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Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*

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Introduction. The purpose of this paper is to obtain conditions for the validity of statements on interpolation between the L^1 and the L^∞ of a measure space, and for analogous statements under the hypothesis of the theorem of Marcinkiewicz. To describe our aim more precisely, let us discuss briefly some basic notions concerning interpolation of linear operations. Given a topological vector space V and two Banach spaces A_1 and A_2 which are contained and continuously embedded in V, we will call the pair (A_1, A_2) an *interpolation pair*. The space A_1+A_2 consisting of elements of V of the form $x+y$ with $x\in A_1$ and $y\in A_2$ with the norm $\|x+y\| = \inf(\|x\|_{A_1}+\|y\|_{A_2})$ is also a Banach space and its embedding in V is continuous. Given two interpolation pairs (A_1, A_2) and (B_1, B_2), a linear mapping $T: A_1+A_2 \to B_1+B_2$ will be called *admissible* if it maps A_j continuously into B_j, $j = 1, 2$. The largest of the corresponding norms will be called the *norm of the admissible mapping* T. The class of admissible mappings with this norm is a Banach space. Given two Banach spaces A and B contained and continuously embedded in A_1+A_2 and B_1+B_2 respectively, we will say that A and B are *associated* if every admissible mapping T maps A into B. It is a consequence of the closed graph theorem that T does so continuously. If $A_j = B_j$, $j = 1, 2$, and A is associated with itself, we will say that A is *intermediate* between A_1 and A_2. If in addition every admissible T of norm 1 maps A into A with norm less than or equal to 1, A will be said to be *strictly intermediate* between A_1 and A_2. Every intermediate space can be renormed so as to become strictly intermediate. A pair of associated spaces A and B will be called *optimal* if whenever A' and B' are associated and $A \subset A'$, $B \supset B'$ it follows that $A = A'$ and $B = B'$. In other words, if the pair of associated spaces A and B is optimal, the statement that every admissible T maps A into B cannot be strengthened by either enlarging A or making B smaller. According to a result of N. Aronszajn if the pair A, B is optimal, then A is intermediate between A_1 and A_2 and B is intermediate between B_1 and B_2. This result of N. Aronszajn says even more,

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namely, if A is continuously embedded in A_1+A_2, then among the spaces that are continuously embedded in B_1+B_2 there is a minimal one B such that A and B are associated, and B is intermediate between B_1 and B_2 and conversely, given a B continuously embedded in B_1+B_2 there exists a maximal A among those spaces which are continuously embedded in A_1+A_2 such that A and B are associated, and A is intermediate between A_1 and A_2. The theorem of Marcinkiewicz, in its sharper version due to E. M. Stein and G. Weiss (see [8]) considers, except in one extreme case, two interpolation pairs consisting of Lorentz spaces and asserts that certain pairs of spaces are associated. Our aim is to obtain conditions, and in certain cases necessary and sufficient conditions, for pairs of spaces to be associated, and for associated pairs to be optimal. In view of what has been said above, it is clear that one need only consider spaces which are intermediate between Lorentz spaces, and since these are intermediate between L^1 and L^∞, such spaces are also intermediate between L^1 and L^∞. Thus, our first task will be to characterize the spaces intermediate between the L^1 and the L^∞ of a measure space.

1. Spaces intermediate between L^1 and L^∞. In what follows we shall consider totally σ-finite measure spaces $\mathscr{M}$ and the spaces V of equivalence classes of real valued measurable functions on $\mathscr{M}$. The equivalence relation here is that of coincidence almost everywhere. When speaking about functions we shall not distinguish between equivalent ones, so that, strictly speaking all our statements will be about equivalence classes rather than functions. With the topology of convergence in measure on sets of finite measure V becomes a metric vector space in which $L^1(\mathscr{M})$ and $L^\infty(\mathscr{M})$ are continuously embedded. We shall make systematic use of the functions f^* and f^{**} associated with a measurable function f. The function f^* is defined as follows: given a measurable f with the property that the set where $|f|>\lambda$ has finite measure for λ sufficiently large, $f^*(t)$ is the unique non-negative non-increasing left-continuous function on $0<t<\infty$ such that the sets where $|f|>\lambda$ and $f^*>\lambda$ have the same measure for all $\lambda>0$. The function f^* is usually called the *non-increasing rearrangement* of the function f. The following properties of the functions f^* are readily verified:

i) if E is the set where $|f|>\lambda>0$ and s is its measure, then

$$\int_E |f|\,d\mu = \int_0^s f^*(t)\,dt$$

where μ is the measure of the space on which f is defined; if E is any set and s is its measure, then

$$\int_{E'} |f|\,d\mu \leqslant \int_E |f|\,d\mu = \int_0^s f^*(t)\,dt;$$

ii) if $|f_1|\leqslant|f_2|$, then $f_1^*\leqslant f_2^*$;

iii) if $|f_n|\leqslant|f|$ and $f_n\to f$ in measure on sets of finite measure, then $f_n^*\to f^*$ at all points of continuity of f^*.

If the function f is integrable on sets of finite measure, then f^* is integrable on every finite interval and we define

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds.$$

The following properties of the functions f^{**} are important and will be used later:

i) $f^*\leqslant f^{**}$;

ii) $|f_1|\leqslant|f_2|$ implies $f_1^{**}\leqslant f_2^{**}$;

iii) $(f+g)^{**}\leqslant f^{**}+g^{**}$;

iv) if $f_n\to f$ in measure on sets of finite measure, then $f^{**}\leqslant \underline{\lim} f_n^{**}$;

v) if f_n converges to f with respect to the norm of $L^1(\mathscr{M})+L^\infty(\mathscr{M})$, then $tf_n^{**}(t)\to tf^{**}(t)$ uniformly on every finite interval.

Properties i) and ii) need no explanation. To see that the remaining ones are valid, we assume first that the space on which the functions under consideration are defined is non-atomic. In this case the function f^{**} can also be defined as

$$f^{**}(t) = \frac{1}{t}\sup_E \int_E |f|\,d\mu$$

where the supremum is taken over all measurable sets in $\mathscr{M}$ of measure no larger than t. This is not difficult to see and the proof is left to the reader. With this definition of f^{**}, iii) is immediate, iv) follows from Fatou's lemma and v) follows from the fact that under the given assumptions $f_n=f+g_n+h_n$ where $g_n\to 0$ in $L^1(\mathscr{M})$ and $h_n\to 0$ uniformly. If the measure space under consideration has atoms, we replace each of its atoms E by a non-atomic measure space E' with $\mu(E)=\mu(E')$ and each function $f(x)$ by a new function $\bar f(x)$ on the measure space thus obtained, where $\bar f(x)=f(x)$ if x does not belong to any E' and $\bar f(x)=a$ if $x\in E'$ and $f=a$ almost everywhere in E. Then we clearly have $f^*=\bar f^*$ for every f and consequently also $f^{**}=\bar f^{**}$, and the general case is thus reduced to the non-atomic case.

LEMMA 1. *Let $f(t)\geqslant 0$ and $g(t)\geqslant 0$ be non-increasing functions defined on the set R^+ of positive reals. Suppose that $g(t)$ is simple, integrable and that $f^{**}(t)>g^{**}(t)$ for all $t>0$. Then there exist finitely many measure preserving transformations ψ_i of R^+ and a convex linear combination T of the induced linear transformations T_i of $L^1(R^+)+L^\infty(R^+)$ such that $Tf\geqslant g$.*

Proof. It is not difficult to see that under the given assumptions there exist a positive ε and two functions $\bar{f}$ and $\bar{g}$ with the following properties: they are non-increasing, $0 \leqslant \bar{f} \leqslant f$, $\bar{g} \geqslant g$, $\bar{g}(t) = 0$ for sufficiently large t,

$$\int_0^t \bar{f}(s)\,ds > \int_0^t \bar{g}(s)\,ds, \quad t > 0,$$

and $\bar{f}$ and $\bar{g}$ are constant on each of the intervals $I_k = \big((k-1)\varepsilon, k\varepsilon\big)$, $k = 1, 2, \ldots$ Let now $n \geqslant 0$ be an integer with the following property: there exist measure preserving transformations ψ_i of R^+ permuting the intervals I_k and a finite convex linear combination T_n of the induced linear transformations of functions such that $T_n\bar{f} = \bar{g}$ for $t \leqslant n\varepsilon$ and

$$\int_0^t T_n\bar{f}\,ds \geqslant \int_0^t \bar{g}\,ds$$

for all t. It is clear that $T_n\bar{f}$ is constant on each I_k and that $T_n\bar{f} \geqslant \bar{g}$ on I_{n+1}, for otherwise the inequality between the integrals above would be violated in this interval. Suppose now that $T_n\bar{f} \geqslant \bar{g}$ for all t, then, since $f \geqslant \bar{f}$ and $g \leqslant \bar{g}$, we have $T_n f \geqslant g$ and T_n has the properties required by our lemma. If, on the other hand, $T_n\bar{f} < \bar{g}$ for some $t > n\varepsilon$, we let I_m, $m > n+1$, be the first interval where $T_n\bar{f} < \bar{g}$. Let a, b and a_1, b_1 be the values of $T_n\bar{f}$ and $\bar{g}$ on I_{n+1} and I_m respectively. Then, since $\bar{g}$ is non-increasing, we have $a \geqslant b \geqslant b_1 > a_1$. Now, let λ, $0 < \lambda \leqslant 1$, be such that $\lambda a + (1-\lambda)a_1 = b$ and ψ a measure preserving transformation of R^+ permuting the intervals I_{n+1} and I_m and leaving all other points of R^+ fixed. Let T' be the linear transformation of functions induced by ψ and

$$T_{n+1} = [\lambda + (1-\lambda)T']T_n.$$

Then, evidently, $T_{n+1}\bar{f} = T_n\bar{f}$ outside the intervals I_{n+1} and I_m, and $T_{n+1}\bar{f} = \bar{g}$ on I_{n+1}. Thus we have $T_{n+1}\bar{f} = \bar{g}$ for $t \leqslant (n+1)\varepsilon$ and $T_{n+1}\bar{f} \geqslant \bar{g}$ for $t \leqslant (m-1)\varepsilon$. On the other hand, we have

$$\int_0^t T_{n+1}\bar{f}\,ds = \int_0^t T_n\bar{f}\,ds$$

for $t \geqslant m\varepsilon$ and consequently

$$\int_0^t T_{n+1}\bar{f}\,ds \geqslant \int_0^t \bar{g}\,ds$$

for $t \geqslant m\varepsilon$. Since $T_{n+1}\bar{f} \geqslant \bar{g}$ for $t \leqslant (m-1)\varepsilon$, the preceding inequality holds for all t not in I_m. But on I_m both sides of the inequality are linear functions of t and consequently the inequality holds for all t.

Thus we have shown that if n has the properties postulated above, so does $n+1$. Since 0 clearly has these properties, we conlude that the same is true for every n, $n \geqslant 0$. Now, if m is so large that $\bar{g}(t) = 0$ for $t > m\varepsilon$, then evidently $T_m\bar{f} \geqslant \bar{g}$ for all t. Setting $T = T_m$, since $f \geqslant \bar{f}$ and $g \leqslant \bar{g}$, we have $Tf \geqslant g$. The lemma is thus established.

Lemma 2. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two totally σ-finite measure spaces and f_1 and f_2 be two measurable non-negative functions on $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively such that $f_1^* = f_2^*$, and f_2 vanishes outside the set where $f_2 > a = \lim_{t\to\infty} f_2^*(t)$. Then there exists a positive admissible linear map T of the interpolation pair $\big(L^1(\mathcal{M}_1), L^\infty(\mathcal{M}_1)\big)$ into the pair $\big(L^1(\mathcal{M}_2), L^\infty(\mathcal{M}_2)\big)$, of norm 1, such that $Tf_1 = f_2$.*

Proof. Let μ_j be the measure on $\mathcal{M}_j$ and $\mathcal{F}_j$ the σ-field of subsets of the set where $f_j > a$ of the form $f_j^{-1}(E)$ where E is a Borel subset of the real line contained in $t > a$. Then, since the sets where $f_1 > \lambda$ and $f_2 > \lambda$ have the same measure for all $\lambda > 0$, it follows that

$$\mu_1[f_1^{-1}(E)] = \mu_2[f_2^{-1}(E)] \leqslant \infty$$

for every Borel subset E of $t > a$. Given a set D in $\mathcal{F}_2$ we associate with it the set D' in $\mathcal{F}_1$, $D' = f_1^{-1}(E)$ where E is a Borel subset of $t > a$ such that $D = f_2^{-1}(E)$. The set D' is defined up to a set of measure 0 for if $D = f_2^{-1}(E_1) = f_2^{-1}(E_2)$, then also $D = f_2^{-1}(E_1 \cap E_2)$ and $f_1^{-1}(E_1 \cap E_2) \subset f_1^{-1}(E_j)$, $j = 1, 2$, and $\mu_1[f_1^{-1}(E_1 \cap E_2)] = \mu_2(D) = \mu_1[f_1^{-1}(E_j)]$. Given a function f in $L^1(\mathcal{M}_1) + L^\infty(\mathcal{M}_1)$ we define a countably additive set function ν on $\mathcal{F}_2$ by

$$\nu(D) = \int_{D'} f\,d\mu_1.$$

If $\mu_2(D) = 0$, then $\mu_1(D') = 0$ and consequently $\nu(D) = 0$, which shows that ν is absolutely continuous with respect to the restriction $\bar{\mu}_2$ of μ_2 to $\mathcal{F}_2$. Now we define Tf to be the Radon-Nikodym derivative of ν with respect to $\bar{\mu}_2$ and $Tf = 0$ outside the set where $f_2 > a$. The operator T just defined is clearly linear and positive. If $f \geqslant 0$ is integrable, then $Tf \geqslant 0$ and denoting by D_1 and D_2 the sets where $f_1 > a$ and $f_2 > a$ respectively, we have

$$\int Tf\,d\mu_2 = \int_{D_2} Tf\,d\mu_2 = \int_{D_2} d\nu = \int_{D_1} f\,d\mu_1 \leqslant \int f\,d\mu_1$$

where the first integral is taken over $\mathcal{M}_2$ and the last over $\mathcal{M}_1$, whence for general integrable f we have

$$\int |Tf|\,d\mu_2 \leqslant \int T|f|\,d\mu_2 \leqslant \int |f|\,d\mu_1.$$

Thus T maps $L^1(\mathcal{M}_1)$ into $L^1(\mathcal{M}_2)$ with norm less than or equal to 1.

If, on the other hand, $|f| \leqslant c$ and D is any set in $\mathscr{F}_2$ we have

$$\Big|\int_D Tf d\mu_2\Big| = \Big|\int_D d\nu\Big| = \Big|\int_{D'} f d\mu_1\Big| \leqslant c\mu_1(D') = c\mu_2(D)$$

which shows that $|Tf| \leqslant c$. Thus T maps $L^\infty(\mathscr{M}_1)$ into $L^\infty(\mathscr{M}_2)$ with norm less than or equal to 1.

Finally, let us show that $Tf_1 = f_2$. Let $\lambda > a$, let D and D' be the sets where $f_2 > \lambda$ and $f_1 > \lambda$ respectively, and t their common measure. Then we have

$$\int_D f_2 d\mu_2 = \int_0^t f_2^*(s)ds = \int_0^t f_1^*(s)ds = \int_{D'} f_1 d\mu_1;$$

this shows that

$$\int_D f_2 d\mu_2 = \int_{D'} f_1 d\mu_1$$

for every D in $\mathscr{F}_2$, and this implies that $Tf_1 = f_2$.

THEOREM 1. *Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two totally σ-finite measure spaces and f_1 an f_2 two functions on $\mathscr{M}_1$ and $\mathscr{M}_2$ respectively such that $f_1^{**} \geqslant f_2^{**}$. Then there exists an admissible linear map T of the pair $(L^1(\mathscr{M}_1), L^\infty(\mathscr{M}_1))$ into the pair $(L^1(\mathscr{M}_2), L^\infty(\mathscr{M}_2))$, of norm less than or equal to 1, such that $Tf_1 = f_2$.*

Proof. Without loss of generality we can assume that the functions f_1 and f_2 are non-negative. Under this assumption there exist two increasing sequences of simple non-negative measurable functions g_n and h_n such that $h_n^{**} < h_{n+1}^{**}$ and converging almost everywhere to f_1 and f_2 respectively. Then $tg_n^{**}(t)$ and $th_n^{**}(t)$ converge to $tf_1^{**}(t)$ and $tf_2^{**}(t)$ uniformly on every finite interval. Furthermore, the functions $tg_n^{**}(t)$ and $th_n^{**}(t)$ are non-negative, non-decreasing, concave, piecewise linear and constant for sufficiently large t, and $th_n^{**}(t) < tf_2^{**}(t) \leqslant tf_1^{**}(t)$. Thus given n there exists an m, $m = m(n)$, such that $th_n^{**}(t) < tg_m^{**}(t)$ for all $t > 0$, and according to lemmas 1 and 2 there exist positive admissible operators $T_{1,n}, T_{2,n}, T_{3,n}$ of the corresponding pairs (L^1, L^∞) such that $T_{1,n}g_m = g_m^*$, $T_{2,n}g_m^* \geqslant h_n^*$, $T_{3,n}h_n^* = h_n$, these operators having norm less than or equal to 1. Now we let φ_n and ψ_n be two measurable functions such that $0 \leqslant \varphi_n \leqslant 1$, $0 \leqslant \psi_n \leqslant 1$ and $g_m = f_1\varphi_n$, $h_n^* = \psi_n T_{2,n}g_m^*$ and define $T_n = T_{3,n}\psi_n T_{2,n}T_{1,n}\varphi_n$, where φ_n and ψ_n here stand for the operators multiplication by φ_n and ψ_n respectively. Clearly T_n is a positive admissible operator of the pair $(L^1(\mathscr{M}_1), L^\infty(\mathscr{M}_1))$ into the pair $(L^1(\mathscr{M}_2), L^\infty(\mathscr{M}_2))$ of norm less than or equal to 1, such that $T_nf_1 = h_n$. Let now λ be a positive linear functional of bounded sequences of real numbers such that $\lambda(a_n) = \lim_{n\to\infty} a_n$ for every convergent sequence a_n. Given a function f in $L^1(\mathscr{M}_1)+L^\infty(\mathscr{M}_1)$, let ν be the set function in $\mathscr{M}_2$ defined by

$$\nu(E) = \lambda\Big[\int_E T_nfd\mu_2\Big]$$

where μ_2 is the measure on $\mathscr{M}_2$. This set function ν is countably additive and absolutely continuous with respect to μ_2. In fact, given a positive number ε there exists c such that $f = g+h$ with

$$\int|g|d\mu_1 < \varepsilon \quad \text{and} \quad |h| < c$$

where μ_1 is the measure on $\mathscr{M}_1$. Consequently, since the norm of the admissible operator T_n is less than or equal to 1, we have

$$\Big|\int_E T_nfd\mu_2\Big| \leqslant \Big|\int_E T_ngd\mu_2\Big| + \Big|\int_E T_nhd\mu_2\Big| \leqslant \varepsilon + c\mu_2(E)$$

and consequently

$$|\nu(E)| \leqslant c\mu_2(E)+\varepsilon$$

which clearly implies the countable additivity of ν and its absolute continuity with respect to μ_2.

Now we define Tf as the Radon-Nikodym derivative of ν with respect to μ_2. Since λ and T_n are positive, T is positive. Furthermore, since $T_nf_1 = h_n$ converges monotonically to f_2, we have

$$\nu(E) = \lim_{n\to\infty}\int_E T_nf_1d\mu_2 = \int_E f_2d\mu_2$$

for every measurable E, which shows that $Tf_1 = f_2$.

Finally, let us show that T is admissible and has norm not exceeding 1. Let f be non-negative and have norm less than or equal to 1 in $L^1(\mathscr{M}_1)$. Then $Tf \geqslant 0$ and

$$\int_E Tfd\mu_2 = \nu(E) \leqslant \lambda\Big[\int T_nfd\mu_2\Big] \leqslant \lambda(1) = 1$$

for any measurable subset E of $\mathscr{M}_2$, which shows that Tf is integrable and its integral does not exceed 1. If, on the other hand, $|f| \leqslant c$, then

$$\Big|\int_E Tfd\mu_2\Big| = |\nu(E)| = \Big|\lambda\Big[\int_E T_nfd\mu_2\Big]\Big| \leqslant c\mu_2(E)$$

which shows that $|Tf| \leqslant c$. The theorem is thus established.

THEOREM 2. *Let T be a quasilinear map of $L^1(\mathscr{M}_1)+L^\infty(\mathscr{M}_1)$ into $L^1(\mathscr{M}_2)+L^\infty(\mathscr{M}_2)$, that is, a map which satisfies the following condi-*

tions

$$|T(\lambda f)| = |\lambda||Tf|, \qquad |T(f_1+f_2)| \leqslant c(|Tf_1|+|Tf_2|)$$

where c is a constant independent of f_1 and f_2. Suppose in addition that T is admissible in the sense that it maps $L^1(\mathcal{M}_1)$ into $L^1(\mathcal{M}_2)$ and $L^\infty(\mathcal{M}_1)$ into $L^\infty(\mathcal{M}_2)$, and that the inequality $\|Tf\| \leqslant c^{-1}\|f\|$ holds between the corresponding norms, where c is the same constant as above. Then $(Tf)^{**} \leqslant f^{**}$.

Proof. Let f, $f \in L^1(\mathcal{M}_1)+L^\infty(\mathcal{M}_1)$, be given and let t_0 be a positive number. Set $a = f^*(t_0)$ and let $g(x)$ be the function on $\mathcal{M}_1$ defined by $g(x) = f(x)-a$ if $f(x) > f^*(t_0)$, $g(x) = f(x)+a$ if $f(x) < -f^*(t_0)$ and $g(x) = 0$ if $|f(x)| \leqslant f^*(t_0)$. Further, let $h(x) = f(x)-g(x)$. Then we have

$$|h(x)| \leqslant f^*(t_0) \quad \text{and} \quad \int |g|\,d\mu_1 \leqslant \int_0^{t_0} f^*(s)\,ds - t_0 f^*(t_0).$$

From this it follows that

$$c|Th| \leqslant f^*(t_0) \quad \text{and} \quad \int c|Tg|\,d\mu_2 \leqslant \int_0^{t_0} f^*(s)\,ds - t_0 f^*(t_0)$$

which implies that

$$(cTh)^{**}(t_0) \leqslant f^*(t_0)$$

$$\text{and} \quad (cTg)^{**}(t_0) \leqslant \frac{1}{t_0}\int c|Tg|\,d\mu_2 \leqslant \frac{1}{t_0}\int_0^{t_0} f^*(s)\,ds - f^*(t_0).$$

Since $|Tf| = |T(h+g| \leqslant c(|Th|+|Tg|)$, this in turn implies that

$$(Tf)^{**}(t_0) \leqslant (cTh)^{**}(t_0)+(cTg)^{**}(t_0) \leqslant \frac{1}{t_0}\int_0^{t_0} f^*(s)\,ds = f^{**}(t_0)$$

as we wished to show.

THEOREM 3. *Let A_1 be a linear space contained in $L^1(\mathcal{M}_1)+L^\infty(\mathcal{M}_1)$; then*

i) *a necessary and sufficient condition in order that every admissible map of the pair $(L^1(\mathcal{M}_1), L^\infty(\mathcal{M}_1))$ into itself map A_1 into itself, is that $g^{**} \leqslant f^{**}$ and $f \in A_1$ imply $g \in A_1$;*

ii) *if the space A_1 has a norm, a necessary and sufficient condition in order that every admissible map or norm less than or equal to 1 map A_1 into itself with norm less than or equal to 1, is that $g^{**} \leqslant f^{**}$, $f \in A_1$ and $\|f\| \leqslant 1$ imply $g \in A_1$ and $\|g\| \leqslant 1$;*

iii) *if A_2 is a linear space contained in $L^1(\mathcal{M}_2)+L^\infty(\mathcal{M}_2)$, then a necessary and sufficient condition in order that every admissible map of the pair $(L^1(\mathcal{M}_1), L^\infty(\mathcal{M}_1))$ into the pair $(L^1(\mathcal{M}_2), L^\infty(\mathcal{M}_2))$ map A_1 into A_2 is that $g^{**} \leqslant f^{**}$ and $f \in A_1$ imply $g \in A_2$;*

iv) *if the condition in* iii) *is satisfied, then every quasilinear map of the first pair into the second which is admissible in the sense of theorem 2 maps A_1 into A_2.*

Proof. The preceding statement is clearly an immediate consequence of theorems 1 and 2.

THEOREM 4. *Let $\mathcal{M}_2$ be a measure space which is either non-atomic, or else purely atomic, all atoms having the same measure. Let A_2 be a Banach space which is continuously embedded in $L^1(\mathcal{M}_2)+L^\infty(\mathcal{M}_2)$ and whose unit sphere is closed with respect to almost everywhere convergence. Then, if $f \in L^1(\mathcal{M}_1)+L^\infty(\mathcal{M}_1)$, a necessary and sufficient condition in order that every admissible map of the pair $(L^1(\mathcal{M}_1), L^\infty(\mathcal{M}_1))$ into the pair $(L^1(\mathcal{M}_2), L^\infty(\mathcal{M}_2))$ map f into A_2 is that the set of all functions $g \in L^1(\mathcal{M}_2)+L^\infty(\mathcal{M}_2)$ such that $g^* \leqslant f^*$ be a bounded subset of A_2.*

Proof. Let us prove first the necessity of the condition. Let $\mathcal{X}$ be the space of all admissible linear maps of the first pair into the second with the norm of admissible maps, and let τ be the linear map of $\mathcal{X}$ into A_2 defined by $\tau(T) = Tf$. Then clearly τ is continuous if A_2 is given the topology induced by the norm of $L^1(\mathcal{M}_2)+L^\infty(\mathcal{M}_2)$ and the graph of τ is closed in $A_2 \oplus \mathcal{X}$ with the corresponding topology. But then the graph of τ is also closed in $A_2+\mathcal{X}$ with the stronger topology induced by the norms of A_2 and $\mathcal{X}$, and consequently, τ maps $\mathcal{X}$ continuously into A_2. Let now S be the image under τ of the unit sphere in $\mathcal{X}$. Then S is a bounded subset of A_2. If g is such that $g^* \leqslant f^*$, then also $g^{**} \leqslant f^{**}$, and according to theorem 2 there exists an admissible linear map of norm less than or equal to 1 such that $Tf = g$. Thus $g \in S$.

To prove the sufficiency of the condition we will show that it implies that if g, $g \in L^1(\mathcal{M}_2)+L^\infty(\mathcal{M}_2)$, is such that $g^{**} \leqslant f^{**}$, then $g \in A_2$, whence the desired result will follow from part iii) of the preceding theorem. For the purposes of this proof, simple functions which are finite linear combinations of characteristic functions of disjoint sets of equal measure will be called *step-functions*. Under our assumptions on the measure space $\mathcal{M}_2$ every non-negative measurable function g on $\mathcal{M}_2$ is the limit almost everywhere of non-negative step-functions h such that $h \leqslant g$. This is clear in the atomic case. In the non-atomic case it is an immediate consequence of the fact that the measures of the subsets of a measurable set E take all values between 0 and the measure of E. Suppose now that g is a non-negative measurable function on $\mathcal{M}_2$ such that $g^{**} \leqslant f^{**}$. Let h be a step-function such that $0 \leqslant h \leqslant g/2$ and let ε be the measure of the disjoint sets on which h is constant. Given an integer m, $m > 0$, we let χ_n be the characteristic functions of the intervals $((n-1)\varepsilon\cdot 2^{-m},$

$n\varepsilon\cdot 2^{-m})$ of the real line and $f_m(t)$ the largest linear combination of these such that $f_m(t) \leqslant f^*(t)$. Then

$$\int_0^t f_m(s)\,ds \to \int_0^t f^*(s)\,ds$$

uniformly in every finite interval, and therefore we will have

$$\int_0^t h^*(s)\,ds < \int_0^t f_m(s)\,ds$$

for m sufficiently large and all $t > 0$. We choose now an m for which the preceding inequality holds. If $\mathscr{M}_2$ is non-atomic, then h can be expressed as

$$h(x) = \sum_1^N \lambda_n \chi_n(x)$$

where the χ_n are characteristic functions of disjoint sets of measure $\varepsilon\cdot 2^{-m}$ and $\lambda_n \geqslant \lambda_{n+1}$. According to lemma 1 there exist measure preserving transformations of R^+ permuting the intervals $\big((n-1)\varepsilon\cdot 2^{-m}, n\varepsilon\cdot 2^{-m}\big)$ and inducing linear maps T_i such that

$$h^* \leqslant \sum a_i T_i f_m, \quad a_i > 0, \quad \sum a_i = 1.$$

Let $\bar{h}_i = T_i f_m$ if $h^* > 0$ and $\bar{h}_i = 0$ otherwise. Then $\bar{h}_i$ is constant on each of the intervals $\big((n-1)\varepsilon\cdot 2^{-m}, n\varepsilon\cdot 2^{-m}\big)$ and can therefore be expressed as

$$\bar{h}_i(t) = \sum_1^N \lambda_n^i \bar{\chi}_n(t).$$

Now we set

$$h_i(x) = \sum_1^N \lambda_n^i \chi_n(x).$$

Then, obviously, $(h_i)^* = (\bar{h}_i)^* \leqslant f_m(t) < f^*(t)$ which implies that $h_i \epsilon A_2$ and that its norm $\|h_i\|$ in A_2 is less than a fixed constant c. On the other hand, we also have

$$h(x) \leqslant \sum a_i h_i(x),$$

whence it follows that $h \epsilon A_2$ and $\|h\| < c$. Now we take a sequence of such functions h converging to $g/2$ almost everywhere and we conclude that, since the sphere of radius c in A_2 is closed with respect to convergence almost everywhere, $g/2$ and g belong to A_2. This concludes the proof in the case when $\mathscr{M}_2$ is non-atomic.

In the case when $\mathscr{M}_2$ is purely atomic we revert to our function h above and assume that ε is the common measure of the atoms of $\mathscr{M}_2$. The preceding argument would be valid with $f_m = f_1$ except for the fact that the inequality

$$\int_0^t h^*(s)\,ds < \int_0^t f_1(s)\,ds$$

may not hold. However, since clearly

$$\int_0^t f_1(s)\,ds \geqslant \int_\varepsilon^t f^*(s)\,ds, \qquad t \geqslant \varepsilon,$$

we have

$$\int_0^t f^*(s)\,ds \leqslant (c+1)\int_0^t f_1(s)\,ds, \qquad t \geqslant \varepsilon,$$

with

$$c = \Big(\int_0^\varepsilon f^*(s)\,ds\Big)\Big/\Big(\int_0^\varepsilon f_1(s)\,ds\Big)$$

which implies that

$$\int_0^t h^*(s)\,ds < (c+1)\int_0^t f_1(s)\,ds, \qquad t \geqslant \varepsilon.$$

But the integral on the left is a linear function of t in the interval $[0, \varepsilon]$ and the one on the right is a concave function there and thus the inequality holds for all positive t. Now we can repeat the argument above with $(c+1)f_1$ replacing f_m. This would establish our assertion in this remaining case.

2. The spaces $L_{p,q}$ and the theorem of Marcinkiewicz. In this section we will discuss conditions for the validity of the theorem of Marcinkiewicz. The spaces $L_{p,q}$, first introduced by G. G. Lorentz, appear naturally in these considerations. They are among the function spaces studied by W. A. J. Luxemburg in his dissertation [5], we will therefore restrict ourselves to discuss some of their special properties which are relevant to our present purposes.

Given a totally σ-finite measure space $\mathscr{M}$ we define $L_{p,q}(\mathscr{M})$, $1 < p < \infty$, $1 \leqslant q < \infty$, as the class of measurable functions f on $\mathscr{M}$ such that

$$\|f\|_{p,q} = \Big\{[(p-1)/p^2]\int_0^\infty [f^{**}(t)\,t^{1/p}]^q \frac{dt}{t}\Big\}^{1/q} < \infty.$$

Here $(p-1)/p^2$ is merely a normalizing factor which ensures the continuity of the norm in the extreme cases.

For $1 < p < \infty$ and $q = \infty$, $L_{p,q}(\mathcal{M})$ is defined as the class of functions f such that

$$\|f\|_{p,\infty} = \sup_t t^{1/p} f^{**}(t) < \infty.$$

Finally, $L_{1,1}(\mathcal{M})$ and $L_{\infty,\infty}(\mathcal{M})$ coincide by definition with $L^1(\mathcal{M})$ and $L^\infty(\mathcal{M})$ respectively, and $L_{\infty,1}(\mathcal{M})$ is the closure in $L^\infty(\mathcal{M})$ of the space of bounded functions vanishing outside sets of finite measure.

If f is bounded and vanishes outside a set of finite measure, integration by parts gives

$$\|f\|_{p,1} = (1/p)\int_0^\infty f^*(t)\, t^{1/p-1} dt, \quad 1 < p < \infty,$$

whence it follows that

$$\lim_{p\to 1} \|f\|_{p,1} = \|f\|_{1,1}, \quad \lim_{p\to\infty} \|f\|_{p,1} = \|f\|_{\infty,1}.$$

On the other hand, we also have

$$\lim_{q\to\infty} \|f\|_{p,q} = \|f\|_{p,\infty}; \quad \lim_{p\to\infty} \|f\|_{p,\infty} = \|f\|_{\infty,\infty}.$$

The spaces $L_{p,q}(\mathcal{M})$ are complete with respect to the norms just introduced and the space of simple functions is dense in $L_{p,q}(\mathcal{M})$ for $q < \infty$. The latter can be readily seen by observing that given a measurable function f there exists a sequence of simple functions f_n such that $|f-f_n|$ converges monotonically and almost everywhere to 0, which implies that $(f-f_n)^{**}$ converges monotonically to zero and thus also the norm of $f-f_n$ converges to 0.

In all cases $L_{p,q}(\mathcal{M})$ is contained in $L^1(\mathcal{M})+L^\infty(\mathcal{M})$ and its embedding in this space is continuous. For if f_n converges to 0 in $L_{p,q}(\mathcal{M})$, then $f_n^{**}(1)$ and $f^*(1)$ converge to 0; thus if we set $h_n = f_n$ if $|f_n| > f_n^*(1)$ and $h_n = 0$ otherwise, and $g_n = f_n - h_n$, we find that g_n converges to 0 uniformly and that

$$\int |h_n|\, d\mu = \int_0^1 f_n^*(s)\, ds = f_n^{**}(1) \to 0.$$

On account of theorem 3, the spaces $L_{p,q}(\mathcal{M})$ are readily seen to be strictly intermediate between $L^1(\mathcal{M})$ and $L^\infty(\mathcal{M})$.

THEOREM 5. *Let $1 < p < \infty$, $q < r$. Then $L_{p,q}(\mathcal{M})$ is contained in $L_{p,r}(\mathcal{M})$ and*

$$\|f\|_{p,r} \leqslant [(p-1)/pq]^{1/r-1/q} \|f\|_{p,q}.$$

Proof. Let f be in $L_{p,q}(\mathcal{M})$ and $\|f\|_{p,q} = a$. Then

$$a^q = [(p-1)/p^2]\int_0^\infty [f^{**}(t)\, t^{1/p}]^q \frac{dt}{t} = [(p-1)/pq]\int_0^\infty f^{**}(t)\, dt^{q/p}$$

and

$$[pq/(p-1)]\, a^q = \int_0^\infty f^{**}(t)\, dt^{q/p}.$$

Since $f^{**}(t)$ is non-increasings we have

$$f^{**}(t)\, t^{q/p} \leqslant \int_0^\infty f^{**}(s)\, ds^{q/p}, \quad 0 < t < \infty.$$

Hence

$$f^{**}(t)\, t^{q/p} \leqslant [pq/(p-1)]\, a^q$$

and

$$[(p-1)/pq]^{1/q} a^{-1} f^{**}(t)\, t^{1/p} \leqslant 1.$$

Consequently, for $r > q$ we have

$$[(p-1)/pq]^{r/q} a^{-r} f^{**}(t)^r t^{r/p} \leqslant [(p-1)/pq]\, a^{-q} f^{**}(t)^q t^{q/p}$$

and integrating we obtain the desired inequality. If $r = \infty$ our result follows directly from

$$f^{**}(t)\, t^{q/p} \leqslant [pq/(p-1)]\, a^q.$$

THEOREM 6. *A necessary and sufficient condition in order that f belong to $L_{p,q}(\mathcal{M})$, $1 < p < \infty$, $1 \leqslant q < \infty$, is that*

$$\int_0^\infty [f^*(t)\, t^{1/p}]^q \frac{dt}{t} < \infty.$$

Furthermore

$$\|f\|_{p,q} < [p/(p-1)]\left\{[(p-1)/p^2]\int_0^\infty [f^*(t)\, t^{1/p}]^q \frac{dt}{t}\right\}^{1/q} \leqslant [p/(p-1)]\, \|f\|_{p,q}.$$

A necessary and sufficient condition in order that f belong to $L_{p,\infty}(\mathcal{M})$ is that $f^*(t)\, t^{1/p}$ be bounded. Furthermore

$$\|f\|_{p,\infty} \leqslant [p/(p-1)] \sup_t f^*(t)\, t^{1/p} \leqslant [p/(p-1)]\, \|f\|_{p,\infty}.$$

Proof. The necessity of the conditions and the right-hand side inequalities are immediate consequences of the fact that $f^* \leqslant f^{**}$. The sufficiency of the condition for the case $q < \infty$ and the corresponding inequality follow from an application of Hardy's inequality (see [9], p. 20) to

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds.$$

In the case $q=\infty$ we let $a = \sup_t t^{1/p} f^*(t)$. Then

$$t^{1/p} f^{**}(t) = t^{1/p-1}\int_0^t f^*(s)\,ds \leqslant t^{1/p-1}\int_0^t a s^{-1/p}\,ds = ap/(p-1),$$

whence the desired result follows.

COROLLARY. *The spaces $L_{p,p}(\mathscr{M})$ and $L^p(\mathscr{M})$ coincide and have equivalent norms.*

If $p=1$ or $p=\infty$ this is clear in view of the corresponding definitions. If $1<p<\infty$ our assertion follows immediately from the fact that

$$\int |f(x)|\,d\mu = \int_0^\infty f^*(s)\,ds.$$

The following result is useful in testing continuity of mappings defined on the spaces $L_{p,1}(\mathscr{M})$.

THEOREM 7. *Let B be a Banach space of measurable functions continuously embedded in the space V of all measurable functions of a totally σ-finite measure space $\mathscr{M}_2$ and with the property that $f\in B$ and $|f|\geqslant|g|$ implies $g\in B$ and $\|f\|_B \geqslant \|g\|_B$. Let T be a continuous map of $L_{p,1}(\mathscr{M}_1)$, $1\leqslant p\leqslant\infty$, into V which is sublinear, that is, such that*

$$|T(f+g)| \leqslant |Tf|+|Tg|, \qquad |T(\lambda f)| = |Tf|\,|\lambda|$$

almost everywhere, for every pair of functions f, g in $L_{p,1}(\mathscr{M}_1)$. Suppose that $T\chi\in B$ for every characteristic function χ of a set of finite measure in $\mathscr{M}_1$, and that there exists a constant c independent of χ such that

$$\|T\chi\|_B \leqslant (c/2)\,\|\chi\|_{p,1}.$$

Then T maps $L_{p,1}(\mathscr{M}_1)$ into B and

$$\|Tf\|_B \leqslant c\,\|f\|_{p,1}$$

for every f in B.

Proof. Let f be a simple non-negative integrable function on $\mathscr{M}_1$ and $f=\sum\lambda_n\chi_n$ where $\lambda_n>0$ and the χ_n are characteristic functions of sets such that $\chi_1\leqslant\chi_2\leqslant\ldots\leqslant\chi_n$. Then, as readily seen, we have

$$f^* = \sum\lambda_n\chi_n^* \quad \text{and} \quad f^{**} = \sum\lambda_n\chi_n^{**},$$

which implies that

$$\|f\|_{p,1} = \sum\lambda_n\|\chi_n\|_{p,1}.$$

On account of the sublinearity of T we have

$$|Tf| \leqslant \sum\lambda_n|T\chi_n|$$

and consequently

$$\|Tf\|_B \leqslant \sum\lambda_n\|T\chi_n\|_B \leqslant \frac{c}{2}\sum\lambda_n\|\chi_n\|_{p,1} \leqslant \frac{c}{2}\|f\|_{p,1}.$$

If now f is simple and integrable but no longer non-negative we set $f=f_1-f_2$ with f_1 and f_2 simple non-negative and such that $|f| = f_1+f_2$. Then

$$\|Tf\|_B \leqslant \|Tf_1\|_B+\|Tf_2\|_B \leqslant \frac{c}{2}\left(\|f_1\|_{p,1}+\|f_2\|_{p,1}\right) \leqslant c\,\|f\|_{p,1}.$$

Suppose now that f is any given function in $L_{p,1}(\mathscr{M}_1)$ and let f_n be a sequence of simple integrable functions converging to f in $L_{p,1}(\mathscr{M}_1)$. Then Tf_n converges to Tf in V. Furthermore, since

$$|Tf_n - Tf_m| \leqslant |T(f_n-f_m)|,$$

we have

$$\|Tf_n-Tf_m\|_B \leqslant \|T(f_n-f_m)\|_B \leqslant c\,\|(f_n-f_m)\|_{p,1}$$

and Tf_n converges in B. Since limits in V are unique, it follows that Tf_n converges to Tf in B and

$$\|Tf\|_B = \lim\|Tf_n\|_B \leqslant c\lim\|f_n\|_{p,1} = c\,\|f\|_{p,1}$$

as we wished to show.

Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two totally σ-finite measure spaces. Let T be an operator defined in some linear space containing $L_{p,1}(\mathscr{M}_1)$ with measurable functions on $\mathscr{M}_2$ as values and which is quasilinear in the sense of theorem 2. We will say that T is of *weak type* (p,q), $1\leqslant q\leqslant\infty$, if T maps $L_{p,1}(\mathscr{M}_1)$ continuously into the space of measurable functions on $\mathscr{M}_2$ and there exists a constant c such that

$$(Tf)^* \leqslant ct^{-1/q}\,\|f\|_{p,1}$$

for every f in $L_{p,1}(\mathscr{M}_1)$. On account of theorem 6 this condition is equivalent with the existence of a constant c' such that $\|Tf\|_{q,\infty} \leqslant c'\,\|f\|_{p,1}$, provided that $q>1$. If $q>1$ and in addition T is sublinear, then according to theorem 7 the preceding inequality will be satisfied by all f in $L_{p,1}(\mathscr{M}_1)$ if it is satisfied by characteristic functions of sets of finite measure.

Thus in the case $q > 1$ this condition is seen to be equivalent with the "restricted weak type" condition of Stein and Weiss [8] ([1]).

Given a closed segment σ in the square $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant 1$, with endpoints (α_1, β_1), (α_2, β_2) and $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$, and given two totally σ-finite measure spaces $\mathscr{M}_1$ and $\mathscr{M}_2$, $\mathscr{W}(\sigma, \mathscr{M}_1, \mathscr{M}_2)$ will denote the class of linear maps of $L_{1/\alpha_1,1}(\mathscr{M}_1) + L_{1/\alpha_2,1}(\mathscr{M}_1)$ into the space of measurable functions on $\mathscr{M}_2$ which are simultaneously of weak types $(1/\alpha_1, 1/\beta_1)$ and $(1/\alpha_2, 1/\beta_2)$. If there is no ambiguity, we will denote this class more briefly by $\mathscr{W}(\sigma)$. On the other hand, $\mathscr{R}(\sigma, \mathscr{M}_1, \mathscr{M}_2)$ or more briefly $\mathscr{R}(\sigma)$ will denote the class of operators T in $\mathscr{W}(\sigma)$ satisfying the inequalities

$$\|Tf\|_{1/\beta_i, 1/\beta_i} \leqslant c\|f\|_{1/\alpha_i, 1/\alpha_i}, \qquad i = 1, 2,$$

for all f in $L_{1/\alpha_i,1}(\mathscr{M}_1)$. If the segment σ is such that $\beta_i < 1$, $i = 1, 2$, $\mathscr{S}(\sigma)$ will denote the class of operators in $\mathscr{W}(\sigma)$ satisfying the inequalities

$$\|Tf\|_{1/\beta_i, \varepsilon_i} \leqslant c\|f\|_{1/\alpha_i, 1/\alpha_i}, \qquad f \in L_{1/\alpha_i,1}(\mathscr{M}_1),$$

where $\varepsilon_i = 1/\alpha_i$ if $\beta_i > 0$ and $\varepsilon_i = \infty$ if $\beta_i = 0$.

With a segment σ as above we associate two functions on $R^+ \times R^+$, namely

$$\psi(t, s) = \min(s^{\alpha_1}/t^{\beta_1}, s^{\alpha_2}/t^{\beta_2}) \qquad \text{and} \qquad \varphi(t, s) = s\frac{d}{ds}\psi(t, s)$$

and an operator $S(\sigma)$ on functions on R^+ defined by

$$S(\sigma)f = \int_0^\infty \varphi(t, s) f(s) \frac{ds}{s},$$

whenever the integral on the right is absolutely convergent.

LEMMA 3. *The operator* $S(\sigma)$ *belongs to* $\mathscr{W}(\sigma, R^+, R^+)$. *If the point* $(1/p, 1/q)$ *is interior to the segment* σ *and* $f \in L_{p,r}(R^+)$, $1 \leqslant r \leqslant \infty$, *is non-negative, non-increasing so is* $S(\sigma)f$ *and* $S(\sigma)f \in L_{q,r}(R^+)$. *Furthermore, we have* $\|S(\sigma)f\|_{q,r} \leqslant c\|f\|_{p,r}$ *with* c *depending on* σ, p *and* q *but not on* r.

Proof. Suppose first that $\alpha_1 > 0$ and let f belong to $L_{1/\alpha_1,1}(R^+)$. Then, since $\varphi(t, s) \leqslant s^{\alpha_1}/t^{\beta_1}$, we have

$$\int_0^\infty \varphi(t, s)|f(s)|\frac{ds}{s} \leqslant t^{-\beta_1}\int_0^\infty f(s)s^{\alpha_1 - 1}ds \leqslant t^{-\beta_1}\int_0^\infty f^*(s)s^{\alpha_1-1}ds$$
$$= t^{-\beta_1}\alpha_1^{-1}\|f\|_{1/\alpha,1},$$

([1]) Observe that if the inequality $(Tf)^* \leqslant ct^{-\beta}\|f\|_{1/\alpha,1}$ is satisfied by characteristic functions of sets and two pairs of values (α_1, β_1), (α_2, β_2) of α and β, then it is also satisfied by any other pair $\alpha = \alpha_1 s + \alpha_2(1-s)$, $\beta = \beta_1 s + \beta_2(1-s)$, $0 \leqslant s \leqslant 1$. In fact, if a is the measure of the set where $f = 1$ we have $\|f\|_{1/\alpha,1} = a^\alpha$ and consequently

$$(Tf)^* \leqslant (ct^{-\beta_1}a^{\alpha_1})^s(ct^{-\beta_2}a^{\alpha_2})^{1-s} = ct^{-\beta}a^\alpha.$$

the last inequality being a consequence of the fact that $s^{\alpha_1 - 1}$ is non-increasing. Thus we have

$$|S(\sigma)f| \leqslant t^{-\beta_1}\alpha_1^{-1}\|f\|_{1/\alpha_1,1} \qquad \text{and} \qquad [S(\sigma)f]^* \leqslant t^{-\beta_1}\alpha_1^{-1}\|f\|_{1/\alpha_1,1}$$

which shows that $S(\sigma)$ is defined on $L_{1/\alpha_1,1}(R^+)$ and is of weak type $(1/\alpha_1, 1/\beta_1)$.

The same conclusion is valid if $\alpha_1 = 0$. For in this case we have $\alpha_2 > 0$ and

$$\varphi(t, s) = \alpha_2 s^{\alpha_2}/t^{\beta_2} \qquad \text{if} \qquad s \leqslant t^{(\beta_2 - \beta_1)/\alpha_2} = \tau$$

and $\varphi(t, s) = 0$ otherwise. Thus

$$|S(\sigma)f| \leqslant \int_0^\infty \varphi(t, s)|f(s)|\frac{ds}{s} \leqslant \alpha_2 t^{-\beta_2}\int_0^\tau f^*(s)s^{\alpha_2 - 1}ds \leqslant \|f\|_{\infty,1}t^{-\beta_1}$$

and the desired result follows. A similar argument shows that $S(\sigma)$ is of weak type $(1/\alpha_2, 1/\beta_2)$.

Let now $f(t)$ be non-increasing bounded and vanish outside a finite interval. Then integration by parts gives

$$S(\sigma)f = \int_0^\infty \varphi(t, s)f(s)\frac{ds}{s} = -\int_0^\infty \psi(t, s)\,df(s)$$

and since $\psi(t, s)$ is a decreasing function of t, it follows that $S(\sigma)f$ is non-increasing. For general non-increasing non-negative functions $f(t)$ the same conclusion is obtained by a passage to the limit.

To prove the last assertion of the theorem we assume that $\alpha_1 < \alpha_2$. Let $m = (\beta_2 - \beta_1)/(\alpha_2 - \alpha_1)$. Then

$$\varphi(t, s) = \begin{cases} \alpha_1 s^{\alpha_1}/t^{\beta_1} & \text{if} \quad t^m \leqslant s, \\ \alpha_2 s^{\alpha_2}/t^{\beta_2} & \text{if} \quad t^m \geqslant s. \end{cases}$$

Setting $S(\sigma)f = g$ and changing dependent and independent variables by means of the substitutions

$$t = \exp(v/m), \qquad s = \exp u, \qquad G(v)\exp(-v/mq) = g[\exp(v/m)],$$
$$F(u)\exp(-u/p) = f(\exp u),$$

we obtain

$$G(v)\exp(-v/mq) = \alpha_2\int_{-\infty}^v F(u)\exp[(\alpha_2 - 1/p)u - \beta_2 v/m]\,du +$$
$$+\alpha_1\int_v^\infty F(u)\exp[(\alpha_1 - 1/p)u - \beta_1 v/m]\,du$$

or

$$G(v) = a_2 \int_{-\infty}^{v} F(u)\exp[(a_2-1/p)u-(\beta_2-1/q)v/m]\,du+$$
$$+a_1\int_{v}^{\infty} F(u)\exp[(a_1-1/p)u-(\beta_1-1/q)v/m]\,du.$$

But, as readily seen

$$(\beta_2-1/q)/m = a_2-1/p, \quad (\beta_1-1/q)/m = a_1-1/p$$

whence

$$G(v) = a_2\int_{-\infty}^{v} F(u)\exp[(a_2-1/p)(u-v)]\,du+$$
$$+a_1\int_{v}^{\infty} F(u)\exp[(a_1-1/p)(u-v)]\,du$$

and Young's theorem on convolutions gives $\|G\|_r \leqslant c\|F\|_r$ where the norms are norms in $L^r(-\infty,\infty)$ and c depends on σ, p and q. Substituting the original variables back in the integrals defining these norms we obtain

$$\|F\|_r = \Big[\int_0^\infty [f(s)s^{1/p}]^r \frac{ds}{s}\Big]^{1/r}, \quad \|G\|_r = |m|^{1/r}\Big[\int_0^\infty [g(s)s^{1/q}]^r\frac{ds}{s}\Big]^{1/r}$$

which combined with theorem 6 and the preceding inequality gives the desired result.

THEOREM 8. *Let T be a quasilinear operator defined on $L_{p_1,1}(\mathscr{M}_1)+$ $+L_{p_2,1}(\mathscr{M}_1)$ with measurable functions on $\mathscr{M}_2$ as values. Suppose that T is simultaneously of weak types $(1/p_1, 1/q_1)$ and $(1/p_2, 1/q_2)$. Let σ be the segment with these endpoints. Then there exists a constant c such that*

$$(Tf)^* \leqslant cS(\sigma)f^*.$$

Proof. Let us begin proving the following inequality which is of independent interest

$$(f+g)^*(t_1+t_2) \leqslant f^*(t_1)+g^*(t_2).$$

It will be enough to prove this of f and g non-negative. Let μ be the measure on the space on which the functions under consideration are defined and let $(f>\lambda)$ denote the set where $f>\lambda$. Then if f is non-negative and $\mu[(f>\lambda)] = t_1 < t_2$ we have $f^*(t_1) \geqslant \lambda \geqslant f^*(t_2)$. Furthermore, $\mu[(f>f^*(t))] \leqslant t$ for all $t>0$. Consequently we have

$$\mu[(f>f^*(t_1))] \leqslant t_1, \quad \mu[(g>g^*(t_2))] \leqslant t_2.$$

Let now

$$\mu[(f+g>f^*(t_1)+g^*(t_2))] = t_3 < t.$$

Then

$$(f+g)^*(t) \leqslant f^*(t_1)+g^*(t_2).$$

But since

$$(f+g>f^*(t_1)+g^*(t_2)) \subset (f>f^*(t_1)) \cup (g>g^*(t_2)),$$

we have $t_3 \leqslant t_1+t_2$ and consequently

$$(f+g)^*(t) \leqslant f^*(t)_1+g^*(t_2)$$

whenever $t>t_1+t_2$, or also

$$(f+g)^*(t_1+t_2) \leqslant f^*(t_1-\varepsilon)+g^*(t_2-\varepsilon)$$

for every $\varepsilon>0$. But f^* and g^* are continuous on the left, whence the desired result follows by a passage to the limit.

We pass now to the proof of the theorem. Let $t>0$ be given and let

$$f_1 = f-f^*(t^m) \text{ if } f>f^*(t^m), \quad f_1 = f+f^*(t^m) \text{ if } f<-f^*(t^m)$$

and $f_1 = 0$ otherwise, where m is the slope of the segment σ. Then clearly, if $f_2 = f-f_1$ we have $f^* = f_1^*+f_2^*$ and $f_1^*(s) = 0$ if $s>t^m, f_2^*(s) = f_1^*(t^m)$ if $s<t^m$. Consequently

$$(Tf)^*(t) \leqslant c(Tf_1+Tf_2)^*(t) \leqslant c(Tf_1)^*(t/2)+c(Tf_2)^*(t/2)$$

and, assuming that $p_1<p_2$,

$$(Tf_1)^*(t/2) \leqslant c_1t^{-1/q_1}\|f_1\|_{p_1,1} \leqslant c_1(1/p_1)t^{-1/q_1}\int_0^{t^m} f_1^*(s)s^{1/p_1}\frac{ds}{s} = c_1S(\sigma)f_1^*,$$

$$(Tf_2)^*(t/2) \leqslant c_2t^{-1/q_2}\|f_2\|_{p_2,1} \leqslant c_2(1/p_2)t^{-1/q_2}\int_0^\infty f_2^*(s)s^{1/p_2}\frac{ds}{s}.$$

But

$$p_i^{-1}t^{-1/q_1}\int_0^{t^m} s^{1/p_1}\frac{ds}{s} = t^{m/p_1-1/q_1} = t^{m/p_2-1/q_2} = p_2^{-1}t^{-1/q_2}\int_0^{t^m} s^{1/p_2}\frac{ds}{s}$$

and therefore, since $f_2^*(s)$ is constant for $s<t^m$, we have

$$p_2^{-1}t^{-1/q_2}\int_0^\infty f_2^*(s)s^{1/p_2}\frac{ds}{s}$$
$$= p_1^{-1}t^{-1/q_1}\int_0^{t^m} f_2^*(s)s^{1/p_1}\frac{ds}{s}+p_2^{-1}t^{-1/q_2}\int_{t^m}^\infty f_2^*(s)s^{1/p_2}\frac{ds}{s} = S(\sigma)f_2^*,$$

whence it follows that

$$(Tf_2)^*(t/2) \leqslant c_2 S(\sigma) f_2^*.$$

Combining these inequalities we obtain

$$(Tf)^*(t) \leqslant cS(\sigma)(f_1^*+f_2^*) = cS(\sigma)f^*$$

which is the desired result.

THEOREM 9. *Let σ be a segment contained in $0 \leqslant \alpha \leqslant 1, 0 \leqslant \beta < 1$, f a function in $L_{1/\alpha_1}(\mathcal{M}_1)+L_{1/\alpha_2,1}(\mathcal{M}_1)$ and g a measurable function on $\mathcal{M}_2$. Then a necessary and sufficient condition for the existence of a linear map T in $\mathcal{W}(\sigma, \mathcal{M}_1, \mathcal{M}_2)$ such that $Tf = g$ is that there exists a constant c such that $g^* \leqslant cS(\sigma)f^*$.*

If $\mathcal{M}_2$ is non-atomic, the preceding statement is still valid if σ is merely contained in $0 \leqslant \alpha \leqslant 1, 0 \leqslant \beta \leqslant 1$.

Proof. The necessity of the condition is an immediate consequence of the preceding theorem.

To prove the sufficiency, let us observe first that under our hypothesis we have $f^*(t) \to 0$ as $t \to \infty$. Furthermore, the same is true of $S(\sigma)f^*$. In fact, integration by parts gives

$$S(\sigma)f^* = \int_0^\infty \varphi(t, s) f^*(s) \frac{ds}{s} = -\int_0^\infty \psi(t, s) df^*(s)$$

and since for each s the function $\psi(t, s)$ decreases and tends to zero at infinity, the desired conclusion follows.

Without loss of generality we may assume that f and g are non-negative. Let now g be such that $g^* \leqslant cS(\sigma)f^*$. Then $g^*(t) \to 0$ as $t \to \infty$. According to lemma 2 there exists an admissible operator T_1 of the pair $\big(L^1(\mathcal{M}_1), L^\infty(\mathcal{M}_1)\big)$ into the pair $\big(L^1(R^+), L^\infty(R^+)\big)$ and an admissible operator T_2 of the pair $\big(L^1(R^+), L^\infty(R^+)\big)$ into the pair $\big(L^1(\mathcal{M}_2), L^\infty(\mathcal{M}_2)\big)$ such that $T_1 f = f^*$ and $T_2 g^* = g$. Then if $h(t)$ is a bounded function such that $g^* = hS(\sigma)f^*$ and H is the operator multiplication by h, we have $g = T_2 HS(\sigma)T_1 f$. Now T_1 maps $L_{1/\alpha_i,1}(\mathcal{M}_1)$ continuously into $L_{1/\alpha_i,1}(R^+)$, $i = 1, 2$, and T_2 maps $L_{1/\beta_1,\infty}(R^+)$ continuously into $L_{1/\beta_i,\infty}(\mathcal{M}_2)$, $\beta_i < 1, i = 1, 2$, and this implies that $T = T_2 HS(\sigma) T_1$ belongs to $\mathcal{W}(\sigma, \mathcal{M}_1, \mathcal{M}_2)$. Thus T has the required properties and the first part of the theorem is established.

If either $\beta_1 = 1$ or $\beta_2 = 1$, then the operator $HS(\sigma)T_1$ belongs to $\mathcal{W}(\sigma, \mathcal{M}_1, R^+)$ but T does not belong to $\mathcal{W}(\sigma, \mathcal{M}_1, \mathcal{M}_2)$ in general. However, if g^* is continuous and strictly decreasing in the interval where it is positive, then a closer examination of the operator T_2 as constructed in lemma 2 shows that $T_2 k$ is equimeasurable with k whenever the support

of k is contained in the support of g^*, and consequently, as readily seen, T belongs to $\mathcal{W}(\sigma, \mathcal{M}_1, \mathcal{M}_2)$. Now if $\mathcal{M}_2$ is non-atomic the reader will have no difficulty in showing that given a function g on $\mathcal{M}_2$ such that $g^* \leqslant cS(\sigma)f^*$ there exists a function $\bar{g}$ such that $\bar{g} \geqslant |g|$ and $\bar{g}^* = cS(\sigma)f^*$ in the interval where g^* is positive. Then, according to what has been said above, there exists an operator T belonging to $\mathcal{W}(\sigma, \mathcal{M}_1, \mathcal{M}_2)$ such that $Tf = \bar{g}$. Now if h is a bounded function such that $g = \bar{g}h$ and H is the operator multiplication by h, then the operator HT belongs to $\mathcal{W}(\sigma, \mathcal{M}_1, \mathcal{M}_2)$ and $g = HTf$. The proof of the theorem is thus complete.

THEOREM 10. *Let σ be a segment contained in $0 \leqslant \alpha \leqslant 1, 0 \leqslant \beta < 1$, A_1 a linear space contained in $L_{1/\alpha_1,1}(\mathcal{M}_1)+L_{1/\alpha_2,1}(\mathcal{M}_1)$ and A_2 a linear space of measurable funtions on $\mathcal{M}_2$. Then a necessary and sufficient condition in order that every T in $\mathcal{W}(\sigma, \mathcal{M}_1, \mathcal{M}_2)$ map A_1 into A_2 is that $f \in A_1$ and $g^* \leqslant S(\sigma)f^*$ imply $g \in A_2$, for every measurable function g on $\mathcal{M}_2$.*

If $\mathcal{M}_2$ is non-atomic, the preceding statement is still valid if σ is merely contained in $0 \leqslant \alpha \leqslant 1, 0 \leqslant \beta \leqslant 1$.

If the segment σ is merely contained in $0 \leqslant \alpha \leqslant 1, 0 \leqslant \beta \leqslant 1$, the condition on the spaces A_1 and A_2 above is a sufficient condition in order that every quasilinear map T defined on $L_{1/\alpha_1,1}(\mathcal{M}_1)+L_{1/\alpha_2,1}(\mathcal{M}_1)$, with values in the space of measurable functions on $\mathcal{M}_2$ and which is simultaneously of weak types $(1/\alpha_1, 1/\beta_1)$, $(1/\alpha_2, 1/\beta_2)$, map A_1 ino A_2.

Proof. This statement is an immediate consequence of theorems 8 and 9.

COROLLARY. *Let σ be a segment contained in $0 \leqslant \alpha \leqslant 1, 0 \leqslant \beta < 1$, $(1/p, 1/q)$ a point interior to σ and T a quasilinear operator defined in $L_{1/\alpha_1,1}(\mathcal{M}_1)+L_{1/\alpha_2,1}(\mathcal{M}_1)$ with values in the space of measurable functions on $\mathcal{M}_2$ and which is simultaneously of weak types $(1/\alpha_1, 1/\beta_1)$, $(1/\alpha_2, 1/\beta_2)$. Then T maps $L_{p,r}(\mathcal{M}_1)$ into $L_{q,r}(\mathcal{M}_2)$ and there exists a constant c such that*

$$\|Tf\|_{q,r} \leqslant c\|f\|_{p,r}$$

for every f in $L_{p,r}(\mathcal{M}_1)$.

Proof. This is an immediate consequence of the preceding theorem and lemma 3, or also, of theorem 8 and lemma 3.

3. Optimal pairs of spaces. We remind the reader of the definitions of the classes $\mathcal{W}(\sigma), \mathcal{R}(\sigma), \mathcal{S}(\sigma)$. The first two are roughly speaking the classes of operators satisfying the hypotheses of the theorems of Marcinkiewicz and Riesz respectively. Given a pair of linear spaces A_1 and A_2 we will say that they are an *optimal pair* with respect to one of these classes if i) A_1 is contained in the common domain of the operators in $\mathcal{W}(\sigma)$, ii) if $g \in A_2$, there exists a T in the class and an f in A_1 such that $Tf = g$, iii) if f is a function in the common domain of the operators in the class and $Tf \in A_2$ for every T in the class, then $f \in A_1$. We will find sufficient conditions for a pair to be optimal with respect to $\mathscr{W}(\sigma)$ and exhibit pairs which are optimal with respect to certain $\mathscr{W}(\sigma)$, $\mathscr{R}(\sigma)$ and $\mathscr{S}(\sigma)$, showing thereby that the conclusions of the theorems of Marcinkiewicz and Riesz in the corresponding cases cannot be strengthened.

The measure spaces $\mathscr{M}$ involved in the discussion that follows will be subject to one of the conditions:

(P) there exists a sequence of sets E_n, $-\infty < n < \infty$, such that

$$E_n \supset E_{n-1}, \quad \mu(E_n) \leqslant c\mu(E_{n-1}),$$
$$\lim_{n\to-\infty} \mu(E_n) = 0, \quad \lim_{n\to\infty} \mu(E_n) = \mu(\mathscr{M}) \leqslant \infty,$$

where μ is the measure on $\mathscr{M}$, c is a constant and $\mu(\mathscr{M})$ is the total measure of the space.

(P′) the space is purely atomic, the atoms have measures bounded away from zero, and there exists a sequence of sets E_n, $1 \leqslant n < \infty$, of finite measure having, except for the third, the same properties as in (P).

THEOREM 11. *Let σ_1 and σ_2 be two segments in $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta < 1$, contained in lines which are symmetrical about the line $\alpha = \beta$. Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two measure spaces and A_1 and A_2 two linear spaces of measurable functions on $\mathscr{M}_1$ and $\mathscr{M}_2$ respectively with the property that*

$$f_2 \in A_2 \text{ implies } S(\sigma_2)f_2^* < \infty,$$
$$f_1 \in A_1 \text{ and } f_2^* \leqslant S(\sigma_1)f_1^* \text{ imply } f_2 \in A_2,$$
$$f_2 \in A_2 \text{ and } f_1^* \leqslant S(\sigma_2)f_2^* \text{ imply } f_1 \in A_1.$$

Let m be the slope of σ_1 and suppose that one of the following conditions is satisfied:

i) *$\mathscr{M}_1$ and $\mathscr{M}_2$ have infinite measure and satisfy condition* (P),

ii) *$\mathscr{M}_1$ and $\mathscr{M}_2$ have finite total measure and satisfy condition* (P), *and $m > 0$,*

iii) *one space has finite total measure and satisfies condition* (P), *the other has infinite measure and satisfies condition* (P′), *and $m < 0$.*

Then, if A_1 is contained in the common domain of the operators in $\mathscr{W}(\sigma_1, \mathscr{M}_1, \mathscr{M}_2)$, the pair A_1, A_2 is optimal with respect to $\mathscr{W}(\sigma_1, \mathscr{M}_1, \mathscr{M}_2)$.

If $\mathscr{M}_2$ is non-atomic, then the preceding statement remains valid if the segments σ_1 and σ_2 are merely contained in $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant 1$.

Proof. Without loss of generality we may assume that if one of the spaces under consideration has finite measure, then its total measure is equal to 1. We may also assume that spaces satisfying condition (P′) have 1 as a lower bound for the measures of their atoms.

We will begin by proving some properties of the operator $S(\sigma)$. Let $f(t)$ be a non-negative non-increasing function on $0 < t < \infty$ such that $S(\sigma)f < \infty$. Let $g = S(\sigma)f$. Then, if c is a positive constant, there is a constant c_1 such that $g(t) \leqslant c_1 g(ct)$. In fact, integration by parts gives

$$S(\sigma)f = \int_0^\infty \varphi(t, s)f(s)\frac{ds}{s} = -\int_0^\infty \psi(t, s)\,df(s)$$

and since $\psi(t, s) \leqslant c_1\psi(ct, s)$ the desired conclusion follows.

Let now f and g be as above and let m be the slope of σ. Then

$$f(t) \leqslant g(t^{1/m})t^{\beta/m-\alpha}, \quad g(t) \geqslant f(t^m)t^{\alpha m-\beta},$$

where (α, β) is any point in the line containing σ. In fact, if $\chi_t(s)$ is the characteristic function of the interval $0 < s \leqslant t^m$, then $f(s) \geqslant f(t^m)\chi_t(s)$ and consequently

$$g(t) \geqslant f(t^m)S(\sigma)\chi_t(s) = -f(t^m)\int_0^\infty \psi(t, s)\,d\chi_t(s) = f(t^m)\psi(t, t^m) = f(t^m)t^{\alpha m-\beta}.$$

The first inequality follows from the second by replacing t by $t^{1/m}$.

Let now f_1 be a measurable function on $\mathscr{M}_1$ such that $S(\sigma_1)f_1^* < \infty$. We will show that there exists a function f_2 on $\mathscr{M}_2$ such that $f_2^* \leqslant S(\sigma_1)f_1^*$ and $f_1^* \leqslant c_1 S(\sigma)f_2^*$. Let $g_1(t) = S(\sigma_1)f_1^*$ and let E_n be the subsets of $\mathscr{M}_2$ postulated in (P) or (P′) and a_n their measures. Define f_2 by $f_2 = g_1(a_n)$ on $E_n - E_{n-1}$ and $f_2 = 0$ elsewhere in the case when $\mathscr{M}_2$ satisfies condition (P), or $f_2 = g_1(a_1)$ on E_1, $f_2 = g_1(a_n)$ on $E_n - E_{n-1}$ and $f_2 = 0$ elsewhere in the case when $\mathscr{M}_2$ satisfies condition (P′). In both cases we will have $f_2^*(t) = g_1(a_n)$ for $a_{n-1} < t \leqslant a_n$. Since $a_n \leqslant ca_{n-1}$, if $a_{n-1} < t \leqslant a_n$, then $ct > a_n$ and therefore $f_2^*(t) \geqslant g_1(ct)$. But, as we saw above, there exists a constant c_1 such that $g_1(t) \leqslant c_1 g_1(ct)$ and therefore we will have

$$c_1 f_2^*(t) \geqslant g_1(t)$$

for all t, $t > 0$, if $\mathscr{M}_2$ has infinite measure and satisfies (P); for $0 < t \leqslant 1$ if $\mathscr{M}_2$ has total measure equal to 1 and satisfies (P) and for $t > 1$ if $\mathscr{M}_2$ has infinite measure and satisfies (P′).

Let now $g_2(t) = S(\sigma_2)f_2^*$. Then if (α, β) is a point in the line containing σ_1, (β, α) is a point in the line containing σ_2, and if m is the slope of σ_1, $1/m$ is the slope of σ_2. Consequently

$$f_1^*(t) \leqslant g_1(t^{1/m})t^{\beta/m-\alpha}, \quad f_2^*(t^{1/m})t^{\beta/m-\alpha} \leqslant g_2(t).$$

Combining these inequalities with the preceding one and assuming that one of the three conditions i), ii), iii) is satisfied we find that

$$f^*(t) \leqslant c_1 g_2(t)$$

for all t in the case i), for $0 < t \leqslant 1$ in the case ii) and for $0 < t \leqslant 1$ in the case iii) if $\mathscr{M}_1$ has measure equal to 1, or for $t > 1$ if the measure of $\mathscr{M}_1$ is infinite. According to our assumptions, if $\mathscr{M}_1$ has finite measure, then its total measure is equal to 1 and consequently $f_1^*(t) = 0$ for $t > 1$. On the other hand, if $\mathscr{M}_1$ satisfies condition (P′), then 1 is a lower bound for the measures of its atoms which implies that $f_1^*(t)$ is constant for $t \leqslant 1$. Consequently in all cases the preceding inequality holds for all t.

Evidently, the preceding argument remains valid if $\mathscr{M}_1$ and $\mathscr{M}_2$ are interchanged. Thus if f_2 is a measurable function on $\mathscr{M}_2$ such that $S(\sigma_2)f_2^* < \infty$, there exists a function f_1 on $\mathscr{M}_1$ such that $f_1^* \leqslant S(\sigma_2)f_2^*$ and $f_2^* \leqslant c_1 S(\sigma_1)f_1^*$.

Let now f_2 be any function in A_2 and f_1 a function on $\mathscr{M}_1$ such that $f_1^* \leqslant S(\sigma_2)f_2^*$ and $f_2^* \leqslant cS(\sigma_1)f_1^*$. Then according to our hypotheses $f_1 \epsilon A_1$ and according to theorem 9 there exists a T in $\mathscr{W}(\sigma_1, \mathscr{M}_1, \mathscr{M}_2)$ such that $Tf_1 = f_2$. Suppose now that f_1 is a function in the common domain of the operators in $\mathscr{W}(\sigma_1)$ such that $Tf_1 \epsilon A_2$ for every T in that class. Then $S(\sigma_1)f_1^* < \infty$ and there exists a function f_2 on $\mathscr{M}_2$ such that $f_2^* \leqslant S(\sigma_1)f_1^*$ and $f_1^* \leqslant cS(\sigma_2)f_2^*$. Since $f_2^* \leqslant S(\sigma_1)f_1^*$, it follows from theorem 9 that there exists a T in $\mathscr{W}(\sigma_1)$ such that $Tf_1 = f_2$. Consequently $f_2 \epsilon A_2$. But $f_1^* \leqslant cS(\sigma_2)f_2^*$ and therefore, according to our hypotheses, it follows that $f_1 \epsilon A_1$. This concludes the proof of the theorem.

THEOREM 12. *Let σ_1 be a segment contained in $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant 1$, $(1/p, 1/q)$ a point interior to σ_1, and $\mathscr{M}_1$, $\mathscr{M}_2$ two measure spaces such that one of the conditions* i), ii), iii) *of the preceding theorem is satisfied. Then the pair of spaces $L_{p,r}(\mathscr{M}_1)$, $L_{q,r}(\mathscr{M}_2)$, $1 \leqslant r \leqslant \infty$, is optimal with respect to $\mathscr{W}(\sigma_1, \mathscr{M}_1, \mathscr{M}_2)$. If in addition σ_1 is contained in the union of the square $0 < \alpha < 1$, $0 < \beta < 1$, and the segment $0 \leqslant \alpha \leqslant 1$, $\beta = 0$, then the pair is optimal with respect to $\mathscr{S}(\sigma_1, \mathscr{M}_1, \mathscr{M}_2)$. If σ_1 is contained in the union of the triangle $0 < \alpha \leqslant \beta < 1$ and the segment $0 \leqslant \alpha \leqslant 1$, $\beta = 0$, then the pair is optimal with respect to $\mathscr{R}(\sigma, \mathscr{M}_1, \mathscr{M}_2)$.*

Proof. The optimality of the pair with respect to $\mathscr{W}(\sigma_1, \mathscr{M}_1, \mathscr{M}_2)$ follows directly from lemma 3 and the preceding theorem. In order to obtain the desired result in the other two cases we let σ_2 be a segment in $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta < 1$, properly containing σ_1, and if σ_1 is in $0 < \alpha < 1$, $0 < \beta < 1$, actually containing σ_1 in its interior. Then it follows from the corollary to theorem 10 that $\mathscr{W}(\sigma_2) \subset \mathscr{S}(\sigma_1) \subset \mathscr{W}(\sigma_1)$, and if σ_1 is contained in the triangle $0 \leqslant \alpha \leqslant \beta \leqslant 1$, theorem 5 shows that $\mathscr{S}(\sigma_1) \subset \mathscr{R}(\sigma_1) \subset \mathscr{W}(\sigma_1)$. But, as we have just seen, the pair $L_{p,r}(\mathscr{M}_1)$, $L_{q,r}(\mathscr{M}_2)$ is optimal with respect to both $\mathscr{W}(\sigma_1)$ and $\mathscr{W}(\sigma_2)$ and thus, since $\mathscr{W}(\sigma_2) \subset \mathscr{S}(\sigma_1) \subset \mathscr{W}(\sigma_1)$ or $\mathscr{W}(\sigma_2) \subset \mathscr{R}(\sigma_1) \subset \mathscr{W}(\sigma_1)$, the pair is also optimal with respect to $\mathscr{S}(\sigma_1)$ or $\mathscr{R}(\sigma_1)$ in the respective cases. This concludes the proof of the theorem

Remarks. The reader interested in further results on spaces intermediate between L^1 and L^∞ should consult E. T. Oklander's dissertation [7]. Theorem 3 is closely related to some of his results. The linear case of the corollary to theorem 10 was obtained and announced by the author some years ago; more recently R. A. Hunt [6] established independently its extension to the quasilinear case. In concluding we would like to point out that in view of theorem 12, the corollary to theorem 10 gives the sharpest result obtainable from the hypotheses of the theorem of M. Riesz for exponents corresponding to points contained in the union of the triangle $0 < \beta \leqslant \alpha < 1$ with the segment $0 \leqslant \alpha \leqslant 1$, $\beta = 0$. What the best result is in the other cases remains an open problem.

Appendix. There is an interesting situation that arises in interpolation theory in connection with some important operators such as the Hilbert transform; namely, that of an operator T satisfying the weak type condition (1,1) and having a transposed T_1 satisfying the same condition. For such operators there is an estimate for $(Tf)^{**}$ analogous to that for $(Tf)^*$ given in theorem 8.

Let T be linear and map $L_{1,1}(\mathscr{M}_1)+L_{2,1}(\mathscr{M}_1)$ continuously into the space of measurable functions on $\mathscr{M}_2$ and satisfying the weak type conditions (1,1) and (2,2) for characteristic functions of sets of finite measure, and suppose that T has a transposed T_1 defined on $L_{1,1}(\mathscr{M}_2)+L_{2,1}(\mathscr{M}_2)$, mapping this space continuously into the space of measurable functions on $\mathscr{M}_1$, and also satisfying the weak type conditions (1,1) and (2,2) for characteristic functions of sets. By saying that T_1 is *transposed* to T we mean that

$$\int \chi_2(T\chi_1)\,d\mu_2 = \int \chi_1(T_1\chi_2)\,d\mu_1$$

for any two characteristic functions χ_1 and χ_2 of subsets of finite measure of $\mathscr{M}_1$ and $\mathscr{M}_2$ respectively. Then if χ is the characteristic function of a set of measure s

$$\text{i)}\quad (T\chi)^{**}(t) \leqslant c\varphi(s,t), \quad \varphi(s,t) = \begin{cases} 2+\ln(s/t) & \text{if } s \geqslant t, \\ (s/t)[2+\ln(t/s)] & \text{if } s \leqslant t. \end{cases}$$

In fact, we have

$$t(T\chi)^{**}(t) \leqslant 2\sup\left|\int \chi_1(T\chi)\,d\mu_2\right| = 2\sup\left|\int \chi(T_1\chi_1)\,d\mu_1\right|$$

where χ_1 ranges over all characteristic functions of subsets of $\mathscr{M}_2$ of measure less than or equal to t.

But on account of our hypothesis we have

$$\Big|\int \chi_1(T\chi)\,d\mu_2\Big| \leqslant c\int_0^t \min[(s/u), (s/u)^{1/2}]\,du,$$

$$\Big|\int \chi(T_1\chi_1)\,d\mu_1\Big| \leqslant c\int_0^s \min[(t/u), (t/u)^{1/2}]\,du,$$

whence a simple calculation gives the desired result.

Suppose now that T is a sublinear operator mapping $L^1(\mathscr{M}_1)$ continuously into the space of measurable functions on $\mathscr{M}_2$ and that T satisfies the condition i) above. Then

$$(1/2c)(Tf)^{**}(t) \leqslant (1/t)\int_0^s f^{**}(s)\,ds + \int_0^\infty f^{**}(s)\frac{ds}{s}.$$

To see this let us rewrite i) as follows:

$$(T\chi)^{**}(t) \leqslant -c\int_0^\infty \varphi(u,t)\,d\chi^*(u).$$

Then integrating twice by parts we obtain

$$(1/c)(T\chi)^{**}(t) \leqslant -\int_0^\infty u\chi^{**}(u)\frac{\partial^2}{\partial u^2}\varphi(u,t)\,du$$
$$= (1/t)\int_0^t \chi^{**}(u)\,du + \int_t^\infty \chi^{**}(u)\frac{du}{u}.$$

If f is now a simple integrable non-negative function we can write

$$f = \sum_1^n \lambda_n\chi_n$$

where $\lambda_n > 0$ and $\chi_1 \leqslant \chi_2 \leqslant \ldots \leqslant \chi_N$ are characteristic functions of sets of finite measure and we will have

$$f^{**} \leqslant \sum_1^N \lambda_n\chi_n^{**}.$$

Substituting in the preceding inequality we obtain

$$(Tf)^{**} = \Big[T\Big(\sum\lambda_n\chi_n\Big)\Big]^{**} \leqslant \Big[\sum\lambda_n|T\chi_n|\Big]^{**} \leqslant \sum\lambda_n(T\chi_n)^{**}$$
$$\leqslant c\sum\lambda_n\Big[(1/t)\int_0^t \chi_n^{**}(s)\,ds + \int_t^\infty \chi_n^{**}(s)\frac{ds}{s}\Big]$$
$$\leqslant c\Big[(1/t)\int_0^t f^{**}(s)\,ds + \int_t^\infty f^{**}(s)\frac{ds}{s}\Big].$$

To extend this result to f non-negative and merely integrable we take a monotone sequence of simple non-negative functions f_n converging to f in $L^1(\mathscr{M}_1)$. Then $f_n^{**}(t)$ converges monotonically to $f^{**}(t)$ and Tf_n converges in measure to Tf. But then $(Tf)^{**} \leqslant \lim(Tf_n)^{**}$ and the desired inequality follows by a passage to the limit. The general case follows now by setting $f = f_1 - f_2$ with f_1 and f_2 non-negative.

Our inequality for $(Tf)^{**}$ is equivalent to

$$(Tf)^{**}(t) \leqslant c\int_0^\infty \frac{f^{**}(s)}{(t^2+s^2)^{1/2}}\,ds$$

which was first obtained for the Hilbert transform by R. O'Neil and G. Weiss.

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