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Reçu par la Rédaction le 29. 6. 1964

Generalized Fréchet varieties

by

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Introduction. The theory of Fréchet varieties and specifically Fréchet surfaces has been a subject of considerable interest not only in its own right but because of its applications to the theory of length and area. Though progress in this area had been made for over sixty years the more recent work of Cesari, Federer, Rado [4] and Youngs [7] has given it a sense of completeness. In the latter work the representation problem is solved for Fréchet surfaces.

This paper deals with the representation problem for two new collections of spaces, called monovarieties and perivarieties, which are each generalizations of the notion of Fréchet variety. In Section 1 we state the representation problem for monovarieties and consider the general situation. In Section 2 we obtain a reduction theorem for some cases of monoequivalence analogous to Youngs' reduction theorem for Fréchet equivalence. Then we show that a monosurface is actually a Fréchet surface. In Section 3 we state the representation problem for perivarieties and consider the general situation. In Section 4 we obtain a reduction theorem for periequivalence. Then we show that a perisurface is also a Fréchet surface. So since the representation problem has been solved for Fréchet surfaces, we have a solution for monosurfaces and perisurfaces.

We attempt to parallel Youngs' notation and development of Fréchet varieties. All manifolds considered are compact and connected.

1. The representation problem for monovarieties.

Notation. If $f_n\colon X \to Y$ is a mapping where X and Y are metric, $n = 0, 1, 2, \ldots$, then the notation $f_n \Rightarrow f_0$ means that f_n *converges uniformly* to f_0; that is, if $\bar{\varrho}\{f_n, f_0\} = \sup \varrho\{f_n(x), f_0(x)\}$, $x \in X$, where ϱ is the distance function in Y, then $\bar{\varrho}\{f_n, f_0\} \to 0$. Also $f\colon X \Rightarrow Y$ means that f maps X onto Y. Throughout this paper $\mathfrak{X}$ will denote the class of Peano spaces (i.e., the class of locally connected (metric) continua) and $\mathfrak{F}$ the class of mappings $f\colon X \to Y$ where $X \in \mathfrak{X}$ and Y is metric. Whenever the

* This research was supported in part by NSF grant 24421; the author wishes to express his gratitude to Professor J. W. T. Youngs for his interest and assistance in the preparation of this paper.

letter X (possibly with subscript) is used to denote a space it is understood that $X \in \mathfrak{X}$.

DEFINITION 1.1. A mapping $f\colon A \to B$ is said to be *monotone* provided that, for each point b in B, the inverse $f^{-1}(b)$ is connected. Two spaces A and B are called *monomorphic* if each is the image of the other under a monotone mapping.

DEFINITION 1.2. A mapping $f_1\colon X_1 \to Y$ is said to be *Fréchet equivalent* to a mapping $f_2\colon X_2 \to Y$ (notation: $f_1 \sim^{\mathrm{F}} f_2$) if and only if there is a sequence, $\{k_n\}$, of homeomorphisms, $k_n\colon X_1 \Rightarrow X_2$, $n = 1, 2, \ldots$, such that $f_2 k_n \Rightarrow f_1$.

DEFINITION 1.3. A mapping $f_1\colon X_1 \to Y$ is said to be *monoequivalent* to a mapping $f_2\colon X_2 \to Y$ (notation: $f_1 \sim^{\mathrm{m}} f_2$) if and only if there are sequences, $\{g_n\}$, $\{h_n\}$, of monotone mappings, $g_n\colon X_1 \Rightarrow X_2$ and $h_n\colon X_2 \Rightarrow X_1$, $n = 1, 2, \ldots$, such that $f_2 g_n \Rightarrow f_1$ and $f_1 h_n \Rightarrow f_2$. Under these circumstances it will be said that there is an *approximate monomatching* between f_1 and f_2. In the event that there are monotone mappings $g_n\colon X_1 \Rightarrow X_2$ and $h_n\colon X_2 \Rightarrow X_1$ such that $\bar{\varrho}(f_1, f_2 g_n) = 0 = \bar{\varrho}(f_1 h_n, f_2)$, we say the monomatching is *exact*.

Note that the above definition is a weakening of Fréchet equivalence (actually the homeomorphisms k_n and k_n^{-1} required for Fréchet equivalence are replaced by monotone mappings g_n and h_n). In view of the fact that being monomorphic is an equivalence relation and standard techniques of double limits it follows that monoequivalence is also an equivalence relation over the class $\mathfrak{F}$ and so partitions it into mutually exclusive equivalence classes, $[f]$. Each equivalence class, $[f]$, will be called a *monovariety*, $\boldsymbol{M}$. Any representative of the equivalence class (that is, any mapping in $[f]$) is said to be a representative of the monovariety, $\boldsymbol{M}$.

It is now possible to state the *representation problem for monovarieties*: *Given one representation of a monovariety, find all of its representations.* The representation problem for monovarieties really asks for suitable criteria with which to test the validity of the statement $f_1 \sim^{\mathrm{m}} f_2$; any such criterion will be called a *M-criterion*.

If $f_1 \sim^{\mathrm{m}} f_2$ then even though one has control over the accuracy of the approximation it is not possible to obtain an exact monomatching. Youngs' example ([7], p. 7) with g_n being the homeomorphism $h_{\varepsilon(n)}$ described there with $\varepsilon(n) = 1/n$ and $h_n = g_n^{-1}$ shows that $f_1 \sim^{\mathrm{m}} f_2$, but the monomatching is not exact for this particular choice of g_n and h_n, nor is it for any other choice (since f_2 is a homeomorphism and $f_1 h$ cannot be a homeomorphism for any map $h\colon X_2 \Rightarrow X_1$).

This example shows that exact monomatching is not a necessary condition for monoequivalence. One necessary condition is that the spaces X_1 and X_2 be monomorphic. Another necessary condition is that

$f_1(X_1) = f_2(X_2)$. Both conditions are immediate consequence of the definition of monoequivalence.

Remark 1.1. If $\boldsymbol{M}$ is a monovariety with representatives $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ where X_1 and X_2 are n-manifolds, then the fact that X_1 and X_2 are monomorphic makes it possible to associate a class $\langle M \rangle$ of n-manifolds, M, each of which is monomorphic to the domain space of any representation of $\boldsymbol{M}$ whose domain space is an n-manifold.

It is possible to distinguish further between monovarieties, $\boldsymbol{M}$, in terms of the associated classes of manifolds, $\langle M \rangle$. In general, a monovariety, $\boldsymbol{M}$, is called a monovariety *of the type of an n-manifold B* if and only if each space (i.e. manifold) in $\langle M \rangle$ is monomorphic to B. Note that in some cases all the elements of $\langle M \rangle$ are homeomorphic. For example, if B is a 1-cell (1-sphere) then any nondegenerate monotone image of B is also a 1-cell (1-sphere). Thus a monovariety, $\boldsymbol{M}$, is called a *monocurve* if and only if $\langle M \rangle$ is the class of 1-cells or 1-spheres. Furthermore, if $\langle M \rangle$ is the class of 1-cells, then the monocurve, $\boldsymbol{M}$, is known as a monocurve of the type of a 1-cell; if $\langle M \rangle$ is the class of 1-spheres, then the monocurve, $\boldsymbol{M}$, is known as the monocurve *of the type of a 1-sphere*. A monovariety, $\boldsymbol{M}$, is called a *monosurface* if and only if $\langle M \rangle$ is the class of 2-manifolds which are monomorphic images of some 2-manifold B. Actually we will show in this case that all the elements of $\langle M \rangle$ are homeomorphic to B (see Theorem 2.4). This statement is not true if B is a higher dimensional manifold, nor if $\langle M \rangle$ is replaced by all the monomorphic images of B even when B is a closed 2-manifold. Moreover, we show in the next example that it is possible for $f_1 \sim^{\mathrm{m}} f_2$ and the domain of f_1 to be a closed 2-manifold and yet the domains of f_1 and f_2 are not homeomorphic.

EXAMPLE 1.1. Denote the sphere in euclidean 3-space with radius 1 and center at the origin by S. Then a point of S can also be described in terms of the spherical coordinates $\langle \varphi, \theta \rangle$, $0 \leqslant \varphi < 2\pi$, $0 \leqslant \theta \leqslant \pi$ (where φ is the geographical longitude, i.e., the angle between the xz-plane and the plane determined by the point and the z-axis, and θ is the polar distance, i.e., the angle between the radius to the point and the positive z-axis).

Suppose now that:

(1) $X_1 = S$,

(2) $X_2 = S \cup T$ where $T = \{(x, y, z) \mid x = y = 0,\ 1 \leqslant z \leqslant 2\}$,

(3) $Y = [0, 1]$ (the closed unit interval),

(4) $f_1\colon X_1 \Rightarrow Y$ is defined by $f_1(\langle \varphi, \theta \rangle) = t$ where $\theta = \pi t$, $0 \leqslant t \leqslant 1$,

(5) $f_2\colon X_2 \Rightarrow Y$ is defined by $f_2(\langle \varphi, \theta \rangle) = t$ where $\theta = \pi t$, $0 \leqslant t \leqslant 1$ (i.e., $(f_2|S) = f_1$), and $f_2[T] = 0$.

Then $f_1 \sim^{\mathrm{m}} f_2$ with respect to the monotone mappings $\{g_n\}$, $\{h_n\}$ defined as follows:

(1) g_n: $X_1 \Rightarrow X_2$ is given by

$$g_n(\langle\varphi, \theta\rangle) = \begin{cases} (0, 0, -2^n t+2) \in T, & 0 \leqslant t \leqslant 2^{-n}, \\ \langle\varphi, \lambda_n(t)\pi\rangle \in S, & 2^{-n} \leqslant t \leqslant 1 \end{cases}$$

where $\theta = \pi t$ and $\lambda_n(t) = (t-2^{-n})/(1-2^{-n})$, and

(2) $h_n = h$, for $n = 1, 2, 3, \ldots$, where h: $X_2 \Rightarrow X_1$ is given by $(h|S) =$ identity and $h[T] = \langle 0, 0\rangle$.

In one direction we have exact monomatching, i.e.,

$$\bar{\varrho}(f_1 h_n, f_2) = \bar{\varrho}(f_1 h, f_2) = 0\,.$$

While in the other direction we have

$$\begin{aligned} \bar{\varrho}(f_1, f_2 g_n) &= \sup\{\varrho(f_1(\langle\varphi, \theta\rangle), f_2 g_n(\langle\varphi, \theta\rangle)) \mid \langle\varphi, \theta\rangle \in X_1\} \\ &= \sup\{\varrho(f_1(\langle\varphi, \pi t\rangle), f_2 g_n(\langle\varphi, \pi t\rangle)) \mid 0 \leqslant \varphi < 2\pi,\ 0 \leqslant t \leqslant 1\} \\ &\leqslant \sup\{\varrho(t, f_2(0, 0, 2^n t+2)) \mid 0 \leqslant t \leqslant 2^{-n}\} \\ &\quad + \sup\{\varrho(t, f_2(\langle\varphi, \lambda_n(t)\pi\rangle)) \mid 2^{-n} \leqslant t \leqslant 1\} \\ &= \sup\{\varrho(t, 0) \mid 0 \leqslant t \leqslant 2^{-n}\} + \sup\{\varrho(t, \lambda_n(t)) \mid 2^{-n} \leqslant t \leqslant 1\} \\ &= 2^{-n} + 2^{-n} = 2 \cdot 2^{-n}. \end{aligned}$$

So $\bar{\varrho}(f_1, f_2 g_n) \to 0$ and we have that $f_1 \sim^{\mathrm{m}} f_2$.

2. Monoequivalence and monosurfaces. After stating some well-known results we obtain a reduction theorem for monoequivalence which reduces the problem of finding M-criteria for general mappings to that of finding M-criteria for monotone mappings. We then prove that a monosurface is actually a Fréchet surface.

DEFINITION 2.1. A mapping f: $A \to B$ is said to be *light* provided that, for each point b in B, the inverse image $f^{-1}(b)$ is totally disconnected (i.e., every component of $f^{-1}(b)$ is a single point if $f^{-1}(b)$ is not empty).

THEOREM 2.1 (Eilenberg-Whyburn Factor Theorem [6], p. 143). *If A is compact and f: $A \Rightarrow B$ is a mapping, then there exists a factorization $f = \lambda\mu$ such that μ: $A \Rightarrow \mathfrak{A}$ is monotone and λ: $\mathfrak{A} \Rightarrow B$ is light.*

In the above theorem $\lambda\mu$ is called the *monotone-light factorization of f*. The space $\mathfrak{A}$ is the *middle space* of the factorization. If $\lambda\mu$ is a monotone-light factorization of f it is usually referred to as being unique while what is meant is that it is unique up to a homeomorphism. That is, if $\lambda_1\mu_1$ and $\lambda_2\mu_2$ are monotone-light factorizations of f with middle spaces $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively, then there is a unique homeomorphism k: $\mathfrak{A}_1 \Rightarrow \mathfrak{A}_2$ such that $\mu_2 = k\mu_1$ and $\lambda_2 = \lambda_1 k^{-1}$ (actually $k = \mu_2\mu_1^{-1}$).

LEMMA 2.1 ([4], p. 47). *Suppose that $\{\mu_n\}$ is a sequence of mappings where μ_n: $X_1 \Rightarrow X_2$ and that λ: $X_2 \to X_3$ is a light mapping. If the sequence $\{\lambda\mu_n\}$ is equicontinuous on X_1, then the sequence $\{\mu_n\}$ is also equicontinuous on X_1.*

LEMMA 2.2 ([4], p. 55). *Suppose that $\{\mu_n\}$ is a uniformly convergent sequence of monotone mappings from X_1 onto X_2. Then the limit function of the sequence $\{\mu_n\}$ is a monotone mapping.*

THEOREM 2.2. *If two mappings f_1: $X_1 \to Y$ and f_2: $X_2 \to Y$ are monoequivalent, then there exist monotone-light factorizations, $\lambda\mu_1$ for f_1 and $\lambda\mu_2$ for f_2.*

Proof. By hypothesis there exist sequences of monotone mappings $\{g_n\}$ and $\{h_n\}$ such that $f_2 g_n \Rightarrow f_1$ and $f_1 h_n \Rightarrow f_2$. Replacing f_1 by its monotone-light factorization $\lambda\mu_1$ we have $\lambda\mu_1 h_n \Rightarrow f_2$. Therefore $\{\lambda\mu_1 h_n\}$ is equicontinuous, so by Lemma 2.1 we have that $\{\mu_1 h_n\}$ is equicontinuous also. Thus it contains a uniformly convergent subsequence. So renumbering if necessary and letting the limit mapping be μ_2 we have $\mu_1 h_n \Rightarrow \mu_2$ which is monotone by Lemma 2.2. From this it follows that $f_1 h_n = \lambda\mu_1 h_n \Rightarrow \lambda\mu_2$. But $f_1 h_n \Rightarrow f_2$ so $f_2 = \lambda\mu_2$ and we have the desired result.

THEOREM 2.3. *Two mappings f_1: $X_1 \to Y$ and f_2: $X_2 \to Y$ are monoequivalent if there are monotone-light factorizations $\lambda\mu_1$ for f_1 and $\lambda\mu_2$ for f_2 such that μ_1 and μ_2 are monoequivalent.*

Proof. Since $\mu_1 \sim^{\mathrm{m}} \mu_2$ there are sequences $\{g_n\}$ and $\{h_n\}$ of monotone mappings such that (1) g_n: $X_1 \Rightarrow X_2$ and h_n: $X_2 \Rightarrow X_1$ for each positive integer n; (2) $\mu_2 g_n \Rightarrow \mu_1$ and $\mu_1 h_n \Rightarrow \mu_2$. Thus $f_2 g_n = \lambda\mu_2 g_n \Rightarrow \lambda\mu_1 = f_1$ and $f_1 h_n = \lambda\mu_1 h_n \Rightarrow \lambda\mu_2 = f_2$. This implies that $f_1 \sim^{\mathrm{m}} f_2$.

LEMMA 2.3. *Suppose*: (1) $\{g_n\}$ *is a sequence of mappings of B onto C such that $g_n \Rightarrow g$*, (2) $\{k_n\}$ *is a sequence of mappings of A onto B such that $g_n k_n \Rightarrow \mu$. Then $g k_n \Rightarrow \mu$.*

Now as a partial converse to Theorem 2.3 we have the following

THEOREM 2.4. *If two mappings f_1: $X_1 \to Y$ and f_2: $X_2 \to Y$ are monoequivalent with respect to $\{g_n\}$ and $\{h_n\}$ and $g_n h_n \Rightarrow r$, a homeomorphism, then there are monotone-light factorization $\lambda\mu_1$ for f_1 and $\lambda\mu_2$ for f_2 such that μ_1 and μ_2 are monoequivalent.*

Proof. Since $f_1 \sim^{\mathrm{m}} f_2$ with respect to the sequences $\{g_n\}$ and $\{h_n\}$ of monotone mappings, we have (1) g_n: $X_1 \Rightarrow X_2$ and h_n: $X_2 \Rightarrow X_1$ for each positive integer n; (2) $f_2 g_n \Rightarrow f_1$ and $f_1 h_n \Rightarrow f_2$. Choose any monotone-light factorization $\lambda\mu_2$ of f_2 and denote the middle space by $\mathfrak{X}$. Then we have $\lambda\mu_2 g_n \Rightarrow f_1$ and so $\{\lambda\mu_2 g_n\}$ is equicontinuous. So applying Lemma 2.1 we have that $\{\mu_2 g_n\}$ is also equicontinuous and, therefore, has a convergent subsequence $\{\mu_2 g_n\}$ whose limit μ_1 is a monotone map of X_1 onto $\mathfrak{X}$ such that $\lambda\mu_1 = f_1$. Now let $k_n = h_n r^{-1}$ where r is the homeomorphism to which the sequence $\{g_n h_n\}$ converges. Since $\mu_2 g_n \Rightarrow \mu_1$ and $\mu_2 g_{n_i} k_{n_i} \Rightarrow \mu_2$ by Lemma 2.3 we have that $\mu_1 k_{n_i} \Rightarrow \mu_2$. Therefore $\mu_1 \sim^{\mathrm{m}} \mu_2$ with respect to $\{g_{n_i}\}$ and $\{k_{n_i}\}$.

CONJECTURE. *Theorem 2.4 remains true if one deletes "$g_n h_n \Rightarrow r$, a homeomorphism".*

DEFINITION 2.2. If M is a topological space and p is a point of M such that $M-p$ is not connected, then p is called a *cut point* of M. A topological space is said to be *cyclic* if it has no cut points. A point p of M is said to be a *local cut point* of M provided there exists an open set U, $p \in U$, such that for each open set V, $p \in V \subset U$, p is a cut point of V. (Note that the cut points of a connected space are also local cut points.) A Peano space is said to be *locally cyclic* if it has no local cut points.

DEFINITION 2.3. A *generalized cactoid* is a Peano space whose every maximal cyclic element (see [6]) is a closed 2-manifold and only a finite number of these are different from 2-spheres. A *mantoid* is a monotone continuous image of a closed 2-manifold.

LEMMA 2.4 ([5], p. 854). *The class of mantoids is the class of Peano spaces each of which can be obtained from a generalized cactoid by making a finite number of 2-point identifications.*

LEMMA 2.5. *If A and B are monomorphic closed 2-manifolds, then they are homeomorphic.*

Proof. Suppose that g: $A \Rightarrow B$ is monotone and let $b' = g^{-1}(b)$, $b \in B$. By Theorem 4 of [5] each component of $A-b'$, $b \in B$ has just one cylinder of approach to b'. Since B has no cut points, $A-b'$ is connected. Therefore, Theorem 2 of [5] applies and we have $R^1(B) \leqslant R^1(A)$ (where $R^n(Y)$ denotes the (mod 2) n-dimensional Betti number of Y) and if A is orientable so is B. Now consider A to be the monotone image of B and we get that $R^1(A) \leqslant R^1(B)$ and if B is orientable so is A. Therefore, $R^1(A) = R^1(B)$ and either A and B are both orientable or both nonorientable. This implies that A and B are homeomorphic.

LEMMA 2.6. *If A and B are monomorphic 2-manifolds with boundary, then they are homeomorphic.*

Proof. Suppose that A has boundary curves $A_1, \ldots, A_m$ and g: $A \Rightarrow B$ is monotone. Now as in [7], p. 15, let A' be the closed 2-manifold obtained by capping the boundary curves with open 2-cells $C_1, \ldots, C_m$. It is possible to adjoin an open 2-cell E_i to $g[A_i]$, $i = 1, \ldots, m$, in such a way that (1) $B^* = B \cup (\bigcup_{i=1}^{m} E_i)$ is a Peano space, (2) the frontier of E_i is $g[A_i]$, $i = 1, \ldots, m$, and (3) $E_i \cap E_j = \emptyset$; $i \neq j$; $i, j = 1, \ldots, m$. Moreover, there exists a mapping g': $A' \Rightarrow B^*$ such that $(g'|A) = g$ and $x = g'^{-1}g'(x)$ for $x \in A'-A$. Now B^* being a monotone image of a closed 2-manifold is by Lemma 2.4 a mantoid. Since B is a 2-manifold with boundary, it must be contained in a maximal locally cyclic element B' of B^*. Now $B' = B \cup (\bigcup_{i=1}^{n} D_i)$ where D_i is a 2-cell capping B_i, $i = 1, \ldots, n$. There is a monotone retraction of B^* onto B' (just send a component of B^*-B' to the point in B' which is its frontier and leave the points of B' fixed). Therefore we have $R^1(A') \geqslant R^1(B')$ and if A' is orientable so is B'. Similarly we get $R^1(B') \leqslant R^1(A')$ and if B' is orientable so is A'. Thus $R^1(A') = R^1(B')$ and either A' and B' are both orientable or they are both nonorientable. Therefore A' and B' are homeomorphic by the fundamental theorem of the topology of closed 2-manifolds. Now letting $\chi(X)$ denote the Euler characteristic of a polyhedron X, we have $\chi(A) = \chi(A') - m$ and $\chi(B) = \chi(B') - n$. Since $\chi(A') = \chi(B')$ it follows that $m - n = \chi(B) - \chi(A)$. Recall that the Euler-Poincaré formula holds for all fields. Then since (1) $R^0(A) = 1 = R^0(B)$, (2) $R^1(A) = R^1(B)$ (since there is a homomorphism taking $H_1(A, Z_2)$ onto $H_1(B, Z_2)$ and *vice versa* by the Vietoris mapping theorem), (3) $R^2(A) = 0 = R^2(B)$, we have that $m - n = 0$, i.e., A and B have the same number of boundary curves. From this and the previously determined structure of A and B it follows that A and B are homeomorphic.

THEOREM 2.5. *If A and B are monomorphic 2-manifolds, then they are homeomorphic.*

Proof. Assume that A is a closed 2-manifold. Then so is B since it is a generalized cactoid. Therefore Lemma 2.5 applies and A and B are homeomorphic. Now assume that A is a 2-manifold with boundary. Then B must also be a 2-manifold with boundary, otherwise A being a generalized cactoid would be a closed 2-manifold. Therefore Lemma 2.6 applies and A and B are homeomorphic.

The 1-dimensional case of the following is to be found in [4], p. 67-68, and the 2-dimensional case in [8].

THEOREM 2.6. *Suppose that g: $A \Rightarrow B$ is a monotone mapping of a 1-manifold or a 2-manifold A onto B where B is homeomorphic to A. Then there exists a sequence of homeomorphisms $\{k_m\}$ such that $k_m \Rightarrow g$.*

THEOREM 2.7. *Suppose that X_1 and X_2 are 1-manifolds or 2-manifolds. Then f_1: $X_1 \to Y$ and f_2: $X_2 \to Y$ are monoequivalent if and only if they are Fréchet equivalent.*

Proof. Obviously Fréchet equivalence implies monoequivalence. Consider the converse. Now X_1 and X_2 are homeomorphic by Theorem 2.5 in the 2-dimensional case and by well-known results on the nondegenerate monotone images of 1-cells and 1-spheres in the 1-dimensional case. Let $\{g_n\}$ be a sequence of monotone mappings such that $g_n: X_1 \Rightarrow X_2$ and $f_2 g_n \Rightarrow f_1$. Now by Theorem 2.6, for each n, there exists a sequence of homeomorphisms $\{k_m^n\}$, $m = 1, 2, 3, \ldots$ such that $k_m^n \Rightarrow g_n$. So $f_2 k_m^n \Rightarrow f_2 g_n$, for each n, and since $f_2 g_n \Rightarrow f_1$, we can choose a subsequence $\{k_m^n\}$, call it $\{k_i\}$, such that $f_2 k_i \Rightarrow f_1$. Therefore f_2 is Fréchet equivalent to f_1.

3. The representation problem for perivarieties. After some definitions we state the representation problem for perivarieties and consider the general situation with respect to these classes.

DEFINITION 3.1. The *norm* of a function $f: X \to Y$ is the sup $\{\operatorname{diam} f^{-1}(y) | y \in Y\}$. An *ε-mapping* of a metric space A onto B is a continuous function, f, of A onto B of norm less than ε. Two spaces A and B are defined to be *quasihomeomorphic* if, for each $\varepsilon > 0$, there exist an ε-mapping of A onto B and an ε-mapping of B onto A. If these ε-mapping of A onto B and B onto A are $\{\varphi_\varepsilon\}$ and $\{\omega_\varepsilon\}$, respectively, then we say that A and B are quasihomeomorphic with respect to $\{\varphi_\varepsilon\}$ and $\{\omega_\varepsilon\}$. If A and B are quasihomeomorphic with respect to ε-mappings $\{\varphi_\varepsilon\}$ and $\{\omega_\varepsilon\}$ and in addition these mappings are all *monotone*, then we say that A and B are *perimorphic* (with respect to $\{\varphi_\varepsilon\}$ and $\{\omega_\varepsilon\}$) or B a *perimorphic image of* A. (Note that two spaces can be monomorphic and quasihomeomorphic without being perimorphic (see Example 4.2).)

DEFINITION 3.2. A mapping $f_1: X_1 \to Y$ is said to be *periequivalent* to a mapping $f_2: X_2 \to Y$ (notation: $f_1 \sim^{\mathrm{P}} f_2$) if and only if f_1 is monoequivalent to f_2 with respect to sequences of monotone mappings $\{\varphi_n\}$, $\{\omega_n\}$ such that φ_n and ω_n are each ε_n-mappings and $\varepsilon_n \to 0$. Under these circumstances it will be said that there is an *approximate perimatching* between f_1 and f_2. If the event that the monotone ε_n-mappings φ_n: $X_1 \Rightarrow X_2$ and ω_n: $X_2 \Rightarrow X_1$ are such that $\bar{\varrho}(f_1, f_2\varphi_n) = 0 = \bar{\varrho}(f_1\omega_n, f_2)$, we say that the perimatching is *exact*.

Note that the above definition is a weakening of Fréchet equivalence (actually the homeomorphisms, k_n and k_n^{-1}, required for Fréchet equivalence are replaced by monotone ε_n-mappings φ_n and ω_n with $\varepsilon_n \to 0$).

In view of compactness the relation $\sim^{\mathrm{P}}$ is easily seen to be an equivalence relation over the class $\mathfrak{F}$ and so partitions it into mutually exclusive equivalence classes, $[f]$. Each equivalence class $[f]$ will be called a *perivariety*, **P**. Any representative of the equivalence class (that is, any mapping in $[f]$) is said to be a *representative of the perivariety*, **P**. We have a representation problem as in the case of monovarieties.

If $f_1 \sim^{\mathrm{P}} f_2$, then even though one has control over the accuracy of the approximation it is not possible to obtain an exact perimatching. Youngs' example cited in Section 1 and the argument given there show that this is the case.

The example just mentioned also shows that exact perimatching is not a necessary condition for periequivalence. One necessary condition is that the spaces X_1 and X_2 be perimorphic. Another necessary condition is that $f_1(X_1) = f_2(X_2)$. Both conditions are immediate consequences of the definition of periequivalence.

Remark 3.1. If **P** is a perivariety with representatives f_1: $X_1 \to Y$ and f_2: $X_2 \to Y$, then the fact that X_1 and X_2 are perimorphic makes it possible to associate a class $[P]$ of Peano spaces, P, each of which is perimorphic to the domain space of any representation of **P**.

It is possible to distinguish further between perivarieties, **P**, in terms of the associated classes, $[P]$. In general, a perivariety, **P**, is called a perivariety *of the type of a B* if and only if each space in $[P]$ is perimorphic to B. Note that in some cases all the elements of $[P]$ are homeomorphic. For example, this is the case if B is a 1-cell or a 1-sphere. Thus a perivariety, **P**, is called a *pericurve* if and only if $[P]$ is the class of 1-cells or 1-spheres. Furthermore, if $[P]$ is the class of 1-cells, then the pericurve, **P**, is known as a pericurve *of the type of a 1-cell*; if $[P]$ is the class of 1-spheres, then the pericurve, **P**, is known as the pericurve *of the type of a 1-sphere*. A perivariety, **P**, is called a *perisurface* if and only if $[P]$ is the class of perimorphic images of a 2-manifold. Actually we will show (Theorem 4.1) that all the elements of $[P]$ are homeomorphic in this case also. It is an open question whether this situation holds for higher dimensional manifolds.

4. Periequivalence and perisurfaces. In this section we show that if A is perimorphic to B a 1-manifold or a 2-manifold, then A and B are homeomorphic. We then obtain our main result which is as follows:

Suppose that X_1 is a 1-manifold or a 2-manifold. Then f_1: $X_1 \to Y$ and f_2: $X_2 \to Y$ are periequivalent if and only if they are Fréchet equivalent.

Note that while it was necessary in the corresponding theorem for monoequivalence to require that X_2 also be a manifold of dimension 1 or 2 this is not the case here.

LEMMA 4.1. *Suppose that there are monotone ε-mappings of X_1 onto X_2, for all $\varepsilon > 0$. If X_2 is locally cyclic, then so is X_1.*

Proof. Suppose not, i.e., there is a connected open subset U of X_1 and a point $p \in U$ such that $U - p = U_1 \cup U_2$ where U_1 and U_2 are mutually separated and each has positive diameter. Now there is a positive number ε such that $\varepsilon < \min(\operatorname{diam} U_1, \operatorname{diam} U_2)$ and the ε-neighborhood of p is contained in U. Consider a monotone ε-map g: $X_1 \Rightarrow X_2$. There is an $\eta > 0$ such that if C is a subset of X_2 and the diam $C < \eta$, then diam $g^{-1}[C] < \varepsilon$. Since X_2 is a Peano space, there is a connected open set V in X_2 which contains $g(p)$ and whose diameter is less than η. Thus diam $g^{-1}[V] < \varepsilon$ and, since $p \in g^{-1}[V]$, we have $g^{-1}[V] \subset U$. By hypothesis $g(p)$ is not a cut point of V, and so $V-g(p)$ is connected. Hence since g is monotone we have that $g^{-1}[V-g(p)] = g^{-1}[V]-g^{-1}g(p)$ is also connected. Therefore, $g[U_1] = g(p)$ or $g[U_2] = g(p)$, otherwise $g^{-1}[V]-g^{-1}g(p)$ would contain points of U_1 and U_2 which would contradict the fact that it is connected. So norm $g \geqslant \min(\operatorname{diam} U_1, \operatorname{diam} U_2)$, but this is a contradiction.

THEOREM 4.1. *If A is perimorphic to B a 1-manifold or a 2-manifold, then A and B are homeomorphic.*

Proof. Since any nondegenerate monotone continuous image of a 1-cell (1-sphere) is a 1-cell (1-sphere), the 1-dimensional case is immediate. Now the 2-dimensional case splits into two parts. The first treats the case when B is closed and the second when B has a boundary.

Suppose that B is a closed 2-manifold. Then A is a mantoid which by Lemma 4.1 is locally cyclic and hence must be a closed 2-manifold. Now, since A and B are monomorphic 2-manifolds, we can apply Theorem 2.5 to get that they are homeomorphic.

Suppose that B is a 2-manifold with boundary curves $B_1, \ldots, B_n$ (each B_i is a 1-sphere). We will first show that, for ε sufficiently small, any monotone ε-map $m: B \Rightarrow A$ is such that $(m|B_i)$ is also monotone for $i = 1, \ldots, n$. Now each B_i has an η_i-neighborhood, N_i, which is an annulus. Let $\delta_i = \operatorname{diam} B_i$ and $\gamma_i = (1/2)\varrho(B_i, B^\bullet - B_i)$, where $B^\bullet$ denotes the boundary of B. Finally, let $\eta = \min(\eta_1, \ldots, \eta_n, \delta_1, \ldots, \delta_n, \gamma_1, \ldots, \gamma_n)$ and for the rest of the proof assume that $\varepsilon < \eta$, $m: B \Rightarrow A$ is a monotone ε-map and $\mu_i = (m|B_i)$.

To prove that each μ_i is monotone we will show that if $a, b \in \mu_i^{-1}(y)$, $y \in A$, then one of the two arcs α_1, α_2 in B_i from a to b is also contained in $\mu_i^{-1}(y)$. Suppose that this is not the case. Then the continuum $K = m^{-1}(y)$ is a cutting of N_i (i.e., $N_i - K$ is the union of two mutually separated sets N_i^1 and N_i^2). It follows that $A - y = m[N_i^1] \cup m[N_i^2]$ where $m[N_i^1]$ and $m[N_i^2]$ are mutually separated and hence y is a cut point of A. But this is impossible since A has no local cut points by Lemma 4.1. Hence each μ_i must be monotone. Moreover, since $\varepsilon < \operatorname{diam} B_i$ we have that $\mu_i[B_i]$ is nondegenerate and so must be a 1-sphere.

Now as in [7], p. 15, let B' be the closed 2-manifold obtained by capping the boundary curves $B_1, \ldots, B_n$ with open 2-cells $D_1, \ldots, D_n$. It is possible to adjoin an open 2-cell C_i to $m[B_i]$, $i = 1, \ldots, n$ in such a way that (1) $A' = A \cup (\bigcup_{i=1}^{n} C_i)$ is a Peano space, (2) the frontier of C_i is the 1-sphere $m[B_i]$, $i = 1, \ldots, n$, and (3) $C_i \cap C_j = \emptyset$; $i \neq j$; $i, j = 1, \ldots, n$. Moreover, there exists a mapping $m': B' \Rightarrow A'$ such that $(m'|B) = m$ and $x = m'^{-1}m(x)$ for $x \in B' - B$. Thus A' is a mantoid and since A is locally cyclic so is A'. Hence A' is a closed 2-manifold. Therefore A and B are both 2-manifolds with boundary and being monomorphic they are homeomorphic by Theorem 2.5.

THEOREM 4.2. *Suppose that X_1 is a 1-manifold or a 2-manifold. Then $f_1: X_1 \to Y$ and $f_2: X_2 \to Y$ are periequivalent if and only if they are Fréchet equivalent.*

Proof. Obviously Fréchet equivalence implies periequivalence. Consider the converse. By Theorem 2.5, X_1 and X_2 are homeomorphic. Let $\{\varphi_n\}$ be a sequence of monotone ε_n-mappings with $\varepsilon_n \to 0$, $\varphi_n: X_1 \Rightarrow X_2$ and $f_2\varphi_n \Rightarrow f_1$. Now by Theorem 2.6, for each n, there exists a sequence of homeomorphisms $\{k_m^n\}$, $m = 1, 2, \ldots$, such that $k_m^n \Rightarrow \varphi_n$ (over m). So $f_2 k_m^n \Rightarrow f_2\varphi_n$, for each n, and since $f_2\varphi_n \Rightarrow f_1$, we can choose a subsequence of $\{k_m^n\}$, call it $\{k_i\}$, such that $f_2 k_i \Rightarrow f_1$. Therefore f_2 is Fréchet equivalent to f_1.

EXAMPLE 4.1. In this example we construct two Peano spaces, X and Y, which are perimorphic but not homeomorphic. For each positive integer i, let T_i denote the solid triangle in the plane with vertices $(1/2^i, 0)$, $(1/2^{i-1}, 0)$ and $(3/2^{i+1}, \sqrt{3}/2^{i+1})$ and A_i denote the segment in euclidean 3-space from $a_i = (3/2^{i+1}, \sqrt{3}/2^{i+2}, 0)$ to $(3/2^{i+1}, \sqrt{3}/2^{i+2}, 1/i)$. We define X and Y as follows:

$$X = \bigcup_{i=1}^{\infty} X_i \cup (0,0,0), \text{ where } X_i = T_i \text{ for } i = 1, 2 \text{ and } X_i = T_i \cup A_i \text{ for } i > 2;$$

$$Y = \bigcup_{i=1}^{\infty} Y_i \cup (0,0,0), \text{ where } Y_1 = T_1 \text{ and } Y_i = T_i \cup A_i \text{ for } i \geqslant 2.$$

Now X and Y are clearly not homeomorphic and so we proceed to construct the necessary monotone ε-maps to show that they are perimorphic. For any $\varepsilon > 0$, we will define a monotone ε-map $\varphi_\varepsilon: X \Rightarrow Y$ as follows. Let D be a circular disc in the interior of X_2 with center p and radius $r < \varepsilon/2$. Let C_t denote the circle with center p and radius t. Then $D = \bigcup\{C_t | 0 \leqslant t \leqslant r\}$. Now φ_ε is defined to be (1) identity map on $X - X_2$, (2) on X_2 a map which sends D onto A_2 by sending C_t to $(3/2^{i+1}, \sqrt{3}/2^{i+2}, (r-t)/2r)$ and (3) a homeomorphism of $X_2 - D$ onto $Y_2 - a_2$ which is the identity map on the boundary of X_2.

To obtain a monotone ε-map $\omega_\varepsilon: Y \Rightarrow X$, for any ε, $0 \leqslant \varepsilon \leqslant 1/4$, let ω_ε be a homeomorphism of Y_i onto X_{i+1} sending $(1/2^i, 0)$ to $(1/2^{i+1}, 0)$ and $(1/2^{i-1}, 0)$ to $(1/2^i, 0)$ for $i \geqslant 2$. Now let K be the segment joining $((1+\varepsilon)/2, 0)$ to $(3/4, \sqrt{3}/4)$ and L the segment joining $((1+\varepsilon)/2, 0)$ to $((2+\varepsilon)/4, \sqrt{3}\varepsilon/4)$. Then ω_ε on Y_1 is defined to be the vertical projection of the points above $K \cup L$ onto $K \cup L$ and the identity elsewhere on Y_1

followed by a homeomorphism onto $X_1 \cup X_2$ which sends $(1/2, 0)$ to $(1/4, 0)$, $((1+\varepsilon)/2, 0)$ to $(1/2, 0)$ and $(1, 0)$ to $(1, 0)$.

EXAMPLE 4.2. The purpose of this example is to show that Theorem 4.1 is false if instead of requiring A and B to be perimorphic we only require that they be quasihomeomorphic with respect to $\{\varphi_\varepsilon\}$ and $\{\omega_\varepsilon\}$ and that each ω_ε be monotone. It also shows that two spaces which are monomorphic and quasihomeomorphic need not be perimorphic.

Let T_1 and T_2 be the solid triangles described in the previous example. Let $A = T_1 \cup T_2$, $B = T_1$ and let ω_ε: $B \Rightarrow A$ be the map $(\omega_\varepsilon | T_1)$ defined in the previous example. Now to obtain φ_ε: $A \Rightarrow B$, for any (x, y) in A, let

$$k_1(x, y) = \begin{cases} (x, y), & \text{if} \quad (2+\varepsilon)/4 \leqslant x \leqslant 1, \\ ((2+\varepsilon)/4, y), & \text{if} \quad (2-\varepsilon)/4 \leqslant x \leqslant (2+\varepsilon)4, \\ (x+\varepsilon/2, y), & \text{if} \quad 1/4 \leqslant x \leqslant (2-\varepsilon)/4 \end{cases}$$

and let k_2: $k_1[A] \Rightarrow B$ be a homeomorphism. Finally φ_ε is defined by $\varphi_\varepsilon = k_2 k_1$. Note that the φ_ε are not monotone and indeed by Lemma 4.1 it is not possible to find monotone ε-mappings, for all $\varepsilon > 0$, of A onto B.

Kuratowski and Ulam [3] asked whether or not two quasihomeomorphic n-manifolds are homeomorphic. We ask a more restrictive question, i.e., are two perimorphic n-manifolds homeomorphic? For $n = 1$ or 2 we showed that the answer to the latter question is in the affirmative (actually we only needed to assume one of the spaces was a 1-manifold or a 2-manifold). Thus for a generalization of the lower dimensional results the question should be as follows: If A is perimorphic to an n-manifold B, are A and B homeomorphic? (Borsuk [1] has shown that there is a non-(absolute neighborhood retract) which is quasihomeomorphic to a 3-cell.) It seems necessary to have an approximation theorem of the type of Theorem 2.6 for n-manifolds to generalize Theorems 2.7 and 4.2. This problem is rather well known.

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Reçu par la Rédaction le 27. 10. 1964