STUDIA MATHEMATICA, T. XXVI. (1965)

To Gabor Szegő
on his seventieth birthday

An example in the theory of singular integrals

by

MARY WEISS and A. ZYGMUND (Chicago)

1. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, etc. be points in the n-dimensional space E_n, Σ — the unit sphere $|x| = 1$, and $K(x)$, $x \neq 0$, a positively homogeneous kernel of degree $-n$, i. e., $K(\lambda x) = \lambda^{-n}K(x)$ for $\lambda > 0$. In particular, $K(x) = |x|^{-n}\Omega(x')$, where $x' = x/|x|$ is the projection of x onto Σ. The function Ω is sometimes called the *characteristic* of K (or of the singular integral (1.1) below).

It is by now a familiar fact that if a) $|\Omega|\log^+|\Omega|$ is integrable over Σ and b) the integral of Ω over Σ is 0, then the convolution integral (singular integral)

$$(K*f)(x) = \lim_{\varepsilon\to 0} \int\limits_{|y|>\varepsilon} f(x-y)K(y)\,dy = \lim_{\varepsilon\to 0}\int\limits_{\varepsilon}^{\infty}\frac{dt}{t}\int\limits_{\Sigma} f(x-ty')\,\Omega(y')\,dy' \tag{1.1}$$

exists almost everywhere for any function f in $L^p(E_n)$, provided p is strictly greater than 1. (There is a corresponding result for $p = 1$, but then condition a) must be considerably strengthened; we leave this case aside).

While the necessity of condition b) in the above-mentioned theorem is obvious, that of a) is much less clear, and it is the main purpose of this note to show that it cannot be weakened.

A precise formulation of the result is given below (see Theorem 1). Here we only observe that if the kernel $K(x)$ is odd, that is, $K(-x) = -K(x)$, and if, as before, $f \in L^p$, $p > 1$, then the limit (1.1) exists almost everywhere under the sole condition that Ω is integrable over Σ; condition b) will then be automatically satisfied. The result holds, and the proof remains unchanged, if Ω is merely an odd mass distribution over Σ, i. e., Ω takes opposite values for sets antipodal on Σ. The integral (1.1) is then

$$\lim_{\varepsilon\to 0}\int\limits_{\varepsilon}^{\infty}\frac{dt}{t}\int\limits_{\Sigma} f(x-ty')\,\Omega(dy'). \tag{1.2}$$

Let $\tilde{f}(x)$ denote the value of the integral (1.1). Since the latter is a convolution, one could anticipate, after a suitable normalization of f, the formula

$$\hat{\tilde{f}} = \hat{f}\cdot\hat{K}, \tag{1.3}$$

and it can be shown that it is actually so if $f \in L^2$ and K satisfies conditions a) and b); $\hat{K}(x)$ is defined as

$$\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} K(y)e^{-i(x\cdot y)}dy.$$

The latter limit exists and equals

$$\int_{\Sigma}\Omega(y')\left\{\log\frac{1}{|\cos\varphi|}-\frac{1}{2}\pi i\operatorname{sign}\cos\varphi\right\}dy', \tag{1.4}$$

where $\cos\varphi = x'\cdot y'$. In view of the exponential integrability of the logarithm, (1.4) is a bounded function of x if $\Omega\log^+|\Omega|$ is integrable. It is also not difficult to see that for any function $\varphi(u)$, $u\geqslant 0$, non-negative, increasing and $o(u\log u)$ for $u\to\infty$ we can find an Ω such that $\varphi(|\Omega|)$ is integrable over Σ, the integral of Ω over Σ is 0 and (1.4) is essentially unbounded as a function of x. It follows that (1.1) is then an unbounded operation in L^2 and the limit, *if it exists*, is not necessarily in L^2 (see [1]). We shall, however, prove the following stronger result:

THEOREM 1. *Let $\varphi(u)$, $u\geqslant 0$, be a non-negative non-decreasing function of u which is $o(u\log u)$ for $u\to\infty$. Then there is an Ω such that $\varphi(|\Omega|)$ is integrable over Σ and a function $f(x)$ which is continuous, tends to 0 at ∞, belongs to $L(E_n)$ (and so also to every L^p, $p>1$) and such that*

$$\tilde{f}_\varepsilon(x) = \int_{|y|>\varepsilon} f(x-y)\frac{\Omega(y')}{|y|^n}dy' \tag{1.5}$$

satisfies

$$\limsup_{\varepsilon\to 0}|\tilde{f}_\varepsilon(x)| = +\infty \tag{1.6}$$

for almost all x.

We will give the proof of the theorem for $n=2$ and show later that this implies the theorem for general $n>2$. If $n=2$, (1.5) can be written

$$\tilde{f}_\varepsilon(z) = \int_\varepsilon^\infty\frac{dt}{t}\int_0^{2\pi} f(z-te^{i\theta})\,\Omega(\theta)\,d\theta, \tag{1.7}$$

where $z=x+iy$ is now a complex number.

Before we pass to the proof suppose first that Ω is a measure consisting of two point masses -1 at the points $\theta=0,\pi$, and two point masses $+1$ at $\theta=\pm\frac{1}{2}\pi$. Then the last integral can be written

$$\int_\varepsilon^\infty\frac{f(z+it)+f(z-it)-2f(z)}{t}dt-\int_\varepsilon^\infty\frac{f(z+t)+f(z-t)-2f(z)}{t}dt.$$

Suppose now that f is a function of the variable x only: $f(z)=g(x)$. Then the first integral disappears and the second becomes

$$\int_\varepsilon^\infty\frac{g(x+t)+g(x-t)-2g(x)}{t}dt. \tag{1.8}$$

Now it is well known that there exists a continuous and integrable function $g(x)$ such that the last integral is unbounded for each x as $\varepsilon\to 0$. The corresponding function $f(z)$, which is independent of the variable y, is only locally integrable in E_2, but by means of this f it is easy to construct another f, continuous and in $L(E_2)$, such that (1.8) does not tend to any limit as $\varepsilon\to 0$.

The proof of Theorem 1 follows a similar line but its details are more involved. Section 7 contains a result completing Theorem 1.

2. The proof of Theorem 1 is based on a series of lemmas.

LEMMA 1. *Let $\varphi(u)$, $0\leqslant u<\infty$, be non-negative, non-decreasing and $o(u\log u)$ for $u\to\infty$. Then there is a convex non-decreasing function $\psi(u)$ satisfying $\psi(u)\geqslant\varphi(u)$, $\psi(u)=o(u\log u)$ $(u\to\infty)$.*

The meaning of Lemma 1 is that in Theorem 1 it is enough to consider convex functions φ. We postpone the proof of the Lemma to Section 6.

LEMMA 2. *Let $n_1, n_2, \ldots$ be a sequence of positive integers satisfying $n_{k+1}/n_k>q>1$. Let $a_1, a_2, \ldots$ be a sequence of real numbers such that $\sum|a_k|<\infty$. Finally, let $d_1(\varepsilon), d_2(\varepsilon), \ldots$ be a sequence of numbers depending continuously on the parameter ε, $0<\varepsilon\leqslant 1$, such that the sequence is bounded for each fixed ε, and each $d_k(\varepsilon)$ is bounded in ε for fixed k; moreover $\sum a_k^2d_k^2(\varepsilon)$ tends to $+\infty$ as $\varepsilon\to 0$. Set*

$$\lambda_\varepsilon(x) = \sum a_k d_k(\varepsilon)\cos n_k x.$$

Then

$$\limsup_{\varepsilon\to 0}|\lambda_\varepsilon(x)| = +\infty \tag{2.1}$$

for almost all x.

This lemma is a simple corollary of the following known result (see [2], p. 231, Ex. 27). Given a number $q>1$ and a set E of positive measure contained in $(0, 2\pi)$ there exist two positive numbers λ_q and μ_q with the following property: for any function

$$f(x) = \sum_{1}^{\infty} a_k \cos n_k x,$$

$n_{k+1}/n_k > q$, $\sum a_k^2 < \infty$, $|f(x)|$ exceeds $\lambda_q(\sum a_k^2)^{1/2}$ in a subset of E of measure $> \mu_q|E|$, provided n_1 is large enough: $n_1 > n_0(q, E)$. For suppose that (2.1) does not hold almost everywhere. We can then find a set E, $|E| > 0$, and numbers M, ε_0 such that

$$\Big|\sum a_k d_k(\varepsilon) \cos n_k x\Big| < M \quad \text{for} \quad x \in E,\ 0 < \varepsilon < \varepsilon_0. \tag{2.2}$$

Dropping a sufficiently large number of initial terms and using the fact that each $d_k(\varepsilon)$ is bounded in ε we may assume that n_1 in (2.2) is sufficiently large (this may require a change of M) so that $|\sum a_k d_k(\varepsilon)\cos n_k x|$ exceeds $\lambda_q(\sum a_k^2 d_k^2(\varepsilon))^{1/2}$ in a subset of E of measure $> \mu_q|E|$. Since $\sum a_k^2 d_k^2(\varepsilon)$ tends to ∞ as $\varepsilon \to 0$, this contradicts (2.2) and proves the lemma.

3. Let $0 < h < \frac{1}{4}\pi$ and let $\chi_h(\theta)$ be the characteristic function of the interval $0 \leqslant \theta \leqslant h$ repeated periodically with period 2π. Let

$$\chi_h'(\theta) = \frac{1}{h}\left\{-\chi_h(\theta) + \chi_h\left(\theta + \frac{1}{2}\pi\right) - \chi_h(\theta+\pi) + \chi_h\left(\theta - \frac{1}{2}\pi\right)\right\}.$$

The function Ω of Theorem 1 will be defined as $\sum \delta_k \chi_{h_k}'(\theta)$, where the numbers δ_k are positive, $\sum \delta_k < \infty$, and $\{h_k\}$ is a sequence of positive numbers tending to 0. It is clear that $\Omega(\theta)$ is integrable and its integral over $(0, 2\pi)$ is 0. We will show later that if the δ_k and h_k are chosen suitably, then also $\varphi(|\Omega(\theta)|)$ is integrable, where φ is the function of Theorem 1.

As before, we shall consider a continuous function $f(z)$ which initially will depend on one variable only, $f(z) = g(x)$, and will be periodic of period 2π. We set (see (1.7))

$$J_\varepsilon = J_\varepsilon(z; f, \chi_h') = \int_\varepsilon^1 \frac{dt}{t}\int_0^{2\pi} f(z - te^{i\theta})\chi_h'(\theta)\,d\theta \tag{3.1}$$

$$= \int_\varepsilon^1 \frac{dt}{t}\left\{\frac{1}{h}\int_0^h [-g(x+t\cos\theta) - g(x-t\cos\theta) + {} \right.$$

$$\left. {} + g(x+t\sin\theta) + g(x-t\sin\theta)]\right\} d\theta.$$

The expression in curly brackets can be written

$$\frac{1}{h}\int_0^h [g(x+t\sin\theta) + g(x-t\sin\theta) - 2g(x)]\,d\theta - \tag{3.2}$$

$$-\frac{1}{h}\int_0^h [g(x+t\cos\theta) + g(x-t\cos\theta) - 2g(x)]\,d\theta,$$

and if we set

$$g(x) = \sum_1^\infty a_k \cos 2n_k x \qquad (\sum |a_k| < \infty,\ n_{k+1}/n_k \geqslant 2)$$

a simple computation shows that (3.1) is

$$\sum a_k \cos 2n_k x \left\{\frac{1}{h}\int_0^h\int_\varepsilon^1 \frac{\sin^2(n_k t\cos\theta)}{t}\,dt\,d\theta - \frac{1}{h}\int_0^h\int_\varepsilon^1 \frac{\sin^2(n_k t\sin\theta)}{t}\,dt\,d\theta\right\} \tag{3.3}$$

$$= \sum a_k \cos 2n_k x\, I(\varepsilon, n_k, h),$$

where

$$I(\varepsilon, n_k, h) = I_1(\varepsilon, n_k, h) - I_2(\varepsilon, n_k, (h)),$$

$$I_1 = \frac{1}{h}\int_0^h\int_\varepsilon^1 \frac{\sin^2(n_k t\cos\theta)}{t}\,dt\,d\theta,$$

$$I_2 = \frac{1}{h}\int_0^h\int_\varepsilon^1 \frac{\sin^2(n_k t\sin\theta)}{t}\,dt\,d\theta. \tag{3.4}$$

Our main task now is to find an estimate for $I(\varepsilon, n_k, h)$.

4. We write n for n_k, keep n and ε fixed and consider $I = I(\varepsilon, n, h)$ as a function of h. We set $\nu = n\varepsilon$ (thus $\nu < n$). The $O(1)$ in the lemma that follows are uniform in n, ε, h.

LEMMA 3. a) *If* $\nu \leqslant 1$, *then*

a$_1$) $$I = \frac{1}{2}\log n + O(1) \quad \textit{for} \quad 0 < h \leqslant 1/n,$$

a$_2$) $$I = \frac{1}{2}\log\frac{1}{h} + O(1) \quad \textit{for} \quad 1/n \leqslant h \leqslant 1.$$

b) *If* $\nu \geqslant 1$, *then*

b$_1$) $$I = \frac{1}{2}\log\frac{1}{\varepsilon} + O(1) \quad \textit{for} \quad 0 < h \leqslant 1/n,$$

b$_2$) $$I = \frac{1}{2}\log\frac{1}{\nu h} + O(1) \quad \textit{for} \quad 1/n \leqslant h \leqslant 1/\nu,$$

b$_3$) $$I = O(1) \quad \textit{for} \quad 1/\nu \leqslant h \leqslant 1.$$

The proof of the lemma is based on the following equation:

$$\int_{\varepsilon}^{1}\frac{\sin^2 ta}{t}\,dt=\frac{1}{2}\min\left\{\log^+ a,\ \log\frac{1}{\varepsilon}\right\}+O(1). \tag{4.1}$$

Its verification is simple. If $a\leqslant 1$, the left-hand side is $O(1)$ (since the integrand is $\leqslant ta^2$) and the formula is obvious. Suppose now that $a>1$, consider the two formulas

$$\int_{1}^{\omega}\frac{\sin^2 s}{s}\,ds=\frac{1}{2}\log\omega+O(1),\qquad \int_{\eta}^{1}\frac{\sin^2 s}{s}ds=O(1)\qquad (0<\eta\leqslant 1\leqslant\omega) \tag{4.2}$$

and write the integral (4.1) in the form $\int_{\varepsilon a}^{a}s^{-1}\sin^2 s\,ds$. If $\varepsilon a\geqslant 1$, i. e., $a\geqslant 1/\varepsilon$, the first formula (4.2) shows that the last integral is $\frac{1}{2}\log(1/\varepsilon)+$ $+O(1)$. If $\varepsilon a<1$, the two formulas (4.2) show that the integral is $\frac{1}{2}\log a+$ $+O(1)$. In either case we have (4.1).

From (4.1) we easily obtain

$$I_1=\frac{1}{h}\int_0^h d\theta\int_{\varepsilon}^{1}\frac{\sin^2(tn\cos\theta)}{t}\,dt=\frac{1}{2}\min\left\{\log n,\ \log\frac{1}{\varepsilon}\right\}+O(1), \tag{4.3}$$

$$I_2=\frac{1}{h}\int_0^h d\theta\int_{\varepsilon}^{1}\frac{\sin^2(tn\sin\theta)}{t}\,dt=\frac{1}{2}\frac{1}{h}\int_0^h\min\left\{\log^+ n\theta,\ \log\frac{1}{\varepsilon}\right\}d\theta+O(1). \tag{4.4}$$

In the remainder of this section $A=B$ means $A=B+O(1)$.

Observe now that, by (4.3) and (4.4),

$$I_1=\frac{1}{2}\log n\quad\text{for}\quad \nu\leqslant 1, \tag{4.5}$$

$$I_1=\frac{1}{2}\log\frac{1}{\varepsilon}\quad\text{for}\quad \nu\geqslant 1, \tag{4.6}$$

independently of h. Since, clearly, $I_2=0$ for $h\leqslant 1/n$, equations a_1) and b_1) follow.

Suppose now that $nh\geqslant 1$, that is, $1/n\leqslant h\leqslant 1$. If $h\leqslant 1/\nu$, then, by (4.4),

$$I_2=\frac{1}{2h}\int_{1/n}^{h}\log n\theta\,d\theta=\frac{1}{2}\log nh, \tag{4.7}$$

and if $h\geqslant 1/\nu$, then

$$I_2=\frac{1}{2h}\left\{\int_{1/n}^{1/\nu}\log n\theta\,d\theta+\int_{1/\nu}^{h}\log\frac{1}{\varepsilon}\,d\theta\right\}=\frac{1}{2h}\left[\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}+\left(h-\frac{1}{\nu}\right)\log\frac{1}{\nu}\right] \tag{4.8}$$
$$=\frac{1}{2}\log\frac{1}{\varepsilon}.$$

Let us consider now separately the two cases $\nu\leqslant 1$ and $\nu\geqslant 1$. If $\nu\leqslant 1$, the equation (4.7) is valid for $1/n\leqslant h\leqslant 1$ and in conjunction with (4.5) gives $I=I_1-I_2=\frac{1}{2}\log(1/h)$, which is our formula a_2). If, however, $\nu\geqslant 1$, (4.6) together with (4.7) show that $I=\frac{1}{2}\log 1/(\nu h)$ for $1/n\leqslant h\leqslant 1/\nu$, and equation b_2) follows. Finally, if $1/\nu\leqslant h\leqslant 1$, the equations (4.6) and (4.8) give $I=0$, which is b_3). This completes the proof of Lemma 3.

5. Let now $f(z)=g(x)$ be a function of the variable x alone and let (cf. (1.7) and (3.1))

$$J_\varepsilon(z;f,\Omega)=\int_{\varepsilon}^{1}\frac{dt}{t}\int_0^{2\pi}f(z-te^{i\theta})\Omega(\theta)\,d\theta\qquad(0<\varepsilon<1).$$

Suppose that

$$g(x)=\sum_{1}^{\infty}a_k\cos 2n_k x,$$

where $\sum|a_k|<\infty$ and $n_1,n_2,\ldots$ are positive integers satisfying the condition $n_{k+1}/n_k\geqslant 2$, and let

$$\Omega(\theta)=\sum_{1}^{\infty}\delta_\nu\chi_\nu'(\theta),$$

where the δ_ν are positive, $\sum\delta_\nu=1$ and $\frac{1}{4}\pi>h_1>h_2>\ldots$, $h_\nu\to 0$. Then, using (3.1), (3.2) and (3.3),

$$\begin{aligned}J_\varepsilon(z;f,\Omega)&=\sum_\nu\delta_\nu\int_\varepsilon^1\frac{dt}{t}\int_0^{2\pi}f(z-te^{i\theta})\chi_{h_\nu}'(\theta)\,d\theta\\&=\sum_\nu\delta_\nu\Big\{\sum_k a_k\cos 2n_k x\,I(\varepsilon,n_k,h_\nu)\Big\}\\&=\sum_k a_k\cos 2n_k x\Big\{\sum_\nu\delta_\nu I(\varepsilon,n_k,h_\nu)\Big\}.\end{aligned}$$

The change in the order of summation is justified since, as we easily see from Lemma 3, $I(\varepsilon,n,h)$ is bounded in n and h for ε fixed (it is majorized by $\frac{1}{2}\log(1/\varepsilon)+O(1)$).

Using Lemma 3 we also see that

$$\sum_\nu\delta_\nu I(\varepsilon,n_k,h_\nu)=\tfrac{1}{2}d_k(\varepsilon)+O(1),$$

where the O is bounded in ε and k, and

$$d_k(\varepsilon)=\Big(\sum_{h_\nu\leqslant 1/n_k}\delta_\nu\Big)\log n_k+\sum_{h_\nu>1/n_k}\delta_\nu\log\frac{1}{h_\nu},\qquad\text{if}\quad n_k\leqslant 1/\varepsilon,$$

$$d_k(\varepsilon)=\Big(\sum_{h_\nu\leqslant 1/n_k}\delta_\nu\Big)\log\frac{1}{\varepsilon}+\sum_{1/n_k<h\leqslant 1/\varepsilon n_k}\delta_\nu\log\Big(\frac{1}{n_k\varepsilon h_\nu}\Big),\qquad\text{if}\quad n_k\geqslant 1/\varepsilon.$$

Assuming that the sequences $\{a_k\}$, $\{n_k\}$, $\{\delta_\nu\}$, $\{h_\nu\}$ have the properties already listed we will show that we can select them in such a way that

$$\sum a_k^2 d_k^2(\varepsilon) \to \infty \qquad (\varepsilon \to 0), \tag{5.1}$$

$$\int_0^{2\pi} \varphi(|\Omega(\theta)|)\, d\theta < \infty, \tag{5.2}$$

where φ is the function of Theorem 1 (and is convex). Since, as we can easily verify, the sequence $\{d_k(\varepsilon)\}$ is bounded for each fixed ε, and each $d_k(\varepsilon)$ is a bounded function of ε, an application of Lemma 2 will give us

$$\limsup |J_\varepsilon(z; f, \Omega)| = +\infty \tag{5.3}$$

for almost all x or, what is the same thing, for almost all z.

Choose for $\{a_k\}$ any sequence such that $a_k \neq 0$, $\sum |a_k| < \infty$. Take for $\{\delta_\nu\}$ and $\{h_\nu\}$ sequences such that

$$\sum \delta_\nu h_\nu \varphi(1/h_\nu) < \infty, \quad \sum \delta_\nu \log \frac{1}{h_\nu} = +\infty. \tag{5.4}$$

This is feasible since, by hypothesis,

$$\varphi\left(\frac{1}{h}\right) \Big/ \frac{1}{h} \log \frac{1}{h} = h\varphi\left(\frac{1}{h}\right) \Big/ \log \frac{1}{h} \to 0 \qquad (h \to 0).$$

Let now $\{n_k\}$ increase so rapidly that

$$\sum_{h_\nu > 1/n_k} \delta_\nu \log \frac{1}{h_\nu} > \frac{1}{|a_k|}.$$

Then $d_k(\varepsilon) > 1/|a_k|$ for $n_k < 1/\varepsilon$ and hence

$$\sum a_k^2 d_k^2(\varepsilon) \geqslant \sum_{n_k < 1/\varepsilon} 1 \to \infty$$

as $\varepsilon \to 0$, which is (5.1).

To prove (5.2) we recall the definition of χ_h' and the assumptions that φ is convex and $\sum \delta_\nu = 1$. Then, using Jensen's inequality, we have

$$\int_0^{2\pi} \varphi(|\Omega(\theta)|)\, d\theta = 4 \int_0^{h_1} \varphi\left(\sum \delta_\nu \frac{1}{h_\nu} \chi_{h_\nu}(\theta)\right) d\theta \tag{5.5}$$

$$\leqslant 4 \sum \delta_\nu \int_0^{h_1} \varphi\left(\frac{1}{h_\nu} \chi_{h_\nu}(\theta)\right) d\theta = 4 \sum \delta_\nu \int_0^{h_\nu} \varphi\left(\frac{1}{h_\nu}\right) d\theta$$

$$= 4 \sum \delta_\nu h_\nu \varphi\left(\frac{1}{h_\nu}\right) < \infty,$$

and (5.2) is established.

6. We shall now complete the proof of Theorem 1.

Changing the notation slightly, let us denote by $f^*(z)$ the function (depending on x only) for which we proved (5.3). Let Q be any square in E_2 with sides parallel to the axes, and let $\lambda_Q(z)$ be a function of the class C', positive in the interior of Q and 0 outside Q. Let $f_Q(z) = f^*(z)\lambda_Q(z)$. It is not difficult to see that $J_\varepsilon(z, f_Q, \Omega) - \lambda_Q(z) J_\varepsilon(z, f^*, \Omega)$ tends to a finite limit in the interior of Q as $\varepsilon \to 0$, so that $\limsup |J_\varepsilon(z, f_Q, \Omega)| = +\infty$ almost everywhere in Q. Decomposing the plane into a union of congruent but non-overlapping squares Q_m ($m = 1, 2, \ldots$) and taking for λ_{Q_m} translates of one another, we easily see that if $\eta_m > 0$, $\sum \eta_m < \infty$, the function $f = \sum \eta_m f_{Q_m}(z)$ has all the properties formulated in Theorem 1,

We shall now prove Lemma 1. Let $\omega(u) = u \log u$ for $u \geqslant 2$, $\omega_n(u) = \omega(u)/n$ ($n = 1, 2, \ldots$). Let $2 \leqslant u_1 < u_2 < u_3 < \ldots$ be any sequence of numbers increasing so rapidly that $\omega_n(u_n) < \omega_{n+1}(u_{n+1})$, $\omega_n'(u_n) < \omega_{n+1}'(u_{n+1})$ ($n = 1, 2, \ldots$). In each of the intervals (u_n, u_{n+1}) we construct an increasing convex function (e. g., a polygonal line) situated between the curves $v = \omega_n(u)$ and $v = \omega_{n+1}(u)$, tangent to the former at the point $u = u_n$ and having the same ordinate as the latter at $u = u_{n+1}$. The totality of these convex curves augmented by the segment $v = \omega_1(u_1)$, $0 \leqslant u \leqslant u_1$, constitutes a single convex and non-decreasing (strictly increasing for $u \geqslant u_1$) curve $v = \psi(u)$. Clearly, $\psi(u) = o(u \log u)$ for $u \to \infty$.

Suppose that $\{u_n\}$ has, in addition, the following properties: $\varphi(u) \leqslant \omega_{n+1}(u)$ for $u \geqslant u_n$. Then, obviously, $\varphi(u) \leqslant \psi(u)$ in each of the intervals (u_n, u_{n+1}), $n = 1, 2, \ldots$ In the interval $(0, u_1)$ we have $\varphi(u) \leqslant \varphi(u_1) \leqslant \omega_2(u_1) < \omega_1(u_1) = \psi(u)$. Hence $\varphi(u) \leqslant \psi(u)$ for $u \geqslant 0$ and Lemma 1 is established. This also completes the proof of Theorem 1 in the case $n = 2$.

The result for $n = 2$ is easily extensible to higher values of n. The case $n = 3$ is typical and we confine our attention to it.

Suppose that a function $f(x, y, z)$ of 3 real variables is continuous and integrable in E_3, and suppose that it is a function of 2 variables only, $f = f(x, y)$, in a cube Q. Suppose also that the characteristic Ω, defined on the surface Σ of the unit spere, is a function of latitude θ only, so that $K = \Omega(\theta)/r^3$. If φ is the function of Theorem 1, then the integrability of $\varphi(|\Omega|)$ over Σ is equivalent to the integrability of $\varphi(|\Omega(\theta)|)$ over $0 \leqslant \theta \leqslant \pi$. It is easy to see that at each point (x, y, z) interior to Q the existence of the 3-dimensional convolution of $f(x, y, z)$ and $K = \Omega/r^3$ is equivalent to the existence of the two-dimensional integral

$$\iint f(x-\xi, y-\eta)\, \Omega(\theta) r^{-2} d\xi\, d\eta$$

near $\xi = \eta = 0$. A routine argument completes the proof.

7. One may ask what can be the "degree of continuity" of the function f in Theorem 1. The theorem that follows gives some information on that score, though not a complete answer. We will return to this question on another occasion.

THEOREM 2. *Let α and β be two positive numbers of sum less than 1. Then there are a function $f(x)$ integrable over E_n, tending to 0 at ∞, having modulus of continuity*

$$\omega(\delta) = O\left\{\frac{1}{(\log 1/\delta)^{\alpha}}\right\}$$

and a function $\Omega(x')$ of the class $L(\log^+L)^{\beta}$ over Σ,

$$\int_{\Sigma} \Omega d\sigma = 0,$$

such that the integral $f * r^{-n}\Omega(x')$ diverges almost everywhere.

The proof of this theorem runs parallel to that of Theorem 1 and we may be brief. It is enough to consider the case $n = 2$. We need the following lemma which is certainly known though it is difficult to give exact reference.

LEMMA 4. *Let $n_1, n_2, \ldots$ be an increasing sequence of positive integers, and let the sequence $a_1, a_2, \ldots$ of real numbers and the function $w(\delta)$ decreasing monotonically to 0 with δ have the following properties:*

(i) $a_k = O\big(w(1/n_{k+1})\big)$,

(ii) $\sum_{N+1}^{\infty} |a_k| = O(a_N)$,

(iii) $\sum_{1}^{N} n_k |a_k| = O(n_N |a_N|)$.

Then the modulus of continuity $\omega(\delta)$ of the funtcion $f(x) = \sum a_k \cos n_k x$ is $O\big(w(\delta)\big)$.

Let $0 < \delta \leqslant 1/n_1$ and let N be such that $\delta n_N \leqslant 1 < \delta n_{N+1}$. Then

$$|f(x+\delta) - f(x)| \leqslant \sum |a_k| \left|2 \sin n_k\left(x + \frac{1}{2}\delta\right) \sin \frac{1}{2} n_k \delta\right|$$

$$\leqslant \delta \sum_{1}^{N} |a_k| n_k + \sum_{N+1}^{\infty} |a_k| = P + Q,$$

say, and

$$P \leqslant \frac{1}{n_N} O(a_N n_N) = O(|a_N|) = O\left(w\left(\frac{1}{n_{N+1}}\right)\right) = O\big(w(\delta)\big),$$

$$Q = O(|a_N|) = O\big(w(\delta)\big).$$

Hence $\omega(\delta) = O\big(w(\delta)\big)$ and the lemma is established.

Let now

$$w(\delta) = \left(\log\frac{1}{\delta}\right)^{-\alpha}, \quad n_k = 2^{2^k}, \quad a_k = 2^{-k\alpha}.$$

It is easy to see that the hypotheses (i), (ii), (iii) of Lemma 4 are satisfied so that the function $f(x) = \Sigma a_k \cos 2 n_k x$ has modulus of continuity $O\left\{\left(\log\frac{1}{\delta}\right)^{-\alpha}\right\}$.

Let us also set

$$\delta_\nu = \frac{1}{\nu^{1+\beta}\log^2\nu}, \quad h_\nu = 2^{-\nu} \quad (\nu = 2, 3, \ldots).$$

Then (see (5.4) and (5.5))

$$\sum \delta_\nu \left(\log\frac{1}{h_\nu}\right)^{\beta} = C \sum \nu^{-1}(\log\nu)^{-2} < \infty,$$

so that the function Ω of Section 5 is in the class $L(\log^+L)^{\beta}$.

Finally, let $0 < \varepsilon < 1$, $n_k < 1/\varepsilon$. Then the function $d_k(\varepsilon)$, considered in Section 5, satisfies the inequality

$$d_k(\varepsilon) \geqslant (\log n_k) \sum_{h_\nu \leqslant 1/n_k} \delta_\nu = 2^k (\log 2) \sum_{\nu=2^k}^{\infty} \frac{1}{\nu^{1+\beta}\log^2\nu} \geqslant c\,\frac{2^k}{2^{\beta k} k^2}.$$

It follows that

$$a_k d_k(\varepsilon) \geqslant c(2^k)^{1-\alpha-\beta} k^{-2},$$

and hence $\sum a_k^2 d_k^2(\varepsilon) \to \infty$ if $\alpha + \beta < 1$. The rest of the proof is the same as in Theorem 1.

References

[1] A. P. Calderón and A. Zygmund, *On singular integrals*, American J. of Math. 78 (1956), p. 289-309.

[2] A. Zygmund, *Trigonometric series*, 2nd ed., vol. I and II, Cambridge 1959.

DE PAUL UNIVERSITY,
UNIVERSITY OF CHICAGO

Reçu par la Rédaction le 12. 11. 1964