In an entirely similar way we can prove

$$(17.35)\qquad \int\limits_{Q_k-P} \frac{|\psi(t)|}{|x-t|^k}\,dt < \infty \Longleftrightarrow \sum_{j=N}^{\infty} \int\limits_{C_j} \frac{|\psi(t)|}{|x-t|^k}\,dt < \infty, \qquad x \in P.$$

(17.32), (17.34) and (17.35) together imply (17.31) and hence (17.30).

We are now practically at the end of the proof. By Lemma 9 we know that

$$\int\limits_{Q_k} \frac{\lambda(\delta(t))}{|x-t|^k}\,dt < \infty$$

for almost every x in P. Hence it follows from (17.30) that

$$\int\limits_{Q_k} \frac{|\psi(t)|}{|x-t|^k}\,dt < \infty$$

for almost every x in P. This proves (17.13) which, as we have already seen, implies (17.3). This completes the proof of condition II stated at the beginning of this section and hence completes the proof of Theorem 1.

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DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF HONG KONG, HONG KONG.

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On quasi-Fredholm ideals

by

D. PRZEWORSKA-ROLEWICZ and S. ROLEWICZ (Warszawa)

Let X be a linear space. Let A be a linear operator (briefly: an operator) mapping the space X into itself. By a *nullity* of the operator A we will call the number $\alpha_A = \dim Z_A$, where

$$Z_A = \{x \in X \colon Ax = 0\}.$$

By a *deficiency* of the operator A we will call the number $\beta_A = \dim X/AX$, where X/AX is a quotient space. The pair of numbers (α_A, β_A) is called the *d-characteristic* of the operator A. We say that the d-characteristic of an operator A is *finite* if numbers α_A, β_A are both finite.

Let an operator T be given. We say that λ is a *d-point* of the operator T if the operator $A = \lambda I - T$ possesses a finite d-characteristic.

Suppose we are given an algebra $\mathscr{X}$ of linear operators mapping space X into itself. Let $\mathscr{I}$ be a two-sided ideal in the algebra $\mathscr{X}$.

We say that the ideal $\mathscr{I}$ is a *quasi-Fredholm ideal* if, for each $T \in \mathscr{I}$, $I+T$ is an operator with a finite d-characteristic. We say that the ideal $\mathscr{I}$ is a *Fredholm ideal* if we have also $\varkappa_{I+T} = \beta_{I+T} = \alpha_{I+T} = 0$.

The aim of this note is a characterisation of quasi-Fredholm ideals in operator algebras. The terminology and notation in this paper are the same as in paper [6].

We say that an operator $A \in \mathscr{X}$ possesses a *simple regularizer* $R_A \in \mathscr{X}$ to the ideal $\mathscr{I}$ if

$$AR_A = I+T_1, \qquad R_A A = I+T_2, \qquad \text{where} \qquad T_1, T_2 \in \mathscr{I}.$$

If A possesses a simple regularizer to a quasi-Fredholm ideal $\mathscr{I}$, then A possessses a finite d-characteristic ([9], proposition 5.7).

PROPOSITION 1. *If T belongs to a quasi-Fredholm ideal I, then each number $\lambda \neq 0$ is a d-point.*

Proof. The operator

$$\lambda I + T = \lambda\left(I + \frac{1}{\lambda}T\right)$$

possesses a finite d-characteristic because $T/\lambda \in \mathscr{I}$.

By the *radical* $R(\mathscr{X}_0)$ of an algebra $\mathscr{X}_0$ with unit e we will call the set of such elements x that for each $a, b \in \mathscr{X}_0$ the element $e+axb$ is invertible. The radical is a two-sided ideal (Jacobson [6]).

Suppose we are given an algebra $\mathscr{X}$ of linear operators. A set of all finite dimensional operators belonging to $\mathscr{X}$ will be denoted by $\mathscr{K}$. It is easy to check that $\mathscr{K}$ is a two-sided ideal.

We divide the algebra $\mathscr{X}$ by the ideal $\mathscr{K}$. Let $\mathscr{X}_0 = \mathscr{X}/\mathscr{K}$ be the quotient algebra. By $R(\mathscr{X}_0)$ we denote the radical of the algebra $\mathscr{X}_0$. By $\mathscr{K}_0$ we denote the set of all operators which belong to the cosets belonging to $R(\mathscr{X}_0)$.

THEOREM 1. *The ideal $\mathscr{K}_0$ is a quasi-Fredholm ideal.*

Proof. Let $U \in \mathscr{K}_0$. From the definition of the radical in the algebra $\mathscr{X}_0$, the element $\tilde{I}+\tilde{U}$ is invertible, where $\tilde{I}$ denotes the coset containing I and $\tilde{U}$ — the coset containing U. Let $\tilde{V}$ be a coset inverse to $\tilde{I}+\tilde{U}$. Then for each $V \in \tilde{V}$

$$(I+U)V = I+K_1, \qquad V(I+U) = I+K_2,$$

where K_1 and K_2 belong to $\mathscr{K}$. Therefore proposition 5.7 of [9] implies that $I+U$ possesses a finite d-characteristic, q. e. d.

COROLLARY. *Operators belonging to $\mathscr{K}_0$ are perturbations of operators with finite d-characteristics belonging to $\mathscr{X}$* (theorem 3.2 of [9]).

THEOREM 2. *If each operator with a finite d-characteristic belonging to $\mathscr{X}$ possesses a simple regularizer to the ideal $\mathscr{K}$, then each quasi-Fredholm ideal $\mathscr{I}$ is contained in $\mathscr{K}_0$.*

Proof. Let $U \in \mathscr{I}$. Then for arbitrary $A, B \in \mathscr{X}$, $AUB \in \mathscr{I}$. The operator $I+AUB$ possesses a finite d-characteristic. Therefore the assumption implies that it possesses a simple regularizer V. But the coset induced by V is inverse in $\mathscr{X}_0$ to the coset induced by $I+AUB$. The arbitrariness of A, B implies that $\tilde{U}$ belongs to the radical $R(\mathscr{X}_0)$. Therefore $U \in \mathscr{K}_0$, q. e. d.

Example 1. Let $\mathscr{X}$ be an algebra of all linear operators. Theorem 6.2 of [9] implies that $\mathscr{X}$ satisfies the assumption of theorem 2.

Example 2. Let $\mathscr{X}$ be an algebra of all linear operators preserving a certain conjugate space Ξ [11]. Then $\mathscr{X}$ satisfies the assumption of theorem 2.

Indeed, in the proof of theorem 6.2 of [9] we can assume that $I-K$ is a Φ_Ξ-operator (see [9], proposition 4.2) and that $\mathfrak{C}$ can be described by a finite system of functionals. Moreover, $E_A = AX$ is also described by a finite system of functionals. Hence we can extend the operator A_1^{-1} to an operator preserving the space Ξ.

Example 3. Let X be a linear metric complete space. Let $\mathscr{X}$ be a set of all continuous linear operators. Then $\mathscr{X}$ satisfies the assumption of theorem 2.

The proof is similar to that of theorem 6.2 of [9]: it makes use of the Banach theorem on the continuity of an inverse operator.

Remark 1. Let $\mathscr{X}$ be an algebra of operators operating in X. Let $\mathscr{I}$ be an arbitrary quasi-Fredholm ideal. Let each operator $A \in \mathscr{X}$ with a finite d-characteristic possess a simple regularizer to the ideal $\mathscr{I}$. Let $\mathscr{X}_1 = \mathscr{X}/\mathscr{I}$. Let $R(\mathscr{X}_1)$ be the radical of the algebra $\mathscr{X}_1$. Let $\mathscr{I}_0$ be the set of operators which belong to the cosets belonging to $R(\mathscr{X}_1)$. Then $\mathscr{I}_0 = \mathscr{K}_0$.

In fact, repeating the arguments of theorems 1 and 2 and replacing $\mathscr{K}$ by $\mathscr{I}$ we find that $\mathscr{I}_0$ is a maximal quasi-Fredholm ideal. Therefore $\mathscr{K}_0 = \mathscr{I}_0$, q. e. d.

PROPOSITION 2. *Let X be a locally bounded space* (Aoki [1], Rolewicz [12]). *Let X be an algebra of bounded operators. Then each quasi-Fredholm ideal is a Fredholm ideal.*

Proof. Basing ourselves on theorem 3.1 of Gochberg and Krein [4] and [10] we find that for each $\lambda \neq 0$ the index $\varkappa_{\lambda I-T} = \beta_{\lambda I-T} - \alpha_{\lambda I-T}$ is constant. But for sufficiently large λ an operator $\lambda I - T$ is invertible. Therefore $\varkappa_{I-T} \equiv 0$, q. e. d.

PROPOSITION 3. *Let X be a locally bounded space. Let $\mathscr{X}$ be an algebra of bounded operators. If $T \in \mathscr{K}_0$, then the set of eigenvalues of T either is finite or forms a sequence tending to 0. Moreover, for each λ the set of corresponding eigenvectors is of finite dimension.*

It is a simple consequence of theorem 3.3 of Gochberg and Krein [4] and [10].

PROPOSITION 4 (1). *Let X be a locally bounded space. Let $\mathscr{X}$ be the algebra of all bounded operators. Let $\mathscr{I}$ be an arbitrary ideal of such operators that the spectrum of $T \in \mathscr{I}$ either is finite or forms a sequence tending to zero and, moreover, for each eigenvalue λ the set of eigenvectors is of finite dimension; then $\mathscr{I} \subset \mathscr{K}_0$.*

The above trivially follows from theorem 2.

PROPOSITION 5. *Let X be a locally bounded space. Let $\mathscr{X}$ be an algebra of all bounded operators. Then the ideal $\mathscr{K}_0$ is closed.*

Proof. Basing ourselves on remark 1 we will construct the ideal $\mathscr{K}_0$ starting from the ideal $\mathscr{I} = \overline{\mathscr{K}}$ (= the closure of $\mathscr{K}$). Algebra $\mathscr{X}_1 = \mathscr{X}/\mathscr{I}$ is a locally bounded algebra, whence (see Żelazko [15]) the radical $R(\mathscr{X}_1)$ of the algebra $\mathscr{X}_1$ is closed. Therefore $\mathscr{I}_0$ is closed. But $\mathscr{I}_0 = \mathscr{K}_0$, q. e. d.

COROLLARY. *If $X = l^p$, $0 < p \leqslant +\infty$, then the ideal $\mathscr{K}_0$ of the algebra $\mathscr{X}$ is an ideal of compact operators.*

Proof. In the space l^p, $1 \leqslant p \leqslant \infty$ (where $l^\infty = c_0$), there is only one closed two-sided ideal (Gochberg, Markus, Feldman [5]) the ideal of com-

(1) See Kleinecke [7] and Yood [14] for Banach spaces.

pact operators $\mathcal{T}(X)$. It is also true for l^p, $0 < p < 1$. Therefore $\mathcal{K}_0 = \mathcal{T}(X)$, q. e. d.

But there are spaces X for which the ideal $\mathcal{K}_0$ is essentially larger than $\mathcal{T}(X)$.

Example 4. Let X be either the space $C(S)$ or the space $L^1(S, \Sigma, \mu)$, where S is a compact set, Σ is a σ-algebra of subsets of S and μ is a measure determined on Σ (2). In these spaces each weak compact operator belongs to $\mathcal{K}_0$.

Proof. The Dunford-Pettis [2] theorem (see also [3]) implies that the square T^2 of each weakly compact operator T is compact. Therefore, since

$$(I+T)(I-T) = I-T^2,$$

the operator $I+T$ possesses a simple regularizer to the ideal of compact operators, whence it possesses a finite d-characteristic. But all weakly compact operators constitute a two-sided ideal. Therefore theorem 2 implies that the ideal of weakly compact operators is contained in $\mathcal{K}_0$, q. e. d.

Since there are weakly compact operators acting in the space $C(S)$ of $L^1(S, \Sigma, \mu)$, which are not compact, it follows that $\mathcal{K}_0$ is essentially larger than the ideal of compact operators.

Theorem 3. *Let X be a space $C(S)$ ($L^1(S, \Sigma, \mu)$). Let $\mathfrak{X}$ be the algebra of all bounded operators. Then $\mathcal{K}_0$ is the ideal of all weakly compact operators.*

Proof. Let $X = C(S)$. If T is not a weakly compact operator, then there are subspaces X_0, $Y_0 \subset C(S)$ such that X and Y_0 are isomorphic to the space c_0 and T transforms X_0 and Y_0 in a one-to-one manner [8]. Obviously the operator T^{-1} determined on Y_0 is continuous. Basing ourselves on Sobczyk theorem [13] the operator T^{-1} can be extended to the operator $\tilde{T}^{-1}$ determined on the whole space $C(S)$. On the space X_0 we have $\tilde{T}^{-1}T = I$. Hence the operator $I-\tilde{T}^{-1}T$ does not possess a finite d-characteristic. Hence $T \notin \mathcal{K}_0$.

If $X = L^1(S, \Sigma, \mu)$ and an operator T is not weakly compact, then there are projections X_0, $Y_0 \subset L^1(S, \Sigma, \mu)$ such that X_0 and Y_0 are isomorphic to the space l and T transforms X_0 on Y_0 in a one-to-one manner [8]. Further proof is the same, q. e. d.

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