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COMPTES RENDUS

CONFÉRENCE SUR L'ALGÈBRE GÉNÉRALE

VARSOVIE 7. IX - 11. IX. 1964

La Conférence a été réalisée par les soins de l'Institut Mathématique de l'Académie Polonaise des Sciences avec concours de E. Marczewski et J. Łoś en qualité d'organisateurs de la Conférence, de A. Hulanicki et M[me] M. Łoś, qui ont été secrétaires du Comité d'organisation, et de A. Birula-Białynicki, M[lle] R. Czaplińska et J. Mycielski, qui ont contribué aux travaux du Comité au cours de la Conférence. Le nombre des participants s'élevait à 66 et 5 personnes les accompagnant. Il y a eu parmi eux 40 participants de la Pologne, les autres étant venu des deux parties de l'Allemagne, de l'Australie, de la Grande Bretagne, de l'Hongrie, de la Tchécoslovaquie, des USA et de l'URSS. La Conférence a été ouverte par K. Kuratowski, le directeur de l'Institut Mathématique. On a donné lecture à 31 communications; 3 autres travaux ont été présentés à la Conférence en résumés polygraphiés.

Voici le texte de l'allocution d'ouverture, la liste des rapports et communications dans leur ordre chronologique, les données bibliographiques qui leur appartiennent et les résumés parvenus à la Rédaction.

7. IX. 1964. E. Marczewski (Wrocław), *Opening address.*

The subject of this conference is general algebra, which is also called universal algebra, or the general theory of algebraic systems, or theory of general algebras.

I am not going to define the scope of this mathematical discipline and I do not wish to outline its rather short thirty year history. I do not feel competent enough. I would not dare to venture my opinion in the presence of such eminent experts. I would only like to make a few remarks and will start with something quite personal.

When I was a young student in the twenties and I first read about the group theory, I felt that some fundamental notions of that theory had nothing to do with the axioms of that theory, and that there should exist a more general theory to which these notions should belong. I am sure many people felt that way. And when in the course of development of mathematics such a theory, the general algebra, came into being, and started to grow. I was glad to see that old postulate was finally met.

The general algebra attracts mathematicians of various fields. For those working in the foundations of mathematics, general algebras are first of all models for various sets of axioms and, in general, a mathematical tool of metamathematics. For algebraists, general algebra is a common part of some of the most frequently pursued algebraic theories, and when they deal with general algebras, they immediately use lattices, semigroups, rings and other algebras of known classic types. For those working in the set theory, the general algebra is mainly the theory of functions of several variables defined on arbitrary sets and of composition of such functions. The general algebra also attracts people working in combinatorial analysis, i. e. in the theory of finite sets and relations in them, simply because algebras containing a finite number of elements, even only two, are a source of many interesting phenomena and difficult problems. The logicians investigating the two-valued or many-valued sentential calculi are also interested in general algebra, since those calculi can be formulated in general algebraic terms. The topologists are also interested in general algebra because of Boolean algebras and groups and because some interesting results on general topological algebras have been recently found. Also those who are attracted by analogies and relations between the notions and methods of various fields of mathematics, for the general algebra is the common background where they all fit.

Perhaps the variety of approach of those who are working in the general algebra, and variety of their taste is the reason why, in spite of many results and many successful notions, we still do not have a theory of a generally accepted form. There is no textbook of general algebra, and no monograph has appeared (1). There are only some mimeographed lectures, and some of us here, and also some who are not, have attempted to write such books, or at least they feel they should have done so. It seems to us that so far there has been no conference on general algebra. Therefore I hope that this conference may be useful.

In this country many papers have appeared long since, which concern the general algebra, or the general algebra is used in them, or at least we, their authors, so believe. A part of those papers deals with the general notions of independence. Recently, such papers have been even more frequent. Therefore the program of the conference emphasises to some extent this particular topic. We will also have some lectures on the notions of independence which no doubt go beyond the limits of general algebra.

(1) A chapter on universal or general algebra is contained in a book by A. N. Kuroš on various branches of modern algebra, entitled *Lectures on General Algebra* (Moscow 1962). Several months after the Conference a book by P. M. Cohn *Universal Algebra* (New York 1965) has appeared.

The role of general algebra is sometimes compared with the role of general topology. I think that this comparison is right in some respect, at least considering some of the dangers which may arise in course of their development. We know from experience that in research in very general branches of mathematics it is easy to get stranded in trivial topics, and caught in the net of overdetailed conditions, of futile generalizations. In this situation it is even more important to exchange information, not only about the results but also about problems, new ideas, plans of future development. I think this conference may serve to this aim.

7. IX. 1964. A. Tarski (Berkeley, Calif.), *Remarks on some basic notions of the general theory of algebras.*

7. IX. 1964. А. Г. Курош (Москва), *Работы московских алгебраистов в теории универсальных алгебр* (voir ce volume, p. 131-133).

7. IX. 1964. R. Baer (Frankfurt on Main), *Group theoretical properties and functions* (voir ce volume, p. 285-327).

7. IX. 1964. B. H. Neumann (Canberra) and E. C. Wiegold (Caerphilly) (presenté par B. H. Neumann), *A semigroup representation of varieties of algebras* (voir ce volume, p. 111-114).

8. IX. 1964. E. Marczewski (Wrocław), *Independence in abstract algebras. Results and problems* (voir ce volume, p. 169-188).

8. IX. 1964. J. Płonka (Wrocław), *Remarks on independence in finite abstract algebras* (voir du même auteur *Diagonal algebras and algebraic independence*, Bulletin de l'Académie Polonaise des Sciences, Série des sc. math., astr. et phys., 12 (1964), p. 729-733; *Diagonal algebras*, Fundamenta Mathematicae, sous presse; *On the number of independent elements in finite abstract algebras having a binary operation*, ce volume, p. 189-201; *Exchange of independent sets in abstract algebras (II)*, ce volume, p. 217-223).

8. IX. 1964. J. Schmidt (Bonn), *A general existence theorem on partial algebras and its special cases* (voir ce volume, p. 73-87).

8. IX. 1964. R. Rado (Reading), *Abstract linear dependence* (voir ce volume, p. 257-264).

8. IX. 1964. V. Dlab (Prague), *General algebraic dependence structures and some applications* (voir ce volume, p. 265-273).

8. IX. 1964. K. Urbanik (Wrocław), *Linear independence in abstract algebras* (voir ce volume, p. 233-255).

9. IX. 1964. B. Jónsson (Minneapolis, Minn.), *Decompositions of relational structures* (voir *The unique factorization problem for finite relational structures*, ce volume, p. 1-32).

9. IX. 1964. J. Łoś (Warszawa), *Direct sums in general algebra* (voir ce volume, p. 33-38).

9. IX. 1964. T. M. Baranovitch (Moscow), *Free decompositions in intersections of primitive classes of algebras.*

For several primitive classes of universal algebras there is a good theory of free decompositions, i. e. holds the theorem that a subalgebra of a free algebra is free, the structure of a subalgebra of the free product of algebras is well described, and, which is usually a consequence of the preceding two facts, the theorem on the existence of isomorphic refinements of any two free decompositions of an algebra is valid. All this is true for the classes of groups, non-associative algebras, loops, algebras with multi-operators.

Our problem is: Given two primitive classes of algebras $K_1 = (\Omega_1, \Lambda_1)$ and $K_2 = (\Omega_2, \Lambda_2)$ for each of which the good theory of free decompositions holds. What can be said about free decompositions in the class $K = (\Omega_1 \cup \Omega_2, \Lambda_1 \cup \Lambda_2)$, which we call the *intersection* of K_1 and K_2?

Here we built a theory of free decompositions for the intersection of classes K_1 and K_2 such that the systems of operations Ω_1, Ω_2 have the 0-ary operation 0 only in common, and so the identities Λ_1 and Λ_2 contain all identities $00\ldots0\omega = 0$ for $\omega \in \Omega_1$ and $\omega \in \Omega_2$, respectively. For simplicity sake all theorems are formulated and proved for the intersection of two primitive classes, though by the same methods they can be obtained for the intersection of any finite number classes.

The theory of the free decompositions of groups with multi-operators ([2]) follows as a particular case from the results we obtain here.

10. IX. 1964 R. C. Lyndon (Ann Arbor, Mich.), *Dependence in groups* (voir ce volume, p. 275-283).

10. IX. 1964. L. Fuchs (Budapest), *On partially ordered algebras, I* (voir ce volume, p. 115-130).

10. IX. 1964. Л. А. Бокуть (Новосибирск), *Теоремы вложения в теории алгебр* (voir ce volume, p. 349-353).

10. IX. 1964. E. Szodoray (Debrecen), *The relation of abstract dependence and its equivalents.*

The concept of the relation of abstract dependence is to be found in van der Waerden's *Moderne Algebra*. H. Whitney takes ([3]) a finite set as starting point and defines the concepts of abstract independence, basis and rank by systems of axioms. R. Rado extends ([4]) the concept of rank-function of Whitney to infinite sets. M. N. Bleicher, G. B. Preston

([2]) See А. Г. Курош, *Свободные суммы мультиоператорных групп*, Acta Scientiarum Mathematicarum (Szeged) 21 (1960), p. 187-196.

([3]) H. Whitney, *On the abstract properties of linear dependence*, American Journal of Mathematics 57 (1935), p. 509-533.

([4]) R. Rado, *A theorem on independence relations*, The Quarterly Journal of Mathematics, Oxford Second Series 13 (1943), p. 83-89.

and A. Kertész investigate the concept of dependence in the case of sets of arbitrary cardinality; M. N. Bleicher and G. B. Preston prove [5] the equivalence of the concepts of dependence and independence, and A. Kertész [6] shows the equivalence of the concepts of dependence and rank-function. A more general concept of independence has been introduced by E. Marczewski [7].

The purpose of this lecture is to introduce the concept of basis by a system of axioms in the case of arbitrary sets, and to prove the equivalence of the basis so defined with dependence, independence, and rank-function.

When we define the concept of basis for arbitrary sets, we must add to the system of axioms of H. Whitney two further axioms:

Let S be an arbitrary, non-empty set. We make correspond to each finite subset A of S a function $F_A(A')$, which is defined on the set of all subsets A' of A, and which has the value 1 or 0. We call the set S a B-set if the following conditions are satisfied:

(B_1) if $F_A(A') = 1$ and $A'' \subset A'$, then $F_A(A'') = 0$;

(B_2) if $F_A(A') = 1$, $F_A(A'') = 1$ and $a' \in A'$, then there exists an element a'' of A'' such that $F_A[(A' \setminus a') \cup a''] = 1$;

(B_3) if $A' \subseteq A$ and $F_{A'}(A'') = 1$, then there exists a subset A^* of A such that $A'' \subseteq A^*$ and $F_A(A^*) = 1$;

(B_4) if $A' \subseteq A$, $F_{A'}(A'') = 1$, $F_A(A^*) = 1$ and $A'' \subseteq A^*$, then $A^* \setminus A'' \subseteq A \setminus A'$.

If for some finite subsets A and A' of a B-set S the conditions $A' \subseteq A$ and $F_A(A') = 1$ are fulfilled, then we say that A' is a ***basis*** of the set A. We call a subset B of S a basis of S, if it is maximal with respect to the following property: every finite subset of B is a basis of itself.

A B-set has always a basis, and every two bases have the same cardinality.

10. IX. 1964 Jan Mycielski (Wrocław), *Independent set in topological algebras* (voir Fundamenta Mathematicae 55 (1964), p. 139-147).

10. IX. 1964 A. Kertész (Debrecen), *Lattice theoretic remarks on completely reducible algebras.*

S. MacLane gives [8] a lattice theoretic interpretation of van der

[5] M. N. Bleicher and G. B. Preston, *Abstract linear dependence relations*, Publicationes Mathematicae Debrecen 8 (1961), p. 55-63.

[6] A. Kertész, *On independent sets of elements in algebra*, Acta Scientiarum Mathematicarum (Szeged) 21 (1960), p. 260-269.

[7] E. Marczewski, *A general scheme of the notions of independence in mathematics*, Bulletin de l'Académie Polonaise des Sciences, Série des sc. math., astr. et phys., 6 (1958), p. 731-736.

[8] S. MacLane, *A lattice formulation for transcendence degrees and p-bases*, Duke Mathematical Journal 4 (1938), p. 455-468.

Waerden's abstract dependence relation, by introducing the concept of exchange lattice:

A complete lattice L is termed *exchange lattice* if the following conditions hold:

(E) If a is an arbitrary element in L, and p and q are atoms of L, then $a < a \cup p \leqslant a \cup q$ implies $q \leqslant a \cup p$.

(G) L is relatively atomic, i. e. if $b < a$ in L, then there is an atom p in L such that $b < b \cup p \leqslant a$.

(F) If P is a set of atoms and q an atom of L with $q \leqslant \bigcup P$, then there exists a finite set of atoms $p_1, \ldots, p_n$ in P such that $q \leqslant p_1 \cup \ldots \cup p_n$.

An abstract algebra is said to be *completely reducible,* if it is the direct union of simple abstract algebras of the same kind. It is easy to see that the lattice of normal subgroups of a completely reducible group or the lattice of ideals of a completely reducible ring is an exchange lattice. In this lecture, in order to clarify the properties of completely reducible abstract algebras, certain exchange lattices are investigated.

A complete lattice L is called an *S-lattice,* if S is a set of compact elements of L such that each element of L is the union of elements of S and the conditions

$$s \leqslant x_1 \cup x_2; \quad x_1 \leqslant s; \quad s \in S; \quad x_1, x_2 \in L$$

imply the existence of elements $s_1 (\leqslant x_1)$, $s_2 (\leqslant x_2)$ with $s = s_1 \cup s_2$ in S. E. g. the lattice of all normal subgroups of a group G is an S-lattice, if we take for S the set of all normal subgroups generated by a single element of G.

THEOREM 1. *Let L be an S-lattice with the following properties:*

(i) $a < b \leqslant a \cup p$ *implies* $b = a \cup p$,

(ii) $m \cap a \leqslant b < a$ *implies* $b = m \cap a$ *for all* $a, b \in L$, *and for every atom* p *and dual atom* m *of* L.

Then the following conditions are equivalent:

(a) *L is relatively atomic;*

(b) *the greatest element of L is the union of atoms;*

(c) *there exist in L dual atoms, the intersection of which is the smallest element of L; furthermore, L satisfies the minimum condition.*

Since a modular lattice satisfies conditions (i) and (ii), then conditions (a), (b) and (c) are equivalent in particular for any modular S-lattice. In order to formulate further equivalent properties, we shall need two definitions. An element b of a modular S-lattice L is called *pure,* if for any element s of S, the element b has a complement in the sublattice $[0, b \cup s]$. An independent subset B of S is said to be a *basis* of L if $\cup B$ is the greatest element of L. Then we have

THEOREM 2. *For a modular S-lattice L the following conditions are equivalent*:

(a) *L is relatively atomic*:

(d) *every element of L is a pure element of L*;

(e) *any maximal independent set of elements of L is a basis of L.*

A maximal independent subset of S in the arbitrary modular S-lattice L is not in general a basis of L. As a basis criterion we have

THEOREM 3. *A maximal independent subset B of S in the modular S-lattice L is a basis of L if and only if $\bigcup B$ is pure in L.*

A detailed treatment of the subject will appear in the Publicationes Mathematicae Debrecen.

10. IX. 1964 A. Goetz (Wrocław), *On weak isomorphisms and weak homomorphisms of abstract algebras* (voir ce volume, p. 163-167).

10. IX. 1964. W. Holsztyński (Warsaw), *Lattices with real numbers as additive operators.*

10. IX. 1964. M. Makkai (Budapest), *On a problem of G. Grätzer concerning endomorphism semigroups.*

Let $\mathfrak{A}$ be a universal algebra (briefly: algebra) and let $E(\mathfrak{A})$, $M(\mathfrak{A})$, $H(\mathfrak{A})$ be the sets of the endomorphisms, monomorphisms, and epimorphisms of $\mathfrak{A}$, respectively. (A monomorphism is a one-to-one endomorphism, and an epimorphism is an onto endomorphism). Let $\cdot\mathfrak{A}$ be the usual product operation for the transformations of the universe of $\mathfrak{A}$ into itself. The problem [9] is to characterize the class K of the relational systems which are isomorphic to $(E(\mathfrak{A}); M(\mathfrak{A}), H(\mathfrak{A}), \cdot\mathfrak{A})$ for some $\mathfrak{A}$.

THEOREM. *The class K defined above is precisely the class of the systems $(E; M, H, \cdot)$ for which the following conditions hold*:

C1. *$(E; \cdot)$ is a semigroup with unit element* 1;

C2. (a) *$a \in M$ and $b \in M$ imply $a \cdot b \in M$,*
(b) *$x \in E$, $y \in E$, $a \in M$, and $xa = ya$ imply $x = y$,*
(c) *$x \in E - M$ and $y \in E$ imply $x \cdot y \in E - M$,*
(d) $1 \in M$;

C3. (a) *$a \in H$ and $b \in H$ imply $a \cdot b \in H$,*
(b) *$x \in E$, $y \in E$, $a \in H$, and $ax = ay$ imply $x = y$,*
(c) *$x \in E - H$ and $y \in E$ imply $y \cdot x \in E - H$,*
(d) $1 \in H$;

C4. *If $a \in M$, $b \in H$, $x \in E$, $y \in E$, and $xa = by$, then there exists an element z in E such that $x = bz$.*

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[9] See Problem 17 in G. Grätzer, *Some results on universal algebras* (mimeographed), 1962.

Corollary. *K is a Horn class* (10), *$K \in HC$. In particular, K is closed under taking direct products of arbitrarily many factors of K.*

The conditions C1-C3 are due to Grätzer (11), C4 was found independently by Ervin Fried and the author.

A detailed treatment of the subject will appear in the Acta Mathematica Academiae Scientiarum Hungaricae.

10. IX. 1964. Z. Semadeni (Poznań), *Free objects in the theory of categories* (voir ce volume, p. 107-110).

10. IX. 1964. K.-H. Diener (Cologne), *Order in absolutely free and related algebras* (voir ce volume, p. 63-72).

11. IX. 1964. P. Freyd (Philadelphia, Penn.), *Algebra valued functors in general and tensor products in particular* (voir ce volume, p. 89-106).

11. IX. 1964. M. Armbrust (Bonn), *Quasi-direct products and decompositions.*

There are various notions of combining algebraic structures subdirectly coinciding with the full direct product in the case of finitely many factors: the direct product itself, the weak direct product (e. g. in group theory), in generalization hereof the weak direct product with respect to a selected family of subalgebras by Karolinskaya (12) (working on algebras with finitary operations) or the L-restricted product by Hashimoto (13), and the direct sum of Hilbert spaces. Looking for a workable decomposition theory it seemed useful to amalgamate all these "products" and to investigate them under a more general point of view.

In the following the term *product* means a universal subdirect combination of algebras which is fully direct in the case of finitely many factors, i. e. a function $\prod$ assigning to every family $(A_t)_{t\in T}$ of algebras of the same species a certain (arbitrary) set $\prod\limits_{t\in T} A_t$ of subdirect products of the A_t, this set consisting of the direct product alone if T is finite. Imposing the directness condition upon finite families only does not make much sense if there is no connection between finite and infinite families. The most natural feed-back is furnished by associativity of the combination in some sense or other. An efficient notion of associativity is that

(10) For definition see T. Frayne, A. C. Morel and D. S. Scott, *Reduced direct products*, Fundamenta Mathematicae 51 (1962), p. 195-228.

(11) G. Grätzer, op. cit.

(12) Л. Н. Каролинская, *Прямые разложения абстрактных алгебр с отмеченными подалгебрами*, Успехи Магематических Наук 14, 5 (89) (1959), p. 230-231.

(13) J. Hashimoto, *Direct, subdirect decompositions and congruence relations*, Osaka Mathematical Journal 9 (1957), p. 87-112.

of *associativity from above*: I call a product $\prod$ *associative from above* iff for every $A \in \prod_{t \in T} A_t$ and for any partition P of T and $S \in P$ the projection

$$A^{(S)} = \{(\mathrm{pr}_t x)_{t \in S} : x \in A\}$$

is in $\prod_{t \in S} A_t$, and

$$A' = \{((\mathrm{pr}_t x)_{t \in S})_{S \in P} : x \in A\}$$

being canonically isomorphic to A, lies in $\prod_{S \in P} A^{(S)}$. The niceness of this sort of associativity is buried in the fact that there is a distinguished product associative in this sense: Calling $\prod_1 \subset \prod_2$ iff for every $(A_t)_{t \in T}$

$$\prod_{t \in T}{}_1 A_t \subset \prod_{t \in T}{}_2 A_t$$

in the set theoretic sense, one has a greatest product $\prod^*$ with this associativity property, namely $A \in \prod^*_{t \in T} A_t$ iff $A \in \prod_{t \in T} A_t$ for some product $\prod$ associative from above. I have called the elements of $\prod^* A_t$ *almost direct products* of the A_t. There is a surprisingly simple characterization of almost direct products:

THEOREM 1. *A subdirect product A of the A_t $(t \in T)$ is almost direct iff for any $x, y \in A$ and $S \subset T$ there is a $z \in A$ such that $\mathrm{pr}_t z = \mathrm{pr}_t x$ for all $t \in S$ and $\mathrm{pr}_t z = \mathrm{pr}_t y$ for all $t \in T - S$; z is uniquely determined.*

All aforementioned practically important products are almost direct products.

Obviously, $\prod^*$ is finitely *associative from below*, i. e. for every family $(A_t)_{t \in T}$, any finite partition P of T, and arbitrary

$$A^{(S)} \in \prod^*_{t \in S} A_t \quad (S \in P) \quad \text{and} \quad A' \in \prod^*_{S \in P} A^{(S)}$$

there is an $A \in \prod^*_{t \in T} A_t$ such that

$$A^{(S)} = \{(\mathrm{pr}_t x)_{t \in S} : x \in A\} \quad \text{and} \quad A' = \{((\mathrm{pr}_t x)_{t \in S})_{S \in P} : x \in A\}.$$

However, this condition does not hold for arbitrary partitions P. The attempt to escape this inconvenience by enlarging $\prod^*$ must necessarily destroy associativity from above; but there is a weak form of associativity from above which leads to full associativity from below and furnishes the most comprehensive product with a smooth decomposition theory: A product $\prod$ is called *finitely associative from above* iff the condition for associativity from above holds for partitions P of T such that there is an $S \in P$ containing all but finitely many elements of T. Here again we have a greatest product $\prod^0$ finitely associative from above — I have called it *quasi direct product* — and the simple characterization is easily seen to run as follows:

THEOREM 2. *A subdirect product A of the $A_t(t \in T)$ is quasi direct iff for any $x, y \in A$ and $t_0 \in T$ there is a $z \in A$ such that* $\mathrm{pr}_{t_0} z = \mathrm{pr}_{t_0} x$ *and* $\mathrm{pr}_t z = \mathrm{pr}_t y$ *otherwise; z is uniquely determined.*

This product $\prod^0$ is associative from below but not fully associative from above. Nevertheless, this notion of product is a suitable framework to construct a workable and comprehensive decomposition theory:

The representations of an algebra A as quasi direct product of other algebras correspond — up to canonical isomorphisms — one-one with the sets R of congruence relations on A with the properties

(i) $\bigcap_{\varrho \in R} \varrho = \mathrm{id}_A$,

(ii) $\varrho \cdot \bigcap_{\substack{\sigma \in R \\ \sigma \neq \varrho}} \sigma = A \times A$ for every $\varrho \in R$,

(iii) $A \times A \notin R$

(condition (iii) is but to exclude trivial factors). The congruence relations in R are the kernels of the natural projections of A onto the factors of the corresponding representation. Decomposition theory is thus a matter of the set A and the congruence lattice C alone (by the way, this is not the case in algebras with partial operations). The notion of refinement between representations is to be translated as follows: Quasi direct decomposition R is *finer* than quasi direct decomposition S iff every $\sigma \in S$ is the meet of some $\varrho \in R$, what is the same as to every $\varrho \in R$ there is a $\sigma \in S$ such that $\varrho \supset \sigma$. The investigation of the order theoretic structure of the system of all quasi direct decompositions of an algebra is not very fruitful without any special information about the algebra; even direct decompositions are not very pleasant in general. However, there is a lattice theoretic aspect of decomposition theory which gives some nice results without specializing the algebra considered.

Let, for example, A be a Hilbert space with a compact symmetric operator: the decomposition of A into the eigenmanifolds of the operator is an almost direct decomposition of A and at the same time induces a direct decomposition of the complete lattice of topologically closed invariant subspaces (it is even the finest direct decomposition of this lattice). Similar examples give rise to the notion of *faithful decompositions*: Let A be an arbitrary set and C a system of equivalence relations on A containing the identity relation and being closed with respect to any intersection. Then I call a quasi direct decomposition $R \subset C$ of A *C-faithful* iff it induces a direct decomposition of C, regarded as complete lattice, in a natural way, that means iff C is canonically isomorphic to the direct product $\bigtimes_{\varrho \in R} [\varrho, A \times A]$ of the intervals $[\varrho, A \times A]$ in C. The characterization of faithful quasi direct decompositions is given by

THEOREM 3. *The quasi direct decomposition* $R \subset C$ *of* A *is* C*-faithful iff*

$$\sigma = \bigcap_{\varrho \in R} (\varrho + \sigma)$$

for every $\sigma \in C$, $\varrho + \sigma$ *being the least upper bound of* ϱ *and* σ *in* C.

Faithful quasi direct decompositions form a semi-lattice with respect to refinement:

THEOREM 4. *There is always a coarsest common refinement for finitely many faithful quasi direct decompositions.*

This is not the case with faithful almost direct decompositions. However, there is at most one nonrefinable faithful almost direct decomposition which is then the finest one; moreover, the system of all faithful almost direct decompositions is conditionally complete:

THEOREM 5. *For every non-void system of faithful almost direct decompositions there is a finest common coarsening.*

In some important special cases — for instance, if C is inductive — the faithful almost direct decompositions form a conditionally complete lattice. If C satisfies the ascending or descending chain condition, then there are only finitely many faithful almost direct decompositions and therefore the finest faithful almost direct decomposition exists which is finite and hence a direct decomposition of A. Without any restrictions on C, the faithful *direct* decompositions of A form a conditionally complete lattice.

A detailed treatment of the first part of this summary is contained in the paper *Die fastdirekten Zerlegungen einer allgemeinen Algebra, I*, which has appeared in this volume (p. 39-62).

11. IX. 1964. Y. Hion (Tartu), *Ω-ringoids, Ω-rings and their representations* (présenté par A. G. Kuroš).

Let $A = \{a, b, \ldots\}$ be an *Ω-algebra*, i. e. a universal algebra with a system $\Omega = \{\omega, \varphi, \ldots\}$ of operations. If ω is an n-ary operation, we denote by $a_1 a_2 \ldots a_n \omega = \sum\limits_i^{\omega} a_i$ the result of ω for the arguments $a_1, \ldots, a_n$. If ν is a nullary operation, 0_ν denotes the unique value of ν.

An Ω-algebra A is called an *Ω-ringoid*, if, additionally, in A is defined an associative multiplication such that

$$b \sum_i^{\omega} a_i = \sum_i^{\omega} b a_i, \qquad b 0_\nu = 0_\nu$$

for any n-ary ($n > 0$) operation ω, any a_i, b, and any nullary operation ν.

An Ω-ringoid, satisfying the analogous right-hand side conditions is called an *Ω-ring*. An Ω-ringoid without multiplication (i. e. the corresponding "additive" Ω-algebra) will be denoted by A^{Ω}.

The following algebraic systems are examples of Ω-ringoids: rings, semigroups, near rings, neo-rings, distributive lattices.

Let $M = \{x, y, \ldots\}$ be an Ω-algebra and $S(M) = \{a, b, \ldots\}$ the set of all mappings $x \to xa$ of M into itself. Let us consider in $S(M)$ the ordinary multiplication of mappings and, moreover, let us introduce by the formulas

$$x\Big(\sum_i^{\omega} a_i\Big) = \sum_i^{\omega} xa_i, \qquad x0_\nu = 0_\nu$$

operations in $S(M)$ corresponding to those belonging to Ω.

So, *for every Ω-algebra M, $S(M)$ is an Ω-ringoid. If Λ is a primitive class of Ω-algebras, amd $M \in \Lambda$, then $S^{\Omega}(M) \in \Lambda$.*

The set $E(M)$ of all endomorphisms of an Ω-algebra M is in general not closed with respect to operations belonging to Ω. An Ω-algebra is called *Abelian* if for any $\omega, \varphi \in \Omega$, $x_{11}, \ldots, x_{1n}, \ldots, x_{m1}, \ldots, x_{mn} \in M$ (ω n-ary operation, φ m-ary operation) we have

$$\sum_i^{\varphi}\Big(\sum_j^{\omega} x_{ij}\Big) = \sum_j^{\omega}\Big(\sum_i^{\varphi} x_{ij}\Big), \qquad 0_\mu = 0_\nu, \qquad \sum_i^{\omega} 0_\nu = 0_\nu .$$

If an Ω-algebra M is Abelian, then $E(M)$ is an Abelian Ω-ring. If $M \in \Lambda$, then $E^{\Omega}(M) \in \Lambda$.

THEOREM 1. *Any Ω-ringoid A is isomorphic with an Ω-subringoid of the Ω-ringoid $S(M)$ for a certain Ω-algebra M. If $A^{\Omega} \in \Lambda$, then it is possible to choose $M \in \Lambda$.*

It turns out that not every Ω-ring A is isomorphic with an Ω-subring of the Ω-ring $E(M)$ for some Ω-algebra M. In this paper there are given some necessary and sufficient conditions for the existence of such a representation of an Ω-ring.

The problem of representation of Ω-ringoids and Ω-rings is related with the problem of adjoing of a unit element to them. Let A be an Ω-ringoid with $A^{\Omega} \in \Lambda$. An Ω-ringoid B (with $B^{\Omega} \in \Lambda$) with a unit element e is called *Λ-free unitary extension* $F(A, \Lambda)$ of A, if 1° there is a monomorphism $\alpha\colon A \to B$, where $A\alpha \not\ni e$, 2° for any homomorphism $\beta\colon A \to C$, where $C^{\Omega} \in \Lambda$ and C has a unit element, there exists a unique homomorphism $\gamma\colon B \to C$ such that $\beta = \alpha\gamma$.

THEOREM 2. *For any Ω-ringoid $A \in \Lambda$ there exists a (unique up to isomorphisms) Λ-free unitary extension $F(A, \Lambda)$.*

It is shown in the paper, how, by the aid of free unitary extensions, a survey of all representations of a given Ω-ringoid A by mappings of Ω-algebras can be obtained.

It turns out that not for every Ω-ring there is a free unitary extension (in the class of Ω-rings). A necessary and sufficient condition for the existence of such an extension is given.

11. IX. 1964. E. T. Schmidt (Budapest), *Congruence lattices of abstract algebras.*

11. IX. 1964. H. J. Hoehnke (Berlin), *Einige neue Resultate über abstrakte Halbgruppen* (voir ce volume, p. 329-348).

11. IX. 1964. L. Budach (Berlin), *Transfinite iterations of functors and their importance for the theory of ideals.*

11. IX. 1964. J. Słomiński (Toruń), *A theory of P-homomorphisms* (voir ce volume, p. 135-162).

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Les communications suivantes ont été présentées hors de séances en résumés polygraphiés:

J. Schmidt (Bonn), *Concerning algebraic independence.*

In the Bourbakian hierarchy of mathematical theories, General Algebra may be considered as the link—hitherto missing or at least underdevelopped — between General Set Theory, the fundament of the entire hierarchy, and the higher and more special algebraic theories like those of groups, rings, fields, moduls. It is with General Topology and its very universal concepts of continuity, compactness, connection etc., that General Algebra may be compared in its tendency of developping and studying algebraic concepts of analogous universality. One of these concepts is no doubt that of independence in the algebraic form, the importance of which has been emphasized by Prof. Marczewski in a short note in 1958, and which was already semi-published, but never actually published, in a lecture of Philip Hall in 1949, as we have been informed in the lecture of Prof. B. H. Neumann held in New York in 1961-1962. It is well known that this concept of independence is closely related to the notion of a free algebraic system that goes back to a famous paper of Birkhoff in 1935. In my communication, I want to give a summary of a part of a general theory of algebraic independence which contains the above notions as special cases.

Let us consider the *species* of algebras $(A, (f_i)_{i\in I})$ *of type* $\Delta = (K_i)_{i\in I}$, i. e. the *fundamental operations* f_i being of types K_i respectively, f_i: $A^K \to A$. A, B being algebras of type Δ, subset $M \subset A$ is *B-independent* iff each *B-valuation* of M, β: $M \to B$, can be extended to a (necessarily unique) homomorphism φ: $\mathrm{C}M \to B$, $\mathrm{C}M$ being the *closure* of M in algebra A, i.e. the subalgebra generated by B. In particular, $\emptyset \subset A$ is B-independent iff there is a (necessarily unique) homomorphism φ: $\mathrm{C}_A\emptyset \to B$ (which is onto $\mathrm{C}_B\emptyset$); this is always the case if type Δ is *without constants*, i.e. $K_i = \emptyset$ for all $i\in I$, $C_A\emptyset$ then being empty for all algebras A of type Δ. $\mathfrak{B}$ being any class of algebras B, one may call M $\mathfrak{B}$-*independent* iff M is B-independent for each $B \in \mathfrak{B}$. The largest class $\mathfrak{B}$ such that M is $\mathfrak{B}$-independent, $\mathrm{ind}M = \{B \mid MB\text{-independent}\}$, is called the *degree of independence of M (with respect to algebra A)*. This class is primitive, i. e. closed with respect to direct products, subalgebras, and homomorphic images; in particular, any algebra B of order $|B| \leqslant 1$ belongs to $\mathrm{ind}M$. These notions are (i) *invariant*, i. e. preserved under isomorphisms; (ii) *absolute*, i. e. not depending on the entire algebra A, but only on subalgebra CM; (iii) *hereditary*, i. e. if M is B-independent, then so is any subset $M' \subset M$ ($M' \subset M$ implies $\mathrm{ind}M' \supset \mathrm{ind}M$).

There are two extreme special cases: (i) $\mathrm{ind}M$ is the entire species (the largest primitive class); then M is called *absolutely independent*. It is well known that this is the case iff the following *Generalized Peano Axioms* hold true:

P1. $f_i(\mathfrak{a}) \notin M$ for any index $i \in I$, for any sequence $\mathfrak{a} \in (CM)^{K_i}$;

P2. $f_i(\mathfrak{a}) = f_j(\mathfrak{b})$ implies $i = j$ and $\mathfrak{a} = \mathfrak{b}$ for any $i, j \in I$, $\mathfrak{a}, \mathfrak{b} \in (CM)^{K_i}$.

(As a matter of fact, adding the *Axiom of* ("complete") *Induction*:

P3. $CM = A$,

we obtain the full set of Peano Axioms for the case of an arbitrary type $\Delta = (K_i)_{i \in I}$, the classical case being the special case $\Delta = (0, 1)$.) Answering a question of mine, Diener has obtained the remarkable result that M is absolutely independent iff M is B-independent in any extension B of A and $\emptyset$ is absolutely independent in A (this indispensable additional condition holding trivially true in the special case without constants). The extremally opposite special case: (ii) $\mathrm{ind}M$ is the smallest primitive class that consists precisely of the algebras B of order $|B| \leqslant 1$; we may call M *absolutely dependent*, the inner characterization of absolutely dependent sets may be put as an open problem.

Then the "normal" case of a set $M \subset A$ lies between those two extremes of absolute independence and absolute dependence. Let us consider the important special case that A itself belongs to $\mathrm{ind}M$; this what has been called *independence* by Prof. Marczewski, *Hall-independence* by Prof. Neumann in honour of the quoted lecture of Philip Hall. There is the remarkable *Marczewski Independence Criterion*: $M \subset A$ is independent iff $g(\mathrm{id}_M) = h(\mathrm{id}_M)$ implies $g = h$, for any *algebraic operations* g, h of type M in algebra A, id_M denoting the *identical sequence* of type M in A. This criterion has been proved by Prof. Marczewski in the case of finitary fundamental operations f_i and has been generalized by me to arbitrary infinitary operations. It may be considered as the combination of the following fundamental facts: (i) *the independence of identity operations*, i. e. the set $E^M(A)$ of *identity operations* of type M in A is an A-independent subset of algebra $O^M(A) = A^{A^M}$ of all operations of type M

in A, or what is the same (due to absoluteness of independence) of the subalgebra $F^M(A) \subset O^M(A)$ of *algebraic operations* of type M, i.e. the subalgebra generated by $E^M(A)$ (this is the reason why $F^M(A)$, Tarski's "function algebra of order M over A", is called the "free algebra in M variables over A" by Birkhoff); (ii) *the representability of elements of* $\mathrm{C}M$ *by means of algebraic operations*, i. e. (as a consequence of (i)) there is one and only one homomorphism of $F^M(A)$ into A, onto $\mathrm{C}M$, which carries identity operation $e_x^M: A^M \to A$ into $x \in M$, namely the restriction of the natural projection of direct power A^{A^M} onto A which belongs to index $\mathrm{id}_M \in A^M$ (in particular, each element $x \in \mathrm{C}M$ may be represented in the from $x = g(\mathrm{id}_M)$ with suitable $g \in F^M(A)$); (iii) the *regularity criterion for homomorphisms*: a homomorphism $\varphi\colon \mathrm{C}M \subset A \to B$ where M is (A-) independent, is injective iff $\varphi | M$ is and $\varphi(M)$ is A-independent (in B) (this is an obvious generalization of a regularity criterion for linear operators).

Closely related to this general regularity criterion is the following *theorem of Philip Hall*: a surjective homomorphisms $\varphi\colon A = \mathrm{C}M \to B$, where M is an (A-) independent subset and therefore a *basis* of A, is *fully invariant* (i. e. its congruence relation $\varphi^{-1} \circ \varphi$ admits all endomorphisms of A or, what is the same, each endomorphism σ of A induces a — necessarily unique — endomorphism τ of B such that $\tau \circ \varphi = \varphi \circ \sigma$) iff $|B| \leqslant 1$ or $\varphi | M$ is injective and $\varphi(M)$ is a (B-) independent subset and therefore a basis of B. There is the following *strengthening of Hall's theorem*: we only assume M to be B- (instead of A-) independent and replace "fully invariant" by "*superinvariant*" (i. e. each homomorphism $\psi\colon A \to B$ induces a — necessarily unique — endomorphism τ of B such that $\tau \circ \varphi = \psi$). There is an even more general theorem which delivers as another special case the following *transitivity of independence*: if M is a B-independent subset of A, N a C-independent subset of B of power $|N| \geqslant |M|$, then M is a C-independent subset of A.

This important property involving cardinals may be strengthened by means of the dimension introduced (in an ordinal form) by Słomiński, the *dimension* $\mathfrak{s}$ of $\Delta = (K_i)_{i \in I}$ being defined here as the least infinite regular cardinal number $>$ all $|K_i|$. First, we have the following strengthening of hereditariness of independence: M is B-independent iff so is any $M' \subset M$ of power $|M'| < \mathfrak{s}$, i. e.

$$\mathrm{ind}\, M = \bigcap_{\substack{M' \subset M \\ |M'| < \mathfrak{s}}} \mathrm{ind}\, M';$$

the exactness of this upper bound being left as an open problem [14].

[14] This problem has been solved in the meantime by Peter Burmeister and the author (*added in proof*).

By combination with transitivity: if M is B-independent in A, N C-independent in B, of power $|N| \geqslant \mathfrak{s}$, then M is C-independent in A. This is an important clue for structural insights like: if B contains a C-independent subset N of power $|N| \geqslant \mathfrak{s}$, then C is the homomorphic image of a subalgebra of a direct power of B, e. g. of subalgebra $F^C(B) \subset B^{B^C}$. This can be used for a structural characterization of the "functionally free" algebras in the sense of Tarski, and for an extreme generalization of Birkhoff's theory of equations.

S. Fajtlowicz (Wrocław), *Properties of the family of independent subsets of a general algebra* (voir ce volume, p. 225-231).

B. Węglorz (Wrocław), *Remarks on compactifications of abstract algebras.*

A *topological algebra* is a general algebra in the set in which a Hausdorff topology is defined in a way that algebraic operations are continuous. By a *compactification* of a topological algebra $\mathfrak{A}$ we mean a compact topological algebra that contains $\mathfrak{A}$ as a dense subalgebra and such that the induced topology coincides with the topology of $\mathfrak{A}$. A compactification of an abstract algebra $\mathfrak{A}$ is a compactification of the discrete topological algebra $\mathfrak{A}$.

THEOREM 1. *Let $\mathfrak{A}$ be a completely regular topological algebra such that the fundamental operations have no more than one variable. Then $\mathfrak{A}$ admits a compactification.*

THEOREM 2. *If $\mathfrak{A}$ is a completely regular topological algebra which, in addition, is $\aleph_0$-compact and locally $\aleph_0$-complete, then $\mathfrak{A}$ admits a compactification.*

Let $\mathfrak{A}$ be an abstract algebra, K an arbitrary class of compactifications of $\mathfrak{A}$. We say that a topological algebra $\mathfrak{K}$ is a *maximal compactification of* $\mathfrak{A}$ with respect to the class K if the following conditions are satisfied:

(i) $\mathfrak{K}$ is a compactification of $\mathfrak{A}$;

(ii) any algebra $\mathfrak{N} \in \mathfrak{K}$ is a continuous homomorphic image of $\mathfrak{K}$ and the homomorphism is a continuous isomorphism on the embedding of $\mathfrak{A}$ into $\mathfrak{K}$ and $\mathfrak{N}$, respectively.

(iii) if a topological algebra $\mathfrak{M}$ satisfies (i) and (ii), then $\mathfrak{K}$ is a continuous homomorphic image of $\mathfrak{M}$ and the homomorphism is a continuous isomorphism on the embedding of $\mathfrak{A}$ into $\mathfrak{M}$ and $\mathfrak{K}$, respectively.

THEOREM 3. *For any non-void class K of compactifications of an abstract algebra $\mathfrak{A}$ there exists a maximal compactification of $\mathfrak{A}$ with respect to the class K. Any two maximal compactifications of $\mathfrak{A}$ with respect to K are topologically isomorphic.*