If in addition $c(u)$ fulfills the assumption from 4.3.1 and φ has the representation 4.3 (*), then from (+) and (**) the inequalities

$$\liminf_{u\to\infty}\frac{c(au)}{c(u)}a^{s-\varepsilon}\leqslant \underline{l}_\varphi(a)\leqslant \bar{l}_\varphi(a)\leqslant a^{\sigma+\varepsilon}\limsup_{u\to\infty}\frac{c(au)}{c(u)}$$

follow. This implies $\bar{l}_\varphi(a)\to 1$ as $a\to 1$, $\underline{l}_\varphi(a)>1$ if

$$\liminf_{u\to\infty} c(au)/c(u)\geqslant 0.$$

If $0<s^1_\varphi<\sigma^1_\varphi<\infty$, then φ can be represented in form 4.3 (*) and $\varepsilon(u)$, $c(u)$ satisfy the conditions of 4.3.1 with $s=s^1_\varphi$, $\sigma=\sigma^1_\varphi$. Whence inequalities (+), where $s=s^1_\varphi$, $\sigma=\sigma^1_\varphi$, for large u, hold. Since the inequalities

$$\varphi(u)a^{s-\varepsilon}\leqslant\varphi(au)\leqslant a^{\sigma+\varepsilon}\varphi(u)$$

are also satisfied for any $a\geqslant 1$ and for sufficiently large u, we get

$$a^{-(\sigma^1_\varphi-s^1_\varphi-2\varepsilon)}\leqslant\frac{c(au)}{c(u)}\leqslant a^{\sigma^1_\varphi-s^1_\varphi+2\varepsilon}\quad\text{for}\quad u\geqslant\bar{u}(a)$$

and (**) follows.

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On Bochner-Riesz summability almost everywhere of multiple Fourier series

by

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I. Introduction

§ 1. The purpose of this paper is to prove the k-dimensional ($k\geqslant 2$) version of the following theorem in Fourier series of one variable due to J. Marcinkiewicz [2]. The author wishes to thank Professor Antoni Zygmund for suggesting the problem and for many useful consultations with him in the preparation of this work.

THEOREM A. *Suppose $f(x)\in L[-\pi,\pi]$, f is periodic with period 2π. If f satisfies, at every point x in a set E of positive measure, the condition*

$$(1.1)\qquad |f(x+h)-f(x)|=O\left(1/\log\frac{1}{|h|}\right)\quad as\quad h\to 0,$$

or even merely

$$(1.2)\qquad \frac{1}{h}\int_0^h|f(x+t)-f(x)|\,dt=O\left(1/\log\frac{1}{|h|}\right)\quad as\quad h\to 0,$$

then the Fourier series $S[f]$ of f converges almost everywhere in E.

It is obvious that at an individual point x condition (1.1) implies (1.2), so that it is enough to prove Theorem A under the weaker assumption (1.2). It may be remarked that condition (1.1) at an individual point x does not imply convergence of the Fourier series $S[f]$ at x. Zygmund [6], p. 303, has pointed out that even the stronger condition

$$(1.3)\qquad |f(x+h)-f(x)|=o\left(1/\log\frac{1}{|h|}\right)\quad \text{as}\quad h\to 0$$

does not always imply convergence of the Fourier series $S[f]$ at x. Thus Theorem A is primarily a theorem of almost everywhere convergence of Fourier series on a set E.

We now introduce notation and definitions in connection with multiple Fourier series. E_k will denote the k-dimensional Euclidean space.

A single letter such as $x, y, u, t, \ldots$ will usually denote a point in E_k. A point $x = (x_1, \ldots, x_k)$ in E_k is called a *lattice point* if its coordinates $x_1, \ldots, x_k$ are integers. The letter n will always denote a lattice point in E_k unless otherwise stated. For any two points $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k)$ in E_k we define a scalar product $x\cdot y = x_1y_1+\ldots+x_ky_k$. The usual Euclidean norm in E_k is then given by $|x| = (x\cdot x)^{1/2} = (x_1^2+ +\ldots+x_k^2)^{1/2}$; $dx = dx_1dx_2\ldots dx_k$ will denote the usual k-dimensional Lebesgue measure in E_k while $d\sigma$ will denote the $(k-1)$-dimensional Lebesgue measure in E_k. For the k-dimensional Lebesgue measure of a set S in E_k we shall use the notation $|S|$. The unit sphere $|x| = 1$ in E_k will be denoted by Σ and $\sigma_k = \int_{\Sigma} d\sigma$ is the surface area of Σ. It is wellknown that $\sigma_k = 2\pi^{k/2}/\Gamma(k/2)$.

The distance between two sets P and Q in E_k will be denoted by $\operatorname{dist}(P, Q)$. ∂S and S^0 denote respectively the boundary and interior of a set S in E_k. The diameter of a set S in E_k, in symbol $\operatorname{diam}(S)$, is defined by $\operatorname{Sup}\{|x-y| \mid x\in S, y\in S\}$.

The term *function* throughout this paper is understood to be complex-valued function unless the contrary is stated. A function $f(x) = f(x_1, \ldots, x_k)$ is said to be *periodic* if f is periodic with period 2π in each of its independent variables $x_1, \ldots, x_k$. Q_k will denote the fundamental cube in E_k consisting of all points $x = (x_1, \ldots, x_k)$ satisfying the inequalities $-\pi \leqslant x_j \leqslant \pi$ $(j = 1, 2, \ldots, k)$. If $f(x)\in L(Q)$, f is periodic, we define the numbers

$$a_n = (2\pi)^{-k} \int_{Q_k} f(x)e^{-in\cdot x}dx, \quad n \text{ being lattice points}, \tag{1.4}$$

and call them the *Fourier coefficients* of $f(x)$. The formal series

$$\sum a_n e^{in\cdot x} \tag{1.5}$$

is called the *multiple Fourier series* of $f(x)$ and will be denoted by $S[f] = S[f(x)]$. Following the notation of the one variable case this relationship is indicated by $f(x) \sim \Sigma a_n e^{in\cdot x}$.

For any $R > 0$ and any complex δ we define

$$S_R^\delta(x) = S_R^\delta(x, f) = \sum_{|x|<R} (1-|n|^2/R^2)^\delta a_n e^{in\cdot x}$$

and call it the *Bochner-Riesz means of order* δ of the Fourier series (1.5). We say that (1.5) is *Bochner-Riesz summable of order* δ, in symbol summable (B-R, δ), to a finite (complex) number s if $\operatorname{Lim}_{R\to\infty} S_R^\delta(x, f) = s$.

Bochner-Riesz summability for multiple Fourier series of k variables is a generalization of Cesàro summability for Fourier series of one variable. In fact, when $k = 1$ summability (B-R, δ) is equivalent to the classical

Cesàro summability (C, δ) of order δ. Bochner-Riesz summability of the special order $\delta = (k-1)/2$ is particularly important and throughout this paper we shall consider only summability (B-R, $(k-1)/2$) for multiple Fourier series of k variables. This particular method of summability is, in fact, the analog of the ordinary convergence of Fourier series of one variable because it has been found that some theorems on the convergence of Fourier series of one variable have their analogs valid for multiple Fourier series when convergence is replaced by summability (B-R, $(k-1)/2$) in multiple Fourier series of k variables. In view of the frequent occurrence of the quantity $(k-1)/2$ in this paper we shall let $\alpha = (k-1)/2$.

$L\log^+L(Q_k)$ will denote the class of all measurable functions $f(x)$ defined on Q_k such that

$$\int_{Q_k} |f(x)|\log^+|f(x)|\,dx < \infty.$$

Similarly, $L(\log^+L)^2(Q_k)$ will denote the class such that

$$\int_{Q_k} |f(x)|\{\log^+|f(x)|\}^2dx < \infty.$$

Because of the frequent occurrence of the function $1/\log(1/t)$, $0 \leqslant t < 1$, throughout this paper we shall denote it by $l(t)$, i. e.

$$l(0) = 0, \quad l(t) = 1/\log\frac{1}{t} \quad \text{if} \quad 0 < t < 1. \tag{1.6}$$

We do not define $l(t)$ for $t > 1$ although $1/\log(1/t)$ is meaningful for $t > 1$. In the notation of $l(t)$ the right-hand side of (1.1) may be denoted by $O(l(|h|))$.

It is clear that $l(t)$ is continuous, concave for $0 \leqslant t \leqslant 1/e^2$, convex for $1/e^2 \leqslant t < 1$ and has a point of inflection at $x = 1/e^2$.

We now give below the statement of the main theorem to be proved in this paper.

THEOREM 1. *Suppose* $f(x) = f(x_1, \ldots, x_k)\in L\log^+L(Q_k)$, $k \geqslant 2$, f *is periodic. If f satisfies, at every point x in a set E of positive measure in* E_k, *the condition*

$$|f(x+h)-f(x)| = O(l(|h|)) \quad \text{as} \quad |h| \to 0, \tag{1.7}$$

or even merely

$$\frac{1}{h^k}\int_{|t|\leqslant h} |f(x+t)-f(x)|\,dt = O(l(h)) \quad \text{as} \quad h \to 0+, \tag{1.8}$$

then the Fourier series $S[f(x)]$ of f is summable (B-R, α) *at almost every point x in the set E.*

II. Preliminary theorems and lemmas

§ 2. A review of the proof of Theorem A (see for example [7], p. 170-172) reveals that the following six theorems (Theorems B, C, D, E, F, G below) on one-dimensional Fourier series have been used.

THEOREM B (Dini's test for convergence). *Suppose $f(x)\in L[-\pi,\pi]$, f is periodic. For any point x_0 and for any real $r>0$ we define*

$$f(x_0; r) = \tfrac{1}{2}\{f(x_0+r)+f(x_0-r)\}.$$

Then the condition

$$\int_0^\delta \frac{|f(x_0; r)-f(x_0)|}{r}\,dr < \infty \text{ for some } \delta>0 \quad (f(x_0) \text{ being finite valued})$$

implies that the Fourier series $S[f]$ of f converges to $f(x_0)$ at $x = x_0$.

THEOREM C. *Suppose $f(x)\in L[-\pi,\pi]$, f is periodic, $f(x_0)$ is finite valued. Then the condition*

$$\int_{|t|\leqslant\delta} \frac{|f(x_0+t)-f(x_0)|}{|t|}\,dt < \infty \quad \text{for some} \quad \delta>0$$

implies that the Fourier series $S[f]$ of f converges to $f(x_0)$ at $x = x_0$.

THEOREM D. *Suppose $f(x)\in L^2[-\pi,\pi]$, f is periodic, $f(x)\sim\sum c_n e^{inx}$. If the Fourier coefficients c_n $(n = 0, \pm1, \pm2, \ldots)$ satisfy the condition $\sum_n{}' |c_n|^2\log|n| < \infty$ (the $\sum_n{}'$ sign here means that the summation extends over all integers n excluding 0) then the Fourier series $S[f(x)]$ of f converges at almost every point x.*

THEOREM E. *Suppose $f(x)\in L[-\pi,\pi]$, f is periodic, $f(x)\sim\sum c_n e^{inx}$. Define*

$$g(x) = \int_{-\pi}^{\pi} \frac{|f(x+t)-f(x-t)|^2}{|t|}\,dt = 2\int_0^\pi \frac{|f(x+t)-f(x)|^2}{t}\,dt, \quad -\infty<x<\infty$$

(hence $g(x)$ is periodic). Then

$$\int_{-\pi}^{\pi} g(x)\,dx < \infty \text{ if and only if } \sum_n{}' |c_n|^2\log|n| < \infty.$$

THEOREM F. *Suppose $f(x)\in L[-\pi,\pi]$, f is periodic, $f(x)\sim\sum c_n e^{inx}$. Let $g(x)$ be defined as in the previous Theorem E. Then the condition*

$$\int_{-\pi}^{\pi} g(x)\,dx < \infty$$

implies that the Fourier series $S[f(x)]$ of f converges at almost every point x.

THEOREM G. *If $f(x)$, $-\infty<x<\infty$, is periodic, continuous everywhere, and there exist two constants $A>0$, $a>\frac{1}{2}$ such that*

$$|f(x+h)-f(x)| \leqslant l^a(|h|)$$

holds for all x in $(-\infty,\infty)$ and for all sufficiently small h, say $|h|<\delta$ where $\delta>0$ does not depend on the point x, then the Fourier series $S[f(x)]$ of f converges at almost every point x in $(-\infty,\infty)$.

Theorem B is the well-known Dini's test for convergence, from which Theorem C follows easily as a corollary. Theorems D, E, F are due to Plessner [3] and Theorem G follows easily from Theorem F. For a handy reference of Theorems D, E, F, G see Zygmund [7], p. 163-164.

The question arises naturally whether or not Theorems B, C, D, E, F, G have their analogs for multiple Fourier series of k variables $(k\geqslant 2)$. This question can be answered affirmatively. We shall establish the following Theorems 2, 3, 4, 5, 6, 7 which are the k-dimensional analogs of Theorems B, C, D, E, F, G above respectively.

Let $f(x)$, $x\in E_k$, be locally integrable, i. e. integrable over every finite sphere in E_k. For any point $x\in E_k$ and any $r>0$ we define

$$f(x_0; r) = \frac{1}{\sigma_k r^{k-1}} \int_{|x-x_0|=r} f(x)\,d\sigma.$$

Here $\sigma_k = 2\pi^{k/2}/\Gamma(k/2)$ so that $\sigma_k r^{k-1}$ is the $(k-1)$-dimensional measure of $|x-x_0| = r$ in E_k. Thus $f(x_0; r)$ is the mean value of f taken over the sphere $|x-x_0| = r$. When $k = 1$, $f(x_0; r)$ reduces to $\frac{1}{2}\{f(x_0+r)+$ $+f(x_0-r)\}$.

THEOREM 2 (Dini's test for summability). *Suppose $f(x)\in L\log^+L(Q_k)$, $k\geqslant 2$, f is periodic. If $f(x_0)$ is finite valued at a point $x_0\in E_k$ and the following Dini's condition*

$$(2.1)\qquad \int_0^\varepsilon \frac{|f(x_0; r)-f(x_0)|}{r}\,dr < \infty \quad \text{for some } \varepsilon>0$$

is satisfied, then the Fourier series $S[f]$ is summable $(\mathrm{B}\text{-}\mathrm{R}, a)$ to $f(x_0)$ at $x = x_0$.

THEOREM 3. *Suppose $f(x)\in L\log^+L(Q_k)$, $k\geqslant 2$, f is periodic, $f(x_0)$ is finite valued. Then the condition*

$$(2.2)\qquad \int_{|t|\leqslant\varepsilon} \frac{|f(x_0+t)-f(x_0)|}{|t|^k}\,dt < \infty \quad \text{for some } \varepsilon>0$$

implies that the Fourier series $S[f]$ is summable $(\mathrm{B}\text{-}\mathrm{R}, a)$ to $f(x_0)$ at $x = x_0$.

THEOREM 4. *The Fourier series of a periodic function $f(x)\in L(\log^+L)^2(Q_k)$, $k\geqslant 2$, is summable $(\mathrm{B}\text{-}\mathrm{R}, a)$ at almost every point x in E_k.*

THEOREM 5. *Suppose* $f(x)\in L(Q_k)$, $k\geqslant 2$, f *is periodic*, $f(x)\sim\sum c_n e^{in\cdot x}$. *Define*

$$g(x)=\int\limits_{Q_k}\frac{|f(x+t)-f(x-t)|^2}{|t|^2}\,dt,\qquad x\in E_k$$

(*hence* $g(x)$ *is periodic*). *Then*

(2.3) $$\int\limits_{Q_k} g(x)\,dx<\infty \text{ if and only if } \sum_n{}' |c_n|^2\log|n|<\infty$$

(*the* $\sum\limits_n{}'$ *sign here means that the summation extends over all lattice points* n *in* E_k *excluding* $(0,\ldots,0)$).

THEOREM 6. *Let* $f(x)$ *and* $g(x)$ *be as in the last theorem. Then the condition*

$$\int\limits_{Q_k} g(x)\,dx<\infty$$

implies that the Fourier series of f *is summable* (B-R, α) *at almost every point* x *in* E_k.

THEOREM 7. *If* $f(x)$, $x\in E_k$, $k\geqslant 2$, *is periodic, continuous everywhere and there exist two constants* $A>0$, $\alpha>\frac{1}{2}$ *such that*

(2.4) $$|f(x+h)-f(x)|\leqslant Al^{\alpha}(|h|)$$

holds for all $x\in E_k$ *and for all sufficiently small* h, *say* $|h|\leqslant\delta$ *where* $\delta>0$ *is independent of* x, *then the Fourier series* $S[f(x)]$ *of* f *is summable* (B-R, α) *at almost every point* x *in* E_k.

Theorems 2 and 4 are due to Stein (see [5], p. 107, and [4], p. 140, respectively). A comparison of Theorem 4 with Theorem D shows that the assumption $\sum\limits_n{}'|c_n|^2\log|n|<\infty$ in Theorem D about f can be considerably relaxed to $f\in L(\log^+L)^2(Q_k)$ in k-dimensional Fourier series ($k\geqslant 2$). For the remaining Theorems 3, 5, 6, 7 we shall give proofs in the following sections.

§ 3. It is enough to show that the assumption (2.2) of Theorem 3 implies the Dini's condition (2.1) of Theorem 2. Using the polar coordinates formula

$$\int\limits_{|t|\leqslant\varepsilon} g(t)\,dt=\int\limits_0^{\varepsilon}\Big\{\int\limits_{|t|=r} g(t)\,d\sigma\Big\}\,dr$$

by setting $g(t)=|f(x_0+t)-f(x_0)|/|t|^k$ we have

(3.1) $$\int\limits_{|t|\leqslant\varepsilon}\frac{|f(x_0+t)-f(x_0)|}{|t|^k}\,dt=\int\limits_0^{\varepsilon} r^{-k}\Big\{\int\limits_{|t|=r}|f(x_0+t)-f(x_0)|\,d\sigma\Big\}\,dr.$$

Taking absolute value on the identity

$$f(x_0;r)-f(x_0)=\frac{1}{\sigma_k r^{k-1}}\int\limits_{|t|=r}\big(f(x_0+t)-f(x_0)\big)\,d\sigma$$

and then dividing throughout by r we have

$$\frac{|f(x_0;r)-f(x_0)|}{r}\leqslant\sigma_k^{-1}r^{-k}\int\limits_{|t|=r}|f(x_0+t)-f(x_0)|\,d\sigma.$$

Integrating the above inequality with respect to r from 0 to ε and using (3.1) we get

$$\int\limits_0^{\varepsilon}\frac{|f(x_0;r)-f(x_0)|}{r}\,dr\leqslant\sigma_k^{-1}\int\limits_0^{\varepsilon}r^{-k}\Big\{\int\limits_{|t|=r}|f(x_0+t)-f(x_0)|\,d\sigma\Big\}\,dr$$
$$=\sigma_k^{-}\int\limits_{|t|\leqslant\varepsilon}\frac{|f(x_0+t)-f(x_0)|}{|t|^k}\,dt$$

which shows that (2.2) implies the Dini's condition (2.1).

§ 4. An important step in the proof of Theorem E (see [7], p. 163) lies in the asymptotic formula

$$\int\limits_{-\pi}^{\pi}\frac{\sin^2 nt}{|t|}\,dt=2\int\limits_0^{\pi}\frac{\sin^2 nt}{t}\,dt=\log n+O(1)\simeq\log n,$$

as $n\to\infty$, $n=1,2,3,\ldots$

Actually what is really needed in the proof is the inequality

(4.1) $$\gamma\log n\leqslant\int\limits_{-\pi}^{\pi}\frac{\sin^2 nt}{|t|}\,dt\leqslant\beta\log n,\qquad \text{for } n=2,3,4,\ldots,$$

where β, γ are constants. In the proof of Theorem 5 in § 9 of this paper we shall need the k-dimensional version of (4.1) which reads: there exist two constants A_k, B_k depending on k only such that

(4.2) $$A_k\log|n|\leqslant\int\limits_{Q_k}\frac{\sin^2(n\cdot t)}{|t|^k}\,dt\leqslant B_k\log|n|$$

for all lattice points n in E_k with $|n|^2\geqslant 2$ (i. e. $|n|>1$). For simplicity we write

(4.3) $$I(n)=\int\limits_{Q_k}\frac{\sin^2(n\cdot t)}{|t|^k}\,dt=\int\limits_{Q_k}\frac{\sin^2(n_1t_1+\ldots+n_kt_k)}{(t_1^2+\ldots+t_k^2)^{k/2}}\,dt_1\ldots dt_k$$

so that (4.2) can be written as $A_k\log|n|\leqslant I(n)\leqslant B_k\log|n|$.

For any $R>0$ we let Δ_R denote the triangular set in E_k defined by

$$\Delta_R = \{(x_1, \ldots, x_k) \,|\, x_i \geqslant 0 \text{ for } i = 1, 2, \ldots, k;\ x_1+\ldots+x_k \leqslant R\}.$$

Let P_k denote the restriction of Q_k to the first quadrant in E_k, i. e.

$$P_k = \{(x_1, \ldots, x_k) \,|\, 0 \leqslant x_i \leqslant \pi \text{ for } i = 1, 2, \ldots, k\}.$$

(4.2) will be established via the following lemmas.

LEMMA 1. *There exists a constant B_k depending on k such that*

$$I(n) \leqslant B_k \log|n|$$

for all lattice points n in E_k with $|n|^2 \geqslant 2$.

LEMMA 2. *Suppose $f(u)$ is a complex-valued function defined for all $u \geqslant 0$ and continuous there, $q_i > 0$ for $i = 1, 2, \ldots, k$. If f satisfies the two conditions*

(i) $\dfrac{f(u)}{u} \in L[0, h]$ *for some $h>0$ (hence for every $h>0$),*

(ii) $\dfrac{f(x_1+\ldots+x_k)}{(q_1x_1+\ldots+q_kx_k)^k}$ *is integrable over Δ_h for some $h>0$ (hence for every $h>0$),*

then, for every $R>0$, the following formula holds:

$$\int\limits_{\Delta_R} \frac{f(x_1+\ldots+x_k)}{(q_1x_1+\ldots+q_kx_k)^k}\,dx = \frac{1}{q_1\ldots q_k\Gamma(k)} \int\limits_0^1 \frac{f(Ru)}{u}\,du. \tag{4.4}$$

LEMMA 3. *There exists an absolute constant C (not even depending on k) such that the inequality*

$$\int\limits_{Q_k} \frac{\sin^2(n_1t_1+\ldots+n_kt_k)}{(t_1+\ldots+t_k)^k}\,dt \geqslant \frac{C}{\Gamma(k)} \log(n_1+\ldots+n_k) \tag{4.5}$$

holds for all lattice points $n = (n_1, \ldots, n_k)$ with $n_i > 0$ for all $i = 1, 2, \ldots, k$.

LEMMA 4. *There exists a constant A_k depending on k such that*

$$I(n) = \int\limits_{Q_k} \frac{\sin^2(n\cdot t)}{|t|^k}\,dt \geqslant A_k \log|n| \tag{4.6}$$

for all lattice points $n \neq (0, \ldots, 0)$ in E_k.

Lemmas 4 and 1 are the first and second halves of the inequality (4.2) respectively. We now prove Lemmas 1, 2, 3, 4 in the sections that follow.

§ 5. Proof of Lemma 1. Since Q_k is contained in the spheere $|t| \leqslant k^{1/2}\pi$ and the integrand in (4.6) is non-negative we have

$$I(n) = \int\limits_{Q_k} \frac{\sin^2(n\cdot t)}{|t|^k}\,dt \leqslant \int\limits_{|t|\leqslant k^{1/2}\pi} \frac{\sin^2(n\cdot t)}{|t|^k}\,dt$$

$$= \int\limits_{|t|\leqslant 1/|n|} + \int\limits_{1/|n|\leqslant t\leqslant k^{1/2}\pi} = \mathrm{I}_1 + \mathrm{I}_2, \quad \text{say}.$$

Now $\sin(n\cdot t)$ satisfies the following estimates:

$$|\sin(n\cdot t)| \leqslant |(n\cdot t)| \leqslant |n|\,|t|, \qquad |\sin(n\cdot t)| \leqslant 1.$$

Using the first and second estimates for I_1 and I_2 respectively we have

$$\mathrm{I}_1 = \int\limits_{|t|\leqslant 1/|n|} |n|^2|t|^{2-k}dt = |n|^2\sigma_k \int\limits_0^{1/|n|} r\,dr = |n|^2\sigma_k \frac{1}{2|n|^2} = \frac{\sigma_k}{2},$$

$$\mathrm{I}_2 \leqslant \int\limits_{1/|n|\leqslant|t|\leqslant k^{1/2}\pi} \frac{1}{|t|^k}\,dt = \sigma_k \int\limits_{1/|n|}^{k^{1/2}\pi} \frac{dr}{r} = \sigma_k(\log k^{1/2}\pi + \log|n|).$$

Hence

$$I(n) = \mathrm{I}_1 + \mathrm{I}_2 \leqslant \sigma_k(\tfrac{1}{2} + \log k^{1/2}\pi) + \sigma_k \log|n| \leqslant B_k \log|n|$$

for all lattice points n in E_k with $|n| \geqslant 2^{1/2}$ and for some constant B_k depending on k only.

§ 6. Proof of Lemma 2. We quote a formula from Fichtengolz [1], p. 481, which asserts the following:

If $f(u)$ is a complex-valued function defined and continuous in $0 \leqslant u \leqslant 1$ and if $p_i > 0$, $q_i \geqslant 0$ $(i = 1, 2, \ldots, k)$, $r > 0$, then

$$\int\limits_{\Delta_1} f(x_1+\ldots+x_k) \frac{x_1^{p_1-1}\ldots x_k^{p_k-1}}{(q_1x_1+\ldots+q_kx_k)^{p_1+\ldots+p_k}}\,dx$$

$$= \frac{\Gamma(p_1)\ldots\Gamma(p_k)}{\Gamma(p_1+\ldots+p_k)^k} \int\limits_0^1 \frac{f(u)\,u^{p_1+\ldots+p_k-1}}{(q_1u+r)^{p_1}\ldots(q_ku+r)^{p_k}}\,du.$$

Setting $p_i = 1$ $(i = 1, 2, \ldots, k)$ in this formula we obtain

$$\int\limits_{\Delta_1} \frac{f(x_1+\ldots+x_k)}{(q_1x_1+\ldots+q_kx_k)}\,dx = \frac{1}{\Gamma(k)} \int\limits_0^1 \frac{f(u)\,u^{k-1}du}{(q_1u+r)\ldots(q_ku+r)}. \tag{6.1}$$

From (6.1) we can derive a more general formula: If $f(u)$ is a complex-valued function defined and continuous in $0 \leqslant u \leqslant R$ $(R>0)$ and if

$q_i \geqslant 0$ $(i = 1, 2, \ldots, k)$, $r > 0$, then

$$(6.2)\qquad \int\limits_{\Delta_R} \frac{f(x_1+\ldots+x_k)}{(q_1x_1+\ldots+q_kx_k+Rr)^k}\,dx = \frac{1}{\Gamma(k)}\int\limits_0^1 \frac{f(Ru)u^{k-1}du}{(q_1u+r)\ldots(q_ku+r)}.$$

(6.2) is obtained by making the change of variables $y_1 = Rx_1, \ldots, y_k = Rx_k$ to the integral on the left-hand side of (6.1) and by setting $g(u) = f(u/R)$. Thus we obtain

$$\int\limits_{\Delta_R} \frac{g(y_1+\ldots+y_k)}{(q_1y_1+\ldots+q_ky_k+Rr)^k}\,dy = \frac{1}{\Gamma(k)}\int\limits_0^1 \frac{g(Ru)u^{k-1}du}{(q_1u+r)\ldots(q_ku+r)}$$

which is (6.2) in another set of notations, i. e. y_i instead of x_i and g instead of f.

For $m = 1, 2, 3, \ldots$ we define two sequences of functions $\{F_m(x)\}$, $\{f_m(u)\}$ by

$$F_m(x) = F_m(x_1, \ldots, x_k)$$

$$= f(x_1+\ldots+x_k)\Big/\left(q_1x_1+\ldots+q_kx_k+\frac{R}{m}\right)^k \quad \text{for} \quad x_1 \geqslant 0, \ldots, x_k \geqslant 0,$$

$$f_m(u) = f(Ru)u^{k-1}\Big/\left(q_1u+\frac{1}{m}\right)\ldots\left(q_ku+\frac{1}{m}\right) \quad \text{for} \quad u \geqslant 0.$$

Setting $r = 1/m$ $(m = 1, 2, 3, \ldots)$ in (6.2) we obtain

$$(6.3)\qquad \int\limits_{\Delta_R} F_m(x)\,dx = \frac{1}{\Gamma(k)}\int\limits_0^1 f_m(u)\,du, \qquad m = 1, 2, 3, \ldots$$

Now for each $m = 1, 2, 3, \ldots$ we have

$$|F_m(x)| = \frac{|f(x_1+\ldots+x_k)|}{\left(q_1x_1+\ldots+q_kx_k+\dfrac{R}{m}\right)^k}$$

$$\leqslant \frac{|f(x_1+\ldots+x_k)|}{(q_1x_1+\ldots+q_kx_k)^k} \quad \text{for all} \quad x \in \Delta_R,\ x \neq (0, \ldots, 0),$$

$$|f_m(u)| = \frac{|f(Ru)|u^{k-1}}{(q_1u+1/m)\ldots(q_ku+1/m)} \leqslant \frac{1}{q_1\ldots q_k}\,\frac{|f(Ru)|}{u} \quad \text{for} \quad 0 < u \leqslant 1.$$

Thus each member of the sequence $\{F_m(x)\}$ is dominated by the function

$$\frac{|f(x_1+\ldots+x_k)|}{(q_1x_1+\ldots+q_kx_k)^k}$$

integrable over Δ_R and each member of the sequence $\{f_m(u)\}$ is dominated by the function $|f(Ru)|/u$ integrable over $0 < u \leqslant 1$. Passing to the limit $m \to \infty$ on both sides of (6.3) and using the Lebesgue dominated convergence theorem we obtain the desired result (4.4).

We remark that assumption (ii) of Lemma 2 is actually redundant because it can be proved that assumption (i) implies assumption (ii) but this point is of no importance to our future application of Lemma 2.

§ 7. Proof of Lemma 3. It is enough to prove (4.5) with the domain of integration P_k replaced by Δ_π since $\Delta_\pi \subset P_k$ and the integrand in (4.5) is non-negative for all points $t = (t_1, \ldots, t_k)$ in P_k. Hence we seek to prove

$$(7.1)\qquad J(n) = \int\limits_{\Delta_\pi} \frac{\sin^2(n_1t_1+\ldots+n_kt_k)}{(t_1+\ldots+t_k)^k}\,dt \geqslant \frac{C}{\Gamma(k)}\log(n_1+\ldots+n_k)$$

for all lattice points $n = (n_1, \ldots, n_k)$ with $n_k > 0$ $(i = 1, 2, \ldots, k)$.

Applying the change of variables $x_1 = n_1t_1, \ldots, x_k = n_kt_k$ to the above integral in (7.1) we obtain

$$J(n) = \frac{1}{n_1\ldots n_k}\int\limits_{\Delta_{(n_1+\ldots+n_k)\pi}} \frac{\sin^2(x_1+\ldots+x_k)}{(x_1/n_1+\ldots+x_k/n_k)^k}\,dx.$$

Our next step is to apply Lemma 2 with $f(u) = \sin^2 u$. We now claim that the function $f(u) = \sin^2 u$ satisfies conditions (i), (ii) of Lemma 2. For (i) this is nearly trivial. To see (ii) we let $q = \operatorname{Min}(q_1, \ldots, q_k)$ and note that for all points $x = (x_1, \ldots, x_k) \in \Delta_R$ we have

$$\left|\frac{f(x_1+\ldots+x_k)}{(q_1x_1+\ldots+qx_k)^k}\right| = \frac{|\sin(x_1+\ldots+x_k)|^2}{(q_1x_1+\ldots+q_kx_k)^k} \leqslant \frac{(x_1+\ldots+x_k)^2}{\{q(x_1+\ldots+x_k)\}^k}$$

$$= \frac{1}{q^k}\,\frac{1}{(x_1+\ldots+x_k)^{k-2}} \leqslant \frac{1}{q^k|x|^{k-2}}$$

which proves (ii) since the function $1/|x|^{k-2}$ is integrable in any neighborhood of the origin in E_k.

An application of Lemma 2 with

$$f(u) = \sin^2 u, \qquad R = (n_1+\ldots+n_k)\pi, \qquad q_i = \frac{1}{n_i} \quad (i = 1, 2, \ldots, k)$$

yields

$$(7.2)\qquad J(n) = \frac{1}{n_1\ldots n_k(1/n_1)\ldots(1/n_k)\Gamma(k)}\int\limits_0^1 \frac{\sin^2(Ru)}{u}\,du$$

$$= \frac{1}{\Gamma(k)}\int\limits_0^\pi \frac{\sin^2(n_1+\ldots+n_k)v\,dv}{v}.$$

It is well known that

$$\int_0^{\pi} \frac{\sin^2 mv}{v}\,dv \simeq \frac{1}{2}\log m \quad \text{as} \quad m \to \infty \text{ through } 1, 2, 3, \ldots$$

Hence there exists an absolute constant C such that

$$(7.3) \qquad \int_0^{\pi} \frac{\sin^2 mv}{v}\,dv \geqslant C \log m \quad \text{for all} \quad m = 1, 2, 3, \ldots$$

(7.1) now follows from (7.2) and (7.3) with $m = n_1 + \ldots + n_k$. This completes the proof of (4.5).

§ 8. Proof of Lemma 4. We first make the following three remarks:

1. In proving (4.6) it is enough to prove it for only those lattice points $n = (n_1, \ldots, n_k)$ with $n_1 \geqslant 0, \ldots, n_k \geqslant 0$.

2. In proving (4.6) it is enough to prove it with the domain of integration Q_k here replaced by P_k (recall that P_k is the restriction of Q_k to the first quadrant).

3. In proving (4.6) it is enough to prove it with $|t|$ in the denominator replaced by $\|t\|$ and $|n|$ on the right-hand side of (4.6) replaced by $\|n\|$. Here $\|\cdot\|$ is another norm defined by

$$\|x\| = |x_1| + \ldots + |x_k|, \qquad x = (x_1, \ldots, x_k).$$

Remark 1 is justified by the identity

$$I(n_1, \ldots, n_k) = I(|n_1|, \ldots, |n_k|).$$

To see this identity we define a "modified signum function" $s(r)$, r being a real number, by setting

$$s(r) = \begin{cases} -1 & \text{if} \quad r < 0, \\ \ \ 1 & \text{if} \quad r > 0, \end{cases} \qquad s(0) = 1.$$

With this definition it is clear that $|r| = rs(r)$ for all real r and that $s(r)$ is either 1 or -1. Applying the change of variables

$$t_1 = u_1 s(n_1), \qquad t_2 = u_2 s(n_2), \qquad \ldots, \qquad t_k = u_k s(n_k)$$

to the integral defining $I(n) = I(n_1, \ldots, n_k)$ (see the right-hand side of (4.3)) we have

$$I(n) = I(n_1, \ldots, n_k) = \int_{Q_k} \frac{\sin^2(n_1 t_1 + \ldots + n_k t_k)}{(t_1^2 + \ldots + t_k^2)^{k/2}}\,dt_1 \ldots dt_k$$

$$= \int_{Q_k} \frac{\sin^2(|n_1| u_1 + \ldots + |n_k| u_k)}{(u_1^2 + \ldots + u_k^2)^{k/2}}\,du_1 \ldots du_k = I(|n_1|, \ldots, |n_k|)$$

since $s^2(n_j) = 1$, $n_j t_j = n_j u_j s(n_j) = |n_j| u_j$ for all $j = 1, 2, \ldots, k$.

Remark 2 above is justified by the inclusion relation $P_k \subset Q_k$ and the non-negativeness of the integrand in (4.3).

Remark 3 above is justified as follows. Clearly $\|t\| \geqslant |t|$, $\|n\| \geqslant |n|$ from the definition of the norm $\|\cdot\|$. Suppose we have succeeded in proving (4.6) with $|t|$ replaced by $\|t\|$, $|n|$ replaced by $\|n\|$, i. e.

$$\int_{Q_k} \frac{\sin^2(n \cdot t)}{\|t\|^k}\,dt \geqslant A_k \log \|n\| \quad \text{for} \quad n \neq (0, \ldots, 0).$$

Since $\sin^2(n \cdot t)$ is always non-negative, $\|t\| \geqslant |t|$, $\|n\| \geqslant |n|$ and the logarithm is an increasing function, we have

$$\int_{Q_k} \frac{\sin^2(n \cdot t)}{|t|^k}\,dt \geqslant \int_{Q_k} \frac{\sin^2(n \cdot t)}{\|t\|^k}\,dt, \qquad \log \|n\| \geqslant \log |n|.$$

The preceding three inequalities together imply (4.6).

We now apply the remarks 1, 2, 3 simultaneously with the result that in order to prove (4.6) it is enough to prove

$$\int_{P_k} \frac{\sin^2(n \cdot t)}{\|t\|^k}\,dt \geqslant A_k \log \|n\|$$

for all lattice points $n = (n_1, \ldots, n_k)$ with $n_i \geqslant 0$ $(i = 1, 2, \ldots, k)$ and $n \neq (0, \ldots, 0)$. The above inequality in full notation can be written as

$$(8.1) \qquad \int_{P_k} \frac{\sin^2(n_1 t_1 + \ldots + n_k t_k)}{(t_1 + \ldots + t_k)^k}\,dt_1 \ldots dt_k \geqslant A_k \log(n_1 + \ldots + n_k)$$

where n_i are integers with $n_i \geqslant 0$ for all $i = 1, 2, \ldots, k$ and $n_1 + \ldots + n_k \geqslant 1$ since at least one of $n_1, \ldots, n_k$ is an integer $\geqslant 1$.

We now prove (8.1) by induction on k, i. e. we shall derive (8.1) from

$$(8.2) \qquad \int_{P_{k-1}} \frac{\sin^2(n_1 t_1 + \ldots + n_{k-1} t_{k-1})}{(t_1 + \ldots + t_{k-1})^{k-1}}\,dt_1 \ldots dt_{k-1} \geqslant A_{k-1} \log(n_1 + \ldots + n_{k-1}),$$

where $n_i \geqslant 0$ are integers for all $i = 1, 2, \ldots, k-1$ with $n_1 + \ldots + n_{k-1} \geqslant 1$.

Clearly (8.1) is true for $k = 1$ because of (7.3) so that in proving the implication (8.2) $\Rightarrow$ (8.1) we may assume $k \geqslant 2$. We distinguish between the following two cases:

Case (i). Each of $n_1, \ldots, n_k$ is > 0.

In this case we can derive (8.1) even without using (8.2). It is already in Lemma 3.

Case (ii). At least one of $n_1, \ldots, n_k$ is >0.

Since the value of the integral in (8.1) is symmetric with respect to $n_1, \ldots, n_k$ we may, without loss of generality, assume that $n_k = 0$. Formula (8.1) now becomes

$$(8.3)\qquad I(n_1, \ldots, n_{k-1}, 0) = \int\limits_{P_k} \frac{\sin^2(n_1 t_1 + \ldots + n_{k-1} t_{k-1})}{(t_1 + \ldots + t_k)^k} dt_1 \ldots dt_k \geqslant \log(n_1 + \ldots + n_{k-1}),$$

where $n_i \geqslant 0$ are integers for all $i = 1, 2, \ldots, k-1$ with $n_1 + \ldots + n_{k-1} \geqslant 1$. Here the condition $n_1 + \ldots + n_{k-1} \geqslant 1$ comes from $n_1 + \ldots + n_k \geqslant 1$ and $n_k = 0$.

In the integral in (8.3) we integrate first with respect to t_k, then with respect to $t_1, \ldots, t_{k-1}$ and thus have

$$(8.4)\qquad I(n_1, \ldots, n_{k-1}, 0) = \int\limits_{P_{k-1}} \left\{ \int\limits_0^\pi \frac{dt_k}{(t_1 + \ldots + t_k)^k} \right\} \sin^2(n_1 t_1 + \ldots + n_{k-1} t_{k-1})\, dt_1 \ldots dt_{k-1}.$$

The inner integral above can be evaluated as follows:

$$(8.5)\qquad \int\limits_0^\pi \frac{dt_k}{(t_1 + \ldots + t_k)^k} = -\frac{1}{(k-1)(t_1 + \ldots + t_k)^{k-1}} \Bigg|_{t_k=0}^{t_k=\pi} = \frac{(s+\pi)^{k-1} - s^{k-1}}{(k-1) s^{k-1} (s+\pi)^{k-1}}$$

where we have written s for $t_1 + \ldots + t_{k-1}$ for simplicity. We now claim that the integral (8.5) satisfies the inequality

$$(8.6)\qquad \int\limits_0^\pi \frac{dt_k}{(t_1 + \ldots + t_k)^k} \geqslant \frac{1}{(k-1) k^{k-1} (t_1 + \ldots + t_{k-1})^{k-1}}$$

for all points $(t_1, \ldots, t_{k-1}) \epsilon P_{k-1}$.

In fact when $(t_1, \ldots, t_{k-1}) \epsilon P_{k-1}$ we have certainly

$$(8.7)\qquad s + \pi = t_1 + \ldots + t_{k-1} + \pi \leqslant (\pi + \ldots + \pi) + \pi = k\pi.$$

Now $(s+\pi)^{k-1} \geqslant s^{k-1} + \pi^{k-1}$ since $k \geqslant 2$. Thus

$$(8.8)\qquad (s+\pi)^{k-1} - s^{k-1} \geqslant \pi^{k-1}.$$

Applying inequalities (8.7), (8.8) to the right-hand side of (8.5) we have

$$\int\limits_0^\pi \frac{dt_k}{(t_1 + \ldots + t_k)^k} \geqslant \frac{\pi^{k-1}}{(k-1) s^{k-1} (k\pi)^{k-1}} = \frac{1}{(k-1) k^{k-1} s^{k-1}}$$

which is precisely (8.6). Applying inequality (8.6) to (8.4) and noting (8.2) we obtain

$$I(n_1, \ldots, n_{k-1}, 0) \geqslant \frac{1}{(k-1) k^{k-1}} \int\limits_{P_{k-1}} \frac{\sin^2(n_1 t_1 + \ldots + n_{k-1} t_{k-1})}{(t_1 + \ldots + t_{k-1})^{k-1}} dt_1 \ldots dt_{k-1} \geqslant \frac{A_{k-1}}{(k-1) k^{k-1}} \log(n_1 + \ldots + n_{k-1})$$

which shows that (8.1) is satisfied with $A_k = A_{k-1}/(k-1) k^{k-1}$. The induction argument on k is now complete.

§ 9. Proof of Theorems 5 and 6. To prove Theorem 5 we begin by showing

$$(9.1)\qquad \int\limits_{Q_k} g(x)\, dx = \int\limits_{Q_k} \left\{ \int\limits_{Q_k} \frac{|f(x+t) - f(x-t)|^2}{|t|^k} dt \right\} dx = \int\limits_{Q_k} \left\{ \int\limits_{Q_k} \frac{|f(x+t) - f(x-t)|^2}{|t|^k} dx \right\} dt,$$

$$(9.2)\qquad \int\limits_{Q_k} \left\{ \int\limits_{Q_k} \frac{|f(x+t) - f(x-t)|^2}{|t|^k} dx \right\} dt = 4(2\pi)^k \sum_n{}' |c_n|^2 I(n).$$

The first identity follows from Fubini's theorem by an interchange of the order of integration. To prove the second identity we note that

$$f(x+t) \sim \sum (c_n e^{in\cdot t}) e^{in\cdot x}, \qquad f(x-t) \sim \sum (c_n e^{-in\cdot t}) e^{in\cdot x}$$

for any fixed $t \epsilon E_k$. Subtracting we have

$$f(x+t) - f(x-t) \sim \sum \big(c_n 2i \sin(n\cdot t)\big) e^{in\cdot x}.$$

By Parseval's identity on multiple Fourier series for the integrable function $f(x+t) - f(x-t)$ with $x \epsilon Q_k$ as variable we have

$$(2\pi)^{-k} \int\limits_{Q_k} |f(x+t) - f(x-t)|^2 dx = \sum_n |c_n 2i \sin(n\cdot t)|^2 = 4 \sum_n{}' |c_n|^2 \sin^2(n\cdot t)$$

since $\sin(n\cdot t)=0$ when $n=(0,\ldots,0)$. Here we remark that the Parseval's identity on multiple Fourier series is usually stated for $L^2(Q_k)$ periodic functions but it is likewise valid for all $L^1(Q_k)$ periodic functions. Dividing both sides of the above identity by $|t|^k$ and then integrating with respect to t over Q_k we obtain

$$(2\pi)^{-k}\int_{Q_k}\left\{\int_{Q_k}\frac{|f(x+t)-f(x-t)|^2}{|t|^k}dx\right\}dt = 4\int_{Q_k}\left\{\sum_n{}'\frac{|c_n|^2\sin^2(n\cdot t)}{|t|^k}\right\}dt$$
$$=4\sum_n{}'\left\{\int_{Q_k}\frac{|c_n|^2\sin^2(n\cdot t)}{|t|^k}dt\right\}=4\sum_n{}'|c_n|^2I(n).$$

In the last step of the above process we have interchanged the $\int_{Q_k}$ and $\sum_n'$ signs, which is justified by the non-negativeness of $|c_n|^2\sin^2(n\cdot t)/|t|^k$ for every $t\in Q_k$. This proves (9.2).

We are now practically at the end of the proof of Theorem 5. From (9.1) and (9.2) follows

$$\int_{Q_k}g(x)\,dx = 4(2\pi)^k\sum_n{}'|c_n|^2I(n).$$

Hence

$$\int_{Q_k}g(x)\,dx<\infty \text{ if and only if } \sum_n{}'|c_n|^2I(n)<\infty.$$

Because of the inequality $A_k\log|n|\leqslant I(n)\leqslant B_k\log|n|$ for all lattice points n with $|n|^2\geqslant 2$ (see (4.2)) we have now

$$\sum_n{}'|c_n|^2I(n)<\infty \text{ if and only if } \sum_n{}'|c_n|^2\log|n|<\infty.$$

The desired result (2.3) follows now immediately.

The proof of Theorem 6 is now almost a triviality. In fact, we have

$$\int_{Q_k}g(x)\,dx<\infty \Rightarrow \sum_n{}'|c_n|^2\log|n|<\infty\Rightarrow\sum_n{}'|c_n|^2<\infty$$

$\Rightarrow$ (by the Riesz-Fischer theorem) $f(x)\in L^2(Q_k)\Rightarrow f\in L(\log^+L)^2(Q_k)$

$\Rightarrow$ (by Theorem 4) $S[f(x)]$ is summable (B-R, α) at almost every x in E_k.

§ 10. Proof of Theorem 7. By Theorem 6 it is enough to prove

$$\int_{Q_k}g(x)\,dx<\infty \quad\text{where}\quad g(x)=\int_{Q_k}\frac{|f(x+t)-f(x-t)|^2}{|t|^k}dt.$$

From hypothesis (2.4) we have

$$|f(x+t)-f(x-t)|\leqslant Al^a(2|t|)\quad\text{for all}\quad x\in E_k,\ t\in E_k \text{ with } |t|\leqslant\frac{\delta}{2},$$

and hence

(10.1)
$$|f(x+t)-f(x-t)|^2\leqslant A^2l^{2a}(2|t|)\quad\text{for all}\quad x\in E_k,\ t\in E_k \text{ with } |t|\leqslant\frac{\delta}{2}.$$

Now the function $f(x)$, being continuous everywhere and periodic, is certainly bounded. Thus there exists a constant $M>0$ such that

(10.2) $$|f(x+t)-f(x-t)|\leqslant M\quad\text{for all}\quad x\in E_k,\ \text{all } t\in E_k.$$

Write

$$g(x)=\int_{Q_k}\frac{|f(x+t)-f(x-t)|^2}{|t|^k}dt=\int_{|t|\leqslant\delta/2}+\int_{\substack{t\geqslant\delta/2\\ t\in Q_k}}=\mathrm{I}_1+\mathrm{I}_2,\ \text{say}.$$

Applying the estimates (10.1) and (10.2) to the integrals I_1, I_2 respectively we obtain

$$\mathrm{I}_1=\int_{|t|\leqslant\delta/2}\frac{|f(x+t)-f(x-t)|^2}{|t|^k}dt\leqslant A^2\int_{|t|\leqslant\delta/2}\frac{l^{2\alpha}(2|t|)}{|t|^k}dt$$
$$=A\sigma_k\int_0^{\delta/2}\frac{l^{2\alpha}(2r)}{r}dr=\text{some constant } C_1 \text{ not depending on } x.$$

(here we have used the fact that $\int_0^{\delta/2}r^{-1}l^{2\alpha}(2r)\,dr<\infty$ if $2\alpha>1$),

$$\mathrm{I}_2=\int_{\substack{|t|\geqslant\delta/2\\ t\in Q_k}}\frac{|f(x+t)-f(x-t)|^2}{|t|^k}dt\leqslant\int_{\substack{|t|\geqslant\delta/2\\ t\in Q_k}}M\left(\frac{2}{\delta}\right)^k dx<\int_{Q_k}M\left(\frac{2}{\delta}\right)^k dx$$
$$=\text{some constant } C_2 \text{ not depending on } x.$$

Thus

$$g(x)=\mathrm{I}_1+\mathrm{I}_2\leqslant C_1+C_2$$
$$=\text{some constant } C_3 \text{ not depending on } x, \text{ for all } x\in E_k.$$

The boundedness and measurability of $g(x)$ implies its integrability over Q_k, i. e.

$$\int_{Q_k}g(x)\,dx<\infty.$$

This completes the proof of Theorem 7.

III. Further lemmas

§ 11. In this section we list the statements of some additional lemmas which will be needed in the proof of Theorem 1 in § 17.

A set C in E_k is said to be *periodic* if $(x_1, \ldots, x_k) \in C$ implies that each of the following $2k$ points

$$(x_1 \pm 2\pi, x_2, \ldots, x_k), (x_1, x_2 \pm 2\pi, x_3, \ldots, x_k), \ldots, (x_1, \ldots, x_{k-1}, x_k \pm 2\pi)$$

also belongs to C. From this definition it follows that all points of the form $(x_1, \ldots, x_k) \pm 2\pi n$, n being any lattice point in E_k, also belong to C.

Lemma 5 (Extension Lemma). *Suppose that*

(i) *C is a closed and periodic set in E_k. $f(x)$ is a periodic complex-valued function defined on C.*

(ii) *$\omega(t)$ is a real-valued function defined for all $t \geqslant 0$ such that $\omega(0) = 0$, $\omega(t)$ is continuous, monotone-increasing (i. e. non-decreasing) and concave for all $t \geqslant 0$.*

(iii) *$f(x)$ satisfies the inequality*

$$|f(x) - f(y)| \leqslant \omega(|x-y|) \quad \text{for any} \quad x, y \in C. \tag{11.1}$$

Then there exist a periodic extension $\bar{f}(x)$ of $f(x)$ to the whole space E_k such that the following inequality is satisfied:

$$|\bar{f}(x) - \bar{f}(y)| \leqslant \sqrt{2}\,\omega(|x-y|) \quad \text{for any} \quad x, y \in C. \tag{11.2}$$

Before passing to the statement of the next lemma let us revert to the function $l(t)$, $0 \leqslant t < 1$, defined earlier at the end of § 1 by $l(0) = 0$, $l(t) = 1/\log(1/t)$ for $0 < t < 1$. We have seen that $l(t)$ is concave for $0 \leqslant t \leqslant 1/e^2$, convex for $1/e^2 \leqslant t < 1$. In certain steps in the proof of our main Theorem 1 in § 17 the convexity property of $l(t)$ for $1/e^2 \leqslant t < 1$ is an obstacle and so it is necessary for us to redefine $l(t)$ for $t \geqslant 1/e^2$ so as to make $l(t)$ concave everywhere. More precisely stated we are seeking a real-valued function $\lambda(t)$ satisfying the following properties:

(i) $\lambda(t)$ is defined for all real $t \geqslant 0$,

(ii) $\lambda(t)$ coincides with $l(t)$ for all sufficiently small t,

(iii) $\lambda(t)$ is strictly increasing and concave for all $t \geqslant 0$.

Such a function $\lambda(t)$ certainly exists and can be constructed in many ways. One way is to define

$$\lambda(t) = \begin{cases} l(t) & \text{if} \quad 0 \leqslant t \leqslant 1/e^2, \\ \alpha + \beta \log t & \text{if} \quad t \geqslant 1/e^2, \end{cases}$$

where α, β are constants to be determined with $\beta > 0$. Property (iii) is clearly satisfied for all $t \geqslant 1/e^2$ since $\log t$ is strictly increasing and concave for all $t \geqslant 1/e^2$ (indeed for $t > 0$).

We now choose α, β in such a way that

a) $l(t)$ and $\alpha + \beta \log t$ give the same value at $t = 1/e^2$,

b) the graphs of $y = l(t)$ and $y = \lambda(t)$ have the same tangent at $t = 1/e^2$, i. e. $\lambda'(t) = l'(t)$ at $t = 1/e^2$.

These conditions uniquely determine $\alpha = 1$, $\beta = 1/4$. Accordingly we now define $\lambda(t)$ by

$$\lambda(0) = 0, \qquad \lambda(t) = \begin{cases} 1/\log \dfrac{1}{t} & \text{if} \quad 0 < t \leqslant 1/e^2, \\ 1 + \dfrac{1}{4}\log t & \text{if} \quad t \geqslant 1/e^2. \end{cases}$$

For any two positive numbers $m > 0$, $a > 0$ we define

$$F(m, a) = \int_0^a (a-t)^{m-1} \lambda(t)\, dt.$$

The above integral clearly exists for all $m > 0$, $a > 0$. The next two lemmas are concerned with certain inequalities satisfied by $F(m, a)$ and an asymptotic expression for $F(m, a)$ as $a \to 0$.

Lemma 6. *We have*

$$F(m, a) \simeq \left(\frac{1}{m}\right) a^m \lambda(a) \quad \text{as} \quad a \to 0. \tag{11.3}$$

Hence there exists a constant $\delta_m > 0$ (depending on m only) such that

$$F(m, a) > \left(\frac{1}{2m}\right) a^m \lambda(a) \quad \text{for} \quad 0 < a < \delta_m. \tag{11.4}$$

Lemma 7. *For any $\theta > 0$ there exists a positive constant $\delta_{m,\theta}$ (depending on m and θ only) such that*

$$F(m, a) < 2\theta^{-m} F(m, \theta a) \quad \text{for} \quad 0 < a < \delta_{m,\theta}. \tag{11.5}$$

The next lemma is concerned with the decomposition of certain sets in E_k. A cube in E_k is defined by the set of all points $(x_1, \ldots, x_k)$ satisfying the inequality

$$a_1 \leqslant x_1 \leqslant b_1, \quad a_2 \leqslant x_2 \leqslant b_2, \quad \ldots, \quad a_k \leqslant x_k \leqslant b_k,$$

where $-\infty < a_j < b_j < \infty$ $(j = 1, 2, \ldots, k)$ with $b_1 - a_1 = b_2 - a_2 = \ldots = b_k - a_k$. The common quantity $s = b_j - a_j$ $(j = 1, 2, \ldots, k)$ is called the *edge* of the cube and $k^{1/2}s$ is called the *diagonal* of the cube. A cube is clearly a closed set. The diameter of a cube is equal to its diagonal (see § 1 for the definition of the diameter of a set). Two cubes are said to be *non-overlapping* if their interiors do not intersect even though the cubes may intersect on the boundary.

LEMMA 8 (Decomposition Lemma). *Let K be a cube in E_k and let K^0 denote the interior of K. Suppose P is a non-empty closed set contained in K^0. Then there exists an infinite sequence of non-overlapping cubes $\{Q_j\}$ $(j = 1, 2, 3, \ldots)$ such that*

$$K-P = \bigcup_{j=1}^{\infty} Q_j, \quad \lim_{j\to\infty} d_j = 0, \quad 1 \leqslant s_j/d_j \leqslant 3 \quad \text{for all } j = 1, 2, 3, \ldots$$

Here d_j = diameter of Q_j and $s_j = \operatorname{dist}(Q_j, P)$.

The next lemma is the k-dimensional version of a theorem of J. Marcinkiewicz in one variable (see [6], p. 129, part (ii)).

LEMMA 9. *Let Q be a cube in E_k and P be a closed set contained in Q. Let $\delta(x) = \operatorname{dist}(x, P)$ for any point $x \in E_k$. Then for almost every point x in P we have*

$$I(x) = \int_Q \frac{\lambda(\delta(t))}{|x-t|^k}\, dt < \infty.$$

§ 12. Proof of Lemma 5. First we claim that $\omega(t)$ satisfies the triangular inequality

$$\omega(u+v) \leqslant \omega(u)+\omega(v) \quad \text{for} \quad u \geqslant 0,\ v \geqslant 0. \tag{12.1}$$

In fact (12.1) is trivial in any one of the following three cases: (i) $u = 0$, (ii) $v = 0$, (iii) $u = v$.

In the case of $u = v$ (12.1) becomes $\omega(2u) \leqslant 2\omega(u)$ which is a direct consequence of $\omega(0) = 0$ and the concavity of ω. Hence it is enough to prove (12.1) under the assumption of $u > 0$, $v > 0$, $u \neq v$. Now because (12.1) is symmetric with respect to u and v it is enough to prove (12.1) by assuming $0 < u < v$.

By the concavity of $\omega(t)$ the point $(u+v), \omega(u+v))$ must lie on or below the line joining the points $(u, \omega(u))$, $(v, \omega(v))$, and also on or below the line joining $(0, 0)$ and $(u, \omega(u))$. These mean the following two inequalities:

$$(v-u)\,\omega(u+v) \leqslant v\omega(v)-u\omega(u),$$
$$u\omega(u+v) \leqslant (u+v)\,\omega(u).$$

Adding these two inequalities and then dividing by v we obtain $\omega(u+v) \leqslant \omega(u)+\omega(v)$.

From (12.1) we deduce the following inequality:

$$|\omega(u_1)-\omega(u_2)| \leqslant \omega(|u_1-u_2|), \quad u_1 \geqslant 0,\ u_2 \geqslant 0. \tag{12.2}$$

In fact, if $u_1 \geqslant u_2$, we set $u = u_2$, $v = u_1-u_2$ in (12.1) and thus obtain

$$\omega(u_1)-\omega(u_2) \leqslant \omega(u_1-u_2), \quad u_1 \geqslant u_2 \geqslant 0.$$

If $u_2 \geqslant u_1$, we set $u = u_1$, $v = u_2-u_1$ in (12.1) and thus obtain

$$\omega(u_2)-\omega(u_1) \leqslant \omega(u_2-u_1), \quad u_2 \geqslant u_1 \geqslant 0.$$

This completes the proof of (12.2).

Reverting to Lemma 5 let us first deal with the special case in which the function $f(x)$ is real-valued by proving the following

PROPOSITION. *If we make the additional assumption that the function $f(x)$ in Lemma 5 is real-valued, then there exists a periodic and real-valued extension $\bar{f}(x)$ of $f(x)$ to the whole space E_k such that*

$$|\bar{f}(x)-\bar{f}(y)| \leqslant \omega(|x-y|) \quad \text{for} \quad x, y \in E_k. \tag{12.3}$$

To prove this we construct the extension $\bar{f}(x)$ by

$$\bar{f}(x) = \operatorname*{Sup}_{y\in C}\{f(y)-\omega(|y-x|)\} = \operatorname*{Max}_{y\in C}\{f(y)-\omega(|y-x|)\} \tag{12.4}$$

and then show that $\bar{f}(x)$ satisfies all the properties mentioned in the above Proposition.

First we show that in the formula (12.4) the symbol "Sup" may be replaced by "Max" because the supremum is actually attained. This follows easily from the fact that the expression $f(y)-\omega(|y-x|)$, with x kept fixed, is a continuous real-valued function in $y \in C$ (since $f(y)$ and $\omega(|y-x|)$ each is a continuous function in $y \in C$) and the set C is a closed and periodic set in E_k; hence $f(y)-\omega(|y-x|)$ attains its supremum at some point in C.

Next we show that $\bar{f}(x)$ is indeed an extension of $f(x)$, i. e. $\bar{f}(x) = f(x)$ for all points $x \in C$. For this purpose it suffices to show the following two conditions:

(a) For some point $y_0 \in C$ we have $f(y_0)-\omega(|y_0-x|) = f(x)$,

(b) $f(y)-\omega(|y-x|) \leqslant f(x)$ for every point $y \in C$.

Clearly (a) is satisfied with the choice of $y_0 = x$. The assumption (11.1), namely $|f(x)-f(y)| \leqslant \omega(|x-y|)$ for any $x, y \in C$, implies

$$f(y)-f(x) \leqslant \omega(|y-x|) \quad \text{for} \quad x, y \in C$$

(since $f(y)-f(x) \leqslant |f(x)-f(y)|$) and so condition (b) is also satisfied.

Let

$$e_j = (0, \ldots, 0, 2\pi, 0, \ldots, 0) \quad (j = 1, 2, \ldots, k)$$

where the non-zero coordinate 2π is in the j-th place. For each $j = 1, 2, \ldots, k$ we have, by (12.4) and using the periodicity of f,

$$\bar{f}(x+e_j) = \sup_{y\in C}\{f(y)-\omega(|y-x-e_j|)\}$$
$$= \sup_{y\in C}\{\text{same expression as above but with } y \text{ replaced by } y+e_j\}$$
$$= \sup_{y\in C}\{f(y+e_j)-\omega(|y-x|)\} = \sup_{y\in C}\{f(y)-\omega(|y-x|)\} = \bar{f}(x).$$

This shows that the extension $\bar{f}(x)$ is a periodic function.

We now come to the proof of (12.3) which is equivalent to

$$\bar{f}(x)-\bar{f}(y) \leqslant \omega(|x-y|), \quad x, y \in E_k, \tag{12.5}$$

and

$$-\omega(|x-y|) \leqslant \bar{f}(x)-\bar{f}(y), \quad x, y \in E_k, \tag{12.6}$$

together.

From the definition of the extension $\bar{f}$ by (12.4) we have

$$\bar{f}(x) = f(u)-\omega(|u-x|) \quad \text{for some point } u \in C,$$

and for this point u we have the inequality

$$\bar{f}(y) \geqslant f(u)-\omega(|u-y|).$$

Subtracting,

$$\bar{f}(x)-\bar{f}(y) \leqslant \omega(|u-y|)-\omega(|u-x|).$$

Now (12.2) implies

$$\omega(u_1)-\omega(u_2) \leqslant \omega(|u_1-u_2|) \quad \text{for} \quad u_1 \geqslant 0,\ u_2 \geqslant 0.$$

Setting $u_1 = |u-y|$, $u_2 = |u-x|$ we have

$$\bar{f}(x)-\bar{f}(y) \leqslant \omega(|u-y|)-\omega(|u-x|) \leqslant \omega(\big||u-y|-|u-x|\big|) \leqslant \omega(|x-y|).$$

In the last step above we have used the fact that $\omega(t)$ is monotone-increasing and that $\big||u-y|-|u-x|\big| \leqslant |(u-y)-(u-x)| = |x-y|$. This proves (12.5).

From the definition of the extension $\bar{f}$ by (12.4) we have

$$\bar{f}(y) = f(v)-\omega(|v-y|) \quad \text{for some point } x \in C,$$

and for this point v we have the inequality

$$\bar{f}(x) \geqslant f(v)-\omega(|v-x|).$$

Subtracting,

$$\bar{f}(x)-\bar{f}(y) \geqslant \omega(|v-y|)-\omega(|v-x|).$$

Now (12.2) implies

$$-\omega(|u_1-u_2|) \leqslant \omega(u_1)-\omega(u_2) \quad \text{for} \quad u_1 \geqslant 0,\ u_2 \geqslant 0.$$

Setting $u_1 = |v-y|$, $u_2 = |v-x|$ we obtain

$$\bar{f}(x)-\bar{f}(y) \geqslant \omega(|v-y|)-\omega(|v-x|) \geqslant -\omega(\big||v-y|-|v-x|\big|) \geqslant -\omega(|x-y|).$$

In the last step above we have used the fact that $-\omega(t)$ is monotone-decreasing and that $\big||v-y|-|v-x|\big| \leqslant |(v-y)-(v-x)| = |x-y|$. This proves (12.6) and hence the above Proposition.

In the general case where $f(x)$ is complex-valued we write

$$f(x) = g(x)+ih(x)$$

where $g(x)$ and $h(x)$ are the real and imaginary parts of $f(x)$ respectively. It is clear that the inequality

$$|f(x)-f(y)| \leqslant \omega(|x-y|) \quad \text{for} \quad x, y \in C$$

implies

$$|g(x)-g(y)| \leqslant \omega(|x-y|), \quad |h(x)-h(y)| \leqslant \omega(|x-y|) \quad \text{for} \quad x, y \in C.$$

Applying the real-valued functions $g(x)$ and $h(x)$ to the above Proposition we see that there exist real-valued, periodic extensions $\bar{g}$ and $\bar{h}$ of g and h respectively to the whole space E_k such that

$$|\bar{g}(x)-\bar{g}(y)| \leqslant \omega(|x-y|), \quad |\bar{h}(x)-\bar{h}(y)| \leqslant \omega(|x-y|) \quad \text{for} \quad x, y \in C.$$

Define $\bar{f}(x) = \bar{g}(x)+i\bar{h}(x)$, then $\bar{f}(x)$ is a periodic extension of $f(x)$ to the whole space E_k. For any two points x, y in E_k we have

$$|\bar{f}(x)-\bar{f}(y)|^2 = |\bar{g}(x)-\bar{g}(y)|^2+|\bar{h}(x)-\bar{h}(y)|^2$$
$$\leqslant \omega^2(|x-y|)+\omega^2(|x-y|) = 2\omega^2(|x-y|)$$

from which (11.2) follows at once.

§ 13. Proof of Lemma 6. First we prove the asymptotic formula

$$\lambda(a) \simeq \lambda(\theta a) \text{ as } a \to 0 \quad \text{for any } \theta > 0. \tag{13.1}$$

In fact by confining a to both $0 < a < 1/e^2$ and $0 < \theta a < 1/e^2$ we have

$$\lambda(a) = 1/\log\frac{1}{a}, \quad \lambda(\theta a) = 1/\log\frac{1}{\theta a} = 1\bigg/\left(-\log\theta+\log\frac{1}{a}\right).$$

Hence

$$\frac{\lambda(a)}{\lambda(\theta a)} = 1-\frac{\log\theta}{\log\dfrac{1}{a}} = 1-(\log\theta)\lambda(a).$$

Passing to the limit $a \to 0$ and noting that $\lambda(a) \to 0$ as $a \to 0$ we get $\lambda(a)/\lambda(\theta a) \to 1$ as $a \to 0$.

Reverting to the statement of Lemma 6, the inequality (11.4) there clearly needs no proof since it follows readily from (11.3). Now the asymptotic formula (11.3) is equivalent to

$$\limsup_{a\to 0}\frac{mF(m,a)}{a^m\lambda(a)}\leqslant 1 \tag{13.2}$$

and

$$\liminf_{a\to 0}\frac{mF(m,a)}{a^m\lambda(a)}\geqslant 1 \tag{13.3}$$

together. We first prove the inequality

$$mF(m,a)<a^m\lambda(a)\quad \text{for all}\quad a>0. \tag{13.4}$$

In fact, since $\lambda(t)\leqslant\lambda(a)$ for all t in $0\leqslant t\leqslant a$, we have

$$mF(m,a)=\int_0^a m(a-t)^{m-1}\lambda(t)\,dt<\lambda(a)\int_0^a m(a-t)^{m-1}dt$$
$$=\lambda(a)\{-(a-t)^m|_{t=0}^{t=a}\}=a^m\lambda(a)\quad \text{for all } a>0.$$

This proves (13.4) from which (13.2) follows at once.

To prove (13.3) we first establish the following inequality:

$$\frac{mF(m,a)}{a^m\lambda(a)}>(1-\theta)^m\frac{\lambda(\theta a)}{\lambda(a)}\quad \text{for all } a>0, \text{ all } \theta \text{ in } 0<\theta<1. \tag{13.5}$$

To see this we write

$$mF(m,a)=\int_0^a m(a-t)^{m-1}\lambda(t)\,dt=\int_0^{\theta a}+\int_{\theta a}^a>\int_{\theta a}^a m(a-t)^{m-1}\lambda(t)\,dt$$
$$\geqslant\lambda(\theta a)\int_{\theta a}^a m(a-t)^{m-1}dt=\lambda(\theta a)\{-(a-t)^m|_{t=\theta a}^{t=a}\}$$
$$=\lambda(\theta a)\{(a-\theta a)^m\}=\lambda(\theta a)\{a^m(1-\theta)^m\}$$

from which (13.5) follows at once.

To prove (13.3) it is enough to show that to every $\varepsilon>0$ there exists a $\delta>0$ such that

$$\frac{mF(m,a)}{a^m\lambda(a)}\geqslant 1-\varepsilon\quad \text{for}\quad 0<a<\delta. \tag{13.6}$$

Given $\varepsilon>0$ we can choose a number θ, $0<\theta<1$, such that

$$(1-\theta)^m>1-\varepsilon$$

because $x^m\to 1$ as $x\to 1$ (since $m>0$). Thus we have $1>(1-\varepsilon)/(1-\theta)^m$.

Referring to (13.1) we have $\lambda(\theta a)/\lambda(a)\to 1$ as $a\to 0$. Hence to the number $(1-\varepsilon)/(1-\theta)^m$ which is <1 there exists a number $\delta>0$ such that

$$\frac{\lambda(\theta a)}{\lambda(a)}>\frac{1-\varepsilon}{(1-\theta)^m}\quad \text{for}\quad 0<a<\delta. \tag{13.7}$$

Thus (13.6) follows readily from (13.5) and (13.7) and the proof of (13.3) is now complete.

§ 14. Proof of Lemma 7. By the asymptotic formula of Lemma 6 we have

$$F(m,a)\simeq\left(\frac{1}{m}\right)a^m\lambda(a)\quad \text{as}\quad a\to 0,$$

$$F(m,\theta a)\simeq\left(\frac{1}{m}\right)(\theta a)^m\lambda(\theta a)\ \text{as}\ \theta a\to 0, \text{ i. e.}\quad \text{as}\quad a\to 0.$$

Hence

$$\operatorname*{Lim}_{a\to 0}\frac{F(m,a)}{F(m,\theta a)}=\operatorname*{Lim}_{a\to 0}\frac{m^{-1}a^m\lambda(a)}{m^{-1}(\theta a)^m\lambda(\theta a)}=\theta^{-m}\operatorname*{Lim}_{a\to 0}\frac{\lambda(a)}{\lambda(\theta a)}=\theta^{-m}$$

by (13.1). Therefore to the number $2\theta^{-m}$ (which is $>\theta^{-m}$) there exists a constant $\delta_{\theta,m}$ depending on θ and m, such that

$$\frac{F(m,a)}{F(m,\theta a)}<2\theta^{-m}\quad \text{for all } a \text{ in } 0<a<\delta_{\theta,m},$$

which is the desired result (11.5).

§ 15. Proof of Lemma 8. For any cube C in E_k we say C is *good* if $\operatorname{dist}(C,P)\geqslant\operatorname{diag}(C)$, and *bad* if $\operatorname{dist}(C,P)<\operatorname{diag}(C)$ (here $\operatorname{diag}(C)$ denotes the diagonal of C). The sequence $\{Q_j\}$ of cubes is constructed in the following manner.

Step 1. Let Π_1 be the system of the 2^k cubes obtained by bisecting the edges of K. We select all the good cubes from Π_1 and carry over the bad cubes to the next step.

Step 2. Let Π_2 be the system of all cubes obtained by bisecting the edges of all the bad cubes left over from step 1. We select all the good cubes from Π_2 and carry over the bad cubes to the next step.

Step 3. Let Π_3 be the system of all cubes obtained by bisecting the edges of all the bad cubes left over from step 2. We select all the good cubes from Π_3 and carry over the bad cubes to the next step, and so forth.

We remark that all the cubes in step 1 are always bad since $P\subset K^0$, so that the first good cubes start in step 2. The set of good cubes is certainly non-empty because $P\subset K^0$ implies that P is bounded away from the boundary of K. Thus for some sufficiently large integer m those cubes in the system Π_m touching the boundary of K will be good.

Let $\{Q_j\} = \{Q_1, Q_2, \ldots\}$ denote the infinite sequence of all the good cubes selected in the order of their selection in the above process. Clearly all these cubes are non-overlapping (see the paragraph preceding Lemma 8 for the definition of non-overlapping). Let d be the diagonal of K; then each good cube selected from the system Π_m (i. e. in step m) has its diagonal $= 2^{-m}d$ and hence $d_j \to 0$ as $j \to \infty$.

Since each selected cube is good, we have clearly $d_j \leqslant s_j$ $(j = 1, 2, 3, \ldots)$. We now prove that $s_j \leqslant 3d_j$ $(j = 1, 2, 3, \ldots)$. Let Q be a cube selected in step m and let C be the bad cube in step $m-1$ such that Q is obtained from C by bisecting the edges of C. Let $s = \operatorname{dist}(Q, P)$, $d = \operatorname{diag}(Q)$. We now prove that $s \leqslant 3d$. For this purpose we first prove the inequality

$$\operatorname{dist}(Q, P) \leqslant \operatorname{dist}(C, P) + \operatorname{diag}(Q). \tag{15.1}$$

It is enough to show that there exist a point $x \in Q$ and another point $y \in P$ such that

$$|x-y| \leqslant \operatorname{dist}(C, P) + \operatorname{diag}(Q) \tag{15.2}$$

because $\operatorname{dist}(Q, P) \leqslant |x-y|$ and so (15.2) implies (15.1).

Because C and P are compact sets there exist therefore two points $y \in P$, $z \in C$ such that $\operatorname{dist}(C, P) = |y-z|$. Let x be the center of C. Clearly $x \in Q$ since Q is one of the 2^k cubes obtained by bisecting the edges of C. Now x being the center of C and $z \in C$ imply

$$|z-x| \leqslant \operatorname{diag}(Q).$$

Here we have used the easily proved fact that the distance from any point in a cube C to its center is $\leqslant$ half of the diagonal of C. Hence

$$|x-y| \leqslant |y-z| + |z-x| \leqslant \operatorname{dist}(C, P) + \operatorname{diag}(Q).$$

This proves (15.2) which implies (15.1). Note that the proof of (15.1) here uses only the fact that Q is one of the 2^k cubes obtained by bisecting the edges of C but not the good or bad property of C and Q.

From (15.1) and recalling that $\operatorname{dist}(C, P) < \operatorname{diag}(C)$ (since C is bad) we have

$$\begin{aligned} s = \operatorname{dist}(Q, P) &\leqslant \operatorname{dist}(C, P) + \operatorname{diag}(Q) < \operatorname{diag}(C) + \operatorname{diag}(Q) \\ &= 2\operatorname{diag}(Q) + \operatorname{diag}(Q) = 3\operatorname{diag}(Q) = 3d. \end{aligned}$$

Lastly we come to the proof of

$$L - P = \bigcup_{j=1}^{\infty} Q_j.$$

Since $Q_j \subset K-P$ for each j it is enough to prove $\bigcup_{j=1}^{\infty} Q_j \supset K-P$, i. e. to say any point $x \in K-P$ implies that $x \in Q_j$ for some $j = 1, 2, 3, \ldots$ For this purpose we first prove this assertion: If C is a bad cube in E_k and x is any point in C, then

$$\operatorname{dist}(x, P) < 2\operatorname{diag}(C). \tag{15.3}$$

Since C and P are compact sets there exist two points $y \in P$, $z \in C$ such that $\operatorname{dist}(C, P) = |y-z|$. Now $\operatorname{dist}(x, P) \leqslant |x-y|$ because $y \in P$. Hence

$$\operatorname{dist}(x, P) \leqslant |x-y| \leqslant |x-z| + |y-z| < \operatorname{diag}(C) + \operatorname{diag}(C) = 2\operatorname{diag}(C).$$

Here we have used the following two obvious facts:

$$x \in C \text{ and } z \in C \text{ imply } |x-z| \leqslant \operatorname{diag}(C),$$
$$|y-z| = \operatorname{dist}(C, P) < \operatorname{diag}(C) \quad (\text{since } C \text{ is bad}).$$

Suppose $x \in K-P$. Let $\delta = \operatorname{dist}(x, P)$; then $\delta > 0$ because P is closed. Let $d = \operatorname{diag}(K)$. We select the least positive integer m such that $2(d/2^m) < \delta$. Let $\{Q_1, Q_2, \ldots, Q_{m'}\}$ be all the good cubes selected in steps 1 through m (here we assume that there are altogether m' such selected good cubes). Let $B_1, B_2, \ldots, B_{m''}$ be all the bad cubes in Π_m. From the process of selection we have clearly

$$K = Q_1 \cup Q_2 \cup \ldots \cup Q_{m'} \cup B_1 \cup B_2 \cup \ldots \cup B_{m''}. \tag{15.4}$$

We now claim that

$$x \notin B_1, x \notin B_2, \ldots, x \notin B_{m''}; \tag{15.5}$$

for if $x \in B$ for some bad cube B in $\{B_1, B_2, \ldots, B_{m''}\}$, then it would follow from (15.3) that

$$\delta = \operatorname{dist}(x, P) < 2\operatorname{diag}(B) = 2\left(\frac{d}{2^m}\right)$$

which contradicts to the choice of m for which we had $2(d/2^m) < \delta$.

Since $x \in K-P$ implies $x \in K$, it follows from (15.4) and (15.5) that $x \in Q_1 \cup Q_2 \cup \ldots \cup Q_{m'}$, i. e. x belongs to some one of $Q_1, Q_2, \ldots, Q_{m'}$.

§ 16. Proof of Lemma 9. First we observe that $\delta(t) = 0$ if and only if $t \in P$. Hence $\lambda(\delta(t)) = 0$ if and only if $t \in P$. To prove the conclusion of Lemma 9 it is enough to show that $\int_P I(x)\,dx < \infty$. Now

$$\begin{aligned} \int_P I(x)\,dx &= \int_P \left\{ \int_Q \frac{\lambda(\delta(t))}{|x-t|^k}\,dt \right\} dx = \int_Q \lambda(\delta(t)) \left\{ \int_P \frac{dx}{|x-t|^k} \right\} dt \\ &= \int_{Q-P} \lambda(\delta(t)) \left\{ \int_P \frac{dx}{|x-t|^k} \right\} dt \end{aligned} \tag{16.1}$$

(since $\lambda(\delta(t)) = 0$ for $t \in P$).

Let a be the diagonal ($=$ diameter) of the cube Q. For any point $t\in\{Q-P\}$ it is clear that P is contained in the spherical shell consisting of all points x satisfying the inequality $\delta(t)\leqslant|x-t|\leqslant a$. Hence

$$(16.2)\qquad \int\limits_P \frac{dx}{|x-t|^k}\leqslant \int\limits_{\delta(t)\leqslant|x-t|\leqslant a}\frac{dx}{|x-t|^k}=\int\limits_{\delta(t)\leqslant|x|\leqslant a}\frac{dx}{|x|^k}$$
$$=\sigma_k\int\limits_{\delta(t)}^{a}\frac{r^{k-1}dr}{r^k}=\sigma_k\int\limits_{\delta(t)}^{a}\frac{dr}{r}=\sigma_k\log\frac{a}{\delta(t)}.$$

From (16.1) and (16.2) follows

$$(16.3)\qquad \int\limits_P I(x)\,dx\leqslant\sigma_k\int\limits_{Q-P}\lambda\big(\delta(t)\big)\log\frac{a}{\delta(t)}\,dt.$$

We now decompose $Q-P$ into two parts S_1 and S_2 in such a way that:

S_1 consists of all points t in $Q-P$ for which $0<\delta(t)\leqslant 1/e^2$;

S_2 consists of all points t in $Q-P$ for which $\delta(t)>1/e^2$.

In order to prove

$$\int\limits_P I(x)\,dx<\infty$$

t is enough, because of (16.3), to prove

$$(16.4)\qquad \int\limits_{S_1}\lambda\big(\delta(t)\big)\log\frac{a}{\delta(t)}\,dt<\infty,$$

$$(16.5)\qquad \int\limits_{S_2}\lambda\big(\delta(t)\big)\log\frac{a}{\delta(t)}\,dt<\infty.$$

Now for all $t\in S_2$ we have

$$0<\log\frac{a}{\delta(t)}\leqslant\log\frac{a}{1/e^2}=\log 2+\log a$$

which shows that $\log(a/\delta(t))$ is bounded for $t\in S_2$. Clearly $\lambda\big(\delta(t)\big)$ is also bounded for $t\in S_2$ (indeed for $t\in Q$) since $\delta(t)\leqslant\operatorname{diag}(Q)=a$ and $\lambda(s)$ is an increasing function of the variable s). Thus the integrand in (16.5) is bounded for $t\in S_2$ and so we obtain (16.5).

To prove (16.4) we have, for all $t\in S_1$,

$$\lambda\big(\delta(t)\big)\log\frac{a}{\delta(t)}=\frac{\log\dfrac{a}{\delta(t)}}{\log\dfrac{1}{\delta(t)}}=\frac{\log a+\log\dfrac{1}{\delta(t)}}{\log\dfrac{1}{\delta(t)}}=1+\frac{\log a}{\log\dfrac{1}{\delta(t)}}$$
$$=1+(\log a)\,\lambda\big(\delta(t)\big).$$

The last term $(\log a)\lambda\big(\delta(t)\big)$ above is bounded for $t\in S_1$. In fact, we have

$$(\log a)\,\lambda\big(\delta(t)\big)\leqslant\begin{cases}(\log a)\,\lambda(1/e^2)=\tfrac12\log a & \text{if}\quad a\geqslant 1,\\ 0 & \text{if}\quad a\leqslant 1.\end{cases}$$

Thus

$$\lambda\big(\delta(t)\big)\log\frac{a}{\delta(t)}$$

is bounded for $t\in S_1$ and so we obtain (16.4). (16.4) and (16.5) together imply

$$\int\limits_P I(x)\,dx<\infty$$

which proves the Lemma.

IV. Proof of the main theorem

§ 17. We begin with two trivial remarks. Firstly, because of the periodicity of the function f we may assume, without loss of generality, that the set E is contained in the fundamental cube

$$Q_k=\{(x_1,\ldots,x_k)|-\pi\leqslant x_j\leqslant\pi,\ j=1,2,\ldots,k\}.$$

Secondly, because the conclusion of Theorem 1 concerns almost every point x in E we may remove from E any set of points of measure zero, in particular we may remove from E the intersection of E with the boundary of Q_k. Combining these two remarks we may assume now, without loss of generality, that the set E is contained in Q_k^0, the interior of Q_k.

We shall prove Theorem 1 by showing that the following condition is satisfied:

(*) To every $\eta>0$ there exists a closed set $P\subset E$ with $|E-P|<\eta$ such that $S[f(x)]$ is summable $(\text{B-R},\alpha)$ at almost every point x in P.

Condition (*) implies that $S[f(x)]$ is summable $(\text{B-R},\alpha)$ at almost every point x in P since $\eta>0$ is arbitrarily small. We shall prove (*) by establishing the following:

To every $\eta>0$ there exists a closed set $P\subset E$ with $|E-P|<\eta$ and there exists a decomposition of the function $f(x)$ into two parts, say

$$(17.1)\qquad f(x)=\varphi(x)+\psi(x),\qquad x\in E_k,$$

with $f(x)=\varphi(x)$ for all points $x\in\tilde P$, where $\tilde P$ is the smallest periodic set containing P such that the following two conditions are satisfied:

CONDITION I. *$\varphi(x)$ is periodic, continuous everywhere and satisfies the inequality*

$$|\varphi(x+h)-\varphi(x)| \leqslant Al(|h|) \tag{17.2}$$

for all $x \in E_k$ and for all $h \in E_k$ with $|h| < 1$. Here A is some constant depending on $\eta > 0$ only.

CONDITION II. *There exists a number $\varepsilon > 0$ (ε depending on η only) such that $|\psi(x)| < \infty$ (i. e. $\psi(x)$ is finite-valued) for almost every point x in P, and that the inequality*

$$\int\limits_{|t|\leqslant\varepsilon} \frac{|\psi(x+t)-\psi(x)|}{|t|^k}\,dt = \int\limits_{|t|\leqslant\varepsilon} \frac{|\psi(x+t)|}{|t|^k}\,dt < \infty \tag{17.3}$$

holds for almost every point x in P. (The term $\psi(x)$ in the integrand above may be dropped since $\psi(x) = 0$ at every point $x \in P$).

It is now easy to see that the above two conditions I and II together imply condition (*). In fact, because of condition I and by Theorem 7 it follows that $S[\varphi(x)]$ is summable (B-R, α) at almost every point x in E_k. From condition II and by Theorem 3 it follows that $S[\psi(x)]$ is summable (B-R, α) for almost every point x in P. Thus, by adding up the two Fourier series $S[\varphi(x)]$ and $S[\psi(x)]$ we see that $S[f(x)] = S[\varphi(x)]+S[\psi(x)]$ is summable (B-R, α) for almost every point x in P. The applicability of Theorem 3 is justified by the fact that

$$f(x) \in L\log^+L(Q_k), \qquad \varphi(x) \in L\log^+L(Q_k), \qquad \psi(x) = f(x) - \varphi(x)$$

together imply $\psi(x) \in L\log^+L(Q_k)$.

Suppose $\eta > 0$ has been given. We now show that there exists a closed set $P \subset E$ with $|E-P| < \eta$ and two positive numbers $M > 0$, $\delta > 0$ such that

$$\frac{1}{h^k}\int\limits_{|t|\leqslant h} |f(x+t)-f(x)|\,dt \leqslant Ml(h) \quad \text{for all} \quad x \in P, \quad 0 < h < \delta. \tag{17.4}$$

To see this we define a set E_n, $n = 1, 2, 3, \ldots$, by

(17.5)

$$E_n = \left\{x \mid x \in E,\ \frac{1}{h^k}\int\limits_{|t|\leqslant h} |f(x+t)-f(x)|\,dt \leqslant nl(h),\ \text{for all } h \text{ in } 0 < h < \frac{1}{n}\right\}.$$

It is clear that $E_1 \subset E_2 \subset E_3 \subset \ldots$ We now show that

$$\bigcup_{n=1}^{\infty} E_n = \operatorname*{Lim}_n E_n = E. \tag{17.6}$$

Clearly

$$\bigcup_{n=1}^{\infty} E_n \subset E$$

since $E_n \subset E$ for each n. Hence it is enough to prove that every point $x \in E$ belongs to E_m for some m. By hypothesis (1.8), $x \in E$ implies that there exist two positive numbers $G > 0$, $\alpha > 0$ such that

$$\frac{1}{h^k}\int\limits_{|t|\leqslant h} |f(x+h)-f(x)|\,dt \leqslant Gl(h) \quad \text{for all } h \text{ in } 0 < h < \alpha. \tag{17.7}$$

Choose an integer m so large that $m \geqslant G$, $m \geqslant 1/\alpha$. Inequality (17.7) remains valid if we replace G by m and α by $1/m$. This gives precisely $x \in E_m$ and so (17.6) is proved.

From (17.6) it follows that $|E_n| \to |E|$ as $n \to \infty$. We now choose n_0 so large that $|E|-|E_{n_0}| < \eta/2$. There exists now a closed set $P \subset E_{n_0}$ such that $|E_{n_0}|-|P| < n/2$. Hence

$$|E-P| = |E|-|P| = (|E|-|E_{n_0}|)+(|E_{n_0}|-|P|) < \frac{\eta}{2}+\frac{\eta}{2} = \eta.$$

Now for any point $x \in P$ we have $x \in E_{n_0}$ (since $P \subset E_{n_0}$) and so x satisfies the condition expressed in (17.5) with $n = n_0$, i. e.

$$\frac{1}{h^k}\int\limits_{|t|\leqslant h} |f(x+t)-f(x)|\,dt \leqslant n_0 l(h) \quad \text{for all } h \text{ in } 0 < h < \frac{1}{n_0}.$$

This shows that (17.4) is satisfied with $M = n_0$, $\delta = 1/n_0$.

We may assume that the number $\delta > 0$ in (17.4) is so small that

$$\delta \leqslant 2\operatorname{dist}(P, \partial Q_k), \qquad \delta \leqslant 1/e^2,$$

since (17.4) remains valid if we replace δ by any smaller positive number. The next step is to deduce from (17.4) that there exists a constant M_1 (depending on η) such that

$$|f(x)-f(y)| \leqslant M_1 l(|x-y|) \quad \text{for} \quad x, y \in P \text{ with } h = |x-y| < \delta. \tag{17.8}$$

To prove this we may assume $x \neq y$ since (17.8) is trivial if $x = y$. Let therefore $|x-y| = h > 0$, $z = (x+y)/2$. Let

$$I = \left\{u \mid u \in E_k,\ |u-z| \leqslant \frac{h}{6}\right\}$$

denote the sphere centered at the point z with radius $h/6$. Clearly we have

$$\frac{h}{3} \leqslant |u-x| \leqslant \frac{2h}{3}, \qquad \frac{h}{3} \leqslant |u-y| \leqslant \frac{2h}{3} \quad \text{for any point } u \in I.$$

The k-dimensional volume of I is $|I| = C_k h^k$ where C_k is a constant depending on k only. Define two functions $\xi(u)$, $\eta(u)$ by

$$\xi(u) = |f(u)-f(x)|\log\frac{1}{|u-x|} \text{ for all } u \text{ with } |u-x|<1,\ u\neq x,$$

$$\eta(u) = |f(u)-f(y)|\log\frac{1}{|u-y|} \text{ for all } u \text{ with } |u-y|<1,\ u\neq y.$$

The domains of definition of $\xi(u)$ and $\eta(u)$ certainly contain all points u in I since $h<\delta\leqslant 1/e^2$ and so

$$\frac{2h}{3}<\frac{2}{3e^2}<\frac{1}{10}<1.$$

For any real number $N>0$ we define two subsets H_N, K_N of I by

$$H_N = \{u \mid u\in I,\ \xi(u)>N\},\qquad K_N = \{u \mid u\in I,\ \eta(u)>N\}.$$

The function $\xi(u)$ is non-negative at all points $u\in I$ and so by Tschebychef's inequality we have

$$|H_N| \leqslant N^{-1}\int_I \xi(u)\,du = N^{-1}\int_I |f(u)-f(x)|\log\frac{1}{|u-x|}\,du$$
$$\leqslant N^{-1}\left(\log\frac{3}{h}\right)\int_I |f(u)-f(x)|\,du.$$

In the last step above we have used the inequality $|u-x|\geqslant h/3$ for all $u\in I$.

Since the sphere I is contained in the sphere centered at x with radius h we have, using (17.4),

$$\int_I |f(u)-f(x)|\,du \leqslant \int_{|u-x|\leqslant h}|f(u)-f(x)|\,du$$
$$= \int_{|t|\leqslant h}|f(x+t)-f(x)|\,dt \leqslant h^k M l(h).$$

A combination of the preceding two inequalities gives

$$|H_N| \leqslant N^{-1}\left(\log\frac{3}{h}\right)h^k M l(h) = \frac{Mh^k}{N}\left(\frac{\log\frac{3}{h}}{\log\frac{1}{h}}\right) = \frac{Mh^k}{N}\left(1+\frac{\log 3}{\log\frac{1}{h}}\right)$$
$$\leqslant \frac{Mh^k}{N}\left(1+\frac{\log 3}{\log e^2}\right) = \frac{Mh^k}{N}\left(1+\frac{\log 3}{2}\right) < \frac{2Mh^k}{N}.$$

In the last step above we have used the inequalities

$$0<h<\delta\leqslant 1/e^2,\qquad \frac{1}{h}>e^2,\qquad \log\frac{1}{h}>\log e^2 = 2.$$

Recalling that $|I| = C_k h^k$ we can rewrite the inequality $|H_N| < 2Mh^k/N$ as

$$|H_N| < \frac{2M|I|}{C_k N}.$$

In an entirely similar way we can prove

$$|K_N| < \frac{2M|I|}{C_k N}.$$

Choosing N to be $N_0 = 4M/C_k$ we then have

$$|H_{N_0}| < \frac{|I|}{2},\qquad |K_{N_0}| < \frac{|I|}{2},\qquad |H_{N_0}|+|K_{N_0}| < |I|.$$

From the last inequality and from $H_{N_0}\subset I$, $K_{N_0}\subset I$ it follows that there exists at least a point $u_0\in I$ such that

$$u_0\notin H_{N_0},\qquad u_0\notin K_{N_0}.$$

Hence $\xi(u_0)\leqslant N_0 = 4M/C_k$, $\eta(u_0)\leqslant N_0 = 4M/C_k$ which mean the following two inequalities respectively:

$$|f(u_0)-f(x)| \leqslant \frac{4M}{C_k}\left(1/\log\frac{1}{|u_0-x|}\right) = \frac{4M}{C_k}l(|u_0-x|) < \frac{4M}{C_k}l(|x-y|),$$

$$|f(u_0)-f(y)| \leqslant \frac{4M}{C_k}\left(1/\log\frac{1}{|u_0-y|}\right) = \frac{4M}{C_k}l(|u_0-y|) < \frac{4M}{C_k}l(|x-y|).$$

Here we have used the fact that $l(t)$ is strictly increasing, $|u_0-x| < |x-y|$ and $|u_0-y| < |x-y|$.

Adding up the last two inequalities we have

$$|f(x)-f(y)| \leqslant |f(u_0)-f(x)|+|f(u_0)-f(y)| < \frac{8M}{C_k}l(|x-y|)$$
$$\text{for}\quad x,y\in P \text{ with } 0<|x-y|<\delta,$$

which shows that (17.8) is satisfied with $M_1 = 8M/C_k$.

Let $\tilde{P}$ be the periodic set generated by P, i. e. $\tilde{P}$ is the smallest periodic set containing P. On $\tilde{P}$ we define a function $\varphi(x)$ by

$$\varphi(x) = f(x)\quad\text{for}\quad x\in\tilde{P}.$$

Clearly the function $\varphi(x)$ is periodic since $f(x)$ is so.

Let Σ denote the system of all cubes of edges 2π in E_k obtained by the dividing hyperplanes

$$\ldots, x_j = -3\pi, x_j = -\pi, x_j = \pi, x_j = 3\pi, x_j = 5\pi, \ldots \quad (j = 1, 2, \ldots, k).$$

Because of $\delta \leqslant 2\,\mathrm{dist}(P, \partial Q_k)$ we claim that

$$\text{(17.9)} \quad x \in \tilde{P}, y \in \tilde{P}, |x-y| < \delta \Rightarrow x \text{ and } y \text{ belong to the same cube in } \Sigma.$$

To prove this we let C_1, C_2 be the cubes (from Σ) to which the points x and y belong respectively. We now show that the supposition of C_1, C_2 being distinct cubes would lead to some contradiction.

Suppose C_1, C_2 were distinct cubes. From the assumption of $P \subset Q_k^0$ (see the remarks at the beginning of this section) and from $x \in \tilde{P}$, $y \in \tilde{P}$ it is clear that we would have not only $x \in C_1$, $y \in C_2$ but actually $x \in C_1^0$, $y \in C_2^0$ (here C_1^0, C_2^0 denote the interiors of C_1, C_2 respectively). With C_1, C_2 being distinct cubes from the system Σ the point y is in the exterior of C_1 and the point x is in the exterior of C_2.

Let L be the straight line segment joining the points x and y. As we pass along L from the point x to the point y we first meet a point b_1 on the boundary of C_1 and then a point b_2 on the boundary of C_2 (it could happen that b_1 and b_2 are the same point). Thus by the additivity of the length of the straight line segment L in E_k and noting that

$$|x-b_1| \geqslant \mathrm{dist}(P, \partial Q_k), \qquad |y-b_2| \geqslant \mathrm{dist}(P, \partial Q_k)$$

we have

$$\begin{aligned} |x-y| &= |x-b_1| + |b_1-b_2| + |b_2-y| \geqslant |b_2-y| + |x-b_1| \\ &\geqslant \mathrm{dist}(P, \partial Q_k) + \mathrm{dist}(P, \partial Q_k) = 2\,\mathrm{dist}(P, \partial Q_k) \geqslant \delta \end{aligned}$$

which contradicts to the assumption $|x-y| < \delta$. This completes the proof of (17.9).

Because of (17.9) and the periodicity of the function $\varphi(x)$, inequality (17.8) carries over to the inequality

$$|\varphi(x) - \varphi(y)| \leqslant M_1 l(|x-y|) \quad \text{for} \quad x, y \in \tilde{P} \text{ with } |x-y| < \delta$$

which can also be written as

$$\text{(17.10)} \quad |\varphi(x) - \varphi(y)| \leqslant M_1 \lambda(|x-y|) \quad \text{for} \quad x, y \in \tilde{P} \text{ with } |x-y| < \delta$$

since the two functions $l(t)$, $\lambda(t)$ coincide when $0 \leqslant t \leqslant 1/e^2$ and we certainly have $|x-y| < 1/e^2$ here (since $\delta \leqslant 1/e^2$ by assumption).

Our next step is to show that the restriction $|x-y| < \delta$ in inequality (17.10) can be removed provided that the constant M_1 on the right-hand

side is replaced by $B = \mathrm{Max}(M_1, 2G/\lambda(\delta))$ where

$$G = \mathrm{Sup}\{|\varphi(x)| \mid x \in \tilde{P}\} = \mathrm{Sup}\{|\varphi(x)| \mid x \in P\} < \infty$$

(the finiteness of G follows from the continuity of $\varphi(x)$ in $x \in P$ and the compactness of the set P). In other words, we are claiming validity of the inequality

$$\text{(17.11)} \qquad |\varphi(x) - \varphi(y)| \leqslant B\lambda(|x-y|) \quad \text{for any } x, y \in \tilde{P}$$

without any restriction on $|x-y|$.

In fact, (17.11) is obvious when $|x-y| < \delta$. When $|x-y| \geqslant \delta$, $x \in \tilde{P}$, $y \in \tilde{P}$ we have $\lambda(|x-y|) \geqslant \lambda(\delta)$ (since $\lambda(t)$ increases with t) and so

$$\frac{|\varphi(x) - \varphi(y)|}{\lambda(|x-y|)} \leqslant \frac{|\varphi(x)| + |\varphi(y)|}{\lambda(|x-y|)} \leqslant \frac{G+G}{\lambda(\delta)} \leqslant \mathrm{Max}\left(M_1, \frac{2G}{\lambda(\delta)}\right) = B$$

which proves (17.11) when $|x-y| \geqslant \delta$. The proof of (17.11) is now complete.

Because of (17.11) we can now apply Lemma 5 (Extension Lemma) and conclude that $\varphi(x)$ can be extended to a periodic and continuous function to the whole space E_k such that the extension function, which we now denote by the same symbol $\varphi(x)$, satisfies the inequality

$$\text{(17.12)} \qquad |\varphi(x) - \varphi(y)| \leqslant \sqrt{2}B\lambda(|x-y|) = A\lambda(|x-y|) \quad \text{for any } x, y \in E_k.$$

Since $\lambda(t) \leqslant l(t)$ for all t in $0 \leqslant t < 1$ we can replace the λ function in (17.12) by the l function provided that $|x-y| < 1$. This shows that (17.2) is satisfied with $A = \sqrt{2}B$. The proof of condition I is now complete.

We now come to the proof of condition II.

First we claim that in order to prove (17.3) it is enough to prove

$$\text{(17.13)} \qquad \int_{Q_k} \frac{|\psi(t)|}{|x-t|^k}\,dt < \infty \quad \text{for almost every } x \text{ in } P$$

because (17.13) implies (17.3) provided that the number ε in (17.3) is taken to be $\varepsilon = \mathrm{dist}(P, \partial Q_k)$. To see this we fix on the point $x \in P$ and apply the translation $x + t = t'$ to the integral in (17.3). This gives

$$\int_{|t| \leqslant \varepsilon} \frac{|\psi(x+t)|}{|t|^k}\,dt = \int_{|t'-x| \leqslant \varepsilon} \frac{|\psi(t')|}{|x-t'|^k}\,dt' = \int_{|t-x| \leqslant \varepsilon} \frac{|\psi(t)|}{|x-t|^k}\,dt \leqslant \int_{Q_k} \frac{|\psi(t)|}{|x-t|^k}\,dt.$$

In the last step above we have used the obvious fact that the sphere consisting of all points t satisfying $|t-x| \leqslant \varepsilon$ is contained in the fundamental cube Q_k because $\varepsilon = \mathrm{dist}(P, \partial Q_k)$.

We now apply Lemma 8 (Decomposition Lemma) by setting the cube K there to be the fundamental cube Q_k. This yields the conclusion that there exists an infinite sequence of non-overlapping cubes $\{C_j\}$ $(j = 1, 2, 3, \ldots)$ such that

$$Q_k - P = \bigcup_{j=1}^{\infty} C_j, \quad \operatorname{Lim} d_j = 0 \text{ as } j \to \infty,$$
$$1 \leqslant s_j/d_j \leqslant 3 \quad \text{for all} \quad j = 1, 2, 3, \ldots$$

Here d_j = the diameter (= diagonal) of C_j and $s_j = \operatorname{dist}(C_j, P)$.

Let $\delta(t) = \operatorname{dist}(t, P)$, $t \in E_k$, be the distance function from the set P. Our next step is to establish the following three inequalities: There exist three constants M_2, M_3, M_4 each depending on the numbers η and k (see condition (*) at the beginning of this section for the meaning of the number η) such that the inequalities

$$\int_{C_j} |\psi(t)|\, dt \leqslant M_2 (d_j)^k \lambda(d_j), \tag{17.14}$$

$$(d_j)^k \lambda(d_j) \leqslant M_3 \int_{C_j} \lambda(\delta(t))\, dt, \tag{17.15}$$

$$\int_{C_j} |\psi(t)|\, dt \leqslant M_4 \int_{C_j} \lambda(\delta(t))\, dt \tag{17.16}$$

hold for all those cubes C_j from the sequence $\{C_j\}$ $(j = 1, 2, 3, \ldots)$ with sufficiently small diagonals d_j, say for all d_j satisfying $d_j < \beta$ where $\beta > 0$ depends on k and η.

Let us prove (17.14) first. Consider any cube C_j from the sequence $\{C_j\}$ $(j = 1, 2, 3, \ldots)$. Since C_j and P are compact sets there exist a point $a_j \in P$ and another point $b_j \in C_j$ such that

$$s_j = \operatorname{dist}(C_j, P) = |b_j - a_j|.$$

We now assert that

$$|t - a_j| \leqslant 4d_j \quad \text{for any} \quad t \in C_j$$

which is easily proved by writing

$$|t - a_j| \leqslant |t - b_j| + |b_j - a_j| \leqslant d_j + s_j \leqslant d_j + 3d_j = 4d_j.$$

Noticing that $\psi(a_j) = 0$ (since $a_j \in P$) and from $f = \varphi + \psi$ we have now

$$\begin{aligned} |\psi(t)| = |\psi(t) - \psi(a_j)| &= |\{f(t) - \varphi(t)\} - \{f(a_j) - \varphi(a_j)\}| \\ &\leqslant |f(t) - f(a_j)| + |\varphi(t) - \varphi(a_j)|. \end{aligned} \tag{17.17}$$

By inequality (17.12) we have

$$|\varphi(t) - \varphi(a_j)| \leqslant A\lambda(|t - a_j|) \leqslant A\lambda(4d_j) \leqslant 4A\lambda(d_j) \quad \text{for all} \quad t \in C_j.$$

In the last step above we have used the inequality $\lambda(4t) \leqslant 4\lambda(t)$ for $t > 0$ which follows from the concavity of $\lambda(t)$ for $t \geqslant 0$ and $\lambda(0) = 0$. Integrating the above inequality with respect to t over C_j we get

$$\int_{C_j} |\varphi(t) - \varphi(a_j)|\, dt \leqslant 4A\lambda(d_j) \int_{C_j} dt = 4Ak^{-k/2}(d_j)^k \lambda(d_j). \tag{17.18}$$

Here we have used the simple formula that the volume of a cube in E_k with diagonal d_j is given by $k^{-k/2}(d_j)^k$. The estimate (17.18) is of course valid for all $j = 1, 2, 3, \ldots$

The inequality $|t - a_j| \leqslant 4d_j$ for $t \in C_j$ shows that the cube C_j is contained in the sphere consisting of all points t satisfying $|t - a_j| \leqslant 4d_j$. Hence

$$\int_{C_j} |f(t) - f(a_j)|\, dt \leqslant \int_{|t - a_j| \leqslant 4d_j} |f(t) - f(a_j)|\, dt.$$

The integral on the right-hand side above is equal to

$$\int_{|t| \leqslant 4d_j} |f(t + a_j) - f(a_j)|\, dt$$

by a translation of variables. Applying inequality (17.4) with $h = 4d_j$, $x = a_j$ we have

$$\int_{|t| \leqslant 4d_j} |f(t + a_j) - f(a_j)|\, dt \leqslant (4d_j)^k M l(4d_j) \quad \text{for all} \quad d_j < \frac{\delta}{4}$$

where δ is the number occurring in (17.4). In the above inequality we can replace $l(4d_j)$ by $\lambda(4d_j)$ since $4d_j < \delta \leqslant 1/e^2$. A combination of the last two inequalities together with the concavity inequality $\lambda(4d_j) \leqslant 4\lambda(d_j)$ yields

$$\int_{C_j} |f(t) - f(a_j)|\, dt \leqslant 4^{k+1} M (d_j)^k \lambda(d_j) \quad \text{for all} \quad d_j < \frac{\delta}{4}. \tag{17.19}$$

It now follows from (17.17), (17.18), (17.19) that

$$\begin{aligned} \int_{C_j} |\psi(t)|\, dt &\leqslant \int_{C_j} |f(t) - f(a_j)|\, dt + \int_{C_j} |\varphi(t) - \varphi(a_j)|\, dt \\ &\leqslant (4Ak^{-k/2} + 4^{k+1} M)(d_j)^k \lambda(d_j) \end{aligned}$$

for all cubes C_j with diagonals $d_j < \delta/4$. We have thus shown that (17.14) is satisfied with $M_2 = 4Ak^{-k/2} + 4^{k+1}M$, $\beta = \delta/4$.

Before passing to the proof of the next inequality (17.15) let us revert to Lemmas 6 and 7 which concern some inequalities satisfied by the function $F(m, a)$. In both Lemmas we now specify m to be the dimension k and write $F(a)$ for $F(k, a)$, i. e. to say $F(a)$ is now defined by

$$F(a) = \int_0^a (a - t)^{k-1} \lambda(t)\, dt, \quad a > 0. \tag{17.20}$$

Lemmas 6 now reads: There exists a constant δ_k (depending on k only) such that

$$(17.21)\qquad F(a) > \left(\frac{1}{2k}\right) a^k \lambda(a) \quad \text{for all } a \text{ in } 0 < a < \delta_k.$$

By further specifying the number θ in Lemma 7 to be $\theta = 1/2\sqrt{k}$ Lemma 7 now reads: There exists a constant ε_k (depending on k only) such that

$$(17.22)\qquad F(a) < 2(2k^{1/2})^k F\left(\frac{a}{2\sqrt{k}}\right) \quad \text{for all } a \text{ in } 0 < a < \varepsilon_k.$$

Let $S(x_0, a)$ denote the sphere in E_k centered at the point x_0 and with radius a, i. e. $S(x_0, a) = \{x \mid x \in E_k, |x - x_0| \leqslant a\}$. When we do not wish to specify the center x_0 or when the position of the center x_0 is immaterial we write $S(\cdot, a)$. For any real number $a > 0$ we define a function $\Phi(a)$ by

$$(17.23)\qquad \Phi(a) = \int_{S(\cdot,a)} \lambda(\sigma(x))\,dx, \quad \text{where} \quad \sigma(x) = \operatorname{dist}(x, \partial S(\cdot, a)).$$

Clearly $\Phi(a)$ depends only on a but not on the position of the center of the sphere $S(\cdot, a)$. We now claim that $\Phi(a)$ can be expressed in terms of $F(a)$ by the very simple formula

$$(17.24)\qquad \Phi(a) = \sigma_k F(a) \quad \text{for any} \quad a > 0.$$

In fact, if we specify the center of the sphere $S(\cdot, a)$ in (17.23) to be the origin $(0, \ldots, 0)$ we have $\sigma(x) = a - |x|$ for all points $x \in S(0, a)$. Using polar coordinates we get

$$\Phi(a) = \int_{|x|\leqslant a} \lambda(\sigma(x))\,dx = \int_{|x|\leqslant a} \lambda(a - |x|)\,dx = \sigma_k \int_0^a r^{k-1}\lambda(a-r)\,dr$$
$$= \sigma_k \int_0^a (a-t)^{k-1}\lambda(t)\,dt = \sigma_k F(a).$$

With all these auxiliary remarks we are now ready to prove (17.15). Setting $a = d_j$ in (17.21) and noting (17.24) we have

$$(d_j)^k \lambda(d_j) < \left(\frac{2k}{\sigma_k}\right)\Phi(d_j) \quad \text{for all } d_j \text{ with } 0 < d_j < \delta_k.$$

Setting $a = d_j$ in (17.22) and using (17.24) we have

$$\Phi(d_j) < 2(2k^{1/2})^k \Phi\left(\frac{d_j}{2\sqrt{k}}\right) \quad \text{for all } d_j \text{ with } 0 < d_j < \delta_k.$$

Combining the last two inequalities we have

$$(17.25)\qquad (d_j)^k \lambda(d_j) < M_3 \Phi\left(\frac{d_j}{2\sqrt{k}}\right) \quad \text{for all } d_j \text{ with } 0 < d_j < \beta_k$$

where

$$\beta_k = \operatorname{Min}(\delta_k, \varepsilon_k) \quad \text{and} \quad M_3 = \left(\frac{2k}{\sigma_k}\right) 2(2k^{1/2})^k = (2k^{1/2})^{k+2}/\sigma_k.$$

Our next step is to show

$$(17.26)\qquad \Phi\left(\frac{d_j}{2\sqrt{k}}\right) \leqslant \int_{C_j} \lambda(\delta(t))\,dt \quad \text{for all } j = 1, 2, 3, \ldots$$

By the definition of the function Φ by (17.23) we have

$$(17.27)\qquad \Phi\left(\frac{d_j}{2k}\right) = \int_{S(\cdot, d_j/2\sqrt{k})} \lambda(\sigma_j(t))\,dt,$$

where $\sigma_j(t) = \operatorname{dist}(t, \partial S(\cdot, d_j/2\sqrt{k}))$.

Let x_j be the center of the cube C_j and let $I_j = S(x_j, d_j/2\sqrt{k})$ be the sphere inscribed in the cube C_j. (Clearly the inscribed sphere has radius $= d_j/2\sqrt{k}$.) In the integral in (17.27) we now specify the sphere $S(\cdot, d_j/2\sqrt{k})$ to be the inscribed sphere $I_j = S(x_j, d_j/2\sqrt{k})$. Recalling that $\delta(t) = \operatorname{dist}(t, P)$ and observing that the set P lies in the exterior of the cube C_j and *a fortiori* in the exterior of its inscribed sphere I_j we have clearly $\sigma_j(t) \leqslant \delta(t)$ for all t in the inscribed sphere I_j. Hence $\lambda(\sigma_j(t)) \leqslant \lambda(\delta(t))$ for all $t \in I_j$ and so

$$\Phi\left(\frac{d_j}{2\sqrt{k}}\right) = \int_{I_j} \lambda(\sigma_j(t))\,dt \leqslant \int_{I_j} \lambda(\delta(t))\,dt \leqslant \int_{C_j} \lambda(\delta(t))\,dt, \qquad j = 1, 2, 3, \ldots$$

In the last step above we have used the obvious fact that $I_j \subset C_j$ and that the integrand $\lambda(\delta(t))$ is non-negative for all $t \in C_j$. This proves formula (17.26).

From (17.25) and (17.26) follows

$$\Phi\left(\frac{d_j}{2\sqrt{k}}\right) < M_3 \int_{C_j} \lambda(\delta(t))\,dt \quad \text{for all } d_j \text{ with } 0 < d_j < \beta_k$$

which is precisely (17.25). Incidentally, we have also shown that both M_3 and β_k in the above inequality depend only on k but not on η. This point, however, is of no importance to us here.

Finally the inequality (17.16) is a direct consequence of (17.14) and (17.15) and hence needs no proof.

Our next step is to prove that there exists a constant M_5 depending on η such that the inequality

$$\int_{C_j} \frac{|\psi(t)|}{|x-t|^k}\,dt \leqslant M_5 \int_{C_j} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt \tag{17.28}$$

holds for all points $x \in P$ and for all cubes C_j with sufficiently small diagonals d_j, say for all d_j with $d_j < \beta$ where $\beta > 0$ depends on k and η only.

To prove (17.28) we fix on any point $x \in P$ and let $\varrho_j = \operatorname{dist}(x, C_j)$. We now claim that

$$\varrho_j \leqslant |x-t| \leqslant 2\varrho_j \text{ for all points } t \in C_j, \quad \text{for all } j = 1, 2, \ldots \tag{17.29}$$

The first half of this inequality is obvious since $\varrho_j = \operatorname{Inf}\{|x-t| \mid t \in C_j\}$. To prove the second half we note that C_j is a compact set and so there exists a point $y_j \in C_j$ such that $\varrho_j = \operatorname{dist}(x, C_j) = |x-y_j|$. We then have

$$|x-t| \leqslant |x-y_j| + |y_j-t| \leqslant \varrho_j + d_j \leqslant \varrho_j + \varrho_j = 2\varrho_j \quad \text{for } t \in C_j, \text{ all } j = 1, 2, \ldots$$

Here we have made use of the two inequalities

$$|y_j - t| \leqslant d_j, \qquad d_j \leqslant \varrho_j.$$

The first inequality follows from

$$y_j \in C_j \text{ and } t \in C_j \text{ imply } |y_j - t| \leqslant \operatorname{diam}(C_j) = d_j.$$

The second inequality follows from $d_j \leqslant s_j$ (since C_j is a good cube) and

$$s_j = \operatorname{dist}(P, C_j) \leqslant \operatorname{dist}(x, C_j) = \varrho_j \quad (\text{since } x \in P).$$

Using (17.29) and (17.26) we have

$$\int_{C_j} \frac{|\psi(t)|}{|x-t|^k}\,dt \leqslant \frac{1}{(\varrho_j)^k} \int_{C_j} |\psi(t)|\,dt \leqslant \frac{M_4}{(\varrho_j)^k} \int_{C_j} \lambda(\delta(t))\,dt$$
$$\leqslant 2^k M_4 \int_{C_j} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt$$

for all cubes C_j from the sequence $\{C_j\}$ with $d_j < \beta$ where $\beta > 0$ is the number occurring in (17.16) and depending on k and η. This proves (17.28).

The next step is to prove the following implication:

$$\int_{Q_k} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt < \infty \Rightarrow \int_{Q_k} \frac{|\psi(t)|}{|x-t|^k}\,dt < \infty \quad \text{for any point } x \in P. \tag{17.30}$$

Since $\lambda(\delta(t)) = 0$ and $\psi(t) = 0$ for all $t \in P$ we can, in the two integrals on both sides of (17.30), replace the domain of integration Q_k by $Q_k - P$ without changing the value of the integrals. Thus (17.30) is equivalent to

$$\int_{Q_k-P} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt < \infty \Rightarrow \int_{Q_k-P} \frac{|\psi(t)|}{|x-t|^k}\,dt < \infty, \qquad x \in P. \tag{17.31}$$

We now revert to inequality (17.28) which holds for all cubes C_j from the sequence $\{C_j\}$ with sufficiently small diagonals d_j, say $d_j < \beta$. Because of $d_j \to 0$ as $j \to \infty$ the phrase "for all cubes with sufficiently small diagonals d_j" is equivalent to "for all cubes C_j with all sufficiently large indices j". Hence there exists a positive integer N, where N like β depends on k and η, such that (17.28) holds for all indices $j \geqslant N$. Summing the inequality (17.28) for all $j = N, N+1, N+2, \ldots$ we have

$$\sum_{j=N}^{\infty} \int_{C_j} \frac{|\psi(t)|}{|x-t|^k}\,dt \leqslant M_5 \sum_{j=N}^{\infty} \int_{C_j} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt, \qquad x \in P.$$

Hence

$$\sum_{j=N}^{\infty} \int_{C_j} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt < \infty \Rightarrow \sum_{j=N}^{\infty} \int_{C_j} \frac{|\psi(t)|}{|x-t|^k}\,dt < \infty, \qquad x \in P. \tag{17.32}$$

Using the decomposition $Q_k - P = \bigcup_{j=1}^{\infty} C_j$ we have

$$\int_{Q_k-P} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt = \sum_{j=1}^{\infty} \int_{C_j} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt \tag{17.33}$$
$$= \sum_{j=1}^{N-1} \int_{C_j} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt + \sum_{j=N}^{\infty} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt.$$

Now for each fixed $j = 1, 2, 3, \ldots$ the cube C_j is bounded away from P and so $1/|x-t|^k$ is bounded for all points t in C_j. Therefore

$$\int_{C_j} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt < \infty, \qquad j = 1, 2, 3, \ldots,$$

and so the term

$$\sum_{j=1}^{N-1} \int_{C_j} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt$$

on the right-hand side of (17.33) has finite value. We have thus proved

$$\int_{Q_k-P} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt < \infty \Longleftrightarrow \sum_{j=N}^{\infty} \int_{C_j} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt < \infty, \qquad x \in P. \tag{17.34}$$

In an entirely similar way we can prove

$$(17.35)\qquad \int\limits_{Q_k-P} \frac{|\psi(t)|}{|x-t|^k}\,dt < \infty \Longleftrightarrow \sum_{j=N}^{\infty} \int\limits_{C_j} \frac{|\psi(t)|}{|x-t|^k}\,dt < \infty, \qquad x \in P.$$

(17.32), (17.34) and (17.35) together imply (17.31) and hence (17.30).

We are now practically at the end of the proof. By Lemma 9 we know that

$$\int\limits_{Q_k} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt < \infty$$

for almost every x in P. Hence it follows from (17.30) that

$$\int\limits_{Q_k} \frac{|\psi(t)|}{|x-t|^k}\,dt < \infty$$

for almost every x in P. This proves (17.13) which, as we have already seen, implies (17.3). This completes the proof of condition II stated at the beginning of this section and hence completes the proof of Theorem 1.

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On quasi-Fredholm ideals

by

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Let X be a linear space. Let A be a linear operator (briefly: an operator) mapping the space X into itself. By a *nullity* of the operator A we will call the number $\alpha_A = \dim Z_A$, where

$$Z_A = \{x \in X : Ax = 0\}.$$

By a *deficiency* of the operator A we will call the number $\beta_A = \dim X/AX$, where X/AX is a quotient space. The pair of numbers (α_A, β_A) is called the *d-characteristic* of the operator A. We say that the d-characteristic of an operator A is *finite* if numbers α_A, β_A are both finite.

Let an operator T be given. We say that λ is a *d-point* of the operator T if the operator $A = \lambda I - T$ possesses a finite d-characteristic.

Suppose we are given an algebra $\mathscr{X}$ of linear operators mapping space X into itself. Let $\mathscr{I}$ be a two-sided ideal in the algebra $\mathscr{X}$.

We say that the ideal $\mathscr{I}$ is a *quasi-Fredholm ideal* if, for each $T \in \mathscr{I}$, $I+T$ is an operator with a finite d-characteristic. We say that the ideal $\mathscr{I}$ is a *Fredholm ideal* if we have also $\varkappa_{I+T} = \beta_{I+T} = \alpha_{I+T} = 0$.

The aim of this note is a characterisation of quasi-Fredholm ideals in operator algebras. The terminology and notation in this paper are the same as in paper [6].

We say that an operator $A \in \mathscr{X}$ possesses a *simple regularizer* $R_A \in \mathscr{X}$ to the ideal $\mathscr{I}$ if

$$AR_A = I+T_1, \quad R_A A = I+T_2, \quad \text{where} \quad T_1, T_2 \in \mathscr{I}.$$

If A possesses a simple regularizer to a quasi-Fredholm ideal $\mathscr{I}$, then A possessses a finite d-characteristic ([9], proposition 5.7).

PROPOSITION 1. *If T belongs to a quasi-Fredholm ideal I, then each number $\lambda \neq 0$ is a d-point.*

Proof. The operator

$$\lambda I + T = \lambda\left(I + \frac{1}{\lambda} T\right)$$

possesses a finite d-characteristic because $T/\lambda \in \mathscr{I}$.