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# A generalized group algebra for compact groups

by

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## 1. Introduction and preliminaries.

Definition. An $H^*$-*algebra* $A$ is a Banach algebra, with an involution, that is also a Hilbert space, with the properties

1) $(xy, z) = (x, zy^*)$,

2) $\|x^*\| = \|x\|$,

3) if $x \neq 0$, then $xx^* \neq 0$

for all $x, y, z \in A$.

Perhaps the most important example of an $H^*$-algebra is the group algebra of a compact group, i. e. $L^2(G)$ with convolution as multiplication. In § 3 we shall consider a generalization of $L^2(G)$, viz. the algebra of square-integrable functions on $G$ that have values in an arbitrary $H^*$-algebra. The structure of this algebra will be described in terms of a tensor product of $H^*$-algebras, which is discussed in § 2. In § 4, some of the results of § 3 are extended to non-compact groups via a study of algebra-valued almost periodic functions.

Ambrose [1] proved the following structure theorems for any $H^*$-algebra $A$. $A$ is the orthogonal direct sum of all of its minimal closed two-sided ideals $N$, each of which is isomorphic with a full matrix algebra (possibly infinite dimensional). Each minimal closed ideal $N$ has an orthogonal basis $\{a_{ij}\}$ of "matrix units", with the following properties:

1) $\{a_{ii}\}$ is a maximal collection of orthogonal irreducible self-adjoint idempotents in $N$;

2) $a_{ij}a_{kl} = \begin{cases} a_{il} \text{ if } j = k, \\ 0 \text{ if } j \neq k; \end{cases}$

3) $(a_{ij})^* = a_{ji}$;

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4) $\|a_{ij}\| = \|a_{kl}\|$ for all $i, j, k, l$.

If $A = \Sigma \oplus A_a$ is the decomposition of $A$ as the direct sum of its minimal closed ideals, and if $\{a_{ij}^a\}$ is a basis of matrix units for each $A_a$, then the collection $\{a_{ij}^a\}$, as $a$ is also allowed to vary, is obviously an orthogonal basis for $A$. We shall call such a basis a *canonical basis*; it should be borne in mind that for each fixed $a$ the subscripts of $a_{ij}^a$ vary over an index set that depends on $a$.

Every $H^*$-algebra is semi-simple, in the sense that its Jacobson radical is (0) ([10], p. 272). The concepts of *strong radical* and *strong semi-simplicity* (see [10], p. 59) will be of interest in what follows. The next two theorems provide several characterizations of strong semi-simplicity for $H^*$-algebras.

THEOREM 1.1. *The strong radical $R$ of an $H^*$-algebra $A$ is the orthogonal complement of the (Hilbert space) direct sum of all the finite dimensional minimal closed ideals in $A$.*

Proof. An ideal $M$ in $A$ is maximal if and only if its orthogonal complement $M^\perp$ is a minimal closed ideal, and $M$ is regular if and only if the quotient algebra $A/M$ has an identity. But $A/M$ is easily seen to be isometrically isomorphic with $M^\perp$, and a minimal closed ideal has an identity if and only if it is finite dimensional ([6], p. 105). Thus if $\{N_a\}$ is the collection of all finite dimensional minimal closed ideals in $A$, we have $R = \bigcap_a M_a$, where $M_a = N_a^\perp$, and so $R = \bigcap_a N_a^\perp = (\Sigma_a \oplus N_a)^\perp$, since the ideals $N_a$ are orthogonal.

THEOREM 1.2. *If $A$ is an $H^*$-algebra, the following conditions are equivalent:*

1) *$A$ is strongly semi-simple.*

2) *Each minimal closed ideal of $A$ is finite dimensional.*

3) *Each maximal ideal of $A$ is regular.*

4) *Each closed ideal in $A$ is the intersection of all the regular maximal ideals in which it is included.*

5) *$A$ is a completely continuous algebra (i. e. every left regular representation operator is compact).*

Proof. The equivalence of 1) and 2) is immediate from Theorem 1.1, and the equivalence of 2) and 3) is implicit in the proof of Theorem 1.1. That 3) and 4) are equivalent follows from the fact that each closed ideal in any $H^*$-algebra is the intersection of all the maximal ideals in which it is included ([6], p. 103). Ambrose [1] showed that 1) follows from 5), and the converse was proved by Nakano ([9], p. 20) in the more general context of Hilbert algebras.

**2. Tensor products of $H^*$-algebras.** In their first paper [7] on rings of operators, F. Murray and J. von Neumann defined a tensor product of Hilbert spaces which was again a Hilbert space. In this section we shall consider the corresponding tensor product when the Hilbert spaces under consideration are $H^*$-algebras. The discussion in [7] applies to the tensor product $A_1 \otimes A_2 \otimes \ldots \otimes A_n$ of an arbitrary finite number of Hilbert spaces $A_i$. In order to simplify the notation we shall consider only the tensor product of two $H^*$-algebras. However, the arguments used will apply to the more general situation, and we shall, in the sequel, make occasional use of the results obtained in this section as applied to tensor products of three or more $H^*$-algebras.



Suppose then that $A$ and $B$ are $H^*$-algebras, and that their decompositions into direct sums of minimal closed ideals are given by: $A = \Sigma_a \oplus A_a$, $B = \Sigma_\beta \oplus B_\beta$. We shall denote canonical bases for $A$ and $B$ by $\{a_{ij}^a\}$ and $\{b_{kl}^\beta\}$, respectively.

Definition (following [7]). $A \otimes' B$ is the linear space of all conjugate bilinear functionals on $A \times B$ of the form

$$T = \sum_{i=1}^{n} a_i \otimes b_i, \quad \text{where} \quad T(a, b) = \sum_{i=1}^{n} (a_i, a)(b_i, b)$$

for all pairs $\langle a, b \rangle \in A \times B$. $A \otimes' B$ has an inner product, defined by:

$$\Big(\sum_{i=1}^{n} a_i \otimes b_i, \sum_{j=1}^{m} c_j \otimes d_j\Big) = \sum_{i,j} (a_i, c_j)(b_i, d_j).$$

$A \otimes B$ is the completion of the inner product space $A \otimes' B$ with respect to the metric induced by the norm $\|T\| = \sqrt{(T, T)}$.

The collection $\{a_{ij}^a \otimes b_{kl}^\beta\}$ is an orthogonal basis in $A \otimes B$ ([7], p. 132). We define a multiplication in $A \otimes B$ as follows. If

$$S = \sum \lambda_{ijkl}^{a\beta} (a_{ij}^a \otimes b_{kl}^\beta) \quad \text{and} \quad T = \sum \eta_{ijkl}^{a\beta} (a_{ij}^a \otimes b_{kl}^\beta),$$

then

$$ST = \sum \lambda_{ijkl}^{a\beta} \eta_{mnpq}^{\gamma\delta} (a_{ij}^a a_{mn}^\gamma \otimes b_{kl}^\beta b_{pq}^\delta).$$

By virtue of the properties of canonical bases,

$$ST = \sum \Big(\sum_{j,l} \lambda_{ijkl}^{a\beta} \eta_{jnlq}^{a\beta}\Big) (a_{in}^a \otimes b_{kq}^\beta).$$

If we denote $\|a_{ij}^a\|^2$ and $\|b_{kl}^\beta\|^2$ by $r_a$ and $s_\beta$, respectively, then the Parseval expansions of $\|S\|^2$, $\|T\|^2$, and $\|ST\|^2$, in terms of the orthogonal basis $\{(a_{ij}^a \otimes b_{kl}^\beta)/\sqrt{r_a s_\beta}\}$ are:

$$\|S\|^2 = \sum |\lambda_{ijkl}^{a\beta}|^2 r_a s_\beta, \qquad \|T\|^2 = \sum |\eta_{ijkl}^{a\beta}|^2 r_a s_\beta,$$

and

$$\|ST\|^2 = \sum \Big| \sum_{j,l} \lambda^{\alpha\beta}_{ijkl}\eta^{\alpha\beta}_{jnlq}\Big|^2 r_\alpha s_\beta .$$

By Schwarz's inequality,

$$\|ST\|^2 \leqslant \sum \Big(\sum_{j,l} |\lambda^{\alpha\beta}_{ijkl}|^2\Big)\Big(\sum_{m,p} |\eta^{\alpha\beta}_{mnpq}|^2\Big) r_\alpha s_\beta$$

$$\leqslant \sum |\lambda^{\alpha\beta}_{ijkl}|^2 |\eta^{\gamma\delta}_{mnpq}|^2 r_\alpha r_\gamma s_\beta s_\delta = \|S\|^2\|T\|^2,$$

and so $A\otimes B$ is a Banach algebra.

We define an involution on $A\otimes B$ in the obvious fashion, viz. $(a\otimes b)^* = a^*\otimes b^*$, and the definition is extended by linearity.

THEOREM 2.1. *$A\otimes B$ is an $H^*$-algebra.*

Proof. It remains only to be established that the three technical conditions imposed on $H^*$-algebras hold for $A\otimes B$. The first two involve easy calculations and will be omitted. As for 3), suppose that $T\in A\otimes B$, and $T\neq 0$. We wish to prove that $TT^*\neq 0$. By [1], p. 370, it will suffice to prove that $A\otimes B$ is *proper*, in the sense that the only element $S$ of $A\otimes B$ with the property that $S\cdot(A\otimes B) = (0)$ is $S = 0$. Thus suppose that

$$S = \sum \lambda^{\alpha\beta}_{ijkl}(a^\alpha_{ij}\otimes b^\beta_{kl})$$

annihilates the whole algebra $A\otimes B$ on the left. Then for each $a^\alpha_{jj}\otimes b^\beta_{ll}$ we have

$$0 = S\cdot(a^\alpha_{jj}\otimes b^\beta_{ll}) = \sum_{i,k} \lambda^{\alpha\beta}_{ijkl}(a^\alpha_{ij}\otimes b^\beta_{kl}).$$

Choose $m$, $n$ and $p$, $q$ from the index sets determined by $\alpha$ and $\beta$, respectively. Then

$$0 = \Big(\sum_{i,k} \lambda^{\alpha\beta}_{ijkl}(a^\alpha_{ij}\otimes b^\beta_{kl}),\ a^\alpha_{mn}\otimes b^\beta_{pq}\Big) = \lambda^{\alpha\beta}_{mnpq}\|a^\alpha_{mn}\|^2\|b^\beta_{pq}\|^2,$$

from which it follows that $\lambda^{\alpha\beta}_{mnpq} = 0$, and so $S = 0$. The theorem is proved.

Since it has been established that $A\otimes B$ is an $H^*$-algebra, the next task is to determine the structure of its minimal closed ideals. For a fixed pair $\langle\alpha,\beta\rangle$, let $I$ and $I'$ denote the linear span and closed linear span, respectively, of the set $\{a^\alpha_{ij}\otimes b^\beta_{kl}\}$ in $A\otimes B$. Each element $T$ of $I$ can be considered as an element $T'$ of $A_\alpha\otimes B_\beta$, simply by restricting the functional $T$ to $A_\alpha\times B_\beta$. It is immediately evident that the mapping $f\colon T\to T'$ is well defined, linear, and an isometry, so that it can be extended to an isometry from $I'$ into $A_\alpha\otimes B_\beta$. The extension $f'$ is, in fact, surjective. For suppose that

$$T' = \sum \lambda^{\alpha\beta}_{ijkl}(a^\alpha_{ij}\otimes b^\beta_{kl})\in A_\alpha\otimes B_\beta .$$



Then $T'$ is the limit of a sequence of finite sums of the same form, each of which is an image under $f$ of an element of $I$. Since $f$ is an isometry, the corresponding sequence of elements of $I$ converges to an element $T$ of $I'$. Obviously $f'(T) = T'$. It is easily verified that $I'$ is a closed ideal in $A\otimes B$, and that $f'$ is multiplicative and involution preserving. As a result, we may and shall identify the ideal $I'$ of $A\otimes B$ with the $H^*$-algebra $A_\alpha\otimes B_\beta$.

THEOREM 2.2. *$A\otimes B = \Sigma_{\alpha,\beta}\oplus(A_\alpha\otimes B_\beta)$ is the decomposition of $A\otimes B$ into the direct sum of its minimal closed ideals.*

Proof. It has already been established that each $A_\alpha\otimes B_\beta$ is a closed ideal in $A\otimes B$, and it is obvious that if $\langle\alpha,\beta\rangle\neq\langle\alpha',\beta'\rangle$ then $A_\alpha\otimes B_\beta$ and $A_{\alpha'}\otimes B_{\beta'}$ are orthogonal. It remains only to show that each $A_\alpha\otimes B_\beta$ is minimal, since it is also obvious that the collection $\{A_\alpha\otimes B_\beta\}$ spans $A\otimes B$ (in fact, each element of a canonical basis for $A\otimes B$ lies in some $A_\alpha\otimes B_\beta$).

Thus suppose that $I\neq(0)$ is a closed ideal in $A_\alpha\otimes B_\beta$, and that

$$0\neq T = \sum\{\lambda^{\alpha\beta}_{ijkl}(a^\alpha_{ij}\otimes b^\beta_{kl})\colon i,j,k,l\}\in I.$$

Then some

$$\lambda^{\alpha\beta}_{ijkl}\neq 0,\qquad (a^\alpha_{ii}\otimes b^\beta_{kk})\cdot T\cdot(a^\alpha_{jj}\otimes b^\beta_{ll}) = \lambda^{\alpha\beta}_{ijkl}(a^\alpha_{ij}\otimes b^\beta_{kl})\in I,$$

and so $a^\alpha_{ij}\otimes b^\beta_{kl}\in I$. Thus if $a^\alpha_{mn}\otimes b^\beta_{pq}$ is any basis element in $A_\alpha\otimes B_\beta$, we have

$$(a^\alpha_{mi}\otimes b^\beta_{pk})(a^\alpha_{ij}\otimes b^\beta_{kl})(a^\alpha_{jn}\otimes b^\beta_{lq}) = a^\alpha_{mn}\otimes b^\beta_{pq}\in I,$$

i. e. $I$ contains a basis for $A_\alpha\otimes B_\beta$, and hence equals $A_\alpha\otimes B_\beta$.

COROLLARY. *$A\otimes B$ is strongly semi-simple if and only if both $A$ and $B$ are strongly semi-simple.*

Proof. The dimension of $A_\alpha\otimes B_\beta$ is the product of the dimensions of $A_\alpha$ and $B_\beta$. The corollary follows from Theorem 1.2.

**3. The generalized group algebra.** In this section we shall discuss a generalization, $B^2(G,A)$, of $L^2(G)$, where $G$ is a compact group. The generalization is broad enough to include all $H^*$-algebras, in the trivial sense that an arbitrary $H^*$-algebra $A$ is isometrically isomorphic with $B^2(G,A)$, where $G$ is the one element group. Throughout this section $A$ will denote an $H^*$-algebra and $G$ a compact group with Haar measure $m$ normalized so that $m(G) = 1$.

Definition. $B^2(G,A)$ is the space of all equivalence classes (modulo null functions) of functions $f\colon G\to A$ such that $f$ is Bochner measurable and $\int\|f(x)\|^2dx < +\infty$.

As usual, we shall speak of the functions themselves rather than the equivalence classes to which they belong. If $B^2(G, A)$ is normed by $\|f\|^2 = \int \|f(x)\|^2 dx$, then $B^2(G, A)$ is a Banach space. If multiplication is defined to be convolution,

$$f * g(x) = \int f(y) g(y^{-1}x)\, dx = \int f(xy^{-1}) g(y)\, dy,$$

then it is easily seen that $\|f*g\| \leqslant \|f*g\|_\infty \leqslant \|f\|\,\|g\|$, so that $B^2(G, A)$ is a Banach algebra. Observe that if $f, g \in B^2(G, A)$, or if $f \in L^2(G)$ and $g \in B^2(G, A)$, then $f*g$ is continuous (the proof is the same as the one given in [11], p. 4, for $f, g \in L^2(G)$).

For each $f \in B^2(G, A)$, define the function $f^* \in B^2(G, A)$ by the formula $f^*(x) = f(x^{-1})^*$. It is easily verified that $f^{**} = f$, $(f+g)^* = f^* + g^*$, and $(af)^* = \bar{a}f^*$ for all $f, g \in B^2(G, A)$ and all scalars $a$. In order to show that $(f*g)^* = g^* * f^*$, and hence that the map $f \to f^*$ is an involution, observe that the involution on $A$ is a bounded linear transformation on $A$. Thus we have (by [3], p. 113) $\int f(x)^* dx = (\int f(x)\, dx)^*$ for all $f \in B^2(G, A)$. Applying this fact, we see that

$$(f*g)^*(x) = \Big(\int f(y) g(y^{-1}x^{-1})\, dy\Big)^* = \int \big(f(y) g(y^{-1}x^{-1})\big)^* dy$$
$$= \int g^*(xy) f^*(y^{-1})\, dy = g^* * f^*(x).$$

As a further application of the theorem cited above, note that whenever $f \in B^2(G, A)$ and $a \in A$, we have $\int \big(f(x), a\big) dx = \big(\int f(x)\, dx, a\big)$.

$B^2(G, A)$ is a Hilbert space, since its norm satisfies the parallelogram law. The polarization identity shows that $(f, g) = \int \big(f(x), g(x)\big) dx$ for all $f, g \in B^2(G, A)$. It is not difficult to show directly that $B^2(G, A)$ is, in fact, an $H^*$-algebra, but this will follow from the next theorem, which also provides the key to the minimal ideal structure of $B^2(G, A)$.

THEOREM 3.1. *$B^2(G, A)$ is isomorphic and isometric with $L^2(G) \otimes A$.*

Proof. We shall map $L^2(G) \otimes' A$ into $B^2(G, A)$. If

$$T = \sum_{i=1}^{n} f_i \otimes a_i,$$

define

$$\varphi(T) = \sum_{i=1}^{n} f_i(\cdot) a_i \in B^2(G, A).$$

In order to show that $\varphi$ is well-defined (i. e. independent of the representation chosen for the functional $T$) it suffices to prove that $\varphi(T) = 0$ ecif $T = 0$, sin $\varphi$ is obviously linear. Thus suppose that $T = 0$.



Then

$$(\varphi T, f(\cdot) a) = \sum_i \big(f_i(\cdot) a_i, f(\cdot) a\big) = \sum_i \int \big(f_i(x) a_i, f(x) a\big) dx$$
$$= \sum_i \int f_i(x) \overline{f(x)}\, dx\, (a_i, a) = \sum_i (f_i, f)(a_i, a) = T(f, a) = 0.$$

It follows that $\varphi T$ is orthogonal to all linear combinations

$$\sum_{i=1}^{m} g_i(\cdot) b_i, \quad g_i \in L^2(G), \qquad b_i \in A.$$

These are dense in $B^2(G, A)$ since, for example, the collection contains all simple functions, and so $\varphi T = 0$. Observe also that each simple function is in the range of $\varphi$, so that the range of $\varphi$ is dense.

Routine computations show that $\varphi$ is multiplicative and involution preserving. Thus it remains only to prove that $\varphi$ is an isometry. If

$$T = \sum_{i=1}^{n} f_i \otimes a_i \quad \text{and} \quad S = \sum_{j=1}^{m} g_j \otimes b_j,$$

then

$$(\varphi T, \varphi S) = \sum_{i,j} \big(f_i(\cdot) a_i, g_j(\cdot) b_j\big) = \sum_{i,j} \int \big(f_i(x) a_i, g_j(x) b_j\big) dx$$
$$= \sum_{i,j} (f_i, g_j)(a_i, b_j) = (T, S).$$

The mapping $\varphi$ can now be extended to an isometric isomorphism from $L^2(G) \otimes A$ onto $B^2(G, A)$.

COROLLARY 1. *If $L^2(G) = \Sigma_\alpha \oplus N_\alpha$ and $A = \Sigma_\beta \oplus A_\beta$ are the decompositions of $L^2(G)$ and $A$ into direct sums of minimal closed ideals, then $B^2(G, A) = \Sigma_{\alpha,\beta} \oplus P_{\alpha\beta}$, where $P_{\alpha\beta}$ is isometrically isomorphic with $N_\alpha \otimes A_\beta$, is the decomposition of $B^2(G, A)$ into the direct sum of minimal closed ideals.*

COROLLARY 2. *$B^2(G, A)$ is strongly semi-simple if and only if $A$ is strongly semi-simple.*

We are now in a position to discuss some of the properties of $B^2(G, A)$. Let us first observe that the Plancherel Theorem for a compact group, as stated in [8], p. 436, has a direct generalization to $B^2(G, A)$ when $A$ is strongly semi-simple. In fact, the proof given in [8] applies, with minor modifications, not only to $L^2(G)$ but to any strongly semi-simple $H^*$-algebra. If $\{f_{ij}^\alpha\}$ and $\{a_{kl}^\beta\}$ are canonical bases for $L^2(G)$ and $A$, with $\|f_{ij}^\alpha\| = n_\alpha$ and $\|a_{kl}^\beta\| = r_\beta$, then the statement of the theorem for $B^2(G, A)$ is:

*If $A$ is strongly semi-simple and $f \in B^2(G, A)$, then*

$$\int \|f(x)\|^2 dx = \sum_{a,\beta} n_a r_\beta \operatorname{trace}(T_f^{a\beta} T_{f^*}^{a\beta}),$$

*where $T_f^{a\beta}$ is the restriction of the left regular representation operator $T_f$ to the minimal left ideal $B^2(G, A) * (f_{11}^a(\cdot) a_{11}^\beta)$.*

It should perhaps be emphasized that this is a rather simple consequence of Parseval's Theorem, and that it is of interest only in that it casts the formula of Parseval's Theorem into the same form as the formula in a general Plancherel Theorem for possibly non-abelian and non-compact groups. The formula does not seem to apply when $A$ is not strongly semi-simple, for then, by Theorem 1.2, some $T_f^{a\beta}$ is not a compact operator, from which it follows (see [9]) that $T_f^{a\beta} T_{f^*}^{a\beta}$ is not a compact operator, and hence is not in the trace class of operators (see [12]).

THEOREM 3.2. *A closed left ideal $I$ of $B^2(G, A)$ is a left translation invariant subspace of $B^2(G, A)$, i. e. if $f \in I$ then $f_x \in I$ for every $x \in G$, where $f_x(y) = f(xy)$. A similar statement holds for closed right ideals and right translation.*

Proof. Let $\{e_a\}$ be a maximal collection of irreducible self-adjoint orthogonal idempotents in $B^2(G, A)$, let $\{J\}$ be the collection of all finite subsets of the index set $\{a\}$, and let $u_J = \sum\{e_a : a \in J\}$ for each $J$. Then

$$f = \sum_a f * e_a = \sum_a e_a * f,$$

and so $u_J * f \underset{J}{\to} f$ for every $f \in B^2(G, A)$. If $f \in I$, then we have $(u_J)_x * f = (u_J * f)_x \to f_x \in I$. A similar argument yields the corresponding result for right ideals.

COROLLARY. *If $I$ is a closed left ideal in $B^2(G, A)$ and $h \in L^2(G)$, then $h * f \in I$ for every $f \in I$. Again a similar statement holds for right ideals.*

Proof. Let $g$ be any element of $I^\perp$. Then

$$(h*f, g) = \int \Big( \int h(y) f(y^{-1}x) dy, g(x) \Big) dx = \int\int h(y) \big(f_{y^{-1}}(x), g(x)\big) dy\, dx$$
$$= \int h(y) (f_{y^{-1}}, g) dy = 0,$$

since $f_{y^{-1}} \in I$. Thus $h * f \in I^{\perp\perp} = I$.

The converse of Theorem 3.2 is also true when $A$ is the field $K$ of complex numbers, i. e. for $L^2(G)$ (recall that the converse, as stated for $L^1$, is instrumental in one proof of the Wiener Tauberian Theorem, see [6], p. 148). A simple counter-example shows that the converse may fail to be true for $B^2(G, A)$. Let $G = \{e\}$ be the one element group, and let $A = K \oplus K$, with operations defined pointwise. Observe that then $B^2(G, A)$ can be identified with $A$, and that translation invariance in $B^2(G, A)$ means nothing at all. Thus the subspace $I = \{(c, c) : c \in K\}$ is closed and translation invariant, but $I$ is not an ideal. It is interesting to observe that $B^2(G, A)$, in this example, is isomorphic with $L^2(H)$, where $H$ is the two element group, and that the converse of Theorem 3.2 holds for $L^2(H)$. The difficulty is that the isomorphism is not the identity isomorphism on $K \oplus K$, and that translation invariance does not have the same meaning for the two spaces.



When $B^2(G, A)$ is commutative (i. e. when both $G$ and $A$ are commutative) an analogue of the Wiener Tauberian Theorem can be stated, but it is not always true. The statement is:

*If $f \in B^2(G, A)$ has the property that $\hat{f}$ (the Gelfand transform of $f$) never vanishes, then the linear span of the translates of $f$ is dense in $B^2(G, A)$.*

If we choose $f = (c, c)$, with $c \neq 0$, in the example in the preceding paragraph, it is easy to see that the statement above is false. $A$ has just two maximal ideals, viz. the two copies of $K$, and $\hat{f}(K) = c \neq 0$ for each of these. But the linear span of the translates of $f$ is just the subspace $I$, which is closed and proper. It is interesting to observe that the Wiener Tauberian Theorem for the $L^1$ algebra of a locally compact abelian group $G$ follows from the fact that each proper closed ideal of $L^1(G)$ is included in a regular maximal ideal. It is an immediate consequence of Theorem 1.2 that every proper closed ideal of $B^2(G, A)$ is included in a regular maximal ideal, when $A$ is strongly semi-simple. The reason that the analogue of the Wiener Theorem does not follow from this fact is that closed translation invariant subspaces may fail to be ideals.

Definition. A function $f \in B^2(G, A)$ is called *almost invariant* if and only if the ideal generated by the collection of all translates $f_x^y$ of $f$ is finite dimensional.

THEOREM 3.3. *If $A$ is strongly semi-simple, then every continuous function on $G$ to $A$ can be uniformly approximated by almost invariant functions from $B^2(G, A)$. Conversely, if every continuous function on $G$ to $A$ is a uniform limit of almost invariant functions, then $A$ is strongly semi-simple.*

Proof. Suppose that $f: G \to A$ is continuous, and that $\varepsilon > 0$ is given. The mapping $y \to \|f_{y^{-1}} - f\|_\infty$ is continuous, and so there exists a neighborhood $W$ of $e \in G$ such that $\|f_{y^{-1}} - f\|_\infty < \varepsilon/3$ whenever $y \in W$. Choose a neighborhood $V$ of $e$ such that $V^2 \subseteq W$, and then a continuous function $v$ on $G$ to the complex numbers which is non-negative, has its support in $V$, and is such that $\int v(x) dx = 1$. Let $u = v * v$, and observe that $u$ is a non-negative continuous function with support in $W$ such that $\int u(x) dx = 1$. If $L^2(G) = \Sigma_a \oplus N_a$ is the decomposition of $L^2(G)$ into the direct sum of its minimal ideals, and if $v = \sum_a v_a$, $v_a \in N_a$, then we have

$u = \sum_{a}(v_a * v_a) = \sum_a u_a$, $u_a \in N_a$, and $\sum_a \|u_a\| \leqslant \sum_a \|v_a\|^2 = \|v\|^2 < +\infty$. Furthermore,

$$\|f(x) - u*f(x)\| \leqslant \int_W u(y)\|f(x) - f_{y^{-1}}(x)\|dy < \varepsilon/3,$$

and so $\|f - u*f\|_\infty \leqslant \varepsilon/3$.

If $B^2(G, A) = \Sigma_{a,\beta}\oplus P_{a\beta}$ is the decomposition of $B^2(G, A)$ into the direct sum of its minimal closed ideals, let $f = \sum_{a,\beta} f_{a\beta}$, with $f_{a\beta} \in P_{a\beta}$. There are at most countably many $a$ for which either $u_a \neq 0$ or $f_{a\beta} \neq 0$, and at most countably many $\beta$ for which $f_{a\beta} \neq 0$. Let these be enumerated as $\{a_i\}$ and $\{\beta_j\}$, respectively, and choose $n$, $m$ such that

$$\Big\|u - \sum_{i=1}^{m} u_{a_i}\Big\| < \sqrt{\varepsilon/3},$$

and

$$\Big\|f - \sum_{i=1}^{m}\sum_{j=1}^{n} f_{a_i\beta_j}\Big\| < \min\Big\{\sqrt{\varepsilon/3}, \varepsilon\Big/3\sum_{i=1}^{\infty}\|u_{a_i}\|\Big\}.$$

Observe that

$$\Big(u - \sum_1^m u_{a_i}\Big)*\Big(f - \sum_1^m\sum_1^n f_{a_i\beta_j}\Big) = u*f - \sum_{i=1}^{m}\sum_{j=1}^{n} u_{a_i}*f_{a_i\beta_j} - \sum_{i=1}^{m}\sum_{j=n+1}^{\infty} u_{a_i}*f_{a_i\beta_j}.$$

Thus

$$\begin{aligned}\Big\|u*f - \sum_{i=1}^{m}\sum_{j=1}^{n} u_{a_i}*f_{a_i\beta_j}\Big\|_\infty &\leqslant \Big\|u - \sum_{i=1}^{m} u_{a_i}\Big\|\,\Big\|f - \sum_{i=1}^{m}\sum_{j=1}^{n} f_{a_i\beta_j}\Big\| + \sum_{i=1}^{m}\Big(\|u_{a_i}\|\Big\|\sum_{j=n+1}^{\infty} f_{a_i\beta_j}\Big\|\Big)\\ &< \varepsilon/3 + \sum_{i=1}^{m}\|u_{a_i}\|\Big(\varepsilon/3\Big(\sum\|u_{a_i}\|\Big)\Big) \leqslant 2\varepsilon/3,\end{aligned}$$

and so

$$\Big\|f - \sum_{i=1}^{m}\sum_{i=1}^{n} u_{a_i}*f_{a_i\beta_j}\Big\|_\infty < \varepsilon.$$

By the corollary to Theorem 3.2, each $u_{a_i}*f_{a_i\beta_j} \in P_{a_i\beta_j}$, which is finite dimensional since $A$ is strongly semi-simple, so each $u_{a_i}*f_{a_i\beta_j}$ is almost invariant. Since a finite sum of almost invariant functions is again almost invariant, the first statement in the theorem is proved.



To prove the converse, suppose that $A$ is not strongly semi-simple. Then $B^2(G, A)$ has an infinite dimensional minimal closed ideal $N$. Choose an element $F = f(\cdot)a$ from a canonical basis for $N$, such that $f$ is continuous. Then $F$ is also continuous. Since any almost invariant function $g$ must lie within a finite sum of finite dimensional minimal ideals, $g$ and $F$ are orthogonal. Thus $\|F-g\|^2 = \|F\|^2 + \|g\|^2 \geqslant \|F\|^2 \geqslant 1$.

To conclude this section we shall discuss a generalization of E. Schmidt's class of integral operators.

Definition (see [12]). The *Schmidt Class*, $\mathrm{sc}(H)$, of operators on a Hilbert space $H$ is the collection of all operators $T$ on $H$ with the property that $\|T\|_s^2 = \sum_\lambda \|Tu_\lambda\|^2 < +\infty$ for any complete orthonormal set $\{u_\lambda\}$. The norm $\|\cdot\|_s$ is called the *Schmidt Class norm*.

The Schmidt Class is a simple $H^*$-algebra (see [10], p. 287). It is shown in [12] that when $H = L^2(0,1)$, $\mathrm{sc}(H)$ can be identified as the class of integral operators studied by Schmidt. In fact, the proof given there shows that for any compact group $G$, $\mathrm{sc}\big(L^2(G)\big)$ is exactly the class of operators $\{T_K\}$ defined by

$$(T_K f)(x) = \int K(x, y)f(y)\,dy \quad \text{for all} \quad f \in L^2(G),$$

where $K$ is a measurable and square integrable complex-valued function on $G \times G$.

Since $B^2(G, A)$ is a Hilbert space, $\mathrm{sc}\big(B^2(G, A)\big)$ is a well-defined simple $H^*$-algebra. Because of the special nature of $B^2(G, A)$, however, it is possible to consider a somewhat more direct generalization of the class of operators studied by Schmidt.

Definition. $S(G, A)$ is the set $B^2(G\times G, A)$, with addition, scalar multiplication, and the inner product defined as usual, but with: 1) $(K_1K_2)(x, y) = \int K_1(x, u)K_2(u, y)\,du$, and 2) $K^*(x, y) = K(y, x)^*$. $\mathrm{SC}(G, A)$ is the collection of all operators $T_K$ on $B^2(G, A)$ defined by

$$(T_K f)(x) = \int K(x, y)f(y)\,dy$$

for all $f \in B^2(G, A)$, where $K \in S(G, A)$.

If $\mathrm{SC}(G, A)$ is normed not with the operator norm but by $\|T_K\| = \|K\|$, then it is easy to see that the algebras $S(G, A)$ and $\mathrm{SC}(G, A)$ are isometrically isomorphic under the mapping $K \to T_K$. As a result, we shall sometimes consider $S(G, A)$ rather than the algebra $\mathrm{SC}(G, A)$ of operators, which is actually the algebra of interest. Since $S(G, A)$ and $B^2(G\times G, A)$ are one and the same Hilbert space, it follows immediately from Theorem 3.1 that $S(G, A)$ and $L^2(G\times G)\otimes A$ are isometrically isomorphic Hilbert spaces. By the remarks preceding the definition of $S(G, A)$, $L^2(G\otimes G)$, with multiplication and involution defined as in $S(G, A)$, can be identified with $\mathrm{sc}(L^2(G)) = \mathrm{SC}(G)$. Thus $S(G, A)$ and $\mathrm{SC}(G)\otimes A$ are isometric and isomorphic as Hilbert spaces. It is easily verified that the mapping which identifies them is also multiplicative and involution-preserving, and we have

THEOREM 3.4. *$\mathrm{SC}(G, A)$ is isomorphic and isometric with $\mathrm{SC}(G)\otimes A$.*

COROLLARY 1. *$\mathrm{SC}(G, A)$ is an $H^*$-algebra; its decomposition into the direct sum of minimal closed ideals is given by $\mathrm{SC}(G, A) = \Sigma_a \oplus (\mathrm{SC}(G)\otimes \otimes A_a)$, where $\{A_a\}$ is the collection of minimal closed ideals of $A$, so that $\mathrm{SC}(G, A)$ can be identified with $\Sigma_a \oplus \mathrm{SC}(G, A_a)$.*

COROLLARY 2. *$\mathrm{SC}(G, A)$ is strongly semi-simple if and only if $A$ is strongly semi-simple and $G$ is a finite group; it is simple if and only if $A$ is simple.*

Observe that $B^2(G, A)$ can be embedded in $S(G, A)$ as follows: given $f\in B^2(G, A)$, define $K_f \in S(G, A)$ by $K_f(x, y) = f(xy^{-1})$. This corresponds to the fact that the left regular representation operators on $B^2(G, A)$ are elements of $\mathrm{SC}(G, A)$. This shows immediately that if $A$ is not strongly semi-simple, then $\mathrm{SC}(G, A)$ is not a subset of $\mathrm{sc}(B^2(G, A))$, for then some left regular representation operator is not compact, whereas all Schmidt class operators are compact ([12], p. 32). The opposite inclusion may also fail to hold when $A$ is not strongly semi-simple. For example, let $G$ be the one element group and let $A$ be an infinite dimensional simple $H^*$-algebra. Then $\mathrm{SC}(G, A)$ can be identified with $A$. If $\mathrm{sc}(B^2(G, A))$ were included in $\mathrm{SC}(G, A)$ it would be a proper closed ideal, since it is an ideal in the algebra of all bounded operators on $B^2(G, A)$. This, of course, would contradict the simplicity of $A$.

Suppose now that $A$ is strongly semi-simple. Then each minimal ideal of $A$ is isomorphic with a finite dimensional full matrix algebra. If the matrix algebra is given the Euclidean norm, then there exists a constant $r \geqslant 1$ such that if the matrix algebra is renormed by simply multiplying the Euclidean norm of each matrix by $r$, then the isomorphism is also an isometry. In each minimal ideal $N_a$ of $L^2(G)$, the constant $r = r_a$ is equal to $\sqrt{n_a}$, where $n_a^2$ is the dimension of $N_a$.

Definition. A strongly semi-simple $H^*$-algebra $A$ is said to be *natural* if and only if the constant $r_a$ of the preceding paragraph is equal to $\sqrt{n_a}$ for each minimal ideal $A_a$ of dimension $n_a^2$.

Observe that if $\{a_{ij}^a\}$ is a canonical basis in a natural $H^*$-algebra, then $\|a_{ij}^a\|^2 = n_a$.

THEOREM 3.5. *If $A$ is a natural $H^*$-algebra, then $\mathrm{SC}(G, A)$ is a closed subalgebra of $\mathrm{sc}(B^2(G, A))$, and $\|T_K\|_s = \|K\|$ for every $K\in S(G, A)$.*

Proof. Since $\mathrm{SC}(G, A)$ is complete, it is closed, and so we need only establish the set inclusion and the equality of the norms. Given



$T_K \in \mathrm{SC}(G, A)$, we must show ([10], p. 285) that

$$\sum_{\lambda,\mu} |(T_K u_\lambda, u_\mu)|^2 < +\infty$$

for some complete orthonormal set $\{u_\lambda\}$. Thus let $\{f_{ij}^a\}$ and $\{a_{kl}^\beta\}$ be canonical bases for $L^2(G)$ and $A$, and let

$$c_{ijklmnpq}^{a\beta\gamma\delta} = (T_K f_{ij}^a a_{mn}^\gamma, f_{kl}^\beta a_{pq}^\delta)/\sqrt{n_a n_\beta n_\gamma n_\delta}.$$

We wish to show that

$$\sum |c_{ijklmnpq}^{a\beta\gamma\delta}|^2 = \|K\|^2.$$

Let us first compute $\|K\|^2$. Johnson [5] showed that if $G$ and $H$ are locally compact abelian groups, then $L^1(G\times H)$ is isomorphic and isometric with $B^1(G, L^1(H))$, the algebra of integrable Bochner measurable functions on $G$ with values in $L^1(H)$. His proof carries over with few changes to the present situation, and shows that $L^2(G\times H)$ is isomorphic and isometric with $B^2(G, L^2(H))$, and hence with $L^2(G)\otimes L^2(H)$. Since $S(G, A)$ is the same Hilbert space as $B^2(G\times G, A)$, which can be identified with $L^2(G)\otimes L^2(G)\otimes A$, $S(G, A)$ has as a basis $\{f_{kl}^\beta \overline{f_{ij}^a} a_{mn}^\gamma\}$. The Parseval expansion of $\|K\|^2$ with respect to this basis is

$$\|K\|^2 = \sum \{|(K, f_{kl}^\beta \overline{f_{ij}^a} a_{mn}^\gamma)|^2/n_a n_\beta n_\gamma : a, i, j; \beta, k, l; \gamma, m, n\}$$

$$= \sum \Big\{\Big|\iint f_{ij}^a(y)\overline{f_{kl}^\beta(x)}\big(K(x, y), a_{mn}^\gamma\big)\,dx\,dy\Big|^2\Big/ n_a n_\beta n_\gamma : a, i, j; \beta, k, l; \gamma, m, n\Big\}.$$

Observe that $c_{ijklmnpq}^{a\beta\gamma\delta} = 0$ unless $\gamma = \delta$ and $q = n$. Thus we have

$$\|T_K\|_s^2 = \sum \{|c_{ijklmnpn}^{a\beta\gamma\gamma}|^2 : a, i, j; \beta, k, l; \gamma, m, n, p\}$$

$$= \sum \Big\{\Big|\iint \big(K(x, y) f_{ij}^a(y) a_{mn}^\delta, f_{kl}^\beta(x) a_{pn}^\gamma\big)\,dx\,dy\Big|^2\Big/ n_a n_\beta n_\gamma^2 : a, i, j; \beta, k, l; \gamma, m, n, p\Big\}$$

$$= \sum \Big\{\Big|\iint f_{ij}^a(y)\overline{f_{kl}^\beta(x)}\big(K(x, y), a_{pm}^\gamma\big)\,dx\,dy\Big|^2\Big/ n_a n_\beta n_\gamma^2 : a, i, j; \beta, k, l; \gamma, m, n, p\Big\}.$$

Since the index $n$ does not appear in the latter summands, and the cardinality of its index set is $n_\gamma$, the latter sum is seen to be equal to $\|K\|^2$, and the theorem is proved.

The inclusion $\mathrm{SC}(G, A) \subseteq \mathrm{sc}(B^2(G, A))$ may well be proper. It is always proper when $A$ is not simple, for then, by Theorem 3.4, Corollary 2, $\mathrm{SC}(G, A)$ is not simple, whereas $\mathrm{sc}(B^2(G, A))$ is always simple. Equality may fail to hold even when $A$ is simple. Suppose, for example, that $G$

is a finite group of order $n$, and that $A$ is the natural full matrix algebra of dimension $m^2$, where $1 < m < +\infty$. Then

$$\dim\big(\mathrm{SC}(G, A)\big) = \dim\big(\mathrm{SC}(G)\otimes A\big) = \big(\dim L^2(G)\big)^2 \dim(A) = n^2m^2,$$

whereas

$$\dim\big(\mathrm{sc}(B^2(G, A)\big) = \big(\dim B^2(G, A)\big)^2 = \Big(\dim\big(L^2(G)\otimes A\big)\Big)^2 = \big(\dim L^2(G)\big)^2\big(\dim(A)\big)^2 = n^2m^4,$$

so that $\mathrm{SC}(G, A)$ is a proper subalgebra of $\mathrm{sc}\big(B^2(G, A)\big)$.

**4. Algebra-valued almost periodic functions.** S. Bochner and J. von Neumann have studied (in [2]) almost periodic functions defined on a group $G$ and with values in a complete topological linear space. In this section we shall consider the special case where $G$ is a topological group and the linear space in question is a Banach algebra.

Definition. A bounded continuous function $F$ on a topological group $G$ to a Banach algebra $A$ is called *almost periodic* if and only if the collection of all left and right translates $F_x$ and $F^y$ of $F$ is totally bounded with respect to the uniform norm. The collection of all almost periodic functions $F$ is denoted by $\mathrm{AP}(G, A)$. If $A$ is the field of complex numbers we shall write simply $\mathrm{AP}(G)$.

Bochner and von Neumann proved that $\mathrm{AP}(G, A)$ is a Banach space under the uniform norm. It is easily verified that $\mathrm{AP}(G, A)$ is also a Banach algebra, where multiplication is pointwise multiplication of functions. If $A$ has an involution, then $\mathrm{AP}(G, A)$ has an involution, viz. $F^*(x) = F(x)^*$. Recall that $\mathrm{AP}(G)$ is a commutative $B^*$-algebra with identity ([6], § 41), so that it is isomorphic and isometric with $C(\overline{G})$, the algebra of all continuous complex-valued functions on the maximal ideal space $\overline{G}$ of $\mathrm{AP}(G)$. $\overline{G}$, the *Bohr compactification* of $G$, has the structure of a compact group, and there is a continuous homomorphism $\alpha$ from $G$ into $\overline{G}$, with $\alpha(G)$ dense in $\overline{G}$.

We shall have occasion now to consider another normed tensor product of Banach spaces.

Definition (following [4]). If $A$ and $B$ are Banach spaces, with dual spaces $A'$ and $B'$, then $A\otimes' B$ is the linear space of bilinear functionals $T$ on $A'\times B'$ of the form

$$T = \sum_{i=1}^{n} a_i\otimes b_i, \quad \text{where} \quad T(a', b') = \sum_{i=1}^{n} a'(a_i)b'(b_i)$$

for all $\langle a', b'\rangle \in A'\times B'$. The operator norm of each such functional $T$ is denoted by $\lambda(T)$, and the completion of $A\otimes' B$ with respect to the norm $\lambda$ is denoted by $A\otimes_\lambda B$.



If $X$ is a compact Hausdorff space and $E$ is a Banach space, then Grothendieck ([4], p. 90) has shown that $C(X, E)$ and $C(X)\otimes_\lambda E$ are isomorphic and isometric Banach spaces. In particular, $C(\overline{G}, A)$ and $C(\overline{G})\otimes_\lambda A$ can be so identified. A simple computation shows that the mapping used is also multiplicative. Thus we have an isometric isomorphism between the algebras $\mathrm{AP}(G)\otimes_\lambda A$ and $C(\overline{G}, A)$.

THEOREM 4.1. *$\mathrm{AP}(G, A)$ is isomorphic and isometric with $\mathrm{AP}(G)\otimes_\lambda A$, and hence also with $C(\overline{G}, A)$.*

Proof. We define a mapping $\beta$ on $\mathrm{AP}(G)\otimes' A$ into $\mathrm{AP}(G, A)$, as follows:

$$\beta\Big(\sum_{i=1}^{n} f_i\otimes a_i\Big) = \sum_{i=1}^{n} f_i(\cdot)a_i.$$

It is obvious that $\beta$ is linear. Thus to show that $\beta$ is well defined it will suffice to show that $\beta T = 0$ when $T = 0$. Suppose that

$$T = \sum_{i=1}^{n} f_i\otimes a_i = 0.$$

This means that the linear transformation $T'$ from the dual space of $\mathrm{AP}(G)$ into $A$, defined by

$$T'(f') = \sum_{i=1}^{n} f'(f_i)a_i,$$

is the zero linear transformation. Thus

$$(\beta T)(x) = \sum_i f_i(x)a_i = \sum_i h_x(f_i)a_i = T'(h_x) = 0,$$

where $h_x$ is the multiplicative linear functional on $\mathrm{AP}(G)$ corresponding to the maximal ideal $\alpha(x)\in\overline{G}$. Simple computations show that $\beta$ is multiplicative, and that if $A$ has an involution, then $\beta$ is involution-preserving. Bochner and von Neumann proved that there exists an invariant mean on $\mathrm{AP}(G, A)$, and that it is unique. Using the invariant mean, they proved ([2], p. 37) that if $F\in\mathrm{AP}(G, A)$ then $F$ can be uniformly approximated arbitrarily closely by functions of the form

$$\sum_{i=1}^{n} f_i(\cdot)a_i,$$

where each $a_i\in A$ and each $f_i$ is a matrix element in an irreducible unitary representation of $G$, from which it follows ([13], p. 465) that $f_i\in\mathrm{AP}(G)$. As a result, the image under $\beta$ of $\mathrm{AP}(G)\otimes' A$ in $\mathrm{AP}(G, A)$ is dense.

It remains only to prove that $\beta$ is an isometry. If

$$T = \sum_{i=1}^{n} f_i \otimes a_i,$$

then

$$\lambda(T) = \sup\left\{\left\| \sum_i f'(f_i) a_i \right\| : f' \in \mathrm{AP}(G)', \|f'\| \leqslant 1\right\},$$

and

$$\|\beta T\| = \left\| \sum_i f_i(\cdot) a_i \right\| = \sup\left\{\left\| \sum_i f_i(x) a_i \right\| : x \in G\right\}$$

$$= \sup\left\{\left\| \sum_i h_x(f_i) a_i \right\| : x \in G\right\} \leqslant \lambda(T)$$

since each $h_x \in \mathrm{AP}(G)'$ and $\|h_x\| \leqslant 1$. In the other direction,

$$\lambda(T) = \sup\left\{\left| \sum_i f'(f_i) a'(a_i) \right| : \langle f', a' \rangle \in \text{unit ball of } \mathrm{AP}(G)' \times A'\right\}$$

$$= \sup\left\{\left| f'\left(\sum_i a'(a_i) f_i\right)\right| : \langle f', a' \rangle \in \text{unit ball of } \mathrm{AP}(G)' \times A'\right\}$$

$$\leqslant \sup\left\{\left\| \sum_i a'(a_i) f_i \right\| : a' \in \text{unit ball of } A'\right\}$$

$$= \sup\left\{\sup\left\{\left| \sum_i a'(a_i) f_i(x) \right| : x \in G\right\} : a' \in \text{unit ball of } A'\right\}$$

$$= \sup\left\{\sup\left\{\left| a'\left(\sum_i f_i(x) a_i\right)\right| : x \in G\right\} : a' \in \text{unit ball of } A'\right\}$$

$$\leqslant \sup\left\{\left\| \sum_i f_i(x) a_i \right\| : x \in G\right\} = \|\beta T\|,$$

and the theorem is proved.

Since $\bar{G}$ is a compact group, we can now utilize the fact that $C(\bar{G}, A)$ is a subset of $B^2(G, A)$ to obtain further information about $\mathrm{AP}(G, A)$. If $F \in \mathrm{AP}(G, A)$, let us denote its image in $C(\bar{G}, A)$ by $\hat{F}$, and let us denote a generic element of $\bar{G}$ by $w$. If we define a function $M$ on $\mathrm{AP}(G, A)$ by

$$M(F) = \int_{\bar{G}} \hat{F}(w)\, dw \quad \text{for all} \quad F \in \mathrm{AP}(G, A),$$

integration being with respect to normalized Haar measure on $\bar{G}$, then it is easily verified (using Theorem 17, [2], p. 30) that $M$ is the invariant mean on $\mathrm{AP}(G, A)$ (in this connection, observe that $(F_x)\hat{} = (\hat{F})_{\alpha(x)}$).



If $A$ is an $H^*$-algebra, with

$$B^2(\bar{G}, A) = \Sigma_{\alpha,\beta} \oplus P_{\alpha\beta}$$

the decomposition of $B^2(\bar{G}, A)$ into the direct sum of its minimal closed ideals, then for each $F \in \mathrm{AP}(G, A)$ let us denote by $F_{\alpha\beta}$ the image of the projection of $\hat{F}$ into the minimal closed ideal $P_{\alpha\beta}$ under the isometric isomorphism between $C(\bar{G}, A)$ and $\mathrm{AP}(G, A)$. The next theorem now follows immediately from the discussion in § 3 as applied to $B^2(\bar{G}, A)$.

THEOREM 4.2. *If $G$ is a topological group with Bohr compactification $\bar{G}$, $A$ is an $H^*$-algebra, and $F \in \mathrm{AP}(G, A)$, then $F$ has an expansion*

$$F = \sum_{\alpha,\beta} F_{\alpha\beta}$$

*meaning that*

$$M\left(\left\| F - \sum_{\alpha,\beta}' F_{\alpha\beta} \right\|^2\right) \to 0,$$

*$M$ being the invariant mean on $\mathrm{AP}(G)$ and the net being defined on the directed set of finite sums $\sum_{\alpha,\beta}' F_{\alpha\beta}$. The expansion is unique in the sense that if $F_1$ and $F_2$ have the same expansion, then $F_1 = F_2$. If $A$ is strongly semi-simple, then each $F_{\alpha\beta}$ is a minimal almost invariant function (i. e. $\hat{F}_{\alpha\beta}$ is almost invariant and lies in a minimal ideal in $B^2(\bar{G}, A)$).*

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# Charakterisierung der Quotienten in der zweidimensionalen diskreten Operatorenrechnung

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**1. Einleitung.** J. Mikusiński hat in [1] eine algebraisch begründete Operatorenrechnung entwickelt. Ausgehend von der Menge $\mathfrak{C}$ aller für $t \geqslant 0$ erklärten und dort stetigen, komplexwertigen Funktionen, wird nachgewiesen, daß $\mathfrak{C}$ mit der gewöhnlichen Addition und der Faltung

$$\{f(t)\}\{g(t)\} = \int_0^t f(\tau)g(t-\tau)\,d\tau \tag{1}$$

als Multiplikation einen Integritätsbereich bildet. Der zugehörige Quotientenkörper, der „Körper der Operatoren", dient als Fundament für den Aufbau der Theorie und für die Anwendung auf Differentialgleichungen.

In [2] hat S. Bellert eine „diskrete Operatorenrechnung" aufgebaut. Der Ausgangspunkt ist die Menge $\mathfrak{D}$ aller komplexwertigen Funktionen $\{a_m\}$ der diskreten Variablen $m = k, k+1, \ldots$ mit $k \in \Gamma$ ($\Gamma$ Menge der ganzen Zahlen). In $\mathfrak{D}$ wird die Addition in der gewöhnlichen Weise und die Multiplikation durch

$$\{a_m\}\{b_m\} = \sum_{\mu=-\infty}^{\infty} a_\mu b_{m-\mu} \tag{2}$$

erklärt, woraus folgt, daß $\mathfrak{D}$ nicht nur ein Integritätsbereich, sondern sogar ein Körper, der „Körper der diskreten Operatoren" ist. Würde man sich auf den Fall $k = 0$ beschränken, könnte die Multiplikation (2) in der Form

$$\{a_m\}\{b_m\} = \sum_{\mu=0}^{m} a_\mu b_{m-\mu} \tag{3}$$

als diskretes Analogon zu (1) geschrieben werden. Die Menge dieser speziellen Funktionen wäre aber nur ein Integritätsbereich und kein Körper. In Analogie zu [1] könnte man zum zugehörigen Quotientenkörper übergehen. Von formalen Abweichungen abgesehen, wurde dieser Weg in [3] bzw. [4] beschritten.