that T maps U_1 into $[\mathfrak{V}]$ and is continuous from (U_1, τ_1) into $[\mathfrak{V}]$. Hence, from D of Proposition 1 of [11] it follows that $TU_1 \subset |\bigcap \mathfrak{V}|$. Since $\mathfrak{V}$ is a component of $\mathfrak{Y}$, there must be $(U, \tau) \in \mathfrak{Y}$ coarser than $\bigcap \mathfrak{Y}$ and the Theorem follows.

References

[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[2] J. Dieudonné et L. Schwartz, *La dualité dans les espaces* $(\mathscr{F})$ *et* $(\mathscr{LF})$, Ann. Inst. Four. Grenoble 1 (1949), p. 61.

[3] V. Ptak, *Completeness and the open mapping theorem*, Bull. Soc. Math. Franc. 86 (1958), p. 41-74.

[4] W. Słowikowski, *On continuity of inverse operators*, Bull. Amer. Math. Soc. 67 (1961), p. 467-470

[5] — *Quotient spaces and the open map theorem*, ibid. 67 (1963), p. 498-500.

[6] — *On the continuity of inverse operators in* $(\mathscr{LF})$*-spaces*, ibid. 69 (1963), p. 832-834.

[7] — *The concept of inductive family of* $(\mathscr{F})$*-spaces in connection with solvability of linear equations*. Bull. Acad. Polon. Sci. 11 (1963), p. 517-520.

[8] — *Linear operators in spaces of distributional type*, ibid. 11 (1963), p. 521-524.

[9] — *The concept of* $(\mathscr{F})$*-sequence*, ibid. 12 (1964), p. 465-470.

[10] — $(\mathscr{F})$*-sequences and inductive families of* $(\mathscr{F})$*-spaces*, ibid. 12 (1964), p. 613-615.

[11] — *On the theory of* $(\mathscr{F})$*-sequences*, Studia Mathematica 25 (1965), p. 281-296.

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

Reçu par la Rédaction le 10. 6. 1964

STUDIA MATHEMATICA, T. XXVI. (1965)

On some classes of functions with regard to their orders of growth

by

W. MATUSZEWSKA and W. ORLICZ (Poznań)

The aim of this paper is to investigate some classes of continuous and positive functions φ. Such classes occur in various instances, for example in definitions of such mathematical objects as Orlicz spaces, spaces of sequences strongly summable in a generalized sense, and, more generally, modular spaces etc. Various conditions imposed on moduli of continuity lead also to such classes of continuous positive functions φ. In all the above-mentioned situations some restrictions on functions φ are given which describe, roughly speaking, the growth of φ as $u \to \infty$ (or $u \to 0$) in comparison with the growth of functions from a given functional scale (in most cases the scale of functions t^n). For example, in the theory of Orlicz spaces often occurs the so-called condition Δ_2, and in various problems of analysis functions regularly increasing in the sense of Karamata are of importance.

This paper is a continuation of papers [11], [8], [9] and gives a further development of the ideas of those papers. These simple ideas consist in the application of the so-called indices (compare 3.1 of the present paper), and of a notion of equivalence of functions, more general than that of asymptotical equality. The purpose of the authors is to give a possibly simple and systematic exposition of the problems in question.

1. In this section we shall denote by h a real extended-valued function defined for $\mu \geqslant 0$. The function h is said to be *subadditive* in $\langle 0, \infty)$, if the inequality $h(\mu_1+\mu_2) \leqslant h(\mu_1)+h(\mu_2)$ holds for any non-negative μ_1, μ_2 unless the values $h(\mu_1)$, $h(\mu_2)$ are infinite and of opposite signs. Changing in the above inequality the sign $\leqslant$ into $\geqslant$ we obtain the definition of a *superadditive* function in $\langle 0, \infty)$.

1.1. *Suppose h is monotone and subadditive in $\langle 0, \infty)$, $h(0) = 0$. Under these assumptions the following formulae hold:*

$$(*) \qquad \lim_{\mu\to 0+} \frac{h(\mu)}{\mu} = \sup_{0<\mu<\mu^*} \frac{h(\mu)}{\mu} \quad \textit{for any} \quad 0 < \mu^* \leqslant \infty,$$

$$(**) \qquad \lim_{\mu\to \infty} \frac{h(\mu)}{\mu} = \inf_{\mu\geqslant\mu^*} \frac{h(\mu)}{\mu} \quad \textit{for any} \quad 0 \leqslant \mu^* < \infty.$$

As is well known [3], the formula (**) is valid for an arbitrary finite measurable h which is subadditive in $\langle 0,\infty)$, and so is the formula (*) under some additional assumption on h in the neighbourhood of 0. But we must take into consideration that $h(\mu)$ may be infinite and besides, assuming the monotony of h, the proof can be arranged in a slightly simplified form. Thus, for the sake of completeness, we shall give here the proofs of (*) and (**). Let us write

$$S = \sup_{0<\mu<\mu^*} h(\mu)/\mu.$$

To prove (*) let us first suppose that the function h is non-decreasing on $\langle 0,\infty)$. Then $h(\mu)$ is finite for each $\mu \geqslant 0$ if $h(\mu_0) < \infty$ for some $\mu_0 > 0$, as a direct consequence of the subadditivity of h. Thus we have to consider two cases:

(a) $h(\mu) = \infty$ for any $\mu > 0$,

(b) $0 \leqslant h(\mu) < \infty$ for any $\mu \geqslant 0$.

If (a) occurs, the equality (*) is obviously satisfied and $S = \infty$. If (b) takes place, we choose an arbitrary $\mu^* > 0$ and $\mu^* > \mu_0$. Given a μ, $0 < \mu \leqslant \mu_0$, let us denote by n a non-negative integer such that $\mu_0 = n\mu + \delta(\mu)$, $0 \leqslant \delta(\mu) < \mu$. It follows that

$$(+) \qquad \frac{h(\mu_0)}{\mu_0} \leqslant n\frac{\mu}{\mu_0}\frac{h(\mu)}{\mu} + \frac{h(\delta(\mu))}{\mu_0}.$$

Let us assume that $h(\mu) \to 0$ as $\mu \to 0+$. But $\mu \to 0+$ implies $n\mu/\mu_0 \to 1$ and consequently

$$(++) \qquad \frac{h(\mu_0)}{\mu_0} \leqslant \liminf_{\mu\to 0+} \frac{h(\mu)}{\mu}$$

and the inequality

$$S \leqslant \liminf_{\mu\to 0+}\frac{h(\mu)}{\mu} \leqslant \limsup_{\mu\to 0+}\frac{h(\mu)}{\mu} \leqslant S$$

follows both for finite and for infinite S. If the relation $h(\mu) \to 0$ as $\mu \to 0+$ does not hold, then $h(\mu) \to c$ as $\mu \to 0+$ with a positive c. But then $S = \infty$ and $h(\mu)/\mu \to \infty$ as $\mu \to 0+$, which means that (*) is satisfied. Let us now suppose that h is non-increasing on $\langle 0,\infty)$. Only two cases are possible:

(a) $h(\mu) = -\infty$ for any $\mu > 0$,

(b) $h(\mu) \leqslant 0$ for any $\mu \geqslant 0$ and $h(\mu) \to 0$ as $\mu \to 0+$.

In any case the limit

$$\lim_{\mu\to 0+} h(\mu) = c$$

exists, and $h(\mu) \leqslant 0$. Assuming $c < 0$ we can choose for any $\mu_0 > 0$ a sequence $\mu_n \to 0+$, $\mu_1+\mu_2+\ldots < \mu_0$, $h(\mu_n) \leqslant c$. But owing to the subadditivity of the h we get

$$h(\mu_0) \leqslant h(\mu_1+\mu_2+\ldots+\mu_n) \leqslant nc \quad \text{for} \quad n = 1, 2, \ldots$$

and consequently $h(\mu_0) = -\infty$. Assuming (a) we have $S = -\infty$ and $h(\mu)/\mu \to -\infty$ with μ tending to $0+$; so (*) is then satisfied. In case (b) one can apply the inequality (+), including the limiting case $h(\mu) = -\infty$. Since $h(\mu) \to 0$ as $\mu \to 0+$, the inequality (++) follows and consequently also (*), if $S = -\infty$ as well. To prove (**) let us write

$$s = \inf_{\mu>\mu^*} h(\mu)/\mu.$$

For a non-decreasing function h for which $h(\mu) = \infty$ for any $\mu > 0$, the formula (**) evidently holds. In the second possible case, when $0 \leqslant h(\mu) < \infty$ for any $\mu \geqslant 0$, we choose $\mu_0 > \mu^*$ and an arbitrary $\mu \geqslant \mu_0$. Write $\mu = n\mu_0 + \delta(\mu)$, where $0 \leqslant \delta(\mu) < \mu_0$, n is a non-negative integer, the subadditivity of h gives

$$({}^{+}_{+}{}^{+}) \qquad \frac{h(\mu)}{\mu} \leqslant \frac{n\mu_0}{\mu}\frac{h(\mu_0)}{\mu_0} + \frac{h(\delta(\mu))}{\mu}.$$

Because of $0 \leqslant h(\delta(\mu)) \leqslant h(\mu_0)$, $n\mu_0/\mu \to 1$ as $\mu \to \infty$, we obtain

$$s \leqslant \liminf_{\mu\to\infty}\frac{h(\mu)}{\mu} \leqslant \limsup_{\mu\to\infty}\frac{h(\mu)}{\mu} \leqslant \frac{h(\mu_0)}{\mu_0},$$

and consequently $h(\mu)/\mu \to s$ as $\mu \to \infty$, where $0 \leqslant s < \infty$. Let us suppose now that h is non-increasing in $\langle 0,\infty)$. If $h(\bar\mu) = -\infty$ for a certain $\bar\mu > 0$, then $h(\mu) = -\infty$ for $\mu \geqslant \bar\mu$, and evidently (**) holds. If $h(\mu)$ is finite for every $\mu \geqslant 0$, then $({}^{+}_{+}{}^{+})$ may be applied, and in virtue of

$$-\infty < h(\mu_0) \leqslant h(\delta(\mu)) \leqslant 0$$

this implies (**).

1.2. *Suppose h is monotone and superadditive in $\langle 0,\infty)$, $h(0) = 0$. Under these assumptions the following formulae hold:*

$$(*) \qquad \lim_{\mu\to 0+}\frac{h(\mu)}{\mu} = \inf_{0<\mu<\mu^*}\frac{h(\mu)}{\mu} \quad \textit{for any} \quad 0 < \mu^* \leqslant \infty,$$

$$(**) \qquad \lim_{\mu\to\infty}\frac{h(\mu)}{\mu} = \sup_{\mu>\mu^*}\frac{h(\mu)}{\mu} \quad \textit{for any} \quad 0 \leqslant \mu^* < \infty.$$

This immediately follows from 1.1, for if h is monotone and superadditive in $\langle 0,\infty)$, then $-h$ is monotone and subadditive in $\langle 0,\infty)$ and conversely.

2. Further we will always denote by ϱ a non-negative finite-valued measurable function defined on $(0, \infty)$. The following definitions and notation will be useful in our considerations. We shall say that ϱ_1 is: a) *l-equivalent*, b) *s-equivalent*, c) *a-equivalent* to ϱ_2 respectively if the inequalities

$$(*) \qquad a\varrho_1(k_1 u) \leqslant \varrho_2(u) \leqslant b\varrho_1(k_2 u),$$

hold for: a) $u \geqslant u_0$, b) $0 < u \leqslant u_0$, c) $u > 0$, respectively, where a, b, k_1, k_2 are some positive constants. We will denote l-equivalence, s-equivalence, a-equivalence of ϱ_1 to ϱ_2 by: $\varrho_1 \overset{l}{\sim} \varrho_2$, $\varrho_1 \overset{s}{\sim} \varrho_2$, $\varrho_1 \overset{a}{\sim} \varrho_2$ respectively.

The symbol $\varrho_1 \simeq \varrho_2$ will mean that ϱ_1 and ϱ_2 are *asymptotically equal* for $\mu \to \infty$ (for $u \to 0+$), i. e. that $\varrho_1(u) = h(u)\varrho_2(u)$, where $h(u) \neq 0$, $h(u) \to 1$ as $u \to \infty$ (as $u \to 0+$). Evidently $\varrho_1 \simeq \varrho_2$ for $u \to \infty$ (for $u \to 0+$) implies $\varrho_1 \overset{l}{\sim} \varrho_2$ ($\varrho_1 \overset{s}{\sim} \varrho_2$) but not conversely. Similarly to $\simeq$, $\overset{l}{\sim}$, $\overset{s}{\sim}$, $\overset{a}{\sim}$ have also the properties of the equivalence relation. A function ϱ is called *non-decreasing* (*non-increasing*) *for large u* (*for small u*) if there exists a positive u_0 such that ϱ is non-decreasing (non-increasing) in the interval $\langle u_0, \infty)$ (in the interval $(0, u_0\rangle)$. A function ϱ is called *pseudo-increasing for large u* (*for small u*) if $\varrho \overset{l}{\sim} \varrho_1$ (if $\varrho \overset{s}{\sim} \varrho_1$), where ϱ_1 is non-decreasing for large u (for small u). Let us remark that we can always assume ϱ_1 to be non-decreasing in the whole interval $(0, \infty)$. A function ϱ *pseudo-decreasing* for large u (for small u) is defined in a similar way.

2.1. *If for an $\varepsilon > 0$ the function $\varrho_1(u)/u^{\varepsilon}$ is non-decreasing for large u (for small u), then the inequalities* (*) *and the inequalities*

$$(**) \qquad \varrho_1(\bar{k}_1 u) \leqslant \varrho_2(u) \leqslant \varrho_1(\bar{k}_2 u), \quad \bar{k}_1, \bar{k}_2 > 0,$$

for $u \geqslant \bar{u}_0$ (for $0 < u \leqslant \bar{u}_0$) imply each other.

Indeed, if $0 < b \leqslant 1$ we can assume $\bar{k}_2 = k_2$; if $b > 1$, then because of $\varrho_1(au) \leqslant a^{\varepsilon}\varrho_1(u)$ for $a \geqslant 1$, $u \geqslant \bar{u}$, we get $\varrho_1(\bar{k}_2 u) \geqslant b\varrho_1(k_2 u)$ for large u (for small u), where $\bar{k}_2 = b^{1/\varepsilon}k_2$. Analogously we can define $\bar{k}_1 = k_1 a^{1/\varepsilon}$ if $0 < a \leqslant 1$, $\bar{k}_1 = k_1$ if $a \geqslant 1$.

The notion of l-equivalence, as defined by (**), has been systematically used by Krasnosel'skiĭ and Rutickiĭ [6] under the additional assumption that ϱ is convex, $\varrho(u)/u \to 0$ as $u \to 0+$, $\varrho(u)/u \to \infty$ as $u \to \infty$. But under these hypotheses $\varrho(u)/u$ is non-decreasing in $(0, \infty)$, whence the definition of l-equivalence in the sense of Kranosel'skiĭ and Rutickiĭ coincides with the more general definition of l-equivalence given above (due to W. Matuszewska [7]).

2.2. *Let us assume that the function $u^{\lambda}\varrho(u)$ is non-decreasing for large u (for small u) for a $\lambda > 0$.*

(a) *A function ϱ is pseudo-increasing for large u (for small u) if and only if a constant $0 < k \leqslant 1$ exists such that*

$$(+) \qquad \varrho(u_2) \geqslant k\varrho(u_1),$$

for $u_2 \geqslant u_1 \geqslant u^*$ (for $0 < u_1 \leqslant u_2 \leqslant u^*$).

(b) *A function ϱ is pseudo-decreasing for large u (for small u) if and only if a constant $k \geqslant 1$ exists such that*

$$(++) \qquad \varrho(u_2) \leqslant k\varrho(u_1),$$

for $u_2 \geqslant u_1 \geqslant u^*$ (for $0 < u_1 \leqslant u_2 \leqslant u^*$).

The proof of this theorem for large u can be found in [9]; its proof for small u follows the same lines.

2.3. (a) *If $\varrho \simeq \varrho_1$ for $u \to \infty$ (for $u \to 0+$) where ϱ_1 is non-decreasing for large u (for small u), then inequality 2.2(+) holds for any $0 < k < 1$ and $u \geqslant u^*(k)$ (if $0 < u \leqslant u^*(k)$); conversely, if this property is satisfied, then $\varrho \simeq \varrho_1$ for $u \to \infty$ (for $u \to 0+$), where ϱ_1 is non-decreasing for large u (for small u).*

(b) *If $\varrho \simeq \varrho_1$ for $u \to \infty$ (for $u \to 0+$) where ϱ_1 is non-increasing for large u (for small u), then inequality 2.2(++) holds for any $k > 1$ and $u \geqslant u^*(k)$ (if $0 < u \leqslant u^*(k)$); conversely, if this property is satisfied, then $\varrho \simeq \varrho_1$ for $u \to \infty$ (for $u \to 0+$) where ϱ_1 is non-increasing for large u (for small u).*

If in addition $u^{\lambda}\varrho(u)$ for a $\lambda > 0$ is non-decreasing for large u (for small u), one can define ϱ_1 in such a manner that $u^{\lambda}\varrho_1(u)$ is also non-decreasing for large u (for small u).

Ad (b). Let us consider the case "for large u". If $\varrho \simeq \varrho_1$, where ϱ_1 is non-increasing for large u for any $\varepsilon > 0$, the inequality

$$(1-\varepsilon)\varrho_1(u) \leqslant \varrho(u) \leqslant (1+\varepsilon)\varrho_1(u)$$

holds for large u, und hence

$$\varrho(u_2) \leqslant \frac{1+\varepsilon}{1-\varepsilon}\varrho(u_1)$$

if $u_2 \geqslant u_1 \geqslant u^*$, where u^* is sufficiently large. If condition 2.2(++) for any $k > 1$ and $u \geqslant u^*(k)$ is satisfied, then

$$\varrho(u) \leqslant \sup_{t \geqslant u} \varrho(t) \leqslant (1+\varepsilon)\varrho(u)$$

holds for large u. Assuming $\varrho_1(u_0) < \infty$ and defining

$$\varrho_1(u) = \begin{cases} \sup_{t \geqslant u} \varrho(t) & \text{if} \quad u \geqslant u_0, \\ \varrho_1(u_0) & \text{if} \quad 0 < u \leqslant u_0, \end{cases}$$

we get a non-increasing function which is asymptotically equal to ϱ for large u. The assumption that $u^\lambda\varrho(u)$, $\lambda>0$, is non-decreasing for $u\geqslant\bar{u}$ implies the same property for $u^\lambda\varrho_1(u)$ for large u. In fact, if $u\geqslant u_0$, $\bar{u}$ and $a>1$, then

$$u^\lambda\varrho(t)\leqslant t^\lambda\varrho(t)\leqslant(au)^\lambda\varrho(au)\leqslant(au)^\lambda\varrho_1(au)\quad\text{for}\quad u\leqslant t\leqslant au,$$
$$u^\lambda\varrho(t)\leqslant(au)^\lambda\varrho_1(au)\quad\text{for}\quad t\geqslant au,$$

and consequently

$$u^\lambda\varrho_1(u)\leqslant(au)^\lambda\varrho_1(au).$$

3. According to the terminology of [7] a function φ continuous and non-decreasing for $u\geqslant0$, vanishing only for $u=0$ and tending to infinity as $u\to\infty$ will be called a *φ-function*. We will always denote φ-functions by Greek letters $\varphi,\psi,\chi,\ldots$

Let us define for any φ-function the following extended-valued functions for positive a:

$$\underline{l}_\varphi(a)=\liminf_{u\to\infty}\varphi(au)/\varphi(u),\qquad \bar{l}_\varphi(a)=\limsup_{u\to\infty}\varphi(au)/\varphi(u),$$
$$\underline{l}_{0\varphi}(a)=\liminf_{u\to0+}\varphi(au)/\varphi(u),\qquad \bar{l}_{0\varphi}(a)=\limsup_{u\to0+}\varphi(au)/\varphi(u).$$

3.1. A. *For any φ-function there exist limits (indices)*:

$$s^1_\varphi=\lim_{a\to1+0}\frac{\lg\underline{l}_\varphi(a)}{\lg a}=\inf_{a>1}\frac{\lg\underline{l}_\varphi(a)}{\lg a},\tag{1}$$

$$s_\varphi=\lim_{1<a\to\infty}\frac{\lg\underline{l}_\varphi(a)}{\lg a}=\sup_{a>1}\frac{\lg\underline{l}_\varphi(a)}{\lg a},\tag{2}$$

$$\sigma^1_\varphi=\lim_{a\to1+0}\frac{\lg\bar{l}_\varphi(a)}{\lg a}=\sup_{a>1}\frac{\lg\bar{l}_\varphi(a)}{\lg a},\tag{3}$$

$$\sigma_\varphi=\lim_{1<a\to\infty}\frac{\lg\bar{l}_\varphi(a)}{\lg a}=\inf_{a>1}\frac{\lg\bar{l}_\varphi(a)}{\lg a}.\tag{4}$$

B. *For any φ-function there exist limits (indices) $s^1_{0\varphi}$, $s_{0\varphi}$, $\sigma^1_{0\varphi}$, $\sigma_{0\varphi}$, which we define as above, but replacing $\underline{l}_\varphi(a)$, $\bar{l}_\varphi(a)$ by $\underline{l}_{0\varphi}(a)$, $\bar{l}_{0\varphi}(a)$ respectively.*

As regards the meaning of the above formulae we shall keep the conventions $\lg 0=-\infty$, $\lg\infty=\infty$, and the same conventions are tacitly adopted in analogous situations.

The indices defined by A (2), (4) were first introduced in [11], and these defined by (1), (3) in [10]. We can get a uniform method for proving their existence by applying the following substitutions:

$$f(u)=\lg\varphi(e^u),\qquad e^u=v,\qquad e^\mu=a\quad\text{for}\quad 0<u<\infty,\ 1\leqslant a<\infty.$$

We make use of the remark that

$$\bar{h}(\mu)=\limsup_{u\to\infty}\big(f(u+\mu)-f(u)\big)=\lg\bar{l}_\varphi(a)$$

is non-decreasing and subadditive in $\langle0,\infty)$, and that $\bar{h}(0)=0$,

$$\underline{h}(\mu)=\liminf_{u\to\infty}\big(f(u+\mu)-f(u)\big)=\lg\underline{l}_\varphi(a)$$

is non-decreasing and superadditive in $\langle0,\infty)$, and we apply Lemmas 1.1 and 1.2. We proceed analogously, except for replacing $\bar{l}_\varphi(a)$ by $\bar{l}_{0\varphi}(a)$, $\underline{l}_\varphi(a)$ by $\underline{l}_{0\varphi}(a)$, to prove the existence of the limits mentioned in B.

3.2. The indices σ^1_φ, s^1_φ, resp. $\sigma^1_{0\varphi}$, $s^1_{0\varphi}$ (which may be infinite) are invariants of relation $\simeq$ for $u\to\infty$ (for $u\to0$) but not invariants of the l, s-equivalency; the indices σ_φ, s_φ, resp. $\sigma_{0\varphi}$, $s_{0\varphi}$ are invariants (including the limiting case when they are infinite) of relation $\overset{l}{\sim}$ resp. $\overset{s}{\sim}$ ([11], [10]).

3.3. By I_λ or D_λ respectively, $\lambda>0$, we will denote the class of φ-functions for which the quotient $\varphi(u)/u^\lambda$ is asymptotically equal for $u\to\infty$ to a non-decreasing or non-increasing function on $(0,\infty)$ respectively. I^0_λ or D^0_λ denotes an analogous class but for asymptotical equality for $u\to0+$. $\tilde{I}_\lambda$ or $\tilde{D}_\lambda$ will stand for the class of φ-functions for which the quotient $\varphi(u)/u^\lambda$, $\lambda>0$, is l-equivalent to a non-decreasing or non-increasing function in $(0,\infty)$ (i. e. is pseudo-increasing or pseudo-decreasing for large u) respectively. If in place of l-equivalence we take s-equivalence, we get the definition of class $\tilde{I}^0_\lambda$ or $\tilde{D}^0_\lambda$ respectively.

Let us notice that $\varphi\in I_\lambda$ resp. $\varphi\in D_\lambda$ implies $\varphi\simeq\varrho$, where ϱ is integrable in every interval $(0,b)$ and $\varrho(u)/u^\lambda$ is non-decreasing resp. non-increasing in the whole interval $(0,\infty)$. Of course, analogous remarks can be made with respect to other classes of φ-functions defined previously.

3.3.1. (a) *If $\sigma^1_\varphi<\lambda$, then $\varphi\in D_\lambda$; if $\varphi\in D_\lambda$, then $\sigma^1_\varphi\leqslant\lambda$.*

(b) *If $0<\lambda<s^1_\varphi$, then $\varphi\in I_\lambda$; if $\varphi\in I_\lambda$, then $\lambda\leqslant s^1_\varphi$.*

Analogous theorems are true if we replace σ^1_φ, s^1_φ, D_λ, I_λ by $\sigma^1_{0\varphi}$, $s^1_{0\varphi}$, D^0_λ, I^0_λ respectively.

For example we shall prove (a). Write $\varrho(u)=\varphi(u)/u^\lambda$ and remark that, in virtue of 2.3, $\varphi\in D_\lambda$ if and only if inequality 2.2 (++) for any $k>1$ and $u\geqslant u^*(k)$ is satisfied. Let $\sigma^1_\varphi<\lambda$. This means, in view of 3.1 (3), the inequality $\bar{l}_\varphi(a)<a^\lambda$ for $a>1$, whence

$$\varphi(au)\leqslant a^\lambda\varphi(u)\quad\text{if}\quad u\geqslant u_0(a).\tag{+}$$

For a given $\varepsilon>0$ let us choose a_0 such that $1<a_0<(1+\varepsilon)^{1/\lambda}$. Let $a\geqslant1$ and let for a non-negative integer n the inequalities $a_0^n\leqslant a<a_0^{n+1}$ hold. From (+) it follows that

$$\varphi(au)\leqslant\varphi(a_0^{n+1}u)\leqslant(a_0^{n+1})^\lambda\varphi(u)\leqslant a^\lambda a_0^\lambda\varphi(u)\leqslant a^\lambda(1+\varepsilon)\varphi(u)$$

for $u \geqslant u_0(a_0)$, which means

$$\frac{\varphi(u_2)}{u_2^{\lambda}} \leqslant (1+\varepsilon)\frac{\varphi(u_1)}{u_1^{\lambda}} \quad \text{if} \quad u_2 \geqslant u_1 \geqslant u_0(a_0),$$

and consequently ϱ is asymptotically equal for $u \to \infty$ to a non-increasing function in $(0, \infty)$. Let $\varphi \epsilon D_\lambda$. In virtue of 2.2(++) the inequalities

$$\varphi(au) \leqslant (1+\varepsilon)a^{\lambda}\varphi(u)$$

are satisfied for every $\varepsilon > 0$ and $a \geqslant 1$ if $u \geqslant u^*(\varepsilon)$. But this means $\bar{l}_\varphi(a) \leqslant a^{\lambda}$, $\sigma_\varphi^1 \leqslant \lambda$.

3.3.2. (a) *If $\sigma_\varphi < \lambda$, then $\varphi \epsilon \tilde{D}_\lambda$; if $\varphi \epsilon \tilde{D}_\lambda$, then $\sigma_\varphi \leqslant \lambda$.*

(b) *If $0 < \lambda < s_\varphi$, then $\varphi \epsilon \tilde{I}_\lambda$; if $\varphi \epsilon \tilde{I}_\lambda$, then $\lambda \leqslant s_\varphi$.*

Analogous theorems are true if we replace σ_φ, s_φ, $\tilde{D}_\lambda$, $\tilde{I}_\lambda$, by $\sigma_{0\varphi}$, $s_{0\varphi}$, $\tilde{D}_\lambda^0$, $\tilde{I}_\lambda^0$ respectively.

For example we shall prove (a). In view of 2.2, $\varphi \epsilon \tilde{D}_\lambda$ if and only if inequality 2.2(+) is satisfied for $\varrho(u) = \varphi(u)/u^{\lambda}$ for a certain constant $k \geqslant 1$ and for $u_2 \geqslant u_1 \geqslant u^*$. Assuming $\sigma_\varphi < \lambda$ we have, by 3.1 (4), $\bar{l}_\varphi(\tilde{a}) \leqslant \tilde{a}^{\lambda}$ for an $\tilde{a} > 1$, whence $\varphi(\tilde{a}u) \leqslant \tilde{a}^{\lambda}\varphi(u)$ for $u \geqslant u_0$, $\varphi(au) \leqslant a^{\lambda}\tilde{a}^{\lambda}\varphi(u)$, $a \geqslant \tilde{a}$. But it is easily seen that for $1 \leqslant a \leqslant \tilde{a}$ the same inequality is satisfied for $u \geqslant u_0$; thus inequality 2.2(+) holds with a constant $k = \tilde{a}^{\lambda}$ and for $u \geqslant u_0$. Conversely, if $\lambda \epsilon \tilde{D}_\lambda$ then, owing to 2.2(++), $\varphi(au) \leqslant ka^{\lambda}\varphi(u)$ for $a \geqslant 1$, $u \geqslant u^*$, and it follows that $\bar{l}_\varphi(a) \leqslant ka^{\lambda}$, $\sigma_\varphi \leqslant \lambda$.

3.3.3. *If φ is a strictly increasing φ-function, then the inclusions $\varphi \epsilon D_\lambda$ ($\varphi \epsilon I_\lambda$) and $\varphi^{-1} \epsilon I_{1/\lambda}$ ($\varphi^{-1} \epsilon D_{1/\lambda}$) are equivalent.*

An analogous theorem is valid for the pairs of classes D_λ^0, $I_{1/\lambda}^0$ or I_λ^0, $D_{1/\lambda}^0$ respectively, just as for classes $\tilde{D}_\lambda$, $\tilde{I}_{1/\lambda}$, ...

Evidently the following inequalities imply each other when $v_2 = \varphi(u_2)$, $v_1 = \varphi(u_1)$:

$$\frac{\varphi(u_2)}{u_2^{\lambda}} \leqslant k\frac{\varphi(u_1)}{u_1^{\lambda}} \quad \text{for} \quad u_2 \geqslant u_1 \geqslant u^*(k),$$

$$\frac{\varphi^{-1}(v_1)}{v_1^{1/\lambda}}\frac{1}{k^{1/\lambda}} \leqslant \frac{\varphi^{-1}(v_2)}{v_2^{1/\lambda}} \quad \text{for} \quad v_2 \geqslant v_1 \geqslant \varphi(u^*(k)),$$

and thus by 2.3 the inclusion $\varphi \epsilon D_\lambda$ implies $\varphi^{-1} \epsilon I_\lambda$ and conversely. We proceed analogously with the other classes.

3.3.4. *If φ is a strictly increasing φ-function and $s_\varphi^1 > 0$, then $s_\varphi^1 = 1/\sigma_{\varphi-1}^1$. If $s_\varphi^1 = 0$, then $\sigma_{\varphi-1}^1 = \infty$ and conversely.*

Analogous relations are true also for indices $s_{0\varphi}^1$, $\sigma_{0\varphi}^1$, ... (see [6] for the case of indices s_φ, σ_φ).

If $s_\varphi^1 > 0$, $s_\varphi^1 > \lambda > 0$, then, in virtue of 3.3.1(b) and 3.3.3, $\varphi^{-1} \epsilon D_{1/\lambda}$, $\sigma_{\varphi-1}^1 \leqslant 1/\lambda$ and consequently $1/s_\varphi^1 \geqslant \sigma_{\varphi-1}^1$. If $\lambda > \sigma_{\varphi-1}^1$, then $\varphi \epsilon I_{1/\lambda}$, $s_\varphi^1 \geqslant 1/\lambda$ and $s_\varphi^1 \geqslant 1/\sigma_{\varphi-1}^1$. We have proved $s_\varphi^1 = 1/\sigma_{\varphi-1}^1$ under the assumption $s_\varphi^1 > 0$, but proceeding as before we can find this true also if $\sigma_{\varphi-1}^1 < \infty$, whence equations $s_\varphi^1 = 0$ and $\sigma_{\varphi-1}^1 = \infty$ are equivalent.

3.4. *If ϱ is a non-decreasing function in $\langle a, b\rangle$, where $a > 0$, and $\varrho(u)/u^{\lambda}$ is non-increasing in $\langle a, b\rangle$, then ϱ fulfils the condition of Lipschitz in the interval $\langle a, b\rangle$.*

(An analogous but slightly less general statement can be found in [2]).

Let us assume $u_1, u_2 \epsilon \langle a, b\rangle$, $u_2 \geqslant u_1$, $u_2 = au_1$. Because of $\varrho(au_1) \leqslant a^{\lambda}\varrho(u_1)$ we have

$$0 \leqslant \varrho(u_2) - \varrho(u_1) \leqslant (a^{\lambda}-1)\varrho(u_1).$$

Since

$$a^{\lambda}-1 \leqslant \begin{cases} \lambda a^{\lambda-1}(a-1) & \text{for} \quad \lambda \geqslant 1, \\ \lambda(a-1) & \text{for} \quad 0 < \lambda \leqslant 1, \end{cases}$$

$$a-1 \leqslant (u_2-u_1)/a, \quad 1 \leqslant a \leqslant b/a,$$

we get

$$\varrho(u_2) - \varrho(u_1) \leqslant \begin{cases} \lambda b^{\lambda-1}a^{-\lambda}\varrho(b)(u_2-u_1) & \text{if} \quad \lambda \geqslant 1, \\ \lambda a^{-1}\varrho(b)(u_2-u_1) & \text{if} \quad 0 < \lambda \leqslant 1. \end{cases}$$

Remark. The lemma evidently fails when we assume either ϱ to be non-increasing or the quotient $\varrho(u)/u^{\lambda}$ to be non-decreasing on $\langle a, b\rangle$.

3.4.1. *Let ϱ be positive and non-decreasing on (a, b) ($a = 0$, $b = \infty$ are not excluded). A necessary and sufficient condition for the quotient $\varrho(u)/u^{\lambda}$ to be non-increasing on (a, b) is the absolute continuity of ϱ in any interval $\langle c', c''\rangle$, where $a < c' < c'' < b$, and the fulfilment almost everywhere in (a, b) of the inequality*

$$(+) \qquad \frac{u\varrho'(u)}{\varrho(u)} \leqslant \lambda.$$

This follows immediately from 3.4 and the equation

$$\left(\varrho(u)u^{-\lambda}\right)' = u^{\lambda-1}\left(\varrho'(u)u - \lambda\varrho(u)\right).$$

3.4.2. *Suppose that ϱ is positive and absolutely continuous in any interval $\langle c', c''\rangle$, where $a < c' < c'' < b$. A necessary and sufficient condition for the quotient $\varrho(u)/u^{\lambda}$ to be non-decreasing on (a, b) ($0 \leqslant a < b \leqslant \infty$) is the fulfilment almost everywhere in (a, b) of the inequality*

$$(+) \qquad \lambda \leqslant \frac{u\varrho'(u)}{\varrho(u)}.$$

3.4.3. *Suppose that ϱ is integrable in any interval $(0, b)$, and that the quotient $\varrho(u)/u^{\lambda}$ is non-decreasing (non-increasing) on $(0, \infty)$. If*

$$\varrho_1(u) = \int_0^u \varrho(t)\,dt,$$

then $\varrho_1(u)/u^{\lambda+1}$ is non-decreasing (non-increasing) on $(0, \infty)$, and the inequality $u\varrho(u)/\varrho_1(u) \geqslant \lambda+1$ $(\leqslant \lambda+1)$ holds for every $u > 0$.

Let us assume, for example, that $\varrho(u)/u^{\lambda}$ is non-increasing in $(0, \infty)$. From

$$\varrho_1(u) = \int_0^u \varrho(t)t^{-\lambda}t^{\lambda}dt \geqslant \varrho(u)u^{-\lambda}\int_0^u t^{\lambda}dt = u\varrho(u)/\lambda+1$$

we obtain $u\varrho_1'(u)/\varrho_1(u) \leqslant \lambda+1$ and it suffices to apply 3.4.1. We proceed analogously if $\varrho(u)/u^{\lambda}$ is non-decreasing in $(0, \infty)$.

3.5. *For a φ-function φ let us write*

$$\psi(u) = \int_0^u \varphi(t)\,dt.$$

(a) *If $\varphi \in D_{\lambda}$, then $\psi \in D_{\lambda+1}$; if $\varphi \in I_{\lambda}$, then $\psi \in I_{\lambda+1}$.*

(b) *If $\varphi \in D_{\lambda}^0$, then $\psi \in D_{\lambda+1}^0$; if $\varphi \in I_{\lambda}^0$, then $\psi \in I_{\lambda+1}^0$.*

Analogous theorems are true for classes $\tilde{D}_{\lambda}$, $\tilde{I}_{\lambda}$, $\tilde{D}_{\lambda}^0$, $\tilde{I}_{\lambda}^0$. If $\varphi \in D_{\lambda}$ $(\varphi \in D_{\lambda}^0)$, then $\varphi \simeq \varrho$ for $u \to \infty$; in addition

$$\varrho_1(u) = \int_0^u \varrho(t)\,dt < \infty$$

in any interval $(0, b)$ and $\varrho(u)/u^{\lambda}$ is non-increasing in $(0, \infty)$. By 3.4.3, $\varrho_1(u)/u^{\lambda+1}$ is also non-increasing. On the other hand, $\psi \simeq \varrho_1$ for $u \to \infty$ (for $u \to 0+$), since ψ is a φ-function, and consequently $\psi \in D_{\lambda+1}$. An analogous reasoning can be applied to the other classes, where use is made of the remark that from $\varphi \overset{l}{\sim} \varrho$ $(\varphi \overset{s}{\sim} \varrho)$ follows $\psi \overset{l}{\sim} \varrho_1$ $(\psi \overset{s}{\sim} \varrho_1)$.

4. In this section we shall write for a φ-function φ,

$$\psi(u) = \int_0^u \varphi(t)\,dt,$$

$$h_{\varphi}(u) = \frac{u\varphi(u)}{\psi(u)} \quad \text{for} \quad u > 0.$$

4.1. *The following inequalities hold:*

(a) $1+s_{\varphi}^1 \leqslant \liminf_{u\to\infty} h_{\varphi}(u) \leqslant s_{\psi}^1 \leqslant s_{\psi} \leqslant \sigma_{\psi} \leqslant \sigma_{\psi}^1 \leqslant \limsup_{u\to\infty} h_{\varphi}(u) \leqslant 1+\sigma_{\varphi}^1.$

(b) $1+s_{0\varphi}^1 \leqslant \liminf_{u\to 0+} h_{\varphi}(u) \leqslant s_{0\psi}^1 \leqslant s_{0\psi} \leqslant \sigma_{0\psi} \leqslant \sigma_{0\psi}^1 \leqslant \limsup_{u\to 0+} h_{\varphi}(u) \leqslant 1+\sigma_{0\varphi}^1.$

The intermediate inequalities between the indices in (a) follow immediately from 3.1. To prove the right-hand inequality in (a) let us assume

$$\limsup_{u\to\infty} h_{\varphi}(u) < \lambda.$$

It follows by 3.4.1 that $\psi \in D_{\lambda}$, whence

$$\sigma_{\psi}^1 \leqslant \lambda, \qquad \sigma_{\psi}^1 \leqslant \limsup_{u\to\infty} h_{\varphi}(u).$$

Let $\sigma_{\varphi}^1 < \infty$, $\sigma_{\varphi}^1 < \lambda$. In virtue of 3.3.1, $\varphi \in D_{\lambda}$, which means that $\varphi \simeq \varrho$ for $u \to \infty$, where $\varrho(u)/u^{\lambda}$ is non-increasing for $u > 0$, and

$$\varrho_1(u) = \int_0^u \varrho(t)\,dt$$

is finite for any $u > 0$. In view of 3.4.3 we get

$$h(u) = \frac{u\varrho(u)}{\varrho_1(u)} \leqslant \lambda+1 \quad \text{for} \quad u > 0.$$

But $\varphi \simeq \varrho$, $\psi \simeq \varrho_1$ for $u \to \infty$ implies $h(u) \simeq h_{\varphi}(u)$ for $u \to \infty$; therefore

$$\limsup_{u\to\infty} h_{\varphi}(u) = \limsup_{u\to\infty} h(u) \leqslant \lambda+1$$

and

$$\limsup_{u\to\infty} h_{\varphi}(u) \leqslant \sigma_{\varphi}^1+1.$$

The left-hand inequality in (a) and the inequalities in (b) can be proved by similar arguments.

4.2. A function ϱ, positive for $u > 0$, is said to be *regularly increasing* for $u \to \infty$ (for $u \to 0+$), according to the terminology of [4], [5], if $\varrho(\lambda u)/\varrho(u) \to g(\lambda)$ as $u \to \infty$ (as $u \to 0+$), $g(\lambda) < \infty$, for all $\lambda > 0$ (recently in [1] the term *function of regular asymptotic behaviour* has been adopted for such a function). If $g(\lambda) = 1$ for any $\lambda > 0$, the function ϱ is called *slowly varying* for $u \to \infty$ (for $u \to 0+$). In the sequel it will be assumed that $\varrho = \varphi$ is a φ-function; such an assumption is quite sufficient for many applications. It is easily seen that $g(\lambda)$ is multiplicative in $(0, \infty)$; therefore $g(\lambda) = \lambda^{r_{\varphi}}$. The exponent r_{φ} is called the *index of regularity*; $r_{\varphi} = 0$, if and only if φ is a slowly varying function. It follows directly from 3.1 that φ is regularly increasing for $u \to \infty$ (for $u \to 0+$) and has an index r_{φ} if and only if $s_{\varphi}^1 = \sigma_{\varphi}^1 = r_{\varphi} < \infty$, and is slowly varying for $u \to \infty$ (for $u \to 0+$) if and only if $s_{\varphi}^1 = \sigma_{\varphi}^1 = 0$.

4.2.1. *A necessary and sufficient condition for a φ-function φ to be regularly increasing for $u\to\infty$ (for $u\to 0+$) and have an index r_φ is that: $\varphi\in D_\lambda$ ($\varphi\in D_\lambda^0$) for all $\lambda>r_\varphi$, $\varphi\in I_\lambda$ ($\varphi\in I_\lambda^0$) for all $\lambda<r_\varphi$, $\lambda>0$* (cf. [9], [1]).

A φ-function φ possesses both properties in question if and only if $\lambda\leqslant s_\varphi^1\leqslant\sigma_\varphi^1\leqslant\lambda$; this follows from 3.3.1. Our assertion follows immediately from these inequalities.

4.2.2. (a) *In order that φ be regularly increasing for $u\to\infty$ and have an index r_φ it is necessary and sufficient that*

$$\lim_{u\to\infty} h_\varphi(u)=1+r_\varphi. \tag{+}$$

(b) *In order that φ be regularly increasing for $u\to 0+$ and have an index $r_{0\varphi}$ it is necessary and sufficient that*

$$\lim_{u\to 0+} h_\varphi(u)=1+r_{0\varphi}. \tag{++}$$

If $h_\varphi(u)\to 1+r_\varphi$ for $u\to\infty$, then, by 4.1 (a), $s_\psi^1=\sigma_\psi^1=1+r_\varphi$. This means that ψ is regularly increasing and has an index $1+r_\varphi$, and since $\varphi(u)=\psi(u)u^{-1}h_\varphi(u)$ it follows that φ is also regularly increasing and has an index r_φ. Conversely, if φ is regularly increasing and has an index r_φ, then $s_\varphi^1=\sigma_\varphi^1=r_\varphi$ and, by 4.1 (a), $h_\varphi(u)\to 1+r_\varphi$ for $u\to\infty$. One can prove (b) in a similar way.

4.3. *Every φ-function slowly varying for $u\to\infty$ (for $u\to 0+$) can be represented in the canonical form of Karamata:*

$$\varphi(u)=c(u)\exp\int_{u_0}^{u}\varepsilon(t)t^{-1}dt. \tag{$*$}$$

Here $c(u)$ is a positive and continuous function, $c(u)\to c$, $c>0$, as $u\to\infty$ (as $u\to 0+$), $\varepsilon(u)$ is a non-negative and continuous function, $\varepsilon(u)\to 0$ as $u\to\infty$ (as $u\to 0+$). Conversely, under the hypotheses on $\varepsilon(u)$, $c(u)$ given above, the function defined by the formula $(*)$ is slowly varying for $u\to\infty$ (for $u\to 0+$) (but, of course, not necessarily a φ-function, cf. [4], [5]).

In fact, we have

$$\frac{u\varphi(u)}{\psi(u)}=1+\varepsilon(u),\quad u\varphi(u)\geqslant\psi(u)\quad \text{for}\quad u>0; \tag{+}$$

therefore $\varepsilon(u)$ is non-negative, continuous and tending to 0 as $u\to\infty$ (as $u\to 0+$). Integration on both sides of (+) gives $(*)$ with $c(u)=\psi(u_0)\big(1+\varepsilon(u)\big)$.

4.3.1. *Every φ-function for which $\sigma_\varphi^1<\infty$ (for which $s_\varphi^1>0$) has a representation of the form 4.3 $(*)$, where $c(u)$ is a positive continuous function for which*

$$\limsup_{u\to\infty} c(u)\leqslant k(1+\sigma)\qquad \big(\liminf_{u\to\infty} c(u)\geqslant k(1+s)\big)$$

with a $k>0$, $\sigma\geqslant s>0$, and $\varepsilon(u)$ is a non-negative continuous function,

$$\limsup_{u\to\infty}\varepsilon(u)\leqslant\sigma\qquad \big(\liminf_{u\to\infty}\varepsilon(u)\geqslant s\big).$$

Here one may assume $\sigma=\sigma_\varphi^1$, $s=s_\varphi^1$.

For the indices $\sigma_{0\varphi}^1$, $s_{0\varphi}^1$ an analogous theorem is valid; of course, lim sup, lim inf is to be taken for $u\to 0+$ instead of $u\to\infty$.

From 4.3 (+) we obtain representation 4.3 $(*)$, where

$$c(u)=\psi(u_0)\big(1+\varepsilon(u)\big),$$

and by 4.1 (a) we have

$$\limsup_{u\to\infty}\varepsilon(u)\leqslant\sigma_\varphi^1\quad\text{ or }\quad\liminf_{u\to\infty}\varepsilon(u)\geqslant s_\varphi^1$$

respectively.

4.3.2. Following the notation used in [10] let us denote by K_c the class of φ-functions for which $\bar{l}_\varphi(a)\to 1$ as $a\to 1+0$ and $a\to 1-0$, and by K_c^* a subclass of K_c of all φ-functions for which $\underline{l}_\varphi(a)>1$ for every $a>1$.

4.3.3. *If for a φ-function φ the conditions $s_\varphi^1>0$, $\sigma_\varphi^1<\infty$ are satisfied, then $\varphi\in K_c^*$ and is representable in the form 4.3 $(*)$, where the functions $c(u)$, $\varepsilon(u)$ satisfy all the conditions mentioned in 4.3.1 and in addition the following condition:*

$$\lim_{a\to 1+0}\left(\liminf_{u\to\infty}\frac{c(au)}{c(u)}\right)=\lim_{a\to 1+0}\left(\limsup_{u\to\infty}\frac{c(au)}{c(u)}\right)=1. \tag{$**$}$$

Conversely, if $c(u)$, $\varepsilon(u)$ satisfy the conditions listed in theorem 4.3.1 and $(**)$, then a φ-function of the form 4.3 $(*)$ belongs to K_c. If in addition

$$\liminf_{u\to\infty}\frac{c(au)}{c(u)}\geqslant 1\quad \textit{for}\quad a\geqslant 1,$$

then $\varphi\in K_c^*$, $s_\varphi^1>0$.

Write

$$g(u)=\exp\int_{u_0}^{u}\varepsilon(t)t^{-1}dt.$$

If $\varepsilon(u)$ satisfies the conditions mentioned in 4.3.1, the following inequalities hold for large u and arbitrary ε, $0<\varepsilon<s$,

$$g(u)a^{s-\varepsilon}\leqslant g(au)\leqslant a^{\sigma+\varepsilon}g(u)\quad\text{for}\quad a\geqslant 1. \tag{+}$$

If in addition $c(u)$ fulfills the assumption from 4.3.1 and φ has the representation 4.3 (*), then from (+) and (**) the inequalities

$$\liminf_{u\to\infty}\frac{c(au)}{c(u)}a^{s-\varepsilon}\leqslant \underline{l}_\varphi(a)\leqslant \bar{l}_\varphi(a)\leqslant a^{\sigma+\varepsilon}\limsup_{u\to\infty}\frac{c(au)}{c(u)}$$

follow. This implies $\bar{l}_\varphi(a)\to 1$ as $a\to 1$, $\underline{l}_\varphi(a)>1$ if

$$\liminf_{u\to\infty} c(au)/c(u)\geqslant 0.$$

If $0<s_\varphi^1<\sigma_\varphi^1<\infty$, then φ can be represented in form 4.3 (*) and $\varepsilon(u)$, $c(u)$ satisfy the conditions of 4.3.1 with $s=s_\varphi^1$, $\sigma=\sigma_\varphi^1$. Whence inequalities (+), where $s=s_\varphi^1$, $\sigma=\sigma_\varphi^1$, for large u, hold. Since the inequalities

$$\varphi(u)a^{s-\varepsilon}\leqslant\varphi(au)\leqslant a^{\sigma+\varepsilon}\varphi(u)$$

are also satisfied for any $a\geqslant 1$ and for sufficiently large u, we get

$$a^{-(\sigma_\varphi^1-s_\varphi^1-2\varepsilon)}\leqslant\frac{c(au)}{c(u)}\leqslant a^{\sigma_\varphi^1-s_\varphi^1+2\varepsilon}\quad\text{for}\quad u\geqslant\bar{u}(a)$$

and (**) follows.

References

[1] R. Bojanic and J. Karamata, *On slowly varying functions and asymptotic relations*, Math. Research Center, U. S. Army, Madison, Wis., Tech. Summary Rep. No. 432, 1963.

[2] J. M. Chen, *On two-functional spaces*, Studia Math. 24 (1964), p. 61-88.

[3] E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Providence 1957.

[4] J. Karamata, *Sur un mode de croissance régulière des fonctions*, Mathematica, Cluj, 4 (1930), p. 38-53.

[5] — *Sur un mode de croissance régulière*, Bull. Soc. Math. France 61 (1933), p. 55-62.

[6] M. A. Krasnosel'skiĭ and Ya. B. Rutickiĭ, *Convex functions and Orlicz spaces*, Groningen 1961.

[7] W. Matuszewska, *On generalized Orlicz spaces*, Bull. Acad. Pol. Sci., Sér. sci. math., astr. et phys., 8 (1960), p. 349-353.

[8] — *Some further properties of φ-functions*, ibidem 9 (1961), p. 445-450.

[9] — *Regularly increasing functions in connection with the theory of $L^{*\varphi}$-spaces*, Studia Math. 21 (1962), p. 317-344.

[10] — *On a generalization of regularly increasing functions*, ibidem 24 (1964), p. 271-279.

[11] — and W. Orlicz, *On certain properties of φ-functions*, Bull. Acad. Pol. Sci., Sér. sci. math., astr. et phys., 8 (1960), p. 439-443.

[12] S. Yamamuro, *Exponents of modulared semi-ordered linear spaces*, J. Fac. Sci. Hokkaido Univ. 12 (1953), p. 211-253.

Reçu par la Rédaction le 28. 11. 1964

STUDIA MATHEMATICA, T. XXVI. (1965)

On Bochner-Riesz summability almost everywhere of multiple Fourier series

by

CHAO-PING CHANG (Hong Kong)

I. Introduction

§ 1. The purpose of this paper is to prove the k-dimensional ($k\geqslant 2$) version of the following theorem in Fourier series of one variable due to J. Marcinkiewicz [2]. The author wishes to thank Professor Antoni Zygmund for suggesting the problem and for many useful consultations with him in the preparation of this work.

THEOREM A. *Suppose $f(x)\epsilon L[-\pi,\pi]$, f is periodic with period 2π. If f satisfies, at every point x in a set E of positive measure, the condition*

$$(1.1)\qquad |f(x+h)-f(x)|=O\left(1/\log\frac{1}{|h|}\right)\quad \textit{as}\quad h\to 0,$$

or even merely

$$(1.2)\qquad \frac{1}{h}\int_0^h|f(x+t)-f(x)|\,dt=O\left(1/\log\frac{1}{|h|}\right)\quad \textit{as}\quad h\to 0,$$

then the Fourier series $S[f]$ of f converges almost everywhere in E.

It is obvious that at an individual point x condition (1.1) implies (1.2), so that it is enough to prove Theorem A under the weaker assumption (1.2). It may be remarked that condition (1.1) at an individual point x does not imply convergence of the Fourier series $S[f]$ at x. Zygmund [6], p. 303, has pointed out that even the stronger condition

$$(1.3)\qquad |f(x+h)-f(x)|=o\left(1/\log\frac{1}{|h|}\right)\quad \text{as}\quad h\to 0$$

does not always imply convergence of the Fourier series $S[f]$ at x. Thus Theorem A is primarily a theorem of almost everywhere convergence of Fourier series on a set E.

We now introduce notation and definitions in connection with multiple Fourier series. E_k will denote the k-dimensional Euclidean space.