ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
VIII (1966)

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STATISTICAL SETS OF RANDOM STRUCTURE

Summary

The main aim of this paper is to set forth some fundamental properties of the statistical set, the structure of which is changed at random in relation to a definite characteristic X. In this exposition we shall consider in detail the case where the set of values of the function induced with X is a finite set. In this case the structure in question will be represented by an n-dimensional vector $\boldsymbol{A}$. The results obtained will be applied to a problem of disintegration of radioactive isotopes.

1. The structure of a statistical set. Let us observe the triple $(\Omega = \{\omega\};\ \mathscr{A}, \Psi)$ where Ψ is a probability measure defined over the σ-algebra $\mathscr{A}$ and suppose that the characteristic X of elements ω induce the numerical function $X(\omega)$ defined over Ω and measurable with respect to $\mathscr{A}$. Under the structure u of the statistical set in relation to X the system $u = (\Omega, X, \Psi)$ is understood.

If X induces in a random way the family of functions $\mathscr{E} = \{X\}$ measurable in relation to $\mathscr{A}$, then the structure of the statistical set changes at random in relation to X; for that $\mathscr{E} = \{X\}$ defines the set of structures $U = \{u\}$.

Let us now observe the triple $(U, \mathscr{U}, \mathscr{S})$ where $\mathscr{S}$ is the measure of probability defined over the σ-algebra $\mathscr{U}$ and let us assume that for every u the function X maps Ω on the set of real numbers $\{x_i;\ i = 1, 2, \ldots, n\}$. If we denote by $B_{iu} = \{\omega; X(\omega) = x_i\}$, where it is obvious that for every $u \in U$, $\bigcup_{i=1}^{n} B_{iu} = \Omega$, then $X = \sum_{i=1}^{n} x_i \chi_{B_{iu}}$ where $\chi_{B_{iu}}$ is the indicator of B_{iu}. Further putting $p_i(u) = \Psi(B_{iu})$ we get the n-dimensional vector $\boldsymbol{A}_u = \{p_1(u), p_2(u), \ldots, p_n(u)\}$, whose coordinates satisfy the condition $\sum_{\nu=1}^{n} p_\nu(u) = 1$ for every $u \in U$. The vector $\boldsymbol{A}_u$ is called the *vector of structure of the statistical set in relation to X.*

In the further exposition the random vector $\boldsymbol{A} = (p_1, p_2, \ldots, p_n)$ whose set of realizations is the family $\{\boldsymbol{A}_u;\ u \in U\}$ is called the *vector of random structure of the statistical set with respect to X.*

2. Distribution function of the vector A. On the basis of the preceding exposition the conclusion can be made that the vector A maps the space $U = \{u\}$ on that set of points W of the hyperplane

$$\sum_{\nu=1}^{n} x_\nu = 1 \tag{1}$$

whose coordinates are non-negative. Accordingly, measure $\mathscr{S}$ induces the measure $P(G) = \mathscr{S}\{u;\, A_u \in G\}$, where G is an element of σ-algebra $\mathscr{W}$ of subsets of W.

Let us now observe a sequence of non-negative numbers a_i, $i = 1, 2, \ldots, n$ and the distribution function F of the random vector A

$$F(a_1, a_2, \ldots, a_n) = P\{A_u;\, p_\nu \leqslant a_\nu,\ \nu = 1, 2, \ldots, n\}.$$

Then the following theorems can be proved.

THEOREM 1. *Let* $a_i + a_j > 1$ *for every* $i \neq j$; *then* $F(a_1, a_2, \ldots, a_n) = 1 - \sum_{\nu=1}^{n} D_\nu(a_\nu)$ *where* $D_\nu(a_\nu) = P\{A_u;\, p_\nu > a_\nu\}$.

THEOREM 2. *Let the first s variables* a_i, $i = 1, \ldots, s$, *satisfy the condition* $a_i + a_j \leqslant 1$ *for every* $i \neq j = 1, \ldots, s$, *while the other* $(n-s)$ *ones have the property* $a_\nu + a_q > 1$. *Then*

$$F(a_1, a_2, \ldots, a_n) = 1 - \sum_{\nu=1}^{n} D_\nu(a_\nu) + R_{1s},$$

where $R_{1s} = \sum_{i=1}^{s-1} \sum_{j=1}^{s} P(\beta_{ij})$, $\beta_{ij} \frown \beta_{\nu\mu} = \Theta$, $\beta_{ij} = \{A_u;\, p_i > a_i\} \frown \{A_u;\, p_j > a_j\}$.

Let μ be the Lebesgue measure defined over $\mathscr{W}$ and let P be absolutely continuous in relation to μ; then according to the Radon-Nicodym theorem there is a non-negative function Φ defined over W with the property $P(G) = \int_G \Phi\, d\mu$ $(G \in \mathscr{W})$.

If we have $G = \{A_u;\, a_\nu \leqslant p_\nu \leqslant b_\nu,\ \nu = 1, 2, \ldots, n-1\}$, then the following theorem is valid.

THEOREM 3. *Let* $\sum_{i=1}^{n-1} b_i \leqslant 1$ *and suppose that for every* $i = 1, 2, \ldots, n-1$ *the inequality* $0 \leqslant a_i \leqslant b_i$ *holds. Then*

$$P(G) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \ldots \int_{a_{n-1}}^{b_{n-1}} \Phi(x_1, x_2, \ldots, x_n) \sqrt{n} \prod_{\nu=1}^{n-1} dx_\nu,$$

$$x_n = 1 - \sum_{\nu=1}^{n-1} x_\nu.$$

3. The discrete type. Let us consider vector A and suppose that the set $\{A_u;\, u \in U\}$ is finite; writing $P_r\{A = A_u\} = P_u$ we get $\sum_{u \in U} P_u = 1$.

We shall further consider a random distribution of k elements into n cells. Let us denote each cell by a natural number from 1 to n and by X_ν^* the number of elements which belong to the νth cell. Because of the relation $\sum_{\nu=1}^{n} X_\nu^* = k$, it follows that the end-points of the realization of A belong to the set of points of the hyperplane

$$\sum_{\nu=1}^{n} x_\nu = k \tag{2}$$

whose coordinates are non-negative.

Let us write

$$P_r\{X_\nu^* = i_\nu;\ \nu = 1, \ldots, n\} = p_{i_1 i_2 \ldots i_n};$$

then it is not difficult to see that

$$\sum_{i_\nu=0}^{k} \sum_{i_{\nu-1}=0}^{k-i_\nu} \cdots \sum_{i_1=0}^{k-\tau_{2\nu}} \sum_{i_2=0}^{k-\tau_{1\nu}} \cdots \sum_{i_{\nu+2}=0}^{k-\tau_{1\nu}-\tau_{\nu+3,n}} p_{i_1 i_2 \ldots i_n} = 1, \tag{3}$$

where $\tau_{\mu,m} = \sum_{j=\mu}^{m} i_j$ and $\nu = 1, \ldots, n$. Further, if a_j^*, $j = 1, \ldots, n$, is the sequence of natural numbers such that $a_i^* + a_j^* \geqslant k$ for each $i \neq j$, then the distribution function is

$$F(a_1^*, a_2^*, \ldots, a_n^*) = 1 - \sum_{\nu=1}^{n} D_\nu^*(a_\nu^*),$$

where

$$D_\nu^*(a_\nu^*) = \sum_{i_\nu=a_\nu^*+1}^{k} \sum_{i_{\nu-1}=0}^{k-i_\nu} \cdots \sum_{i_1=0}^{k-\tau_{2\nu}} \cdots \sum_{i_{\nu+2}=0}^{k-\tau_{1\nu}-\tau_{\nu+3,n}} p_{i_1 i_2 \ldots i_n}.$$

If the first s variables a_ν^* have the property $a_i^* + a_j^* < k$, $i \neq j = 1, \ldots, s$ and the other $(n-s)$ ones satisfy the conditions $a_p^* + a_q^* \geqslant k$, $p \neq q = n-s, n-s+1, \ldots, n$, then we have

$$F(a_1^*, a_2^*, \ldots, a_n^*) = 1 - \sum_{\nu=1}^{n} D_\nu^*(a_\nu^*) + R_{1s}^*,$$

where

$$R_{1s}^* = \sum_{\mu=1}^{s-1} \sum_{j=\mu+1}^{s} R_{\mu j}, \qquad R_{\mu j} = \sum_{i_\mu=a_\mu^*}^{k-a_j^*} \sum_{i_{\mu-1}=0}^{k-a_j^*-i_\mu} \cdots \sum_{i_{\mu+2}=0}^{k-a_j^*-\tau_{1\mu}-\tau_{\mu+3,n}} p_{i_1 i_2 \ldots i_n}.$$

Finally, let a_i^* and b_i^* be non-negative integers so that $a_i^* \leqslant b_i^*$, $i = 1, \ldots, n-1$, and $\sum_{\nu=1}^{n-1} b_\nu^* \leqslant k$; then according to Theorem 3

$$P_r\{a_i^* \leqslant X_i^* \leqslant b_i^*;\ i = 1, \ldots, n-1\} = \sum_{i_1=a_1^*}^{b_1^*} \cdots \sum_{i_{n-1}=a_{n-1}^*}^{b_{n-1}^*} p_{i_1 i_2 \ldots i_n}.$$

Let us write $p_{i_1 i_2 \ldots i_n} = \Phi(i_1, i_2, \ldots, i_n)$; then various forms of Φ give various density functions of A. Let us start with the simplest case, i.e. let $\Phi \equiv C$; then according to (3) it is trivially verified that $C = 1\Big/\binom{k+n-1}{n-1}$. Hence it can easily be seen that

$$D_\nu^*(a_\nu^*) = \frac{1}{\binom{k+n-1}{n-1}} \sum_{i_{n-1}=a_\nu^*+1}^{k} \binom{k-i_{n-1}+n-2}{n-2}.$$

From this follows

$$F(a_1^*, a_2^*, \ldots, a_n^*) = 1 - \left\{ n - \frac{1}{\binom{k+n-1}{n-1}} \sum_{\nu=1}^{n} \sum_{i_{n-1}=0}^{a_\nu^*} \binom{k-i_{n-1}+n-2}{n-2} \right\},$$

when $a_i^* + a_j^* > k$.

Let us assume that $\Phi(i_1, i_2, \ldots, i_n) = \prod_{\nu=1}^{n} \Phi_\nu(i_\nu)$; then for various $\Phi_\nu(i_\nu)$ we have various density functions of A. For instance let $\Phi_\nu(i_\nu) = C_\nu \binom{N_\nu}{i_\nu}$ for each $\nu = 1, \ldots, n$; then $\Phi = C \prod_{\nu=1}^{n} \binom{N_\nu}{i_\nu}$ and it can be shown that $C = 1\Big/\binom{N}{k}$ where $\sum_{\nu=1}^{n} N_\nu = N$. Under these conditions we have

$$D_\nu^*(a_\nu^*) = \frac{1}{\binom{N}{k}} \sum_{i_\nu=a_\nu^*+1}^{k} \binom{N_\nu}{i_\nu} \binom{N-N_\nu}{k-i_\nu}.$$

Finally, let $\Phi_\nu(i_\nu) = C_\nu(p_\nu^{i_\nu}/i_\nu!)$, where $\sum_{\nu=1}^{n} p_\nu = 1$ and $0 \leqslant p_\nu$ for $\nu = 1, \ldots, n$. According to the relation

$$\sum_{i_\nu=0}^{k} \sum_{i_{\nu-1}=0}^{k-i_\nu} \cdots \sum_{i_{\nu+2}=0}^{k-\tau_{1\nu}-\tau_{\nu+3,n}} C \prod_{j=1}^{n} \frac{p_j^{i_j}}{i_j!} = 1,$$

as we already known, it follows that $C = k!$.

Theorem 4. *Let* $\sum_{\nu=1}^{n} p_\nu = 1$ *and* $0 \leqslant p_\nu$ *for* $\nu = 1, \ldots, n$; *then*

$$\sum_{i_\nu=a_\nu^*+1}^{k} \sum_{i_{\nu-1}=0}^{k-i_\nu} \cdots \sum_{i_{\nu+2}=0}^{k-\tau_{1\nu}-\tau_{\nu+3,n}} k! \prod_{j=1}^{n} \frac{p_j^{i_j}}{i_j!}$$

$$= 1 - (a_\nu^*+1) \binom{k}{a_\nu^*+1} \int_0^{1-p_\nu} (1-x)^{a_\nu^*} x^{k-a_\nu^*-1} dx.$$

4. Application. In order to apply the above results, let us consider the following problem: a set of N radioactive particles (N is not a statistically large number) and the frequency distribution of the disintegrated isotopes are observed in the following way: Let us divide $(0, \infty)$ into n subintervals (t_i, t_{i+1}), $i = 1, \ldots, n$, $t_1 = 0$, $t_{n+1} = \infty$, and let us consider the frequency distribution of disintegrated particles in relation to those n sub-intervals. Since it is not possible to predict the number of particles which will disintegrate in (t_i, t_{i+1}), it is not possible to predict the form of the frequency distribution. Let us denote the number of disintegrated particles in (t_i, t_{i+1}) by X_ν^*; then $A = (X_1^*, X_2^*, \ldots, X_n^*)$ is a random vector. If the density function of disintegration is $f(t) = \lambda e^{-\lambda t}$, then writing

$$p_i = \int_{t_i}^{t_{i+1}} f(t)\,dt = (e^{-\lambda t_i} - e^{-\lambda t_{i+1}})$$

we get

$$p_{i_1 i_2 \ldots i_n} = N! \prod_{j=1}^{n} \left(\frac{p_j^{i_j}}{i_j}\right), \qquad \sum_{j=1}^{n} i_j = N.$$

We cannot predict the form of the future frequency distribution of these N radioactive particles, but we can compute the probability of a certain set of this distribution. For instance, if $a_i^* + a_j^* > N$ for $i \neq j$ we have

$$P\{A; X_\nu^* \leqslant a_\nu^*, \nu = 1, \ldots, n\}$$
$$= 1 - \sum_{\nu=1}^{n} \left\{1 - (1 + a_\nu^*) \binom{N}{a_\nu^* + 1} \int_0^{1-(e^{-\lambda t_\nu} - e^{-\lambda t_{\nu-1}})} (1-x)^{a_\nu^*} x^{N - a_\nu^* - 1} dx\right\}.$$

5. Appendix. Proof of Theorem 1. Let us prove that $\beta_{ij} = \{A_u; p_i > a_i\} \cap \{A_u; p_j > a_j\} = \Theta$ if $a_i + a_j > 1$; writing $a = (0, 0, \ldots, 0)$ and $b = (1, 1, \ldots, 1)$ we have $W \subset (a, b)$. Since $\{A_u; p_\nu > a_\nu\} \subseteq W$, it is trivially verified that $\{A_u; p_\nu > a_\nu\} \subset (a_\nu, b_\nu)$, where $a_\nu = (0, \ldots, a_\nu, \ldots, 0)$ and $b_\nu = (1 - a_\nu, 1 - a_\nu, \ldots, 1, \ldots, 1 - a_\nu)$. Therefore, under these conditions we have $(a_i, b_i) \cap (a_j, b_j) = \Theta$ for $i \neq j$ and the relation

$$\{A_u; p_\nu \leqslant a_\nu, \nu = 1, \ldots, n\} = \bigcap_{\nu=1}^{n} \{A_u; p_\nu \leqslant a_\nu\}$$

is valid; hence

$$\{A_u; p_\nu \leqslant a_\nu, \nu = 1, \ldots, n\}^c = \bigcup_{\nu=1}^{n} \{A_u; p_\nu \leqslant a_\nu\}^c,$$

$$\{A_u; p_\nu \leqslant a_\nu\}^c = \{A_u; p_\nu > a_\nu\},$$

so that

$$W = \{A_u; p_\nu \leqslant a_\nu, \nu = 1, \ldots, n\} \cup [\bigcup_{\nu=1}^{n} \{A_u; p_\nu > a_\nu\}],$$

which proves the theorem.

Proof of Theorem 2. As we have seen, the following relation

$$W=\{A_u;\,p_\nu\leqslant a_\nu,\ \nu=1,\ldots,n\}\cup[\bigcup_{i=1}^{s}\{A_u;\,p_\nu>a_\nu\}]\cup[\bigcup_{i=s+1}^{n}\{A_u;\ p_i>a_i\}]$$

is valid. Since we have

$$\bigcup_{i=1}^{s}\{A_u;\,p_i>a_i\}=\{A_u;\,p_s>a_s\}\cup[\bigcup_{i=1}^{s-1}(\{A_u;\,p_i>a_i\}-\bigcup_{j=i+1}^{s}\beta_{ij})],$$

it is not difficult to see that

$$1=P\{A_u;\,p_\nu\leqslant a_\nu,\ \nu=1,\ldots,n\}+\sum_{i=s}^{n}P\{A_u;\,p_i>a_i\}+$$
$$+\sum_{i=1}^{s-1}P(\{A_u;\,p_i>a_i\}-\bigcup_{j=i+1}^{s}\beta_{ij})$$

which proves the theorem.

Proof of Theorem 4. Let us assume the function $D_\nu^*(a_\nu^*)$ in following form

$$D_\nu^*(a_\nu^*)=\sum_{i_\nu=0}^{k}\sum_{i_{\nu-1}=0}^{k-i_\nu}\cdots\sum_{i_{\nu+2}=0}^{k-\tau_{1\nu}-\tau_{\nu+3,n}}k!\prod_{j=1}^{n}\left(\frac{p_j^{i_j}}{i_j!}\right)-$$
$$-\sum_{i_\nu=0}^{a_\nu^*}\sum_{i_{\nu-1}=0}^{k-i_\nu}\cdots\sum_{i_{\nu+2}=0}^{k-\tau_{1\nu}-\tau_{\nu+3,n}}k!\prod_{j=1}^{n}\left(\frac{p_j^{i_j}}{i_j!}\right)$$

and denote by B_ν the second of the sums considered; then $D_\nu^*(a_\nu^*)=1-B_\nu$. Further, since

$$B_\nu=\sum_{i_\nu=0}^{a_\nu^*}\binom{k}{i_\nu}p_\nu^{i_\nu}\sum_{i_{\nu-1}=0}^{k-i_\nu}\binom{k-i_\nu}{i_{\nu-1}}p_{\nu-1}^{i_{\nu-1}}\cdots$$
$$\cdots\sum_{i_{\nu+3}=0}^{k-\tau_{1\nu}-\tau_{\nu+3,n-1}}\binom{k-\tau_{1\nu}-\tau_{\nu+3,n-1}}{i_{\nu+3}}p_{\nu+3}^{i_{\nu+3}}\sum_{i_{\nu+2}=0}^{k-\tau_{1\nu}-\tau_{\nu+3,n}}\binom{k-\tau_{1\nu}-\tau_{\nu+3,n}}{i_{\nu+2}}p_{\nu+2}^{i_{\nu+2}}p_{\nu+1}^{i_{\nu+1}}$$

and

$$\sum_{i_{\nu+2}=0}^{k-\tau_{1\nu}-\tau_{\nu+3,n}}\binom{k-\tau_{1\nu}-\tau_{\nu+3,n}}{i_{\nu+2}}p_{\nu+2}^{i_{\nu+2}}p_{\nu+1}^{i_{\nu+1}}=(p_{\nu+2}+p_{\nu+1})^{k-\tau_{1\nu}-\tau_{\nu+3,n}},$$

it is trivially verified that

$$B_\nu=\sum_{i_\nu=0}^{a_\nu^*}\binom{k}{i_\nu}p_\nu^{i_\nu}\Big(\sum_{\tau=1}^{\nu-1}p_\tau+\sum_{\tau=\nu+1}^{n}p_\tau\Big)^{k-i_\nu}=\sum_{i_\nu=0}^{a_\nu^*}\binom{k}{i_\nu}p_\nu^{i_\nu}(1-p_\nu)^{k-i_\nu}$$
$$=(1+a_\nu^*)\binom{k}{a_\nu^*+1}\int_0^{1-p_\nu}(1-x)^{a_\nu^*}\,x^{k-a_\nu^*-1}\,dx.$$

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Received on 12. 9. 1964

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ZBIOROWOŚCI STATYSTYCZNE O LOSOWEJ STRUKTURZE

STRESZCZENIE

Autor rozpatruje w pracy zbiorowość statystyczną o strukturze zmieniającej się losowo ze względu na pewną cechę. W pierwszym paragrafie podane są definicje struktury i losowej struktury zbiorowości statystycznej, która w pewnych szczególnych przypadkach może być przedstawiona jako wektor n-wymiarowy. Autor dowodzi kilku twierdzeń o własnościach rozkładu prawdopodobieństwa tych wektorów. W paragrafach 3 i 4 rozważane są zmienne losowe typu skokowego. Uzyskane wyniki zastosowane są w zagadnieniu rozpadu izotopów radioaktywnych.

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СТАТИСТИЧЕСКИЕ СОВОКУПНОСТИ СО СЛУЧАЙНОЙ СТРУКТУРОЙ

РЕЗЮМЕ

В настоящей работе расматривается статистическая совокупность, структура которой изменяется случайным образом в отношении к определенному признаку. В § 1 определяется понятие структуры и случайной структуры статистической совокупности, которую в некоторых специялных случаях можно представить как n-мерный вектор. В работе детально расматривается этот случай и доказывается ряд теорем указывающих на некоторые свойства функции распределения этого вектора. В § 3 и 4 рассматривается дискретный случай и его применение к проблеме распада радиоактивных изотопов.
