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A remark on reflexivity and summability

by

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Let us recall that a summability method T is a real matrix (c_{mn}), $m=1,2,\ldots,\ n=1,2,\ldots$ The T-means of a sequence $\{z_n\}$ in a Banach space E are

$$t_m = \sum_{n=1}^{\infty} c_{mn} z_n.$$

T is said to be *regular* if z_n real, $z_n \to z$ (finite), implies that t_m exists and $t_m \to x$. According to the Toeplitz-Silverman theorem, T is regular if and only if

1) $\sum_{n=1}^{\infty} |c_{mn}| < M$ for all m,

2) $c_{mn} \to 0$ as $m \to \infty$, for all n, and

3) $\sum_{n=1}^{\infty} c_{mn} \to 1$ as $m \to \infty$.

A regular method T is said to be *essentially positive* [2], if

4) $\sum_{n=1}^{\infty} |c_{mn}| \to 1$ as $m \to \infty$.

A Banach space E is said *to have property* $\mathcal{S}$ $(w\mathcal{S})$ [2] if for every bounded sequence in E there exists a regular method T and a subsequence whose T-means converge strongly (weakly); or, equivalently [2], if for every bounded sequence $\{z_n\}$ in E there exists a regular method T such that the T-means of $\{z_n\}$ converge strongly (weakly).

Recently, T. Nishiura and D. Waterman have proved ([2], theorem 2) that *for a Banach space E the following statements are equivalent*:

(i) *E is reflexive.*

(ii) *E has property $\mathcal{S}$ with essentially positive T.*

(iii) *E has property $w\mathcal{S}$ with essentially positive T.*

The purpose of the present Note is to show that in this result the essential positivity of T can be omitted, i. e. that we have the following

THEOREM. *For a Banach space E the following statements are equivalent*:

(i) *E is reflexive.*

(ii) *E has property $\mathcal{S}$.*

(iii) *E has property $w\mathcal{S}$.*

In the arguments of [2] the essential positivity of T plays a fundamental role. Our proof is different from that of [2], being based on a profound result of A. Pełczyński ([3], theorem 2) concerning basic sequences.

Proof of the theorem. For (i) $\Rightarrow$ (ii), see [2]. (ii) $\Rightarrow$ (iii) is obvious. (iii) $\Rightarrow$ (i). Assume that E has property $w\mathcal{S}$ and let $\{x_n\}$ be an arbitrary basic sequence (i. e. a basis of a closed linear subspace) in E. Then the closed linear subspace $E_1 = [x_n]$ of E has property $w\mathcal{S}$ (by the theorem of S. Mazur [1], according to which the $\sigma(E, E^*)$-limit of any $\sigma(E, E^*)$-convergent sequence in E_1 belongs to E_1). Hence, by [2], theorem 3, the basis $\{x_n\}$ of E_1 must be boundedly complete [1]. Thus every basic sequence in E is boundedly complete, whence, by [3], theorem 2, E is reflexive, which completes the proof.

[1] I. e. for every sequence of scalars $\{a_n\}$ such that $\sup_n \|\sum_{i=1}^{n} a_i x_i\| < \infty$, the series $\sum_{i=1}^{\infty} a_i x_i$ converges.

References

[1] S. Mazur, *Über konvexe Mengen in linearen normierten Räumen*, Studia Math. 4 (1933), p. 70-84.

[2] T. Nishiura and D. Waterman, *Reflexivity and summability*, ibidem 23 (1963), p. 53-57.

[3] A. Pełczyński, *A note on the paper of I. Singer "Basic sequences and reflexivity of Banach spaces"*, ibidem 21 (1962), p. 371-374.

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A remark on the preceding paper of I. Singer

(From a letter to R. Sikorski)

by

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The results of Nishiura and Waterman [2], and Singer [4] suggest the following

THEOREM. *Let W be a weakly closed bounded subset of a Banach space E. Then the following conditions are equivalent*:

(o) *W is weakly compact*;

(oo) *for every sequence (z_n) of elements of W there is a matrix $(c_{m,n})$ such that*

1) $c_{m,n} \geqslant 0$ *and* $c_{m,n} = 0$ *for* $n > n(m)$ $(n, m = 1, 2, \ldots)$,

2) $\sum_{n=1}^{n(m)} c_{m,n} = 1$ $(m = 1, 2, \ldots)$,

3) *the sequence* $\left(\sum_{n=1}^{n(m)} c_{m,n} z_n\right)$ *is convergent*;

(ooo) *for every sequence (z_n) of elements of W there is a regular matrix $(c_{m,n})$ such that the sequence $\left(\sum_{n=1}^{\infty} c_{m,n} z_n\right)$ is weakly convergent to an element of E.*

Proof. (o) $\to$ (oo). Let (z_n) be an arbitrary sequence in W. According to the Eberlein-Šmulian theorem ([1], p. 48) the sequence (z_n) contains a subsequence (z_{n_k}) which is weakly convergent to an element z of W. Then a theorem of Mazur ([1], p. 40) implies the existence of finite averages

$$w_m = \sum_{k=1}^{k(m)} t_{m,k} z_{n_k}$$

such that $\|z - w_m\| < m^{-1}$ $(m = 1, 2, \ldots)$. Let us set $c_{m,n} = t_{m,k}$ for $n = n_k$ $(k = 1, 2, \ldots, k(m);\ m = 1, 2, \ldots)$ and $c_{m,n} = 0$ in the other case. Then the matrix $(c_{m,n})$ has the desired properties 1)-3).

(oo) $\to$ (ooo). This implication is trivial.

non (o) $\to$ non (ooo). It follows from [3] that non (o) implies the existence of a basic sequence (z_n) of elements of W and a linear functio-