In conclusion, we mention an application to a problem in graph coloring. Let $g(n)$ be the largest positive integer for which there exists some way of coloring the edges of a complete graph on $g(n)$ vertices in $n$ colors without forcing the appearance of a monochromatic triangle. That $g(n)$ exists follows from a well known theorem of F. P. Ramsay ([3]), and in fact in [2] it is proved that

$$g(n) \leqslant [n!e]. \tag{11}$$

However, it seems that no lower bound for $g(n)$ appears in the literature. Here we prove that

$$g(n) \geqslant f(n)+1 \tag{12}$$

and hence, in view of (4), that

$$g(n) > 89^{\frac{1}{4}n-c\log n}. \tag{13}$$

In order to prove (12), let $A_1, A_2, \ldots, A_n$ be disjoint sum-free sets containing the integers $1, 2, \ldots, f(n)$. Let $G$ be a complete graph with vertices $P_0, P_1, \ldots, P_{f(n)}$. Color the edges of $G$ in the $n$ colors $C_1, C_2, \ldots, C_n$ by coloring the edge joining $P_s$ and $P_t$ color $C_j$ if $|s-t| \in A_j$. Suppose there results a triangle with vertices $P_s, P_t$ and $P_r$ all of whose edges are colored $C_j$. We may assume $s > t > r$. Then $s-t$, $s-r$, $t-r \in A_j$. But $(s-t)+(t-r) = (s-r)$ and this contradicts the fact that $A_j$ is sum-free.

It is interesting to observe that (11) and (12) afford an independent proof of (1).

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# On the difference $\pi(x) - \mathrm{li}(x)$

by

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**1. Introduction.** The prime number theorem states that $\pi(x)$, the number of primes less than or equal to $x$, is asymptotically equal to $\mathrm{li}(x)$ as $x \to \infty$ where

$$\mathrm{li}(x) = \lim_{\varepsilon\to 0+} \left\{\int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^{x} \frac{dt}{\log t}\right\}.$$

It is a remarkable fact that the difference $\pi(x)-\mathrm{li}(x)$ is negative for all values of $x$ at which $\pi(x)$ has been calculated exactly. In particular, Rosser ([11], p. 72) has shown that the difference is negative for all $x \leqslant 10^8$. Nevertheless, Littlewood ([9]) has proved that there is a positive number $K$ such that

$$\frac{\log x\{\pi(x)-\mathrm{li}(x)\}}{x^{1/2}\log\log\log x}$$

is greater than $K$ for arbitrarily large values of $x$ and less than $-K$ for arbitrarily large values of $x$. Thus the situation represented by the calculations does not continue indefinitely. Skewes ([12]) has obtained a very large upper bound for the first $x$ for which the difference is positive, namely $\exp\exp\exp\exp(7.705)$.

In this paper we first derive an explicit formula for a certain average of the difference $\pi(e^u)-\mathrm{li}(e^u)$. We then describe how this explicit formula can be combined with numerical computations performed by a computer to show that between $1.53\times 10^{1165}$ and $1.65\times 10^{1165}$ there are more than $10^{500}$ successive integers $x$ for which $\pi(x) > \mathrm{li}(x)$.

**2. Explicit formulas.** For background information we refer to Ingham ([4]), [5]).

Throughout this paper $\varrho = \beta+i\gamma$ will denote a zero of the Riemann zeta function $\zeta(s)$ for which $0 < \beta < 1$. We denote by $\vartheta$ a number satisfying $|\vartheta| \leqslant 1$. The number denoted will, in general, be different for different occurrences and may depend on variables.

Let

$$\Pi(x) = \pi(x)+\tfrac{1}{2}\pi(x^{1/2})+\tfrac{1}{3}\pi(x^{1/3})+\ldots,$$

and let

$$\Pi_0(x) = \lim_{\varepsilon\to 0} \tfrac{1}{2}\{\Pi(x+\varepsilon)+\Pi(x-\varepsilon)\}.$$

The Riemann-von Mangoldt explicit formula for $\Pi_0(x)$ (for a proof see [6]) states that for $x>1$

$$\Pi_0(x) = \mathrm{li}(x)-\sum_{\varrho}\mathrm{li}(x^{\varrho})+\int_x^{\infty}\frac{du}{(u^2-1)u\log u}-\log 2, \tag{2.1}$$

where $\mathrm{li}(x^{\varrho}) = \mathrm{li}(e^{\varrho\log x})$ and for $w = u+iv$, $v\neq 0$,

$$\mathrm{li}(e^{w}) = \int_{-\infty+iv}^{u+iv}\frac{e^z}{z}\,dz. \tag{2.2}$$

If the terms in the sum are arranged according to increasing absolute value of $\gamma = \mathrm{Im}\,\varrho$, then the series converges boundedly in every finite interval $1\leqslant a\leqslant x<b$.

Rosser and Schoenfeld ([11], p. 69) have shown that for $x>1$

$$\pi(x) = \frac{x}{\log x}+\frac{\frac{3}{2}\vartheta x}{\log^2 x}.$$

Using this estimate and the more elementary estimate $\pi(x)<2x/\log x$, we obtain for $x>1$

$$\frac{1}{2}\pi(x^{1/2})+\frac{1}{3}\pi(x^{1/3})+\ldots = \frac{x^{1/2}}{\log x}+\vartheta\left(\frac{3x^{1/2}}{\log^2 x}+\frac{2x^{1/3}}{\log x}\left[\frac{\log x}{\log 2}\right]\right).$$

Estimating the integral in (2.1), we have for $x\geqslant e$

$$0<\int_x^{\infty}\frac{du}{(u^2-1)u\log u}<2\int_x^{\infty}\frac{du}{u^3} = \frac{1}{x^2}<\log 2.$$

Since $2/\log 2+\log 2<4$,

$$\pi(x) = \mathrm{li}(x)-\frac{x^{1/2}}{\log x}-\sum_{\varrho}\mathrm{li}(x^{\varrho})+\vartheta\left(\frac{3x^{1/2}}{\log^2 x}+4x^{1/3}\right) \tag{2.3}$$

for $x\geqslant e$.

The formula (2.3) can be used to explain heuristically why $\pi(x)$ is usually smaller than $\mathrm{li}(x)$. If the Riemann hypothesis is true, then the zeros of $\zeta(s)$ occur in conjugate pairs $\varrho = \frac{1}{2}+i\gamma$, $\bar{\varrho} = \frac{1}{2}-i\gamma$; and

$$\mathrm{li}(x^{\varrho})+\mathrm{li}(x^{\bar{\varrho}}) = \frac{x^{1/2}}{\log x}\left\{\frac{2\gamma\sin(\gamma\log x)+\cos(\gamma\log x)}{\frac{1}{4}+\gamma^2}\right\}+O\left(\frac{x^{1/2}}{\log^2 x}\right).$$



Since $|\gamma|>14$ for every zero $\varrho$, each term of the sum is small in magnitude compared to the term $-x^{-1/2}/\log x$. Thus, to have $\pi(x)>\mathrm{li}(x)$ it is necessary for many terms in the sum to combine to overpower the term $-x^{-1/2}/\log x$.

It does not seem feasible to use (2.3) directly to determine by numerical computation a number $x$ for which $\pi(x)>\mathrm{li}(x)$. Instead, we shall derive an explicit formula for $ue^{-u/2}\{\pi(e^u)-\mathrm{li}(e^u)\}$ averaged by a Gaussian kernel.

THEOREM. *Let $A$ be a positive number such that $\beta = \frac{1}{2}$ for all zeros $\varrho = \beta+i\gamma$ of $\zeta(s)$ for which $0<\gamma\leqslant A$. Let $a$, $\eta$, and $\omega$ be positive numbers such that $\omega-\eta>1$ and the conditions*

$$4A/\omega\leqslant a\leqslant A^2 \tag{2.4}$$

*and*

$$2A/a\leqslant\eta<\omega/2 \tag{2.5}$$

*hold. Let*

$$K(y) = \sqrt{\frac{a}{2\pi}}\,e^{-ay^2/2}. \tag{2.6}$$

*Then for $2\pi e<T\leqslant A$*

$$\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)ue^{-u/2}\{\pi(e^u)-\mathrm{li}(e^u)\}\,du = -1-\sum_{0<|\gamma|\leqslant T}\frac{e^{i\gamma\omega}}{\varrho}e^{-\gamma^2/2a}+R$$

*where*

$$|R|\leqslant\frac{3.05}{\omega-\eta}+4(\omega+\eta)e^{-(\omega-\eta)/6}+\frac{2e^{-a\eta^2/2}}{\sqrt{2\pi a}\,\eta}+0.08\sqrt{a}\,e^{-a\eta^2/2}+$$
$$+e^{-T^2/2a}\left\{\frac{a}{\pi T^2}\log\frac{T}{2\pi}+\frac{8\log T}{T}+\frac{4a}{T^3}\right\}+$$
$$+A\log A\,e^{-A^2/2a+(\omega+\eta)/2}\{4a^{-1/2}+15\eta\}.$$

*If the Riemann hypothesis holds, then conditions* (2.5) *and* (2.4) *and the last term in the estimate for $R$ may be omitted.*

**3. Some lemmas.** Let $N(T)$ be the number of zeros for which $0<\gamma\leqslant T$. Backlund ([1]) proved that for $T\geqslant 2$

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+\frac{7}{8}+Q(T) \tag{3.1}$$

where

$$|Q(T)|<0.137\log T+0.443\log\log T+4.35.$$

From this it follows that for $T \geqslant 2\pi e$

$$N(T) = \frac{1}{2\pi}\int_{2\pi e}^{T} \log\frac{t}{2\pi}\,dt + \frac{7}{8} + 2\vartheta\log T. \tag{3.2}$$

Lemma 1. *If $\varphi(t)$ is a continuous function which is positive and monotone decreasing for $2\pi e \leqslant T_1 \leqslant t \leqslant T_2$, then*

$$\sum_{T_1<\gamma\leqslant T_2}\varphi(\gamma) = \frac{1}{2\pi}\int_{T_1}^{T_2}\varphi(t)\log\frac{t}{2\pi}\,dt + \vartheta\left\{4\varphi(T_1)\log T_1 + 2\int_{T_1}^{T_2}\frac{\varphi(t)}{t}\,dt\right\}.$$

Proof. Using Stieltjes integrals, we have

$$\sum_{T_1<\gamma\leqslant T_2}\varphi(\gamma) = \int_{T_1}^{T_2}\varphi(t)\,dN(t) = \frac{1}{2\pi}\int_{T_1}^{T_2}\varphi(t)\log\frac{t}{2\pi}\,dt + \int_{T_1}^{T_2}\varphi(t)\,dQ(t).$$

Also, by (3.2)

$$\left|\int_{T_1}^{T_2}\varphi(t)\,dQ(t)\right| = |\varphi(T_2)Q(T_2) - \varphi(T_1)Q(T_1)| - \int_{T_1}^{T_2}Q(t)\,d\varphi(t)$$

$$\leqslant 2\varphi(T_2)\log T_2 + 2\varphi(T_1)\log T_1 - 2\int_{T_1}^{T_2}\log t\,d\varphi(t)$$

$$\leqslant 4\varphi(T_1)\log T_1 + 2\int_{T_1}^{T_2}\varphi(t)\,d(\log t).$$

Lemma 2. *If $T \geqslant 2\pi e$, then*

$$\sum_{\gamma>T}\frac{1}{\gamma^n} < T^{1-n}\log T \qquad (n = 2, 3, \ldots).$$

Proof. Applying Lemma 1, we obtain

$$\sum_{>T}\frac{1}{\gamma^n} = \frac{1}{2\pi}\int_T^\infty t^{-n}\log\frac{t}{2\pi}\,dt + \vartheta T^{-n}\left(4\log T + \frac{2}{n}\right)$$

$$= \frac{T^{1-n}}{2\pi}\left(\frac{\log(T/2\pi)}{n-1} + \frac{1}{(n-1)^2}\right) + \vartheta T^{-n}\left(4\log T + \frac{2}{n}\right)$$

$$\leqslant T^{1-n}\log T\left\{\frac{1}{2\pi} + \frac{1}{2\pi\log T} + \frac{4}{T} + \frac{1}{T\log T}\right\}$$

$$< T^{1-n}\log T.$$



Lemma 3. *We have*

$$\sum_{0<\gamma<\infty}\frac{1}{\gamma^2} < 0.025.$$

Proof. See [10], p. 28.

Lemma 4. *If $a > 0$ and $\varphi(t)$ is positive and monotone decreasing for $t \geqslant T > 0$, then*

$$\int_T^\infty \varphi(t)e^{-t^2/2a}\,dt < \frac{a}{T}\varphi(T)e^{-T^2/2a}.$$

Proof. Since

$$\frac{d}{dt}\left\{\frac{ae^{-t^2/2a}}{t}\right\} = -\frac{ae^{-t^2/2a}}{t^2} - e^{-t^2/2a},$$

we have

$$\int_T^\infty \varphi(t)e^{-t^2/2a}\,dt < -\int_T^\infty \varphi(t)\frac{d}{dt}\left(\frac{ae^{-t^2/2a}}{t}\right)dt \leqslant \frac{a}{T}\varphi(T)e^{-T^2/2a}.$$

**4. Proof of theorem.** In the following discussion we assume that $a$, $\omega$, and $\eta$ are positive numbers such that $\omega - \eta > 1$. With $K(y)$ defined by (2.6) we have for any real number $\gamma$

$$\int_{-\infty}^{\infty}K(y)e^{i\gamma y}\,dy = e^{-\gamma^2/2a}\sqrt{\frac{a}{2\pi}}\int_{-\infty}^{\infty}e^{-a(y-i\gamma/a)^2/2}\,dy \tag{4.1}$$

$$= \frac{e^{-\gamma^2/2a}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-t^2/2}\,dt = e^{-\gamma^2/2a},$$

and, in particular,

$$\int_{-\infty}^{\infty}K(y)\,dy = 1. \tag{4.2}$$

Consider the integral

$$I(\omega,\eta) = \int_{\omega-\eta}^{\omega+\eta}K(u-\omega)ue^{-u/2}\{\pi(e^u) - \mathrm{li}(e^u)\}\,du.$$

By (2.3) we have for $u > 1$

$$ue^{-u/2}\{\pi(e^u) - \mathrm{li}(e^u)\} = -1 - \sum_{\varrho}ue^{-u/2}\mathrm{li}(e^{\varrho u}) + \vartheta\left(\frac{3}{u} + 4ue^{-u/6}\right). \tag{4.3}$$

In view of the positivity of the kernel $K$, we obtain by (4.2)

$$\left|\int_{\omega-\eta}^{\omega+\eta}\vartheta\left(\frac{3}{u} + 4ue^{-u/6}\right)K(u-\omega)\,du\right| \leqslant \frac{3}{\omega-\eta} + 4(\omega+\eta)e^{-(\omega-\eta)/6}.$$

Also, by Lemma 4

$$\int_{-\infty}^{\omega-\eta} K(u-\omega)\,du = \int_{\omega+\eta}^{\infty} K(u-\omega)\,du = \int_{\eta}^{\infty} K(y)\,dy$$
$$= \sqrt{\frac{\alpha}{2\pi}}\int_{\eta}^{\infty} e^{-\alpha t^2/2}\,dt < \frac{e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\,\eta}.$$

Consequently

$$I(\omega,\eta) = -1-\sum_{\varrho}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\,ue^{-u/2}\,\mathrm{li}(e^{\varrho u})\,du+ \tag{4.4}$$
$$+\vartheta\left(\frac{3}{\omega-\eta}+4(\omega+\eta)e^{-(\omega-\eta)/6}+\frac{2e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\,\eta}\right).$$

The interchange of summation and integration is justified because the series in (4.3) converges boundedly in the interval $\omega-\eta\leqslant u\leqslant\omega+\eta$.

By (2.2)

$$\mathrm{li}(e^{\varrho u}) = \int_{-\infty+i\gamma u}^{\varrho u}\frac{e^z}{z}\,dz = e^{\varrho u}\int_0^{\infty}\frac{e^{-t}}{\varrho u-t}\,dt. \tag{4.5}$$

Integration by parts yields

$$\int_0^{\infty}\frac{e^{-t}}{\varrho u-t}\,dt = \frac{1}{\varrho u}+\int_0^{\infty}\frac{e^{-t}}{(\varrho u-t)^2}\,dt = \frac{1}{\varrho u}+\int_0^{\infty}\frac{\vartheta e^{-t}}{(\gamma u)^2}\,dt.$$

Consequently

$$\mathrm{li}(e^{\varrho u}) = \frac{e^{\varrho u}}{\varrho u}+\frac{\vartheta e^{\beta u}}{\gamma^2u^2}. \tag{4.6}$$

Let $A$ be a positive number such that $\beta=\frac{1}{2}$ for $|\gamma|\leqslant A$, i.e., such that the Riemann hypothesis holds for $|\gamma|\leqslant A$. We break the sum in (4.4) into three parts and use (4.6) for $|\gamma|\leqslant A$ to obtain

$$-\sum_{\varrho}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\,ue^{-u/2}\,\mathrm{li}(e^{\varrho u})\,du = S_1+S_2+S_3, \tag{4.7}$$

where

$$S_1 = -\sum_{0<|\gamma|\leqslant A}\frac{1}{\varrho}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\,e^{i\gamma u}\,du,$$
$$S_2 = -\sum_{0<|\gamma|\leqslant A}\int_{\omega-\eta}^{\omega+\eta}\frac{\vartheta}{\gamma^2u}K(u-\omega)\,du,$$
$$S_3 = -\sum_{|\gamma|>A}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\,ue^{-u/2}\,\mathrm{li}(e^{\varrho u})\,du.$$



We begin by considering $S_1$. By (4.1)

$$S_1 = -\sum_{|\gamma|\leqslant A}\frac{e^{i\gamma\omega}}{\varrho}\int_{-\eta}^{\eta}K(y)\,e^{i\gamma y}\,dy$$
$$= -\sum_{|\gamma|\leqslant A}\frac{e^{i\gamma\omega}}{\varrho}e^{-\gamma^2/2\alpha}+4\vartheta\sum_{0<\gamma\leqslant A}\frac{1}{\gamma}\left|\int_{\eta}^{\infty}K(y)\,e^{i\gamma y}\,dy\right|.$$

Integrating by parts, we obtain

$$\int_{\eta}^{\infty}K(y)\,e^{i\gamma y}\,dy = \int_{\eta}^{\infty}K'(y)\,\frac{(e^{i\gamma\eta}-e^{i\gamma y})}{i\gamma}\,dy.$$

Hence, because $K(y)$ is monotone decreasing for $y>0$,

$$\left|\int_{\eta}^{\infty}K(y)\,e^{i\gamma y}\,dy\right| \leqslant \frac{2}{\gamma}\int_{\eta}^{\infty}|K'(y)|\,dy = \frac{2}{\gamma}K(\eta) = \frac{2}{\gamma}\sqrt{\frac{\alpha}{2\pi}}\,e^{-\alpha\eta^2/2}.$$

Now we use Lemma 3 and the inequality $(2\pi)^{-1/2}<0.4$ to obtain

$$S_1 = -\sum_{|\gamma|\leqslant A}\frac{e^{i\gamma\omega}}{\varrho}e^{-\gamma^2/2\alpha}+8\vartheta\sqrt{\frac{\alpha}{2\pi}}\sum_{0<\gamma\leqslant A}\frac{1}{\gamma^2} \tag{4.8}$$
$$= -\sum_{|\gamma|\leqslant A}\frac{e^{i\gamma\omega}}{\varrho}e^{-\gamma^2/2\alpha}+0.08\vartheta\sqrt{\alpha}\,e^{-\alpha\eta^2/2}.$$

The sum can be taken over just the zeros for which $0<|\gamma|\leqslant T$ if we add another error term. By Lemma 1

$$\left|\sum_{T<|\gamma|\leqslant A}\frac{e^{i\gamma\omega}}{\varrho}e^{-\gamma^2/2\alpha}\right| \leqslant 2\sum_{T<\gamma<\infty}\frac{e^{-\gamma^2/2\alpha}}{\gamma}$$
$$\leqslant \int_T^{\infty}\frac{e^{-t^2/2\alpha}}{\pi t}\log\frac{t}{2\pi}\,dt+\frac{8e^{-T^2/2\alpha}\log T}{T}+4\int_T^{\infty}\frac{e^{-t^2/2\alpha}}{t^2}\,dt$$

provided $T\geqslant 2\pi e$. Applying Lemma 4 to estimate the integrals, we obtain for $2\pi e\leqslant T\leqslant A$

$$\left|\sum_{T<|\gamma|\leqslant A}\frac{e^{i\gamma\omega}}{\varrho}e^{-\gamma^2/2\alpha}\right| < e^{-T^2/2\alpha}\left\{\frac{\alpha}{\pi T^2}\log\frac{T}{2\pi}+8\,\frac{\log T}{T}+\frac{4\alpha}{T^3}\right\}. \tag{4.9}$$

The sum $S_2$ can be estimated by using Lemma 3. We have

$$|S_2| \leqslant \sum_{\varrho}\frac{1}{\gamma^2}\int_{-\eta}^{\eta}\frac{|K(y)|}{\omega+y}\,dy \leqslant \frac{0.05}{\omega-\eta}. \tag{4.10}$$

Observe that as yet we have made no use of the conditions (2.4) and (2.5). Thus, if we assume the Riemann hypothesis, we can combine (4.4) and (4.7)-(4.10) and then let $A \to \infty$ to obtain the conclusion of the theorem with the last term in the estimate for $R$ omitted.

**5. Estimate without the Riemann hypothesis.** In order to complete the proof of the theorem it is sufficient to show

$$(5.1)\qquad |S_3| \leqslant A\log A e^{-A^2/2\alpha+(\omega+\eta)/2}\{4\alpha^{-1/2}+15\eta\}$$

when $A$, $\alpha$, $\omega$, and $\eta$ satisfy the conditions of the theorem.

We begin by considering the function

$$f_\varrho(s) = \varrho s e^{-\varrho s}\mathrm{li}(e^{\varrho s})e^{-\alpha(s-\omega)^2/2}$$

in the sector $-\frac{\pi}{4} \leqslant \arg s \leqslant \frac{\pi}{4}$. The inequality $\frac{5}{12}\pi < |\arg\varrho| < \frac{\pi}{2}$ holds for every zero $\varrho$ because $0<\beta<1$ and $|\gamma|>14$. It follows from (2.2) that $f_\varrho(s)$ is a regular analytic function in the sector since $\frac{\pi}{6} < |\arg(\varrho s)| < \frac{3}{4}\pi$. Also, by (4.5)

$$(5.2)\qquad |f_\varrho(s)| = \left|\varrho s e^{-\alpha(s-\omega)^2/2}\int_0^\infty \frac{e^{-t}}{\varrho s - t}dt\right|$$

$$\leqslant \frac{|\varrho s|\,|e^{-\alpha(s-\omega)^2/2}|}{|\mathrm{Im}(\varrho s)|}\int_0^\infty e^{-t}dt \leqslant 2|e^{-\alpha(s-\omega)^2/2}|.$$

In the sum

$$S_3 = -\sqrt{\frac{\alpha}{2\pi}}\sum_{|\gamma|>A}\frac{1}{\varrho}\int_{\omega-\eta}^{\omega+\eta} e^{(\varrho-\frac12)u}f_\varrho(u)\,du$$

we transform the terms by repeated integration by parts. We obtain

$$\int_{\omega-\eta}^{\omega+\eta} e^{(\varrho-\frac12)u}f_\varrho(u)\,du = \sum_{n=0}^{N-1}\frac{(-1)^n e^{(\varrho-\frac12)\omega}}{(\varrho-\frac12)^{n+1}}\{e^{(\varrho-\frac12)\eta}f_\varrho^{(n)}(\omega+\eta) - e^{-(\varrho-\frac12)\eta}f_\varrho^{(n)}(\omega-\eta)\}+$$

$$+\frac{(-1)^N}{(\varrho-\frac12)^N}\int_{\omega-\eta}^{\omega+\eta} e^{(\varrho-\frac12)u}f_\varrho^{(N)}(u)\,du,$$

where $N$ is a positive integer which will be fixed later.

We estimate $f_\varrho^{(n)}(u)$ for $\omega-\eta \leqslant u \leqslant \omega+\eta$ by using a contour integral around a circle of radius $r \leqslant \omega/4$ about the point $u$. If $s$ is on this circle, then $\mathrm{Re}\, s \geqslant \omega-\eta-\omega/4 > \omega/4$ because of (2.5), and $|\mathrm{Im}\, s| \leqslant \omega/4$.



Thus the circle lies in the sector $|\arg s| \leqslant \pi/4$ where $f_\varrho(s)$ is regular and satisfies (5.2). Consequently, for $\omega-\eta \leqslant u \leqslant \omega+\eta$

$$f_\varrho^{(n)}(u) = \frac{n!}{2\pi i}\oint \frac{f_\varrho(s)}{(s-u)^{n+1}}ds,$$

and hence

$$|f_\varrho^{(n)}(u)| \leqslant \frac{2n!}{r^n}\max_{|s-u|=r}|e^{-\alpha(s-\omega)^2/2}|.$$

If $s=\sigma+it$, then on the circle $(\sigma-u)^2+t^2=r^2$

$$|e^{-\alpha(s-\omega)^2/2}| = e^{\alpha(t^2-(\sigma-\omega)^2)/2} = e^{\alpha(r^2-(\sigma-u)^2-(\sigma-\omega)^2)/2} \leqslant e^{\alpha r^2/2}.$$

If $N \leqslant \alpha\omega^2/16$, then we can choose $r=\sqrt{N/\alpha}$ and obtain

$$(5.3)\qquad |f_\varrho^{(N)}(u)| \leqslant 2N!\,N^{-N/2}\alpha^{N/2}e^{N/2}$$

for $\omega-\eta \leqslant u \leqslant \omega+\eta$. To estimate the derivatives at $\omega\pm\eta$ we let $r=\eta/2$, which because of (2.5) is less than $\omega/4$. On the circle $|s-(\omega\pm\eta)|=r$ we have

$$|e^{-\alpha(s-\omega)^2/2}| = e^{\alpha(\eta^2/4-(\sigma-(\omega\pm\eta))^2-(\sigma-\omega)^2)/2} \leqslant e^{-\alpha\eta^2/8},$$

and therefore

$$(5.4)\qquad |f_\varrho^{(n)}(\omega\pm\eta)| \leqslant 2n!(\eta/2)^{-n}e^{-\alpha\eta^2/8}.$$

Using the estimates (5.3) and (5.4) and the fact that for all of the zeros $0<\beta<1$, we obtain

$$|S_3| \leqslant 2\sqrt{\frac{\alpha}{2\pi}}\,e^{(\omega+\eta)/2}\sum_{\gamma>A}\left\{\frac{4e^{-\alpha\eta^2/8}}{\gamma^2}\sum_{n=0}^{N-1}\frac{n!}{(\gamma\eta/2)^n}+\frac{4\eta N!}{\gamma^{N+1}}\left(\frac{\alpha e}{N}\right)^{N/2}\right\}$$

provided $1 \leqslant N \leqslant \alpha\omega^2/16$. We now choose $N=[A^2/\alpha]$. By (2.4) we have $N \geqslant 1$, and by (2.5) we have $N \leqslant A^2/\alpha \leqslant \alpha\omega^2/16$, as required. Applying Lemma 2 and observing that by (2.5) $\eta \geqslant 2N/A$, we obtain

$$\sum_{\gamma>A}\frac{4e^{-\alpha\eta^2/8}}{\gamma^2}\sum_{n=0}^{N-1}\frac{n!}{(\gamma\eta/2)^n} \leqslant 4e^{-\alpha\eta^2/8}\log A\sum_{n=0}^{N-1}\frac{N^n}{(\eta/2)^n A^{n+1}}$$

$$\leqslant 4e^{-\alpha\eta^2/8}NA^{-1}\log A \leqslant 4e^{-\alpha\eta^2/8}\alpha^{-1}A\log A.$$

Also, since $A^2/\alpha-1 < N \leqslant A^2/\alpha$,

$$\sum_{\gamma>A}\frac{4\eta N!}{\gamma^{N+1}}\left(\frac{\alpha e}{N}\right)^{N/2} \leqslant 4\eta e^{1-N}N^{N+\frac12}(\alpha e/N)^{N/2}A^{-N}\log A$$

$$\leqslant 4\eta e^{1-N/2}N^{1/2}(A^2/N\alpha)^{-N/2}\log A \leqslant 4e^{3/2}\eta e^{-A^2/2\alpha}A\alpha^{-1/2}\log A.$$

From $(2\pi)^{-1/2} < 0.4$ and $e^{3/2} < 4.5$ it follows that

$$|S_3| \leqslant 4a^{-1/2}A\log A\cdot e^{-a\eta^2/8+(\omega+\eta)/2}+15\eta A\log A\cdot e^{-A^2/2a+(\omega+\eta)/2}.$$

Using (2.5) we obtain (5.1), thereby completing the proof of the theorem.

**6. Numerical computations.** The numerical computations described in this section were all made on an IBM 7090 at the Computer Center of the University of California at Berkeley. The first computation was a search for a place where on heuristic grounds $\pi(x)$ could be expected to be greater than $\mathrm{li}(x)$. The sum

$$S_T(u) = -\sum_{0<|\gamma|\leqslant T}\frac{e^{i\gamma u}}{\varrho}$$

is a partial sum which occurs in the explicit formula for $\psi(e^u)e^{-u/2}$, where

$$\psi(x) = \sum_{p^m\leqslant x}\log p,$$

the sum being taken over powers of primes. In a neighborhood where $S_T(u)$ is somewhat larger than 1 one can expect that at some points, $\pi(e^u) > \mathrm{li}(e^u)$.

The sum $S_{100}(u)$ was computed at values of $u$ at a distance 0.01 apart for $15 < u < 5400$ in all intervals where the contribution of the first pair of zeros

$$-\left\{\frac{e^{i\gamma_1 u}}{\frac{1}{2}+i\gamma_1}+\frac{e^{-i\gamma_1 u}}{\frac{1}{2}-i\gamma_1}\right\} \qquad (\gamma_1 = 14.13\ldots)$$

is positive. The increment 0.01 seems to be small enough so that $S_{100}(u)$ behaves smoothly between the sample points. Whenever $S_{100}(u)$ was above a threshold of 0.5, the sum $S_{1000}(u)$ was evaluated at 10 nearby points at a distance 0.001 apart. If any of these results exceeded a given parameter, it was printed. Proceeding in this fashion, we hoped to find values of $u$ for which $S_{1000}(u)$ exceeded 1. In this we were not successful, but we did locate 32 neighborhoods where $S_{1000}(u)$ is greater than 0.8 and 5 neighborhoods where it exceeds 0.9. Near the three points

$$727.952, \quad 853.853, \quad 2682.977$$

$S_{1000}(u)$ is approximately 0.96. Since no higher value was found, we concentrated our attention on neighborhoods of these points. Later, after some computations of $S_{3000}(u)$ we considered only neighborhoods of the last two points.



It is perhaps worth noting that the value $u = 853.853$ has turned up previously in connection with another problem. This value was one used by Haselgrove ([2]) in his proof that there are arbitrarily large $x$ for which

$$T(x) = -\sum_{n\leqslant x}\frac{\lambda(n)}{n} > 0,$$

where $\lambda(n)$ is Liouville's function. The value 853.853 was determined by J. Leech in such a way as to make the contributions of the first, second, and seventh pairs of zeros in an explicit formula for $T(x)$ as large as possible. It seems mysterious that this should produce a number which is exceptionally good for the present problem, where a large number of zeros must cooperate in a different explicit formula. Unfortunately, our computations are not sufficient to prove that $\pi(x) > \mathrm{li}(x)$ in a neighborhood of $e^{853.853}$; but this probably could be proved if enough zeros of $\zeta(s)$ were calculated.

In order to prove that $\pi(e^u) > \mathrm{li}(e^u)$ for a $u$ near 2682.977 it was necessary to perform two major computations. In one extensive computation (see [7]) a verification of the Riemann hypothesis was made for the first 250000 zeros of $\zeta(s)$, i.e., for all of the zeros for which $0 < \gamma < 170571.35$. Haselgrove and Miller ([3]) have computed the zeros of $\zeta(s)$ for which $0 < \gamma < 2090.4$ to 6 decimal places. For our purposes we needed a slightly more accurate computation of many more zeros. Consequently, in the other major computation we computed the zeros of $\zeta(s)$ for which $0 < \gamma < 12000$ to about 7 decimal places. This computation required approximately 6 hours of machine time.

For $T < 12000$ the formula (3.1), which gives the number of zeros of $\zeta(s)$ for which $0 < \gamma \leqslant T$, holds with $|Q(T)| < 2$. Consequently, it is not difficult to separate the zeros in this range. A combination of the regula falsi and bisection was used iteratively to compute an approximation to each zero. The program used the Riemann-Siegel asymptotic formula as a method for evaluating $\zeta(\frac{1}{2}+it)$. If the asymptotic expansion is truncated after four terms and the coefficients calculated by Lehmer ([8]) are used, the formula is sufficiently accurate to obtain the zeros to about 7 decimal places when $100 < \gamma < 12000$. However, rigorous estimates for the remainder in the Riemann-Siegel formula sufficient to prove this accuracy have never been obtained. Hence, in a rigorous computation another method must be used.

The Euler-Maclaurin sum formula provides an alternative method of evaluating $\zeta(s)$, which requires much more calculation, but for which it is easy to estimate the remainder. For each zero we used the Euler-Maclaurin formula to evaluate either $\mathrm{Re}\,\zeta(\frac{1}{2}+it)$ or $\mathrm{Im}\,\zeta(\frac{1}{2}+it)$ at two points, one of which was the approximation to the zero obtained using the Riemann-Siegel formula. A single application of the regula falsi was then made to obtain an improved approximation to the zero. The resulting value was then rounded to 9 decimal places for subsequent use. With a bound on $|\zeta''(s)|$ and estimates of all truncation and rounding errors we were able to prove that the errors for the computed zeros could not exceed the bounds given by the following table:

| Range | Error Bound |
|---|---|
| $14 < \gamma < 31$ | $2.2\times 10^{-8}$ |
| $31 < \gamma < 500$ | $1.5\times 10^{-8}$ |
| $500 < \gamma < 1000$ | $2.2\times 10^{-8}$ |
| $1000 < \gamma < 2000$ | $4\times 10^{-8}$ |
| $2000 < \gamma < 4000$ | $5\times 10^{-8}$ |
| $4000 < \gamma < 8000$ | $1.4\times 10^{-7}$ |
| $8000 < \gamma < 12000$ | $2\times 10^{-7}$ |

Usually the value for a zero calculated by the second method agreed quite closely with the approximation obtained by the first method. For $\gamma > 50$ the difference was always less than $10^{-6}$, and for $\gamma > 144$ the difference was always less than $3.5\times 10^{-7}$. For $144 < \gamma < 5000$ it exceeded $10^{-7}$ only 7 times, and for $5000 < \gamma < 12000$ it exceeded $2\times 10^{-7}$ only 8 times. The calculated zeros were compared for $0 < \gamma < 2090.4$ with the tabulation of Haselgrove and Miller ([3]). All differences were smaller than $7\times 10^{-7}$. For $0 < \gamma < 144$ the calculated values were also compared with a 50 decimal place table of the first 50 zeros computed by M. D. Bigg. The largest difference encountered was less than $5.1\times 10^{-10}$. These checks effectively exclude the possibility of a machine error in this computation serious enough to affect our result concerning $\pi(x)-\mathrm{li}(x)$.

**7. Application of the theorem.** In order to show that $\pi(x) > \mathrm{li}(x)$ for a value of $x$ in a neighborhood of $e^{2682.977}$ we use the result of a computation of the finite sum

$$H = -\sum_{0<|\gamma|\leqslant T} \frac{e^{i\gamma\omega}}{\varrho} e^{-\gamma^2/2a}$$

with

$$T = 12000, \quad a = 10^7, \quad \omega = 2682+16005\cdot 2^{-14} = 2682.9768\ldots$$

Using the estimates of § 6 for the errors in the zeros and taking into account errors made in computing the sum, we were able to prove that the calculated value could not exceed the true value by more than $6.8\times 10^{-4}$. Since the value calculated was 1.00201, we have

$$H \geqslant 1.00133.$$



We apply the theorem of § 2 with the above values of $a$, $T$, and $\omega$ and

$$A = 170000, \quad \eta = 0.034.$$

The inequalities (2.4) and (2.5) are easily verified. Moreover

$$\frac{3.05}{\omega-\eta} < 0.001137,$$

$$e^{-T^2/2a}\left\{\frac{a}{\pi T^2}\log\frac{T}{2\pi}+8\frac{\log T}{T}+\frac{4a}{T^3}\right\} < 0.00013.$$

The other terms in the estimate for $R$ are all quite small, less than $10^{-30}$, in fact. Hence

$$\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)ue^{-u/2}\{\pi(e^u)-\mathrm{li}(e^u)\}du > 0.00006.$$

Because of the positivity of $K$, there must then be a value of $u$ between $\omega-\eta$ and $\omega+\eta$ where $\pi(e^u)-\mathrm{li}(e^u) > 0$. Also, by (4.2)

$$\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)ue^{-u/2}\{e^{u/2}/u\}du < 1.$$

Hence for some $u$ between $\omega-\eta$ and $\omega+\eta$

$$\pi(e^u)-\mathrm{li}(e^u) > 0.00006\,e^{u/2}/u > 10^{500}.$$

From this it follows that there are more than $10^{500}$ successive integers $x$ between $1.53\times 10^{1165}$ and $1.65\times 10^{1165}$ for which the difference $\pi(x)-\mathrm{li}(x)$ is positive.

Although our method permits the localization of a place where the difference $\pi(x)-\mathrm{li}(x)$ is positive, it does not furnish a way to obtain a lower bound for the smallest $x$ for which the difference is positive. We remark, however, that in the search described in § 6 the highest value of $S_{1000}(u)$ found for $u < 46.05\ldots = \log 10^{20}$ was $S_{1000}(43.893) = 0.70$. This value is so low that it appears likely that the difference is always negative for $x$ less than $10^{20}$.

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# On the divisibility properties of sequences of integers (I)

by

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Let $a_1 < a_2 < \ldots$ be a sequence $A$ of integers. Put $A(x) = \sum_{a_i \leqslant x} 1$. The sequence is said to *have positive lower density* if

$$\varliminf_{x=\infty} \big(A(x)/x\big) > 0,$$

it is said to *have positive upper logarithmic density* if

$$\varlimsup_{x=\infty} \frac{1}{\log x} \sum_{a_i \leqslant x} \frac{1}{a_i} > 0.$$

The definition of upper density and lower logarithmic density is selfexplanatory.

Besicovitch ([2]) was the first to construct a sequence of positive upper density no term of which divides any other. Behrend ([1]) and Erdös ([4]) on the other hand proved that in a sequence of positive lower density there are infinitely many couples satisfying $a_i | a_j$, Behrend in fact proved this if we only assume that the upper logarithmic density is positive.

Davenport and Erdös ([3]) proved that if $A$ has positive upper logarithmic density there is an infinite subsequence $a_{i_j}$, $1 \leqslant j < \infty$ satisfying $a_{i_j} | a_{i_{j+1}}$.

Put

$$f(x) = \sum_{\substack{a_i | a_j \\ a_j < x}} 1.$$

It is reasonable to conjecture that if $A$ has positive density then

$$\lim \frac{f(x)}{x} = \infty. \tag{1}$$

We have proved (1) and in fact obtained a fairly accurate determination of the speed with which $f(x)/x$ has to tend to infinity, this