Generalized idempotence in cardinal arithmetic

by

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1. Introduction. Set theories including the axiom of choice are dominated by absorption laws which completely trivialize the problem of idempotence and idemmultiplicity of infinite cardinals. Using the usual algebraic nomenclature we say that a cardinal $\mathfrak{m}$ is *idempotent* if $\mathfrak{m} = \mathfrak{m}^2$ and *idemmultiple* if $\mathfrak{m} = 2\mathfrak{m}$. A direct application of Zorn's lemma shows that every infinite cardinal is both idempotent and idemmultiple. Conversely, Tarski has shown (cf. [9]) that the idempotence of every infinite cardinal implies the axiom of choice. For the moment, let us understand by set theory the usual axioms of Gödel-Bernays, excluding the axiom of choice. Since the axiom of choice is relatively independent of set theory (cf. [1]), the existence of infinite non-idempotent cardinals is relatively consistent with set theory. It is therefore natural to inquire into the various pathologies concerning idempotence and idemmultiplicity which are relatively consistent with set theory. As a consequence of these observations we see that our theorems will have the character of relative consistency results, i.e., will have the form, "If set theory is consistent, then no inconsistency obtains the additional hypothesis φ.", where φ expresses some unusual property of cardinal arithmetic.

There are regularities in the additive theory of cardinals which do not appear in its multiplicative counterpart unless use is made of the axiom of choice. Of chief importance is the fact that without using the axiom of choice we can prove that if k is a non-zero integer, and $\mathfrak{m}$ and $\mathfrak{n}$ are cardinals, then

$$(1) \qquad k\mathfrak{m} \leqslant k\mathfrak{n} \rightarrow \mathfrak{m} \leqslant \mathfrak{n} \qquad \text{(cf. [10])}.$$

That we do not have a multiplicative analogue of this theorem follows from a result of Tarski (cf. [9]) which asserts that the cancellation

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law $\mathfrak{m}^2=\mathfrak{n}^2\to\mathfrak{m}=\mathfrak{n}$ for all cardinals $\mathfrak{m}$ and $\mathfrak{n}$ implies the axiom of choice.

Now suppose that for some cardinal $\mathfrak{m}$, and integers $0<j<k$ we have $j\mathfrak{m}=k\mathfrak{m}$. Then we will also have $j\mathfrak{m}\leqslant(j+1)\mathfrak{m}\leqslant k\mathfrak{m}=j\mathfrak{m}$. Since the Cantor-Bernstein theorem holds without the axiom of choice, it follows that $j\mathfrak{m}=(j+1)\mathfrak{m}$. Mathematical induction on the integer l will then give us $j\mathfrak{m}=l\mathfrak{m}$ for every $l>j$. In particular if $l=2j$, then $j\mathfrak{m}=2j\mathfrak{m}$. Hence as a consequence of (1) we have $\mathfrak{m}=2\mathfrak{m}$. Induction on the integer l will then give us $\mathfrak{m}=l\mathfrak{m}$ for every $l>0$. Thus the terms of the sequence of successive multiples $\mathfrak{m},2\mathfrak{m},3\mathfrak{m},\ldots$ are either all distinct or all identical, and one of the following two cases must arise:

$$\mathfrak{m}<2\mathfrak{m}<3\mathfrak{m}<\ldots \tag{2}$$

$$\mathfrak{m}=2\mathfrak{m}=3\mathfrak{m}=\ldots \tag{3}$$

Which does is completely determined by the initial relation between $\mathfrak{m}$ and $2\mathfrak{m}$, i.e., by the idemmultiplicity of $\mathfrak{m}$.

Considerably less can be said about the sequence of successive powers $\mathfrak{m},\mathfrak{m}^2,\mathfrak{m}^3,\ldots$ If for some integers $0<j<k$ we have $\mathfrak{m}^j=\mathfrak{m}^k$, then as in the preceding paragraph the Cantor-Bernstein theorem and induction on the integer l gives us $\mathfrak{m}^j=\mathfrak{m}^l$ for every $l>j$. Thus one of the following two cases must arise:

$$\mathfrak{m}<\mathfrak{m}^2<\mathfrak{m}^3<\ldots \tag{4}$$

or for some integer $k>0$

$$\mathfrak{m}<\mathfrak{m}^2<\ldots<\mathfrak{m}^{k-1}<\mathfrak{m}^k=\mathfrak{m}^{k+1}=\ldots \tag{5}$$

There are several easily obtained relations between powers of a cardinal and multiples of those powers. If for some integer $j>0$, $\mathfrak{m}^j=2\mathfrak{m}^j$, then for every integer $l>j$, $\mathfrak{m}^l=2\mathfrak{m}^l$. This follows by multiplying both sides of the former equation by $\mathfrak{m}^{l-j}$. Let us assume in the sequel that $\mathfrak{m}>1$. If for some integer $j>0$, $\mathfrak{m}^j=\mathfrak{m}^{j+1}$, then an application of the Cantor-Bernstein theorem gives $\mathfrak{m}^j=2\mathfrak{m}^j$. Thus one of the following three cases must arise:

$$\begin{array}{ccccc} 2\mathfrak{m} & < & 2\mathfrak{m}^2 & < & \ldots \\ \vee & & \vee & & \ldots \\ \mathfrak{m} & < & \mathfrak{m}^2 & < & \ldots \end{array} \tag{6}$$

or there is an integer $k>0$ such that

$$\begin{array}{ccccccccccc} 2\mathfrak{m} & < & 2\mathfrak{m}^2 & < & \ldots & < & 2\mathfrak{m}^{k-1} & < & 2\mathfrak{m}^k & < & \ldots \\ \vee & & \vee & & \ldots & & \vee & & \| & & \ldots \\ \mathfrak{m} & < & \mathfrak{m}^2 & < & \ldots & < & \mathfrak{m}^{k-1} & < & \mathfrak{m}^k & < & \ldots \end{array} \tag{7}$$

or there are integers $0<k\leqslant l$ such that

$$\begin{array}{ccccccccccccccc} 2\mathfrak{m} & < & 2\mathfrak{m}^2 & < & \ldots & < & 2\mathfrak{m}^{k-1} & < & 2\mathfrak{m}^k & < & \ldots & < & 2\mathfrak{m}^l & = & 2\mathfrak{m}^{l+1} = \ldots \\ \vee & & \vee & & \ldots & & \vee & & \| & & \ldots & & \| & & \| \quad \ldots \\ \mathfrak{m} & < & \mathfrak{m}^2 & < & \ldots & < & \mathfrak{m}^{k-1} & < & \mathfrak{m}^k & < & \ldots & < & \mathfrak{m}^l & = & \mathfrak{m}^{l+1} = \ldots \end{array} \tag{8}$$

Let min_n when applied to a predicate containing the free integer variable n be the least integer operator when it exists and equal to ω otherwise. For any cardinal $\mathfrak{m}$ define $ch_0(\mathfrak{m})=min_n(\mathfrak{m}^n=2\mathfrak{m}^n)$, $ch_1(\mathfrak{m})=min_n(\mathfrak{m}^n=\mathfrak{m}^{n+1})$, and $ch(\mathfrak{m})=\langle ch_0(\mathfrak{m}),ch_1(\mathfrak{m})\rangle$. $ch(\mathfrak{m})$ is called the *character* of $\mathfrak{m}$, and generally for $\mathfrak{m}>1$, $0<ch_0(\mathfrak{m})\leqslant ch_1(\mathfrak{m})\leqslant\omega$. Thus for (6), $ch(\mathfrak{m})=\langle\omega,\omega\rangle$, for (7) $ch(\mathfrak{m})=\langle k,\omega\rangle$, and for (8), $ch(\mathfrak{m})=\langle k,l\rangle$. In terms of these notions our principal result is that except for the restrictions developed in the preceding paragraphs, the existence of cardinals having arbitrary character is relatively consistent with set theory. More precisely:

THEOREM. *If set theory is consistent, then no inconsistency obtains under the additional hypothesis that, "For all α,β, with $0<\alpha\leqslant\beta\leqslant\omega$, there is a cardinal $\mathfrak{m}$ having the character $\langle\alpha,\beta\rangle$."*

We prove this theorem by exhibiting a model of set theory which contains cardinals having the desired characters. Generally there are two methods available for the construction of such models. First are the by now classical Fraenkel-Mostowski (henceforth FM) models of set theories containing urelemente (cf. [4]). Second are the more recent Cohen models of Gödel-Bernays set theory (cf. [1]). Although it is possible to obtain our theorem using models of either type, FM methods encompass fewer technical difficulties. Consequently urelemente set theories will be exclusively used throughout the remainder of this paper.

2. Preliminaries. Let $\mathfrak{S}$ be a set theory differing from that of [4] only in regard to an additional axiom:

(1) *There is an infinite set* $\mathbf{K}$ *of all urelemente.* [1]

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[1] Urelemente are a lowest layer of objects out of which it is possible to construct a set theory. Although individually they contain no elements, they are distinct from one another as well as from the empty set (henceforth denoted by Λ). $\mathfrak{S}$ is essentially a variant of the Gödel-Bernays axiom system with a weakened axiom of extensionality so as to allow for the possible existence of urelemente. On a first classification the objects of $\mathfrak{S}$ are divided into the categories of 'individuals' and 'classes'. x is an *individual* if for some, y, $x\in y$. x is a *class* if it is either Λ or for some y, $y\in x$. Individuals (classes) are denoted by lower (upper) case Latin letters unless expressly stated otherwise. Objects which belong to both categories are called *sets*. Our resulting theory is symmetrical in that it provides for proper individuals (the urelemente) as well as for proper classes. Between these two extremes lie the sets which are the natural domain of mathematical discourse.

$\mathfrak{S}^0$ is obtained from $\mathfrak{S}$ by adding the axiom of choice for sets of finite sets, $\mathfrak{S}^+$ is obtained from $\mathfrak{S}$ by adding the principle of linear ordering, and $\mathfrak{S}^{++}$ is obtained from $\mathfrak{S}$ by adding the axiom of choice. It is known that $\mathfrak{S}^{++}, \mathfrak{S}^+, \mathfrak{S}^0, \mathfrak{S}$ in that order are of strictly decreasing strengths. Each of the following lemmas, theorems, and corollaries will be labeled as to the theory in which it occurs.

We denote ordinal numbers by lower case Greek letters and define them in such a way that each *ordinal number* is the set of all its predecessors. An ordinal number is an *integer* if both it and each of its predecessors contains a largest element. Integers are denoted by lower case Latin letters. ω is the first ordinal which is not an integer, and as a set, is the set of all integers. Λ is the empty set.

The notion of a cardinal number may be defined by the method of Scott (cf. [8]). First we recursively define a hierarchy of sets $R_0 = \mathbf{K}$, $R_{\xi+1} = R_\xi \cup \mathfrak{P}(R_\xi)$ (where $\mathfrak{P}$ is the power set operation), and $R_\lambda = \bigcup \{R_\xi: \xi < \lambda\}$ for limit ordinal λ. An application of the axiom of regularity shows that each individual x is an element of some R_ξ. Then we define the *rank* of x, $\tau(x)$ as the smallest ordinal number ξ such that $x \in R_\xi$. In particular the urelemente are exactly those individuals of rank 0, and sets all have positive rank. We denote *equivalence* between sets by $\cong$ (in context $A \cong B$) and take it to have the usual meaning that there exists a one to one correspondence mapping A onto B. Using these concepts we define the *cardinal number* of a set M as

$$\{A:\ A \cong M \wedge (\forall B)\big(B \cong M \to \tau(A) \leqslant \tau(B)\big)\}. \tag{2}$$

According to this definition the cardinal number of M is itself a set which we shall denote by $|M|$. Let Γ be the class of all cardinal numbers. Members of Γ will usually be denoted by lower case German letters unless we specifically wish to refer to the set of which it is a cardinal number.

We say that a set is *finite* if it is equivalent to an integer. A cardinal is *finite* if it is the cardinal number of a finite set, or what amounts to the same thing, if it is the cardinal number of an integer. Distinct integers have distinct cardinal numbers and the ensuing correspondence preserves order in both the ordinal and cardinal sense. For this reason we shall sometimes identify the notions of an integer and a finite cardinal. ω is the first ordinal which is not finite. We say that a set is *denumerably infinite* if it is equivalent to ω, and take $\aleph_0 = |\omega|$.

A *function* is a class of ordered pairs ([2]) satisfying the usual many oneness condition. By requiring that a function be a class we insure

([2]) The ordered pair (x, y) is the set $\{\{x\}, \{x, y\}\}$. This is noted because in the sequel we shall distinguish between ordered pairs and 2-tuples.

the uniqueness of the function with empty domain. If F is a function and x is an individual then by $F(x)$ we mean the value of F for the argument x if x is in the domain of F, and the value Λ otherwise. If A is a class then by $F[A]$ we mean $\{F(x):\ x \in A\}$. Let F be a function and A a set. Indicate a restricted quantifier by appending the restriction to the quantifier bracket. Then we define the *general direct product* by

$$\times \{F(x):\ x \in A\} = \{f:\ f \text{ is a function with domain } A \wedge (\forall x)_A\big(f(x) \in F(x)\big)\}. \tag{3}$$

Let M and N be any two sets. Take $F(0) = M$, $F(1) = N$ and $A = \{0, 1\}$. The special case of (3) which results for this F and A is called the *direct product* of M and N and is denoted by $M \times N$. Let M be any set and k an integer. Take $F(i) = M$ for $i < k$ and $A = k$. The special case of (3) which results for this F and A is called the *direct power* of M and is denoted by M^k. It is not difficult to show that if $M \cong M_1$ and $N \cong N_1$ then $M \times N \cong M_1 \times N_1$ and $M^k \cong M_1^k$. Consequently we are justified in defining the *cardinal product* and *cardinal power* operations by $\mathfrak{m}\mathfrak{n} = |M \times N|$ and $\mathfrak{m}^k = |M^k|$ respectively, where M and N are any sets with $\mathfrak{m} = |M|$ and $\mathfrak{n} = |N|$. We shall systematically abuse this notation only in the case where either $\mathfrak{m}$ or $\mathfrak{n}$ is an integer by failing to distinguish between the integer (an ordinal) and its cardinal number. Thus, for example, we write $k\mathfrak{m}$ in place of $|k|\mathfrak{m}$ for any integer k and cardinal $\mathfrak{m}$. A function x whose domain is an integer k is called an *ordered k-tuple*. In this case we let $x_i = x(i)$ denote the ith *component* of x and write $x = \langle x_0, \ldots, x_{k-1}\rangle$. Ordered k-tuples of cardinal numbers will also be denoted by lower case German letters.

For any set M, let us define

$$E(M) = \{A:\ A \subseteq M \wedge |A| < \aleph_0\}, \tag{4}$$

$$P(M) = \{\Lambda\} \cup \bigcup \{M^k:\ 0 < k < \omega\}. \tag{5}$$

It is not difficult to show that if $M \cong M_1$ then $E(M) \cong E(M_1)$ and $P(M) \cong P(M_1)$. Consequently we are justified in defining $e(\mathfrak{m}) = |E(M)|$ and $p(\mathfrak{m}) = |P(M)|$ ([3]) where M is any set with $\mathfrak{m} = |M|$. In terms of these operations we have the following three lemmas, the proof of the first being quite elementary and consequently omitted.

$\mathfrak{S}$-Lemma 1. *For any cardinals* $\mathfrak{m}$ *and* $\mathfrak{n}$, (i) $e(\aleph_0) = \aleph_0$, (ii) $e(\mathfrak{m}+\mathfrak{n}) = e(\mathfrak{m})e(\mathfrak{n})$, (iii) $e(\mathfrak{m}+\aleph_0) = \aleph_0 e(\mathfrak{m})$.

([3]) The author is grateful to the referee for suggesting the operation p and for the statements of lemma 2, and (i) of lemma 3. Further, the use of p has considerably shortened many of the original proofs of lemmas found throughout the remainder of this paper.

$\mathfrak{S}$-Lemma 2. *For any cardinal* $\mathfrak{m}>0$, $p(\mathfrak{m})=p\big(p(\mathfrak{m})\big)=\aleph_0 p(\mathfrak{m})$.

Proof. We prove this lemma by giving individual demonstrations of three parts corresponding to the inequalities $p(\mathfrak{m})\leqslant p\big(p(\mathfrak{m})\big)$, $p\big(p(\mathfrak{m})\big)\leqslant\aleph_0 p(\mathfrak{m})$, and $\aleph_0 p(\mathfrak{m})\leqslant p(\mathfrak{m})$ respectively.

Part 1. Obviously $\mathfrak{m}\leqslant p(\mathfrak{m})$ and it is not difficult to show that p is an increasing function. Hence $p(\mathfrak{m})\leqslant p\big(p(\mathfrak{m})\big)$.

Part 2. Let M be a set with $\mathfrak{m}=|M|$. We shall construct a one-one function f mapping $P\big(P(M)\big)$ into $\omega\times P(M)$. Let $x\in P\big(P(M)\big)$. If $x=\Lambda$ let $f(x)=\langle 0,\Lambda\rangle$. Otherwise for some integer $k>0$, $x=\langle x_0,\ldots,x_{k-1}\rangle$. If every component of x is Λ let $f(x)=\langle k,\Lambda\rangle$. Otherwise let $A=\{i\colon x_i\neq\Lambda\}$. Take y to be the sequence obtained by amalgamating the sequences x_i, $i\in A$, naturally ordered by the index i, and let n be an integer whose prime power representation indicates how x may be uniquely obtained from y. In this case let $f(x)=\langle n,y\rangle$.

Part 3. If $\mathfrak{m}=1$ then $p(\mathfrak{m})=\aleph_0$ and we are done. Therefore let us suppose that $\mathfrak{m}>1$. Let M be a set with $\mathfrak{m}=|M|$, containing two distinct members a and b. We shall construct a one-one function mapping $\omega\times P(M)$ into $P(M)$. Let $x\in\omega\times P(M)$. If $x=\langle n,\Lambda\rangle$ let $f(x)$ be the sequence consisting of $n+1$ a's. Otherwise x will have the form $\langle n,u\rangle$ where u is a sequence of elements of M. Let c be the first element of a,b (in that order) which is not the last member of the sequence u. Take y to be the sequence consisting of u followed by $n+1$ of the c's. In this case let $f(x)=y$. Q.E.D.

$\mathfrak{S}^0$-Lemma 3. (i) *For any cardinal* $\mathfrak{m}>0$, $p(\mathfrak{m})=\aleph_0 e(\mathfrak{m})$. (ii) *For any integer* $k>0$ *and cardinal* $\mathfrak{m}$, $\aleph_0 e(\mathfrak{m})=\aleph_0 e(k\mathfrak{m})$.

Proof. We prove (i) by giving individual demonstrations of two parts corresponding to the inequalities $\aleph_0 e(\mathfrak{m})\leqslant p(\mathfrak{m})$ and $p(\mathfrak{m})\leqslant\aleph_0 e(\mathfrak{m})$ respectively. Let M be a set with $\mathfrak{m}=|M|$. Each element $x\in E(M)$ is a finite set and consequently is equivalent to some integer k. Since there are only finitely many such equivalences mapping x onto k, we may use the axiom of choice for sets of finite sets to choose one, and then use the chosen equivalence to induce an ordering on x. Thus there exists a one-one function h taking each $x\in E(M)$ into some k-tuple composed of all the elements of x.

Part 1. We shall construct a one-one function f mapping $\omega\times E(M)$ into $\omega\times P(M)$. Let $x\in\omega\times E(M)$. If $x=\langle n,\Lambda\rangle$ take $f(x)=x$. Otherwise x has the form $\langle n,u\rangle$ where u is a finite non-empty subset of M. In this case take $f(x)=\langle n,h(u)\rangle$. Thus $\aleph_0 e(\mathfrak{m})\leqslant\aleph_0 p(\mathfrak{m})$. But by the preceding lemma $p(\mathfrak{m})=\aleph_0 p(\mathfrak{m})$.

Part 2. We shall construct a one-one function f mapping $P(M)$ into $\omega\times E(M)$. Let $x\in P(M)$. If $x=\Lambda$ take $f(x)=\langle 0,\Lambda\rangle$. Otherwise x will be a sequence of elements of M. Let y be the set of distinct terms of x and n an integer whose prime power representation indicates how x may be uniquely obtained from $h(y)$. In this case take $f(x)=\langle n,y\rangle$. (ii) of the lemma now follows by simple algebra. If $\mathfrak{m}=0$ then $e(\mathfrak{m})=e(k\mathfrak{m})$ and we are done. Otherwise by lemma 2 and the monotonicity of p, $p(\mathfrak{m})\leqslant p(k\mathfrak{m})\leqslant p\big(p(\mathfrak{m})\big)=p(\mathfrak{m})$. Hence $p(\mathfrak{m})=p(k\mathfrak{m})$. But then by (i) of this lemma $\aleph_0 e(\mathfrak{m})=\aleph_0 e(k\mathfrak{m})$. Q.E.D.

Before resuming our study of cardinal character it is necessary to introduce the notion of a Dedekind finite cardinal. A set is *Dedekind finite* if it is not equivalent to a proper subset of itself. A cardinal is *Dedekind finite* if it is the cardinal number of a Dedekind finite set. Let Δ be the class of all Dedekind finite cardinals. Tarski (cf. [10]) states that for every cardinal $\mathfrak{x}$ the following conditions are equivalent:

$$(6)\qquad\begin{cases}\text{(i)} & \mathfrak{x}\in\Delta\,,\\ \text{(ii)} & \mathfrak{x}\neq\mathfrak{x}+1\,,\\ \text{(iii)} & \sim\aleph_0\leqslant\mathfrak{x}\,,\\ \text{(iv)} & (\forall\mathfrak{m},\mathfrak{n})(\mathfrak{x}+\mathfrak{m}=\mathfrak{x}+\mathfrak{n}\rightarrow\mathfrak{m}=\mathfrak{n})\,,\\ \text{(v)} & (\forall\mathfrak{m},\mathfrak{n})(\mathfrak{x}+\mathfrak{m}\leqslant\mathfrak{x}+\mathfrak{n}\rightarrow\mathfrak{m}\leqslant\mathfrak{n})\,.\end{cases}$$

These equivalences are obtained in $\mathfrak{S}$ and serve to provide alternative definitions of Δ. For our purposes (iii) is extremely useful. In terms of sets it asserts that a set is Dedekind finite if it has no denumerably infinite subset. It is well known that Δ is closed under the cardinal operations of plus and times. It is of considerable importance that Δ also has the following closure property.

$\mathfrak{S}^0$-Lemma 4. *If* $\mathfrak{m}\in\Delta$ *then* $e(\mathfrak{m})\in\Delta$.

Proof. Suppose that $\mathfrak{m}\in\Delta$ and M is a set with $\mathfrak{m}=|M|$. If $E(M)$ is not Dedekind finite then it has a denumerably infinite subset $\{A_i\colon i<\omega\}$ of distinct elements. Without loss of generality we may suppose that each $B_k=A_k-\bigcup\{A_i\colon i<k\}$ is non-empty. Then $\{B_i\colon i<\omega\}$ is a denumerably infinite disjoint family of non-empty subsets of M. By the axiom of choice for sets of finite sets we may choose elements $x_i\in B_i$. But then $\{x_i\colon i<\omega\}$ is a denumerably infinite subset of M. This is a contradiction. Therefore $E(M)$ is a Dedekind finite set. Q.E.D.

3. Main construction.

With these preliminary lemmas out of the way we will proceed to construct cardinals having specified character. In order to have this character the cardinal must satisfy certain equalities and certain inequalities. In this section we shall present several cardinal polynomials which by virtue of their form automatically satisfy the proper equalities. In the next section we shall show that in an appropriate model of set theory the proper inequalities are satisfied as well.

Let $\mathfrak{x} > 0$ be any cardinal and $\mathfrak{p}(\mathfrak{x}) = \mathfrak{x} + \aleph_0$. Unless we wish to accent the cardinal $\mathfrak{x}$ we will generally write $\mathfrak{p}$ instead of $\mathfrak{p}(\mathfrak{x})$. The following lemma is quite elementary and is consequently offered without proof.

$\mathfrak{S}$-Lemma 5. *For each integer* $s > 0$, $\mathfrak{p}^s = \mathfrak{x}^s + \aleph_0 \mathfrak{x}^{s-1}$.

$\mathfrak{S}$-Corollary. *Let* $\mathfrak{x} \in \Delta$ *and* $s > 0$ *be any integer. Then*

(i) *if* $\mathfrak{p}^s = 2\mathfrak{p}^s$ *then* $\mathfrak{x}^s \leqslant \aleph_0 \mathfrak{x}^{s-1}$,

(ii) *if* $\mathfrak{p}^s = \mathfrak{p}^{s+1}$ *then* $\mathfrak{x}^s \leqslant \aleph_0 \mathfrak{x}^{s-1}$.

Proof. If $\mathfrak{p}^s = 2\mathfrak{p}^s$ then $\mathfrak{x}^s + \aleph_0\mathfrak{x}^{s-1} = 2\mathfrak{x}^s + \aleph_0\mathfrak{x}^{s-1}$. Since $\mathfrak{x} \in \Delta$, $\mathfrak{x}^s \in \Delta$ as well and we may therefore use the Tarski cancellation theorem (cf. (6) (iv) of section 1) to obtain $\aleph_0\mathfrak{x}^{s-1} = \mathfrak{x}^s + \aleph_0\mathfrak{x}^{s-1}$. But this implies our result. (ii) is treated in a similar fashion. Q.E.D.

Let $k > 0$ be any integer and $\mathfrak{x} = \langle \mathfrak{x}_0, \ldots, \mathfrak{x}_{k-1}\rangle$ any k-tuple of cardinal numbers with $\mathfrak{x}_{k-1} > 0$. Consider the expression

$$\mathfrak{q}_k(\mathfrak{x}) = e(\mathfrak{x}_0) + \ldots + e(\mathfrak{x}_{k-2}) + \aleph_0 \mathfrak{x}_{k-1}\,. \tag{1}$$

Unless we wish to accent the k-tuple $\mathfrak{x}$ we will simply denote this expression by $\mathfrak{q}$. Let $\sum$ and $\prod$ be the repeated sum and product operations as applied to cardinal numbers. Now for each integer $j \leqslant k$ let us define $\mathfrak{t}_j(\mathfrak{x}, 0) = \aleph_0$ and for $1 \leqslant i \leqslant j$

$$\mathfrak{t}_j(\mathfrak{x}, i) = \aleph_0 \sum \{\prod \{e(\mathfrak{x}_i):\ i \in A\}:\ A \subseteq j \wedge |A| = i\} \tag{2}$$

and $\mathfrak{t}_j(\mathfrak{x}, i) = \mathfrak{t}_j(\mathfrak{x}, j)$ for integers $i > j$. We will often omit the argument $\mathfrak{x}$ and simply write this function as $\mathfrak{t}_j(i)$.

$\mathfrak{S}^0$-Lemma 6. *For each integer* $s \geqslant k$, $\mathfrak{q}^s = \sum_{j=1}^{s} \mathfrak{t}_{k-1}(s-j)\,\mathfrak{x}_{k-1}^j$.

Proof. Consider the multinomial expansion of $\mathfrak{q}^s$. Clearly the coefficient of $\mathfrak{x}_{k-1}^s$ is $\mathfrak{t}_{k-1}(0) = \aleph_0$. For any $0 < j < s$ the coefficient of $\mathfrak{x}_{k-1}^j$ is a sum of terms each having the form $\mathfrak{t} = \aleph_0 \prod \{e(a_i\mathfrak{x}_i):\ i < k-1\}$ where the a_i's are integers which sum to $s-j$. Hence by lemma 3 there is a set $A \subseteq k-1$ containing $s-j$ or fewer elements such that $\mathfrak{t} = \aleph_0 \prod \{e(\mathfrak{x}_i):\ i \in A\}$. We can expand A to a set $B \subseteq k-1$ containing exactly $\min\{s-j, k-1\}$ elements such that $\mathfrak{t} \leqslant \aleph_0 \prod \{e(\mathfrak{x}_i):\ i \in B\}$. But this implies that the coefficient of $\mathfrak{x}_{k-1}^j$ is dominated by $\mathfrak{t}_{k-1}(s-j)$. The coefficient of $\mathfrak{x}_{k-1}^0$ is a sum of terms each having the form $\mathfrak{t} = \prod \{e(a_i\mathfrak{x}_i):\ i < k-1\}$ where the a_i's are integers which sum to s. Hence by lemma 3

$$\mathfrak{t} \leqslant \aleph_0 \prod \{e(a_i\mathfrak{x}_i):\ i < k-1\} \leqslant \aleph_0 \prod \{e(\mathfrak{x}_i):\ i < k-1\}\,.$$

Thus we see that the coefficient of $\mathfrak{x}_{k-1}^0$ is dominated by $\mathfrak{t}_{k-1}(k-1) = \mathfrak{t}_{k-1}(s)$. By combining these results we see that $\mathfrak{q}^s$ is dominated by

$$\sum_{j=0}^{s} \mathfrak{t}_{k-1}(s-j)\,\mathfrak{x}_{k-1}^j\,. \tag{3}$$

Now since $s-1 \geqslant k-1$, $\mathfrak{t}_{k-1}(s) = \mathfrak{t}_{k-1}(s-1)$. Consequently for the first two terms of (3) we have

$$\mathfrak{t}_{k-1}(s-1)\mathfrak{x}_{k-1} \leqslant \mathfrak{t}_{k-1}(s) + \mathfrak{t}_{k-1}(s-1)\mathfrak{x}_{k-1} \leqslant 2\mathfrak{t}_{k-1}(s-1)\mathfrak{x}_{k-1}\,.$$

But $\mathfrak{t}_{k-1}(s-1)$ is idemmultiple and hence by the Cantor-Bernstein theorem the first term of (3) is absorbed by the second term. Consequently $\mathfrak{q}^s$ is dominated by

$$\sum_{j=1}^{s} \mathfrak{t}_{k-1}(s-j)\,\mathfrak{x}_{k-1}^j\,. \tag{4}$$

Conversely it is clear that every term of (4) is dominated by some term in the multinomial expansion of $\mathfrak{q}^s$. Hence by the Cantor-Bernstein theorem (4) equals $\mathfrak{q}^s$. Q.E.D.

$\mathfrak{S}^0$-Corollary. (i) $\mathfrak{q}^k = 2\mathfrak{q}^k$, (ii) *if for some integer* $s \geqslant k$, $\mathfrak{q}^s = \mathfrak{q}^{s+1}$ *then*

$$\mathfrak{x}_{k-1}^{s+1} \leqslant \sum_{j=1}^{s} \mathfrak{t}_{k-1}(s-j)\,\mathfrak{x}_{k-1}^j\,.$$

$\mathfrak{S}^0$-Lemma 7. $\mathfrak{q}^{k-1} = \prod \{e(\mathfrak{x}_i):\ i < k-1\} + \sum_{j=1}^{k-1} \mathfrak{t}_{k-1}(k-1-j)\,\mathfrak{x}_{k-1}^j$.

Proof. The coefficients of $\mathfrak{x}_{k-1}^j$ for $0 < j \leqslant k-1$ in the multinomial expansion of $\mathfrak{q}^{k-1}$ are treated exactly as in the preceding lemma. The coefficient of $\mathfrak{x}_{k-1}^0$ consists of $\prod \{e(\mathfrak{x}_i):\ i < k-1\}$ as well as terms of the form $\mathfrak{t} = \prod \{e(a_i\mathfrak{x}_i):\ i < k-1\}$ where the a_i's are integers summing to $k-1$ and some $a_i > 1$. But then there is some set $A \subseteq k-1$ with fewer than $k-1$ elements such that $\aleph_0 \prod \{e(a_i\mathfrak{x}_i):\ i < k-1\} = \aleph_0 \prod \{e(\mathfrak{x}_i):\ i \in A\}$. Suppose that A has $k-1-j$ elements. Then $\mathfrak{t} \leqslant \aleph_0 \prod \{e(\mathfrak{x}_i):\ i \in A\} \leqslant \mathfrak{t}_{k-1}(k-1-j)$. Consequently $\mathfrak{t}_{k-1}(k-1-j)\mathfrak{x}_{k-1}^j \leqslant \mathfrak{t} + \mathfrak{t}_{k-1}(k-1-j)\mathfrak{x}_{k-1}^j \leqslant 2\mathfrak{t}_{k-1}(k-1-j)\mathfrak{x}_{k-1}^j$. But the idemmultiplicity of $\mathfrak{t}_{k-1}(k-1-j)$ implies that $\mathfrak{t}$ is absorbed. Q.E.D.

$\mathfrak{S}^0$-Corollary. *If* $\mathfrak{x}_0, \ldots, \mathfrak{x}_{k-2} \in \Delta$ *and* $\mathfrak{q}^{k-1} = 2\mathfrak{q}^{k-1}$ *then*

$$\prod \{e(\mathfrak{x}_i):\ i < k-1\} \leqslant \sum_{j=1}^{k-1} \mathfrak{t}_{k-1}(k-1-j)\,\mathfrak{x}_{k-1}^j\,.$$

Proof. Since each $\mathfrak{x}_i \in \Delta$, $e(\mathfrak{x}_i) \in \Delta$ as well by lemma 4. Hence $\prod \{e(\mathfrak{x}_i):\ i < k-1\} \in \Delta$. Now use the same procedure as we did for the corollary to lemma 5. Q.E.D.

Let $l \geqslant k > 0$ be any integers and $\mathfrak{x} = \langle \mathfrak{x}_0, \ldots, \mathfrak{x}_{l-1}\rangle$ any l-tuple of cardinal numbers. Consider the expression

$$\mathfrak{r}_{k,l}(\mathfrak{x}) = e(\mathfrak{x}_0) + \ldots + e(\mathfrak{x}_{k-2}) + \aleph_0\big(e(\mathfrak{x}_{k-1}) + \ldots + e(\mathfrak{x}_{l-1})\big)\,. \tag{5}$$

Unless we wish to accent the l-tuple $\mathfrak{x}$ we will simply denote this expression by $\mathfrak{r}$.

$\mathfrak{S}^0$-Lemma 8. *For any integer* $s \geqslant k$, $\mathfrak{r}^s = \mathfrak{t}_l(s)$.

Proof. Consider the multinomial expansion of $\mathfrak{r}^s$. Terms occuring in it will have the form $\mathfrak{t} = \mathfrak{a}\prod\{e(a_i\mathfrak{x}_i)\colon i < l\}$ where $\mathfrak{a}$ is either 1 or $\aleph_0$ and the a_i's are integers summing to s. By lemma 3 there is some set $A \subseteq l$ with s or fewer elements such that $\mathfrak{t} \leqslant \aleph_0\prod\{e(a_i\mathfrak{x}_i)\colon i < l\} = \aleph_0\prod\{e(\mathfrak{x}_i)\colon i \in A\}$. We can expand A to a set $B \subseteq l$ containing exactly $min\{s, l\}$ elements such that $\mathfrak{t} \leqslant \aleph_0\prod\{e(\mathfrak{x}_i)\colon i \in B\} \leqslant \mathfrak{t}_l(s)$. But then by the idemmultiplicity of $\mathfrak{t}_l(s)$, $\mathfrak{r}^s \leqslant \mathfrak{t}_l(s)$. Conversely it is clear that any term of $\mathfrak{t}_l(s)$ is dominated by some term in the multinomial expansion of $\mathfrak{r}^s$. Hence $\mathfrak{r}^s = \mathfrak{t}_l(s)$. Q.E.D.

$\mathfrak{S}^0$-Corollary. (i) $\mathfrak{r}^k = 2\mathfrak{r}^k$, (ii) $\mathfrak{r}^l = \mathfrak{r}^{l+1}$, (iii) *if* $\mathfrak{r}^{l-1} = \mathfrak{r}^l$ *then* $\aleph_0\prod\{e(\mathfrak{x}_i)\colon i < l\} \leqslant \mathfrak{t}_l(l-1)$.

For our final lemma we need the additional function

$$(6)\qquad \mathfrak{u}_{k-1}(\mathfrak{x}) = \aleph_0\sum\Big\{\prod\{e(\mathfrak{x}_i)\colon i \in A\}\colon A \subseteq l \wedge A \neq k-1 \wedge |A| = k-1\Big\}.$$

Unless we wish so accent the l-tuple $\mathfrak{x}$, we will simply write this function as $\mathfrak{u}_{k-1}$.

$\mathfrak{S}^0$-Lemma 9. $\mathfrak{r}^{k-1} = \prod\{e(\mathfrak{x}_i)\colon i < k-1\} + \mathfrak{u}_{k-1}$.

Proof. Consider the multinomial expansion of $\mathfrak{r}^{k-1}$. A term occurring in it will either be $\prod\{e(\mathfrak{x}_i)\colon i < k-1\}$ or else have the form $\mathfrak{t} = \mathfrak{a}\prod\{e(a_i\mathfrak{x}_i)\colon i < l\}$ where $\mathfrak{a}$ is either 1 or $\aleph_0$ and the a_i's are integers summing to $k-1$ with either $a_i > 1$ for some $i < k-1$ or $a_i > 0$ for some $i \geqslant k-1$. Hence by lemma 3 there is a set $A \subseteq l$, $A \neq k-1$ with $k-1$ or fewer elements such that $\mathfrak{t} \leqslant \aleph_0\prod\{e(a_i\mathfrak{x}_i)\colon i < l\} = \aleph_0\prod\{e(\mathfrak{x}_i)\colon i \in A\}$. We can expand A to a set $B \subseteq l$, $B \neq k-1$ containing exactly $k-1$ elements such that $\mathfrak{t} \leqslant \aleph_0\prod\{e(\mathfrak{x}_i)\colon i \in B\} \leqslant \mathfrak{u}_{k-1}$. Conversely it is clear that any term of $\mathfrak{u}_{k-1}$ is dominated by some term in the expansion of $\mathfrak{r}^{k-1}$. Our lemma follows by the Cantor-Bernstein theorem. Q.E.D.

$\mathfrak{S}^0$-Corollary. *If* $\mathfrak{x}_0, \ldots, \mathfrak{x}_{k-2} \in \Delta$ *and* $\mathfrak{r}^{k-1} = 2\mathfrak{r}^{k-1}$ *then*

$$\prod\{e(\mathfrak{x}_i)\colon i < k-1\} \leqslant \mathfrak{u}_{k-1}.$$

4. The model. Let $\mathfrak{S}^+(\mathrm{ch})$ be obtained from $\mathfrak{S}^+$ by adding the hypothesis:

(1) *For any ordinals* α *and* β, $1 \leqslant \alpha \leqslant \beta \leqslant \omega$, *there exists a cardinal* $\mathfrak{m} \notin \Delta$, *having character* $\langle\alpha, \beta\rangle$.

In this section we will show that if $\mathfrak{S}$ is consistent, then so is $\mathfrak{S}^+(\mathrm{ch})$. We do so by the technique of FM models. We have seen in the last section how the cardinals $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{r}$ satisfy the equalities necessary to have specified character. Further if they do not satisfy the necessary inequalities, then certain dependencies are introduced between the cardinals $\mathfrak{x}_i$. Hence our problem is to find a model of set theory and of the principle of linear ordering, which contains cardinals that are extremely independent in some imprecisely defined sense.

Before proving our theorem we shall give a brief resume of the FM technique (cf. [4]). It is well known that if $\mathfrak{S}$ is consistent then so is the theory $\mathfrak{S}^{++}$ which is obtained from $\mathfrak{S}$ by adding the axiom of choice. In [4] a model $\mathfrak{W}^+$ is constructed in $\mathfrak{S}^{++}$ which satisfies the axioms of $\mathfrak{S}$, the principle of linear ordering (and consequently the axiom of choice for sets of finite sets), but which does not satisfy the full axiom of choice. Roughly speaking we construct $\mathfrak{W}^+$ in the following way. By the axiom of choice there exists a dense unbordered ordering $\precsim$ of the infinite set $\boldsymbol{K}$ of all urelemente. Let $\mathfrak{G}^+$ be the set of all $\precsim$-monotone permutations of $\boldsymbol{K}$ and let $M^+ = E(\boldsymbol{K})$. For $\varphi \in \mathfrak{G}^+$ and $x \in \boldsymbol{K}$, let $|\varphi, x| = \varphi(x)$. If $x \notin \boldsymbol{K}$ let $|\varphi, x| = \{|\varphi, y|\colon y \in x\}$. This will inductively extend the action of permutations in $\mathfrak{G}^+$ to the whole universe of individuals. For $A \in M^+$, let

$$(2)\qquad \mathfrak{G}^+(A) = \big\{\varphi \in \mathfrak{G}^+\colon (\forall x)_A\big(\varphi(x) = x\big)\big\};$$

$$(3)\qquad \Sigma^+ = \{x\colon (\exists A)_{M^+}(\forall\varphi)_{\mathfrak{G}^+(A)}(|\varphi, x| = x)\}.$$

Σ^+ is essentially the class of those individuals which are symmetrical with respect to M^+. $\mathfrak{W}^+$ will then consist of all individuals which together with every element of their transitive closure belong to Σ^+. We define the primitive notions of our model as follows. The individuals of the model are just the members of $\mathfrak{W}^+$. The classes of the model are just those classes $B \subseteq \mathfrak{W}^+$ such that $(\exists A)_{M^+}(\forall\varphi)_{\mathfrak{G}^+(A)}(\forall x)(x \in B \equiv |\varphi, x| \in B)$. The $\in$ of our model is the restriction of the ordinary $\in$ relation to the individuals and classes of the model. We shall use the same symbol $\mathfrak{W}^+$ for the composite notion of the model as well as for its domain of individuals. It is not difficult to verify that $\mathfrak{W}^+$ satisfies the axioms of $\mathfrak{S}$. A set $A \in M^+$ is said to *support* an $x \in \mathfrak{W}^+$ if $(\forall\varphi)_{\mathfrak{G}^+(A)}(|\varphi, x| = x)$. The crux of the rather involved proof that $\mathfrak{W}^+$ satisfies the principle of linear ordering consists in showing that every individual in the model has a unique minimal support. The notion of support and the theorem on minimal support extends to classes of the model in the obvious way.

Many notions are absolute with respect to $\mathfrak{W}^+$, i.e., they assume the same meaning with respect to $\mathfrak{W}^+$ as they do with respect to the universe of $\mathfrak{S}^{++}$. Besides the absolute notions mentioned in [4], we can easily show that the ordinals belong to $\mathfrak{W}^+$, and are absolute (are exactly the ordinals in the sense of the model). Further, each ordinal has empty support. ω is absolute and consequently so is the notion of finite. $\boldsymbol{K} \in \mathfrak{W}^+$, has empty support, and since the notion of being a set is absolute, the elements of $\boldsymbol{K}$ are precisely the members of $\mathfrak{W}^+$ which are urelemente in the sense of the model. Let us indicate notions relativized to the model by appending a superscript '+' to the symbol for that notion.

$\mathfrak{S}^{++}$-LEMMA 10. ($\boldsymbol{K}$ *is an infinite Dedekind finite set.*)$^+$

Proof. Since finite is absolute, $\boldsymbol{K}$ is clearly infinite in $\mathfrak{W}^+$. Suppose that $\boldsymbol{K}$ were not Dedekind finite in $\mathfrak{W}^+$. Then by the absoluteness theorems of [4] there would exist a one-one function $f \in \mathfrak{W}^+$, mapping ω into $\boldsymbol{K}$. Let $A \in M^+$ support f. Since the range of f is infinite and A is finite, we may choose an element x in the range of f but not in A. Then for some $n \in \omega$, $(n, x) \in f$. Choose $\varphi \in \mathfrak{G}^+(A)$ such that $\varphi(x) \neq x$. Since permutations commute with ordered pairs and integers have empty support $|\varphi, (n, x)| = (|\varphi, n|, |\varphi, x|) = (n, \varphi(x)) \in |\varphi, f| = f$. Thus $(n, \varphi(x)) \in f$. But f is a function, consequently $\varphi(x) = x$ which contradicts our choice of φ. Q.E.D.

For some integer $l > 0$, let $\mathfrak{x} = \langle \mathfrak{x}_0, \ldots, \mathfrak{x}_{l-1} \rangle$ be an l-tuple of cardinal numbers. Consider terms constructed from the $\mathfrak{x}_i$ having the form

$$\mathfrak{s}(\mathfrak{x}) = \prod \{e(\mathfrak{x}_i)\colon\ i \in A\} \prod \{\mathfrak{x}_i^{h(i)}\colon\ i \in B\} \tag{4}$$

where A and B are disjoint subsets of l, and h is a function mapping B into ω. In this case we say that $\mathfrak{s}(\mathfrak{x})$ has the *type* $\langle A, B, h \rangle$. If $\mathfrak{s}_0(\mathfrak{x})$ and $\mathfrak{s}_1(\mathfrak{x})$ are terms having types $\langle A_0, B_0, h_0 \rangle$ and $\langle A_1, B_1, h_1 \rangle$ respectively, we say that the type of $\mathfrak{s}_1(\mathfrak{x})$ is *greater than or equal to* that of $\mathfrak{s}_0(\mathfrak{x})$ if $A_0 \subseteq A_1$, $B_0 \subseteq B_1$, and $(\forall i)_{B_0}(h_0(i) \leqslant h_1(i))$.

Let K_i, $i < l$, be a partition of $\boldsymbol{K}$ into l non-empty disjoint intervals (with respect to $\lessdot$) without endpoints. Take $\mathfrak{k} = \langle \mathfrak{k}_0, \ldots, \mathfrak{k}_{l-1} \rangle$ where each $\mathfrak{k}_i = |K_i|$. For some integer $m > 1$ consider m terms $\mathfrak{s}_0(\mathfrak{x}), \ldots, \mathfrak{s}_{m-1}(\mathfrak{x})$ of the form (4) having types $\langle A_i, B_i, h_i \rangle$ respectively and where $B_0 \subseteq B_i$ for each $0 < i < m$. Then

$\mathfrak{S}^{++}$-LEMMA 11. *If* $\big(\mathfrak{s}_0(\mathfrak{k}) \leqslant \aleph_0 \sum_{i=1}^{m-1} \mathfrak{s}_i(\mathfrak{k})\big)^+$ *then at least one of the* $\mathfrak{s}_i$, $0 < i < m$, *has a type greater than or equal to that of* $\mathfrak{s}_0$.

Proof. Let us represent the cardinal $\mathfrak{s}_i(\mathfrak{k})$ by the set $S_i = \times\{L(K_j)\colon j \in A_i \cup B_i\}$ where $L(K_j) = E(K_j)$ for $j \in A_i$ and $L(K_j) = K_j^{h_i(j)}$ for $j \in B_i$. Now suppose that $\big(\mathfrak{s}_0(\mathfrak{k}) \leqslant \aleph_0 \sum_{i=1}^{m-1} \mathfrak{s}_i(\mathfrak{k})\big)^+$. Then there exists a one-one function $f \in \mathfrak{W}^+$ mapping S_0 into $\omega \times \bigcup_{i=1}^{m-1} S_i$. Let $R \in M^+$ support f as well as each of the K_i's and take $N = \max\{h_j(i)\colon i \in B_j \wedge j < m\}$. Define an element $x \in S_0$ by $x_j \in E(K_j - R)$, $|x_j| > N$, for $j \in A_0$ and $x_j \in (K_j - R)^{h_0(j)}$, the components of x_j distinct, for $j \in B_0$. Then $f(x) = \langle a, y \rangle \in \omega \times S_j$ for some $0 < j < m$. Without loss of generality we may take $j = 1$. Now suppose that the type of $\mathfrak{s}_1$ is not greater than or equal to that of $\mathfrak{s}_0$.

This can happen in either of two ways. First there is some $j \in A_0 - A_1$, or second $A_0 \subseteq A_1$ but for some $j \in B_0$, $h_0(j) > h_1(j)$. In the former case, if y_j is defined then $j \in B_1$. Since $|x_j| > N$ there is some element $z \in x_j$ which does not occur in the sequence y_j. Similarly if y_j is not defined. In the latter case the elements of the $h_0(j)$-tuple x_j are distinct and $h_0(j) > h_1(j)$. Hence there is some element z occurring in the sequence x_j which does not occur in the sequence y_j. In either event choose a permutation $\varphi \in \mathfrak{G}^+(R)$ which leaves K_i pointwise fixed for $i \neq j$, which leaves each element occurring in y_j fixed, but for which $\varphi(z) \neq z$. Then $|\varphi, x| \neq x$ and $|\varphi, y| = y$. Now $f(x) = \langle a, y \rangle$ and $a \in \omega$ hence $|\varphi, f(x)| = f(x)$. Since $(x, f(x)) \in f$, $|\varphi, (x, f(x))| = (|\varphi, x|, |\varphi, f(x)|) = (|\varphi, x|, f(x)) \in |\varphi, f| = f$ and consequently $(|\varphi, x|, f(x)) \in f$. This contradicts the one oneness of f and therefore $\mathfrak{s}_1$ has a type greater than or equal to that of $\mathfrak{s}_0$. Q.E.D.

Now consider the cardinals $\mathfrak{p}$, $\mathfrak{q}_k$, $\mathfrak{r}_{k,l}$ of the last section. Let $\mathfrak{a} = \mathfrak{k}_0$, $\mathfrak{b} = \langle \mathfrak{k}_0, \ldots, \mathfrak{k}_{k-1} \rangle$, and $\mathfrak{c} = \langle \mathfrak{k}_0, \ldots, \mathfrak{k}_{l-1} \rangle$.

$\mathfrak{S}^{++}$-COROLLARY. ($\mathfrak{p}(\mathfrak{a})$, $\mathfrak{q}_k(\mathfrak{b})$, $\mathfrak{r}_{k,l}(\mathfrak{c}) \notin \Delta$ *and have characters* $\langle \omega, \omega \rangle$, $\langle k, \omega \rangle$, $\langle k, l \rangle$ *respectively.*)$^+$

Proof. Since $\aleph_0$ explicitly occurs in their construction, none of these cardinals are members of Δ^+. $\mathfrak{W}^+$ satisfies the axiom of choice for sets of finite sets and by lemma 10 each $\mathfrak{k}_i \in \Delta^+$. Hence we can apply the corollaries of lemmas 5, 6, 7, 8 and 9 which however all contradict Lemma 11. Q.E.D.

Theorem 1 below, our main result, is consequently the corollary to Lemma 11 reformulated as an assertion about relative consistency. Its content is entirely syntactical and it is intended to be a theorem of elementary arithmetic.

THEOREM 1. *If* $\mathfrak{S}$ *is consistent then so is* $\mathfrak{S}^+(\mathrm{ch})$ [4].

Since $\mathfrak{S}^0$ is weaker than $\mathfrak{S}^+$ we have, a fortiori, demonstrated the relative consistency of $\mathfrak{S}^0(\mathrm{ch})$, the theory obtained from $\mathfrak{S}^0$ by taking (1) as an additional axiom.

5. Applications. In this section we will apply theorem 1 to a restricted decision problem for the arithmetic of cardinals in the theory $\mathfrak{S}^0$. Before doing so we study further algebraic properties of cardinal numbers and relate them to the notion of character. Since we are primarily interested in cardinals $\mathfrak{m} \notin \Delta$, we shall not always state our lemmas in full generality, particularly when doing so necessitates a more detailed proof. The strongest form of the lemma, however, will be stated as a footnote.

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(4) In order to replace $\mathfrak{S}$ in theorem 1 by the Gödel-Bernays axioms Σ, we add countably many generic sets of integers to a countable complete model of Σ (cf. [1]) and then use permutations of these generic sets in much the same manner as we used permutations of urelemente in this paper.

$\mathfrak{S}^0$-Lemma 12. *For any cardinal* $\mathfrak{m}$ *and integer* k, (i) *if* $\mathfrak{m} \notin \Delta$ *then* $\mathfrak{m}^k \leqslant e(\mathfrak{m})$ [5], (ii) *if* $\mathfrak{m} > 1$ *and* $\mathfrak{m}^k = \mathfrak{m}^{k+1}$ *then* $\mathfrak{m}^k = e(\mathfrak{m})$.

Proof. If $\mathfrak{m} \notin \Delta$ then $\mathfrak{m} + \aleph_0 = \mathfrak{m}$. Consequently (i) follows easily from $\mathfrak{m}^k \leqslant p(\mathfrak{m}) = \aleph_0 e(\mathfrak{m}) = e(\mathfrak{m}+\aleph_0)$. In order to prove (ii) first note that if $\mathfrak{m} > 1$ and $\mathfrak{m}^k = \mathfrak{m}^{k+1}$ then $\mathfrak{m}^k \leqslant \mathfrak{m}^k+1 \leqslant 2\mathfrak{m}^k \leqslant \mathfrak{m}^{k+1}$. Hence $\mathfrak{m}^k = \mathfrak{m}^k+1$. But this implies $\mathfrak{m}^k \notin \Delta$. Consequently $\mathfrak{m} \notin \Delta$ as well. Now $\mathfrak{m}^k \leqslant \aleph_0 \mathfrak{m}^k \leqslant \mathfrak{m}^{k+1}$. Therefore $\mathfrak{m}^k = \aleph_0\mathfrak{m}^k$. Since $\mathfrak{m}^k = \mathfrak{m}^{k+1}$ an induction on the integer l gives $\mathfrak{m}^k = \mathfrak{m}^l$ for $l > k$, all equivalences being effectively constructed from the one where $l = k+1$. Hence it is not hard to show that $p(\mathfrak{m}) = 1 + \mathfrak{m} + \ldots + \mathfrak{m}^{k-1} + \aleph_0 \mathfrak{m}^k = \aleph_0 \mathfrak{m}^k$. By combining these results we have $e(\mathfrak{m}) = e(\mathfrak{m}+\aleph_0) = \aleph_0 e(\mathfrak{m}) = p(\mathfrak{m}) = \aleph_0 \mathfrak{m}^k = \mathfrak{m}^k$. Q.E.D.

$\mathfrak{S}^0$-Corollary. *For any cardinal* $\mathfrak{m} \notin \Delta$, (i) *if* $ch_1(\mathfrak{m}) = k$ *then* $\mathfrak{m}^k = e(\mathfrak{m})$, (ii) *if* $ch_1(\mathfrak{m}) = \omega$ *then* $\mathfrak{m}^k < e(\mathfrak{m})$ *for every integer* k.

Thus we see that for $\mathfrak{m} \notin \Delta$, the position of $e(\mathfrak{m})$ among the various integer powers of $\mathfrak{m}$ explicitly depends upon $\mathfrak{m}$'s character.

Now for each integer k, let us define the set operations

(1) $$\binom{M}{k} = \{A\colon\ A \subseteq M \wedge |A| = k\},$$

(2) $$\begin{bmatrix} M \\ k \end{bmatrix} = \{x\colon\ x \in M^k \wedge x \textit{ has no repetitions}\}.$$

It is not difficult to show that if $M \cong M_1$ then $\binom{M}{k} \cong \binom{M_1}{k}$ and $\begin{bmatrix} M \\ k \end{bmatrix} \cong \begin{bmatrix} M_1 \\ k \end{bmatrix}$. Consequently we are justified in defining $\binom{\mathfrak{m}}{k} = \left|\binom{M}{k}\right|$ and $\begin{bmatrix} \mathfrak{m} \\ k \end{bmatrix} = \left|\begin{bmatrix} M \\ k \end{bmatrix}\right|$, where M is any set with $\mathfrak{m} = |M|$. We easily see that $\binom{\mathfrak{m}}{k}$ ($\begin{bmatrix} \mathfrak{m} \\ k \end{bmatrix}$) is an extension to cardinals of the ordinary notion of the number of combinations (permutations) of $\mathfrak{m}$ things taken k at a time. Let $k! = (k)(k-1)\ldots(2)(1)$, $0! = 1$ be the usual factorial function. Then

$\mathfrak{S}^0$-Lemma 13. *For any cardinal* $\mathfrak{m}$ *and integer* k, $k!\binom{\mathfrak{m}}{k} = \begin{bmatrix} \mathfrak{m} \\ k \end{bmatrix}$.

Proof. Let M be any set with $\mathfrak{m} = |M|$ and H be the set of all permutations of k. H contains $k!$ elements. Each $A \in \binom{M}{k}$ is equivalent to k and since there are only finitely many such equivalences, we may use the axiom of choice for sets of finite sets to choose one. This equivalence uniquely determines a k-tuple $g(A)$ composed of the distinct elements of A. Take $H \times \binom{M}{k}$ as representative of $k!\binom{\mathfrak{m}}{k}$ and consider any element $x \in H \times \binom{M}{k}$. x will have the form $x = \langle h, A\rangle$. If $g(A) = \langle a_0, \ldots, a_{k-1}\rangle$ take $f(x) = \langle a_{h(0)}, \ldots, a_{h(k-1)}\rangle$. f is a one-one function mapping $H \times \binom{M}{k}$ onto $\begin{bmatrix} M \\ k \end{bmatrix}$. Q.E.D.

[5] It can be shown that this result holds in $\mathfrak{S}^0$ even when we only require that $\mathfrak{m}$ be infinite in the ordinary sense.

$\mathfrak{S}$-Lemma 14. *For any cardinal* $\mathfrak{m} \notin \Delta$ *and integer* k, $\mathfrak{m}^k = \begin{bmatrix} \mathfrak{m} \\ k \end{bmatrix}$.

Proof. Let M be any set with $\mathfrak{m} = |M|$. We may assume without loss of generality that M is disjoint from ω. Since $\begin{bmatrix} M \\ k \end{bmatrix} \subseteq M^k$ half of our theorem is obvious. Conversely consider any $x \in M^k$. Generally x will have repetitions. For $j < k$, let $q(j) = min_i\{i\colon x_i = x_j\}$. Define $f(x) = y$ by $y_j = x_j$ if $q(j) = j$ and $y_j = 2^{q(j)}3^j$ if $q(j) < j$. f is a one-one function mapping M^k into $\begin{bmatrix} M \cup \omega \\ k \end{bmatrix}$. Hence $\mathfrak{m}^k \leqslant \begin{bmatrix} \mathfrak{m}+\aleph_0 \\ k \end{bmatrix}$. But $\mathfrak{m} \notin \Delta$, hence $\mathfrak{m} + \aleph_0 = \mathfrak{m}$. Q.E.D.

$\mathfrak{S}^0$-Corollary. *For any cardinal* $\mathfrak{m} \notin \Delta$ *and integer* k, $k!\binom{\mathfrak{m}}{k} = \mathfrak{m}^k$.

A consequence of this corollary is that for $\mathfrak{m} \notin \Delta$, $\mathfrak{m}^k$ is divisible by $k!$. By (1) of section 1 the result of division by a finite multiple is unique. Hence we may rewrite the corollary as $\binom{\mathfrak{m}}{k} = \mathfrak{m}^k/k!$.

In [5] Myhill introduced the notion of combinatorial functions. These functions were invaluable in [2], where they were used to give a general theory of Dedekind finite cardinals. For the purposes of this section it is convenient to identify the integers with the finite cardinals. Every function $f\colon\ \omega \to \omega$ can be expressed in the form

(3) $$f(x) = \sum c(i)\binom{x}{i}$$

where the summation is extended over all $i < \omega$ and the $c(i)$ belong to the ring of positive and negative integers. The $c(i)$ are uniquely determined by f and are called the *Stirling coefficients* of f. If they are all non-negative f is called a *combinatorial function*. The functions x^k, $\binom{x}{i}$ and $e(x)$ are combinatorial, further, sums, products and compositions of combinatorial functions are combinatorial. With each combinatorial function f having the expansion (3) we associate a unary function Φ, called the *normal combinatorial operator* inducing f, and given by

(4) $$\Phi(M) = \{\langle A, n\rangle\colon\ A \in E(M) \wedge n < c(|A|)\}.$$

Φ induces f in the sense that $f(|M|) = |\Phi(M)|$ for each finite set M. Since Φ preserves equivalence of arbitrary sets we are justified in defining a new function $f_\Gamma\colon\ \Gamma \to \Gamma$ given by $f_\Gamma(|M|) = |\Phi(M)|$ for any set M. This function f_Γ, which agrees with f on ω, is called the *canonical extension* of f to Γ. Although combinatorial functions provide an extremely rich tool for the investigation of Δ (cf. [2]), the following theorem shows that they degenerate rather badly for arguments $\mathfrak{m} \notin \Delta$.

$\mathfrak{S}^0$-THEOREM 2. *For every combinatorial function f, as $\mathfrak{m}$ ranges over $\Gamma - \Delta$, f_Γ reduces to either* (i) *a constant,* (ii) *a function of the form* $\dfrac{i}{k!}\mathfrak{m}^k$, *or* (iii) *the function* $e(\mathfrak{m})$.

Proof. If $c(i) = 0$ for $i > 0$ then clearly $f_\Gamma(\mathfrak{m}) = c(0)$. If only finitely many of the $c(i)$, but at least one for $i > 0$, differ from 0 let k be the largest integer such that $c(k) \neq 0$. In this case it is not difficult to show that $f_\Gamma(\mathfrak{m}) = \sum c(i)\binom{\mathfrak{m}}{i}$. By the corollary of lemma 14 we may replace $\binom{\mathfrak{m}}{i}$ by $\mathfrak{m}^i/i!$. Consequently $f_\Gamma(\mathfrak{m})$ can be written as $Q(\mathfrak{m})/k!$, where $Q(\mathfrak{m})$ is a polynomial function of degree k and leading coefficient $c(k)$. Since $\mathfrak{m} \notin \Delta$, $\mathfrak{m} + i = \mathfrak{m}$ for every integer i. Hence by successively factoring and using this absorption law we can reduce $Q(\mathfrak{m})$ to $c(k)\mathfrak{m}^k$. But then $f_\Gamma(\mathfrak{m}) = \dfrac{c(k)}{k!}\mathfrak{m}^k$. Finally suppose that infinitely many of the $c(i)$ differ from 0. We easily see from (4) that $\Phi(M) \subseteq E(M) \times \omega$. Hence $f_\Gamma(\mathfrak{m}) \leqslant \aleph_0 e(\mathfrak{m}) = e(\mathfrak{m})$ since $\mathfrak{m} \notin \Delta$. For the converse inequality there is no loss of generality in assuming that M is disjoint from ω. Let h be a one-one increasing function mapping ω onto $\{i\colon c(i) \neq 0\}$. Clearly $i \leqslant h(i)$ for every $i < \omega$. For $A \in E(M)$, define $g(A)$ to be the pair $\langle B, 0\rangle$ where B is obtained from A by adding the first $h(|A|) - |A|$ integers to A. We readily see that g is one-one function mapping $E(M)$ into $\Phi(M \cup \omega)$. Hence $e(\mathfrak{m}) \leqslant f_\Gamma(\mathfrak{m} + \aleph_0)$. But $\mathfrak{m} + \aleph_0 = \mathfrak{m}$. Q.E.D.

For integers i, j, k, and l, let us define predicates

$$\mathrm{R}_{ijkl}(\mathfrak{m}) \equiv i\binom{\mathfrak{m}}{j} = k\binom{\mathfrak{m}}{l}, \tag{5}$$

$$\mathrm{Q}_{ij}(\mathfrak{m}) \equiv i\binom{\mathfrak{m}}{j} = e(\mathfrak{m}). \tag{6}$$

By theorem 2 any equality of the form $f_\Gamma(\mathfrak{m}) = g_\Gamma(\mathfrak{m})$, where f and g are combinatorial, reduces to either an identity or to one of the predicates (5) or (6). Consequently we shall not pursue the notion of combinatorial function, but focus our attention on expressions built up from the R's and Q's.

$\mathfrak{S}^0$-LEMMA 15. *For any cardinal $\mathfrak{m} \notin \Delta$ having character $\langle \alpha, \beta\rangle$,* (i) $\mathrm{R}_{ijkl}(\mathfrak{m}) \equiv \big(j = l \wedge (i = k \vee \alpha \leqslant j)\big) \vee (\beta \leqslant j \wedge \beta \leqslant l)$, (ii) $\mathrm{Q}_{ij}(\mathfrak{m}) \equiv \beta \leqslant j$.

Let $\mathcal{A}$ be a language, the formulae of which are constructed from the various R's and Q's (all with the same $\mathfrak{m}$) by means of the sentential connectives, and the sentences of which are obtained by prefixing a formula by one of the restricted quantifiers $(\forall \mathfrak{m})_{\notin\Delta}$ or $(\exists \mathfrak{m})_{\notin\Delta}$. Let $\mathcal{B}$ be a language (with the same connectives) for the elementary theory of the model $\langle \omega+1, \leqslant\rangle$ containing constant symbols for each element of $\omega+1$. For every formula $\mathfrak{A}(\mathfrak{m})$ of $\mathcal{A}$, associate a formula $\overline{\mathfrak{A}}(\alpha, \beta)$ of $\mathcal{B}$ by replacing each occurrence of an $\mathrm{R}_{ijkl}(\mathfrak{m})$ or of a $\mathrm{Q}_{ij}(\mathfrak{m})$ in $\mathfrak{A}(\mathfrak{m})$ by the corresponding formula of $\mathcal{B}$ as given in lemma 15. Then

$\mathfrak{S}^0$-COROLLARY 1. *For any cardinal $\mathfrak{m} \notin \Delta$ having character $\langle \alpha, \beta\rangle$,* $\mathfrak{A}(\mathfrak{m}) \equiv \overline{\mathfrak{A}}(\alpha, \beta)$.

$\mathfrak{S}^0$-COROLLARY 2. $(\forall \alpha, \beta)_{1\leqslant\alpha\leqslant\beta\leqslant\omega}\overline{\mathfrak{A}}(\alpha, \beta) \to (\forall \mathfrak{m})_{\notin\Delta}\mathfrak{A}(\mathfrak{m})$.

$\mathfrak{S}^0$(ch)-COROLLARY 3. $(\forall \alpha, \beta)_{1\leqslant\alpha\leqslant\beta\leqslant\omega}\overline{\mathfrak{A}}(\alpha, \beta) \equiv (\forall \mathfrak{m})_{\notin\Gamma}\mathfrak{A}(\mathfrak{m})$.

THEOREM 3. (i) *The set of sentences of $\mathcal{A}$ which are theorems of $\mathfrak{S}^0$(ch) is a complete and decidable theory.* (ii) *The set of sentences of $\mathcal{A}$ which are theorems of $\mathfrak{S}^0$ is a decidable theory.*

Proof. By Corollaries 1, 2, 3 and the fact that the elementary theory of $\langle \omega+1, \leqslant\rangle$ is complete and decidable. Q.E.D.

6. Some examples.

The present work started with an attempt to decide whether Tarski's cancellation theorem (cf. (1) of section 1) could be strengthened to a cancellation law having the form

$$(\forall \mathfrak{m}, \mathfrak{n})\big(f_\Gamma(\mathfrak{m}) = f_\Gamma(\mathfrak{n}) \to \mathfrak{m} = \mathfrak{n}\big) \tag{7}$$

where f is some non-linear combinatorial function. As it stands (7) does not come under the heading of theorem 3, however the following immediate consequence of (7) does.

$$(\forall \mathfrak{m})\big(f_\Gamma(\mathfrak{m}) = f_\Gamma(2\mathfrak{m}) \to \mathfrak{m} = 2\mathfrak{m}\big). \tag{8}$$

If f_Γ reduces to $\dfrac{i}{k!}\mathfrak{m}^k$ on $\Gamma - \Delta$, where $k \geqslant 2$, then the hypothesis of (8) becomes $\dfrac{i}{k!}\mathfrak{m}^k = \dfrac{i2^k}{k!}\mathfrak{m}^k$. Thus any cardinal $\mathfrak{m}$ having character $\langle 2, 2\rangle$ will serve as a counterexample to (8). If f_Γ reduces to $e(\mathfrak{m})$ on $\Gamma - \Delta$, then the hypothesis of (8) becomes $e(\mathfrak{m}) = e(2\mathfrak{m}) = e(\mathfrak{m})^2$. Thus any cardinal $\mathfrak{m}$ having the character $\langle 2, 2\rangle$ will also serve as a counterexample to (8). Therefore with the single exception of the Tarski theorem, $\mathfrak{S}^0$ contains no other cancellation laws which can be framed as examples of (7).

Another interesting question is whether every idemmultiple cardinal is also idempotent. Clearly every cardinal of character $\langle 1, \omega\rangle$ will serve as a counterexample to this conjecture [6]. In conclusion, we hope that the reader will be able to use the notion of cardinal character to settle other interesting questions in the arithmetic of $\Gamma - \Delta$.

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[6] It has recently come to our attention that this question has previously been answered by A. Lévy (cf. [3]).

Note: In our definition of $\mathfrak{W}^+$ on page 251, add the following requirements. Besides being a dense unbordered ordering of K, $\prec$ must also be weakly homogeneous in the sense that for every $A, B \in E(K)$, $A \simeq B$, there exists a $\prec$-monotone permutation of K which maps A onto B. The existence of such a $\prec$ can easily be proved with the aid of the axiom of choice. This note also applies to [2].

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Несколько теорем о произведении топологических пространств

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Настоящая статья посвящена исследованию некоторых весовых характеристик топологических пространств и структуры отображений произведений топологических пространств. Ещё в 1949 году Есенин-Вольпин доказал в [11], что всякий диадический бикомпакт с первой аксиомой счётности метризуем. Отсюда следует, что если образ Y произведения дискретных двоеточий при непрерывном отображении удовлетворяет первой аксиоме счётности, то отображение можно разложить в композицию проекции произведения двоеточий на его грань, а именно, произведение счётного множества двоеточий, и некоторого отображения этой грани на пространство Y. Если мы уже заранее такое разложение осуществим, то метризуемость бикомпакта Y вытекает из того, что образ компакта при непрерывном отображении является компактом, а значит, метризуемым пространством.

Б. Ефимов в работах [9] и [12] дал некоторые достаточные условия, когда указанное выше разложение отображения возможно, а значить диадический бикомпакт метризуем. Задача об оценке веса образа произведения пространств разбивается на две части: 1) Найти условия разложения отображения в композицию проекции на грань меньшей мощности и отображения этой грани на образ, 2) Найти условия, чтобы вес образа не превосходил веса прообраза. Мы в данной статье займёмся первой задачей.

В настоящей статье находятся более широкие (почти окончательные в смысле, указанном ниже) достаточные условия, налагаемые на пространства X_α, чтобы отображение f произведения $\prod_{\alpha\in A} X_\alpha$ на пространство Y, локальный псевдовес которго не превосходит $\mathfrak{m}$, разлагалось в композицию двух отображений: проекции на грань мощности $\mathfrak{m}$ и отображения этой грани на пространство Y. Оказывается, таким свойством пространств X_α является наличие топологических калибров, введённых Шаниным в [4]. Если же хотя бы для одного из сомножителей X_α это свойство не имеет места, а локальный псевдовес пространства X_α не превосходит $\mathfrak{m}$, то можно построить отображение $\prod_{\alpha\in A} X_\alpha \to Y$, не разлагающееся в указанную композицию двух отображений, хотя локальный псевдовес пространства Y не превосходит $\mathfrak{m}$. В этом смысле достаточные условия являются почти необходимыми.