

As it happens the methods presented here are rather distant from those of the well-known Ptak's and related papers [3]. Nevertheless, the briliant Ptak's analysis of the kind of problems had its reflection already in [11] through the accepted terminology.

Though there is an easy to follow natural dependence between that what is understand as the open mapping and closed graph topic in [3] and what is done here, we are not going to discuss it in this paper waiting for some other occasion to do it. This is because this paper is to be followed by the one which will discuss Example I of this paper and we restrict ourselves here only for what is necessary for that coming paper.

The reader should notice that the only statements that make the theory presented here applicable to some inductive family $\mathfrak{Y} \epsilon (\mathscr{I}\mathscr{F})$ are those stating about existence of an overwhelming set of components in $\mathfrak{Y}$. Since in this paper the only distinguished subclass of $(\mathscr{I}\mathscr{F})$ is that of $(\mathscr{L}_2\mathscr{F})$-families, the said role of linking the general approach with $(\mathscr{L}_2\mathscr{F})$-families is played by Propositions 1 and 2 of this paper.

It is obvious that there are classes much wider than the class of σ^2-families, for which propositions parallel to Propositions 1 and 2 of this paper can be proved. However, it goes beyond the limit set for this paper.

We mention, finally, that the terminology and notation accepted here agree with that of [11].

Let us recall at the beginning a certain definition given in [11]. For linear topological spaces (U, τ) and (V, σ) we write $(U, \tau) \leqslant (V, \sigma)$ iff V is a subspace of U and the identical injection of (V, σ) into (U, τ) is continuous.

A family $\mathfrak{Y}$ of linear topological spaces is said to be an *inductive family* (cf. [7]) iff $\mathfrak{Y}$ is directed by the relation $\leqslant$ as follows. To every $(U_i, \tau_i) \epsilon \mathfrak{Y}$, $i = 1, 2$, there corresponds $(U_3, \tau_3) \epsilon \mathfrak{Y}$ such that $(U_3, \tau_3) \leqslant (U_i, \tau_i)$ for $i = 1, 2$.

An inductive family $\mathfrak{Y}_1$ is said to be *coarser* than the inductive family $\mathfrak{Y}_2$, which we write $\mathfrak{Y}_1 \leqslant \mathfrak{Y}_2$, iff to every $(U_2, \tau_2) \epsilon \mathfrak{Y}_2$ there corresponds $(U_1, \tau_1) \epsilon \mathfrak{Y}_1$ such that $(U_1, \tau_1) \leqslant (U_2, \tau_2)$; $\mathfrak{Y}_1$ is equivalent to $\mathfrak{Y}_2$ iff $\mathfrak{Y}_1 \leqslant \mathfrak{Y}_2$ and $\mathfrak{Y}_2 \leqslant \mathfrak{Y}_1$; $\mathfrak{Y}_1$ is *cofinal* with $\mathfrak{Y}_2$ iff $\mathfrak{Y}_1 \subset \mathfrak{Y}_2$ and $\mathfrak{Y}_1 \leqslant \mathfrak{Y}_2$.

With any inductive family $\mathfrak{Y}$ we associate the linear space

$$|\mathfrak{Y}| = \bigcup_{(U, \tau) \epsilon \mathfrak{Y}} U.$$

The reader will notice that this definition and the analogous one given in [11] on the occasion of defining pre-$(\mathscr{F})$-sequences coincide one with the other on objects being simultaneously subject to both of the definitions.

For any subspace $Y \subset |\mathfrak{Y}|$, where $\mathfrak{Y}$ is an inductive family, we define

$$Y \cap \mathfrak{Y} = \{(Y \cap U, \tau) : (U, \tau) \epsilon \mathfrak{Y}\},$$

where the topology induced by τ on $Y \cap U$ is denoted by the same letter τ.

We introduce the notion of convergence in inductive families following the well-known pattern. Let $\mathfrak{Y}$ be an inductive family. A sequence $\{x_n\} \subset |\mathfrak{Y}|$ *tends to zero* in $\mathfrak{Y}$ iff there exists $(U, \tau) \epsilon \mathfrak{Y}$ such that $\{x_n\} \subset U$ and $\{x_n\}$ tends to zero in (U, τ); $\{x_n\}$ tends to $x \epsilon |\mathfrak{Y}|$ iff $\{x_n - x\}$ tends to zero.

Any $|\mathfrak{Y}_1|$-valued function T defined on $|\mathfrak{Y}_2|$, where $\mathfrak{Y}_i$, $i = 1, 2$, are given inductive families, is said to be *sequentially continuous from* $\mathfrak{Y}_2$ *to* $\mathfrak{Y}_1$ iff for every sequence $\{x_n\} \subset |\mathfrak{Y}_2|$ we have $\{Tx_n\}$ tending to Tx in $\mathfrak{Y}_1$ whenever $\{x_n\}$ tends to x in $\mathfrak{Y}_2$. If an inductive family $\mathfrak{X}$ consists of countable number of elements arranged in the form of a sequence $\mathfrak{X} = \{(X_n, \tau_n)\}$ with $(X_n, \tau_n) \geqslant (X_{n+1}, \tau_{n+1})$ for $n = 1, 2, \ldots$, then $\mathfrak{X}$ is called an *inductive sequence*. An inductive family of $(\mathscr{F})$-spaces is called an $(\mathscr{I}\mathscr{F})$-*family* and an inductive sequence of $(\mathscr{F})$-spaces is called an $(\mathscr{I}\mathscr{F})$-*sequence*.

We shall describe now some special kind of $(\mathscr{I}\mathscr{F})$-families leading to several important examples.

Let $\mathcal{N}$ denote once for all the set of all increasing sequences of natural numbers. Consider a double sequence $\{(Y_{k,n}, \|\cdot\|_{k,n})\}$ of linear spaces provided with single pseudonorm each. Assume that all $Y_{k,n}$ are subspaces of a fixed linear space Y.

For arbitrary $\boldsymbol{k} = \{k_n\} \epsilon \mathcal{N}$ we define

$$Y_{\boldsymbol{k}} = \bigcap_{n=1}^{\infty} Y_{k_n, n},$$

$\tau_{\boldsymbol{k}} = $ the topology induced in $Y_{\boldsymbol{k}}$ by $\{\|\cdot\|_{k_n, n} : n = 1, 2, \ldots\}$.

Let the following conditions be satisfied:

1. $(Y_{k,n}, \|\cdot\|_{k,n}) \geqslant (Y_{k+1,n}, \|\cdot\|_{k+1,n})$ for $k, n = 1, 2, \ldots$,

2. $\bigcup_{k=1}^{\infty} Y_{k,n} = Y$ for every n.

We assume additionally that every $(Y_{\boldsymbol{k}}, \tau_{\boldsymbol{k}})$, $\boldsymbol{k} \epsilon \mathcal{N}$, is complete, i. e. is an $(\mathscr{F})$-space.

It is easy to see that for every $\{(Y_{k,n}, \|\cdot\|_{k,n})\}$ satisfying the conditions given above the set

$$\mathfrak{Y} = \{(Y_{\boldsymbol{k}}, \tau_{\boldsymbol{k}}) : \boldsymbol{k} \epsilon \mathcal{N}\}$$

constitutes an $(\mathscr{I}\mathscr{F})$-family.

The family $\mathfrak{Y}$ is called a σ^2-*family* ([1]) and the double sequence $\{(Y_{k,n}, \|\cdot\|_{k,n})\}$ — a *decomposition* of the family $\mathfrak{Y}$. It is obvious that a given $(\mathscr{I}\mathscr{F})$-family may have several different decompositions indicating that it is a σ^2-family. It is left to the reader to check that countable $(\mathscr{I}\mathscr{F})$-families are equivalent to some very particular kind of σ^2-families.

An $(\mathscr{I}\mathscr{F})$-family is said to be an $(\mathscr{L}_1\mathscr{F})$-*family* (an $(\mathscr{L}_2\mathscr{F})$-*family*) iff it contains a cofinal $(\mathscr{I}\mathscr{F})$-sequence (a cofinal σ^2-family). The class of $(\mathscr{L}_1\mathscr{F})$-families will be denoted by $(\mathscr{L}_1\mathscr{F})$ and the class of $(\mathscr{L}_2\mathscr{F})$-families by $(\mathscr{L}_2\mathscr{F})$.

Examples which will come now correspond with that of [11] taken in the order III, IV, V, VI of the numeration used in [11].

EXAMPLES. I. Consider an $(\mathscr{I}\mathscr{F})$-sequence $\mathfrak{X} = \{(X_n, \tau_n)\}$ and let for every n the sequence of pseudonorms $\{\|\cdot\|_{k,n} : k = 1, 2, \ldots\}$ be pointwise non-decreasing defined on X_n and inducing the topology τ_n in X_n. Denote by X' the linear space of all functionals defined and linear on $|\mathfrak{X}|$ with restrictions continuous in every (X_n, τ_n) separately.

Let

$$\|x\|_{k,n}^* = \sup \{|x'x| : x \epsilon X_n, \|x\|_{k,n} \leqslant 1\}$$

for $x \epsilon |\mathfrak{X}|$ and

$$X^*_{k,n} = \{x' \epsilon X' : \|x'\|^*_{k,n} < \infty\}.$$

It is easy to see that $\{(X^*_{k,n}, \|\cdot\|^*_{k,n})\}$ represents a certain σ^2-family which we shall denote by $\mathfrak{X}^*$. Clearly $|\mathfrak{X}^*| = X'$. It will trivially follow from Corollary 3 of this paper that $\mathfrak{X}^*$ does not depend on the choice of $\{\|\cdot\|_{k,n}\}$ up to the relation of equivalence introduced in $(\mathscr{IF})$. Moreover, equivalent $(\mathscr{IF})$-sequences $\mathfrak{X}$ lead to equivalent $\mathfrak{X}^*$ which belong to the so called *adjoint* class of equivalence.

A separate paper will be devoted exclusively to more extend studies of this important example.

II. Consider a topological space R and suppose that there exists a double sequence $\{R_{k,n}\}$ of open subsets of R such that

$$R_{k,n} \supset R_{k+1,n} \quad \text{for} \quad k, n = 1, 2, \ldots,$$

and

$$\bigcup_{n=1}^{\infty} R_{k_n, n} = R \quad \text{for} \quad \{k_n\} \epsilon \mathscr{N}.$$

For every scalar-valued function f defined on R we introduce a pseudonorm

$$\|f\|_{k,n} = \sup \{|f(r)| : r \epsilon R_{k,n}\}$$

and we put an additional assumption stating that for every continuous function f we have

$$\inf \{\|f\|_{k,n} : k = 1, 2, \ldots\} < \infty$$

for every natural n.

The double sequence $\{(\mathscr{C}_{k,n}, \|\cdot\|_{k,n})\}$, where $\mathscr{C}_{k,n} = \{f \epsilon \mathscr{C}(R) : \|f\|_{k,n} < \infty\}$ and $\mathscr{C}(R)$ denotes the space of all continuous functions on R, is a decomposition of a certain σ^2-family which we denote by $\mathfrak{C}(R)$. It is clear that $|\mathfrak{C}(R)| = \mathscr{C}(R)$.

Let, for instance, R be a separable σ-compact space. Denoting by $\{R_{k,n} : k = 1, 2, \ldots\}$ the sequence of neighbourhoods of a compact set K_n, we find that $\{R_{k,n}\}$ satisfies the above listed requirements whenever $K_n \subset K_{n+1}$ for $n = 1, 2, \ldots$ and $R = \bigcup_{n=1}^{\infty} K_n$.

III. Let (X_i, τ_i), $i = 1, 2$, be two $(\mathscr{F})$-spaces. Suppose that topologies τ_i are given by pointwise non-decreasing sequences of pseudonorms $\{|\cdot|_{i,n} : n = 1, 2, \ldots\}$ for $i = 1, 2$, respectively.

Denote by $\mathscr{L}$ the set of all continuous linear mappings of (X_2, τ_2) into (X_1, τ_1). Define

$$\|A\|_{k,n} = \sup \{|Ax|_{1,n} : |x|_{2,k} \leqslant 1\}$$

for $A \epsilon \mathscr{L}$ and

$$\mathscr{L}_{k,n} = \{x \epsilon \mathscr{L} : \|A\|_{k,n} < \infty\}.$$

It is clear that $\{(\mathscr{L}_{k,n}, \|\cdot\|_{k,n})\}$ is a decomposition of a σ^2-family. Denoting this family by $\mathfrak{L}$ we can write $|\mathfrak{L}| = \mathscr{L}$.

IV. Consider an $(\mathscr{L})$-space (X, τ), where τ is induced by a non-decreasing sequence of pseudonorms $\{|\cdot|_n\}$, and the inductive limit (Y, π) of an $(\mathscr{IF})$-sequence of Banach spaces $\{(Y_n, \||\cdot|\|_n)\}$.

Denote by $\mathscr{B}$ the linear space of all bilinear functionals $f(x, y)$ defined on $X \times Y$ and continuous with respect to product topology $\tau \times \{\||\cdot|\|_n\}$ while restricted to $X \times Y_n$ for $n = 1, 2, \ldots$

Define

$$\|f\|_{k,n} = \sup \{|f(x, y)| : \||y|\|_n < 1, \|x\|_k < 1\}$$

for $f \epsilon \mathscr{B}$.

Let further

$$\mathscr{B}_{k,n} = \{f \epsilon \mathscr{B} : \|f\|_{k,n} < \infty\}.$$

Here again the double sequence $\{(\mathscr{B}_{k,n}, \|\cdot\|_{k,n})\}$ is a decomposition of a σ^2-family. As in the previous case we have $|\mathfrak{B}| = \mathscr{B}$ if $\mathfrak{B}$ denotes the discussed σ^2-family.

We do not intend to quote any more examples of σ^2-families nor we propose to present any detailed discussion of the already presented examples. The examples are supposed to give to the reader only the proper intuition as to which kind of linear spaces with which kind of convergence can be considered as inductive families of that special character. However, as it was already mentioned, a separate paper will be devoted exclusively for discussion of the Example I and related questions.

Consider an $(\mathscr{IF})$-family $\mathfrak{Y}$. An $(\mathscr{F})$-sequence $\mathfrak{B}$ is said to be a *component* [2] of $\mathfrak{Y}$ iff the following conditions hold:

a. $|\mathfrak{B}|$ is a subspace of $|\mathfrak{Y}|$,

b. If $\{x_n\}$ tends to 0 in $[\mathfrak{B}]$, then it tends to 0 in $\mathfrak{Y}$.

c. There is $(U, \tau) \epsilon \mathfrak{Y}$ such that $(U, \tau) \leqslant \bigcap \mathfrak{B}$.

Notice, that a component of any $\mathfrak{Y} \epsilon (\mathscr{IF})$ is always a component of every $\mathfrak{Y}_1 \epsilon (\mathscr{IF})$ containing $\mathfrak{Y}$ or cofinal with $\mathfrak{Y}$. The converse is not true. Indeed, whatever is $\mathfrak{Y} \epsilon (\mathscr{IF})$, we can always produce $\mathfrak{Y}_0 \epsilon (\mathscr{IF})$ with $|\mathfrak{Y}_0| = |\mathfrak{Y}|$ and $(U, \tau) \epsilon \mathfrak{Y}_0$ iff (U, τ) is finite dimensional. Clearly $\mathfrak{Y}$ is coarser than $\mathfrak{Y}_0$ but it is quite obvious that, in general, a component of $\mathfrak{B}$ need not be a component of $\mathfrak{Y}_0$.

A set $\varXi$ of components of $\mathfrak{Y} \epsilon (\mathscr{IF})$ is said to be *overwhelming* in $\mathfrak{Y}$ iff to every pre-$(\mathscr{F})$-sequence $\mathfrak{B}$ and every linear mapping T of a subspace Y of $|\mathfrak{Y}|$ onto a second category subspace of $[\mathfrak{B}]$ there corresponds a component $\{(W_n, \|\cdot\|_n)\} \epsilon \varXi$ such that every $R(Y \cap W_n)$, $n = 1, 2, \ldots$, is of the second category in $[\mathfrak{B}]$.

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[2] This should not be confused with "projective component" introduced in [6] and [8] which is a different notion.

Again, if Ξ is a set of components that overwhelms in some $\mathfrak{Y} \in (\mathscr{I}\mathscr{F})$, then Ξ overwhelms in any $\mathfrak{Y}_1$ containing $\mathfrak{Y}$ and such that $|\mathfrak{Y}_1| = |\mathfrak{Y}|$. Knowing already that components of $\mathfrak{Y}$ are not necessarily components of $\mathfrak{Y}_1$ contained in $\mathfrak{Y}$, we find that the converse is not true.

We shall study the notions of component and overwhelming set of components using the introduced special case of σ^2-families.

With any decomposition $\{(Y_{k,n}, \|\cdot\|_{k,n})\}$ of a given σ^2-family $\mathfrak{Y}$ and with any $\boldsymbol{k} = \{k_n\} \in \mathscr{N}$ we associate a pre-$(\mathscr{F})$-sequence

3. $\mathfrak{Y}_{\boldsymbol{k}} = \{(Y_{\boldsymbol{k},n}, \|\cdot\|_{\boldsymbol{k},n})\}$, where $Y_{\boldsymbol{k},n} = \bigcap\limits_{i=1}^{n} Y_{k_i,i}$ and $\|\cdot\|_{\boldsymbol{k},n} = \max\{\|\cdot\|_{k_i,i}:$

$i = 1, \ldots, n\}$.

Notice that

$$(Y_{\boldsymbol{k}}, \tau_{\boldsymbol{k}}) = \bigcap \mathfrak{Y}_{\boldsymbol{k}}.$$

PROPOSITION 1. *If* $\{(Y_{k,n}, \|\cdot\|_{k,n})\}$ *is a decomposition of a* σ^2-*family* $\mathfrak{Y}$, *then every* $\mathfrak{Y}_{\boldsymbol{k}}$, $\boldsymbol{k} \in \mathscr{N}$, *given by 3, is a component of* $\mathfrak{Y}$.

Proof. We are supposed to show that for any $\boldsymbol{k} \in \mathscr{N}$ and any sequence $\{y_n\} \subset |\mathfrak{B}_{\boldsymbol{k}}|$, such that $\{y_n\}$ satisfies the Cauchy condition in $[\mathfrak{Y}_{\boldsymbol{k}}]$, there correspond $\boldsymbol{h} \in \mathscr{N}$ and $y \in Y_{\boldsymbol{h}} \cap |\mathfrak{Y}_{\boldsymbol{k}}|$ such that $\{y_n\} \subset Y_{\boldsymbol{h}}$ and $\{y_n\}$ tends to y both in $(Y_{\boldsymbol{h}}, \tau_{\boldsymbol{h}})$ and $[\mathfrak{Y}_{\boldsymbol{k}}]$. From A of Proposition 1 of [11] it follows that if $\{y_n\}$ satisfies the Cauchy condition in $[\mathfrak{Y}_{\boldsymbol{k}}]$, then to every p there correspond m_p such that $y_n - y_m \in Y_{\boldsymbol{k},p}$ for $n, m \geqslant m_p$. Then, using 1 and 2 we find that to every p there correspond $h_p \geqslant k_p$ such that $\boldsymbol{h} = \{h_n\} \in \mathscr{N}$ and $\{y_1, \ldots, y_{m_p}\} \subset Y_{h_p,p}$. Since $y_n - y_{m_p} \in Y_{k_p,p} \subset Y_{h_p,p}$ and $y_{m_p} \in Y_{h_p,p}$, we have $y_n = (y_n - y_{m_p}) + y_{m_p} \in Y_{h_p,p}$ for $n \geqslant m_p$. Then $\{y_n\} \subset Y_{\boldsymbol{h}}$, using 1 we find that

$$\lim_{n,m} \|y_n - y_m\|_{\boldsymbol{h},p} = 0$$

for every p, and finally there exists $y \in Y_{\boldsymbol{h}}$ such that $\{y_n\}$ tends to y in $(Y_{\boldsymbol{h}}, \tau_{\boldsymbol{h}})$. Now, for any fixed p we put $\boldsymbol{k}_p = \{k_1, \ldots, k_p, h_{p+1}, h_{p+2}, \ldots\} \in \mathscr{N}$. We have $\{y_n - y_{m_p} : n \geqslant m_p\} \subset Y_{\boldsymbol{k}_p}$ satisfying the Cauchy condition in $(Y_{\boldsymbol{k}_p}, \tau_{\boldsymbol{k}_p})$ and so it tends to some $z_p \in Y_{\boldsymbol{k}_p}$. Since $(Y_{\boldsymbol{k}_p}, \tau_{\boldsymbol{k}_p}) \geqslant (Y_{\boldsymbol{h}}, \tau_{\boldsymbol{h}})$, the considered sequence tends to z_p in $(Y_{\boldsymbol{h}}, \tau_{\boldsymbol{h}})$ as well and we have $y - y_{m_p} = z_p \in Y_{\boldsymbol{k}_p} \subset Y_{\boldsymbol{k},p}$. Thus, for $n \geqslant m_p$ we have $y_n - y = (y_n - y_{m_p}) - (y_{m_p} - y) \in Y_{\boldsymbol{k},p}$ and from the continuity of $\|\cdot\|_{\boldsymbol{k},p}$ in $(Y_{\boldsymbol{k}_p}, \tau_{\boldsymbol{k}_p})$ we obtain $\lim \|y_n - y\|_{\boldsymbol{k},p} = \lim \|(y_n - y_{m_p}) - z_p\|_{\boldsymbol{k},p} = 0$. Hence $\{y_n\}$ tends to y in $\mathfrak{Y}_{\boldsymbol{k}}$ and the Proposition follows.

PROPOSITION 2. *Consider a* σ^2-*family* $\mathfrak{Y}$ *and a decomposition* $\{(Y_{k,n}, \|\cdot\|_{k,n})\}$ *of* $\mathfrak{Y}$. *The set* $\Xi = \{\mathfrak{Y}_{\boldsymbol{k}} : \boldsymbol{k} \in \mathscr{N}\}$, *where* $\mathfrak{Y}_{\boldsymbol{k}}$ *are defined according to 3, is overwhelming in* $\mathfrak{Y}$.

Proof. Take a pre-$(\mathscr{F})$-sequence $\mathfrak{B}$ and let T map $Y \subset |\mathfrak{Y}|$ onto a second category subspace of $[\mathfrak{B}]$. It follows from 2 that $\bigcup\limits_{k=1}^{\infty} T(Y \cap Y_{k,1})$

$= TY$ and then for at least one k_1 the set $T(Y \cap Y_{k_1,1})$ is of the second category in $[\mathfrak{B}]$. Suppose we have found $k_1 < k_2 < \ldots < k_{n-1}$ such that $T(Y \cap Y_{\boldsymbol{k},n-1})$ is of the second category in $[\mathfrak{B}]$, where for any $k_1, \ldots, k_p$ we write $Y_{\boldsymbol{k},p} = Y_{k_1,1} \cap \ldots \cap Y_{k_p,p}$. We have $T(Y \cap Y_{\boldsymbol{k},n-1})$

$= \bigcup\limits_{k=1}^{\infty} T(Y \cap Y_{\boldsymbol{k},n-1} \cap Y_{k,n})$ and then for at least one k_n the set $T(Y \cap Y_{\boldsymbol{k},n-1} \cap Y_{k_n,n}) = T(Y \cap Y_{\boldsymbol{k},n})$ is of the second category in $[\mathfrak{B}]$. In view of the inclusion $Y_{k,n} \subset Y_{k+1,n}$ we can take $k_n > k_{n-1}$. We have produced $\boldsymbol{k} \in \mathscr{N}$ with $T(Y \cap Y_{\boldsymbol{k},n})$ of the second category in $[\mathfrak{B}]$ for every natural n. This finishes the proof of Proposition 2.

This paper is arranged in such a manner that all the Theorems are proved for arbitrary $(\mathscr{I}\mathscr{F})$-families. This is achieved by introduction of the notion of an overwhelming set of components. With that kind of set up Propositions 1 and 2 are needed exclusively for showing the applicability of the general theory to $(\mathscr{L}_2\mathscr{F})$-families.

However, it has been indicated and practically even shown in [4] that the class of $(\mathscr{I}\mathscr{F})$-families admitting overwhelming sets of components is much wider than that of $(\mathscr{L}_2\mathscr{F})$-families. This justifies the introduction of overwhelming sets of components.

It has been explained in [11] what it is a complete-closed mapping of pre-$(\mathscr{F})$-sequences.

Now, having introduced the notion of inductive families we need a certain additional definition.

Suppose that $(X_i, \varkappa_i)$, $i = 1, 2$, are two linear spaces each provided with a sequential topology by way of establishing sets of sequences convergent to zero $\varkappa_i$, $i = 1, 2$, respectively. We call $\{z_n\}$ *convergent to zero* iff $\{z_n - z\}$ converges to zero.

A mapping T of a subspace $X_0 \subset X_2$ into X_1 is said to be *closed relative to* $(X_2, \varkappa_2)$ iff for every $\{x_n\} \subset X_0$ with $\{x_n\}$ tending to x and $\{Tx_n\}$ tending to y we have $x \in X_0$ and $Tx = y$.

In the case X_0 coincides with the whole X_2, we drop for the sake of brevity the word "relative" calling the mapping *closed*.

In all concrete cases appearing later we shall accept as the sets of sequences convergent to zero those which have been already defined as such in the considered cases.

For any component $\mathfrak{B}$ of a $\mathfrak{Y} \in (\mathscr{I}\mathscr{F})$ and any subspace $Y \in |\mathfrak{Y}|$ we write

$$Y \cap \mathfrak{B} \overset{\text{df}}{=} \{(Y \cap |n\mathfrak{B}|, \|\cdot\|_{\mathfrak{B},n})\}.$$

The following Proposition connects complete-closed mappings and mappings closed relative to some $\mathfrak{Y}$:

PROPOSITION 3. *Let* $\mathfrak{Y} \in (\mathscr{I}\mathscr{F})$, $\mathfrak{B}_1$ *be a pre-$(\mathscr{F})$-sequence and* T — *a linear mapping of a subspace* $Y \subset |\mathfrak{Y}|$ *into* $|\mathfrak{B}|$. *If* T *is closed relative to* $\mathfrak{Y}$, *then*

for every component $\mathfrak{B}$ of $\mathfrak{Y}$ the mapping T restricted to a mapping of $[Y \frown \mathfrak{B}]$ into $[\mathfrak{B}_1]$, is complete-closed.

Proof. If $\{y_n\}$ is a Cauchy sequence in $[Y \frown \mathfrak{B}]$ and $\{Ty_n\}$ tends to some z in $[\mathfrak{B}_1]$, then from the definition of a component of $\mathfrak{Y}$ it follows that $\{y_n\}$ tends to some $y \in |\mathfrak{Y}| \frown |\mathfrak{B}|$ in both $\mathfrak{Y}$ and $[\mathfrak{B}]$. Hence $y \in Y$ and $Ty = z$ which proves that T is complete-closed while restricted to $[Y \frown \mathfrak{B}]$. The proposition is proved.

In [11] there have been proved the Open Mapping Theorem in two different versions and the Closed Graph Theorem in one version. In this paper we shall prove some other versions, two of the Open Mapping Theorem and two of the Closed Graph Theorem.

THEOREM 1 (The Open Mapping Theorem III, cf. [4], [11]). *Consider an $(\mathscr{IF})$-family $\mathfrak{Y}$ provided with a set $\varXi$ of components, overwhelming in Y, a pre-$(\mathscr{F})$-sequence $\mathfrak{B}_1$ and a linear mapping T of a subspace $Y \subset |\mathfrak{Y}|$ into $|\mathfrak{B}_1|$. Assume that the image of Y is of the second category in $[\mathfrak{B}_1]$ (it happens, for instance, if $\mathfrak{B}_1$ is an $(\mathscr{F})$-sequence and TY contains $|n\mathfrak{B}_1|$ for some n).*

If T is closed relative to $\mathfrak{Y}$, then there exists $\mathfrak{B} \in \varXi$ such that the mapping restricted to the mapping of $[Y \frown \mathfrak{B}]$ into $[\mathfrak{B}_1]$ is open.

Proof. Since TY is of the second category in $[\mathfrak{B}_1]$, we can find $\mathfrak{B} \in \varXi$ such that every $T(Y \frown |n\mathfrak{B}|)$ is of the second category in $[\mathfrak{B}_1]$. Furthermore, T is closed relative to $\mathfrak{Y}$ and then from Proposition 3 it follows that T is complete-closed restricted to the mapping of $[Y \frown \mathfrak{B}]$ into $[\mathfrak{B}_1]$. Hence, we can apply Theorem 2 of [11] and find that T restricted to the mapping of $[Y \frown \mathfrak{B}]$ into $[\mathfrak{B}_1]$ must be open. This finishes the proof of Theorem 1.

THEOREM 2 (The Closed Graph Theorem II, cf. [5], [11]). *Consider an $(\mathscr{IF})$-family $\mathfrak{Y}$, an overwhelming in $\mathfrak{Y}$ set of components $\varXi$, an $(\mathscr{F})$-sequence $\mathfrak{B}_1$ and a linear mapping T of $|\mathfrak{B}_1|$ into $|\mathfrak{Y}|$. If T is closed, then there exists $\mathfrak{B} \in \varXi$ such that T is a continuous mapping of some shift $[m_0\mathfrak{B}_1]$ into $[\mathfrak{B}]$.*

Proof. Let $L = \{x \in |\mathfrak{B}_1| : Tx = 0\}$. Since T is closed, L must be a $[\mathfrak{B}_1]$-closed subspace of $|\mathfrak{B}_1|$ and we can produce the quotient $(\mathscr{F})$-sequence $\mathfrak{B}_1/L$. Let $\tilde{T}$ denote the factorization of T to a mapping of $|\mathfrak{B}_1/L|$ into $|\mathfrak{Y}|$. Clearly $\tilde{T}$ is closed and $[\mathfrak{B}_1]/L = [\mathfrak{B}_1/L]$ is complete. If we denote by Y the image of $|\mathfrak{B}_1|$ in $\mathfrak{Y}$, then $B = \tilde{T}^{-1}$ is a linear mapping of Y onto $|\mathfrak{B}_1/L|$ closed relative to $\mathfrak{Y}$. Applying Theorem 1 we find $\mathfrak{B} \in \varXi$ such that B restricted to $Y \frown |\mathfrak{B}|$ is open as a mapping from $[Y \frown \mathfrak{B}]$ into $[\mathfrak{B}_1/L]$. Hence, from Proposition 5 of [11] it follows that there exists m_0 such that $B(Y \frown |\mathfrak{B}|) \supset |m_0(\mathfrak{B}_1/L)| = |(m_0\mathfrak{B}_1)/L|$. Further, B is open from $[Y \frown \mathfrak{B}]$ into $[\mathfrak{B}_1/L]$ and then $\tilde{T} = B^{-1}$ considered on $B(Y \frown |\mathfrak{B}|)$ is continuous from $[\mathfrak{B}_1/L]$ into $[\mathfrak{B}]$. Then $\tilde{T}$ is continuous from $[m_0\mathfrak{B}_1/L]$ into $[\mathfrak{B}]$

and finally T restricted to $|m_0\mathfrak{B}_1|$ is continuous from $[m_0\mathfrak{B}_1]$ into $[\mathfrak{B}]$. This finishes the proof of Theorem 2.

From Theorem 2 we obtain the following important properties of components and overwhelming sets of components:

COROLLARY 1. *Suppose $\varXi$ is a set of components of some $\mathfrak{Y} \in (\mathscr{IF})$ and $\mathfrak{B}_1$ is an arbitrary component of $\mathfrak{Y}$. If $\varXi$ is overwhelming in $\mathfrak{Y}$, then there exists $\mathfrak{B} \in \varXi$ such that $\mathfrak{B}_1$ is finer than $\mathfrak{B}$.*

Proof. It follows directly from Theorem 2 if T is the identical injection of $|\mathfrak{B}_1|$ into $|\mathfrak{Y}|$.

COROLLARY 2. *Consider two $(\mathscr{IF})$-families $\mathfrak{Y}$ and $\mathfrak{Y}_1$ such that $\mathfrak{Y} \subset \mathfrak{Y}_1$ and $|\mathfrak{Y}| = |\mathfrak{Y}_1|$. If $\mathfrak{Y}$ admits a set of components, overwhelming in $\mathfrak{Y}$, then every component of $\mathfrak{Y}_1$ is a component of $\mathfrak{Y}$ as well.*

Proof. If $\varXi$ is overwhelming in $\mathfrak{Y}$, it must be also overwhelming in $\mathfrak{Y}_1$ and then, applying Corollary 1, we find $\mathfrak{B} \in \varXi$ with $\mathfrak{B}_1$ finer than $\mathfrak{B}$ and so $\mathfrak{B}_1$ must be a component of $\mathfrak{Y}$.

COROLLARY 3. *If $\mathfrak{Y}_1 \in (\mathscr{IF})$ contains an $(\mathscr{IF})$-family $\mathfrak{Y}$, $|\mathfrak{Y}_1| = |\mathfrak{Y}|$ and $\mathfrak{Y}$ admits an overwhelming set of components, than $\mathfrak{Y}$ is cofinal with $\mathfrak{Y}_1$.*

Proof. It follows from Corollary 3 if we put $(U, \tau) = \bigcap \mathfrak{B}_1 = [\mathfrak{B}_1]$ for each $(U, \tau) \in \mathfrak{Y}_1$.

To complete the list of the open mapping and the closed graph theorems that go together with the kind of approach we have developed here, it is necessary to add two more theorems. Though the theorems are actually simple corollaries of Theorems 1 and 2, we shall quote them as Theorems 2 and 3 because of their independent character and importance.

THEOREM 3 (The Open Mapping Theorem IV). *Consider two $(\mathscr{IF})$-families $\mathfrak{Y}_i$, $i = 1, 2$, let $\mathfrak{Y}_2$ admit an overwhelming set of components $\varXi$ and let T be a linear mapping of $Y \subset |\mathfrak{Y}_2|$ onto $|\mathfrak{Y}_1|$. If T is closed relative to $\mathfrak{Y}_2$, then to every component $\mathfrak{B}_1$ of $\mathfrak{Y}_1$ there corresponds a component $\mathfrak{B} \in \varXi$ such that T restricted to $Y_{\mathfrak{B}_1} \frown |\mathfrak{B}|$, where $Y_{\mathfrak{B}_1} = \{y \in Y : Ty \in |\mathfrak{B}_1|\}$, is open as a mapping of $[Y_{\mathfrak{B}_1} \frown \mathfrak{B}]$ into $[\mathfrak{B}_1]$.*

Proof. Take any component $\mathfrak{B}_1$ of $\mathfrak{Y}_1$. Restricting T to $Y_{\mathfrak{B}_1}$ we obtain a mapping of $Y_{\mathfrak{B}_1}$ onto $|\mathfrak{B}_1|$ closed relative to $\mathfrak{Y}_2$ and applying Theorem 1 we find $\mathfrak{B} \in \varXi$ satisfying the requirements of the Theorem.

THEOREM 4 (The Closed Graph Theorem III). *Consider two $(\mathscr{IF})$-families $\mathfrak{Y}$ and $\mathfrak{Y}_1$, $\mathfrak{Y}$ admitting an overwhelming set of components $\varXi$, and a linear mapping T of $|\mathfrak{Y}_1|$ into $|\mathfrak{Y}|$. If T is closed, then to every $(U_1, \tau_1) \in \mathfrak{Y}_1$ there corresponds $(U, \tau) \in \mathfrak{Y}$ such that $TU_1 \subset U$ and T restricted to the mapping of (U_1, τ_1) into (U, τ) is continuous.*

Proof. Take $(U_1, \tau_1) \in \mathfrak{Y}_1$ and consider the $(\mathscr{F})$-sequence defined as follows. Put $|nU_1| = U_1$ for every n and arrange $\{\|\cdot\|_{\mathfrak{B}_1, n}\}$ to induce the topology τ_1 in U_1. From Theorem 2 it follows that there exists $\mathfrak{B} \in \varXi$ such

that T maps U_1 into $[\mathfrak{B}]$ and is continuous from (U_1, τ_1) into $[\mathfrak{B}]$. Hence, from D of Proposition 1 of [11] it follows that $TU_1 \subset |\bigcap \mathfrak{B}|$. Since $\mathfrak{B}$ is a component of $\mathfrak{Y}$, there must be $(U, \tau)\epsilon\mathfrak{Y}$ coarser than $\bigcap \mathfrak{Y}$ and the Theorem follows.

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On some classes of functions with regard to their orders of growth

by

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The aim of this paper is to investigate some classes of continuous and positive functions φ. Such classes occur in various instances, for example in definitions of such mathematical objects as Orlicz spaces, spaces of sequences strongly summable in a generalized sense, and, more generally, modular spaces etc. Various conditions imposed on moduli of continuity lead also to such classes of continuous positive functions φ. In all the above-mentioned situations some restrictions on functions φ are given which describe, roughly speaking, the growth of φ as $u \to \infty$ (or $u \to 0$) in comparison with the growth of functions from a given functional scale (in most cases the scale of functions t^n). For example, in the theory of Orlicz spaces often occurs the so-called condition Δ_2, and in various problems of analysis functions regularly increasing in the sense of Karamata are of importance.

This paper is a continuation of papers [11], [8], [9] and gives a further development of the ideas of those papers. These simple ideas consist in the application of the so-called indices (compare 3.1 of the present paper), and of a notion of equivalence of functions, more general than that of asymptotical equality. The purpose of the authors is to give a possibly simple and systematic exposition of the problems in question.

1. In this section we shall denote by h a real extended-valued function defined for $\mu \geqslant 0$. The function h is said to be *subadditive* in $\langle 0, \infty\rangle$, if the inequality $h(\mu_1+\mu_2) \leqslant h(\mu_1)+h(\mu_2)$ holds for any non-negative μ_1, μ_2 unless the values $h(\mu_1)$, $h(\mu_2)$ are infinite and of opposite signs. Changing in the above inequality the sign $\leqslant$ into $\geqslant$ we obtain the definition of a *superadditive* function in $\langle 0, \infty\rangle$.

1.1. *Suppose h is monotone and subadditive in* $\langle 0, \infty\rangle$, $h(0) = 0$. *Under these assumptions the following formulae hold*:

$$(*) \qquad \lim_{\mu \to 0+} \frac{h(\mu)}{\mu} = \sup_{0 < \mu \leqslant \mu^*} \frac{h(\mu)}{\mu} \quad \text{for any} \quad 0 < \mu^* \leqslant \infty,$$

$$(**) \qquad \lim_{\mu \to \infty} \frac{h(\mu)}{\mu} = \inf_{\mu \geqslant \mu^*} \frac{h(\mu)}{\mu} \quad \text{for any} \quad 0 \leqslant \mu^* < \infty.$$