$R_1' \supset x'$, $Q'' \supset x''$. The requirements (a) and (b) eliminate only a subset of E of measure 0. Dividing both sides of (7.14) by $|Q''|$ and making the passage to the limit $|Q''| \to 0$, which is justified by (a), we see that

$$\int_{R_1'} g(\xi', x'')\, d\xi' \leqslant \varepsilon w^{kp} |R_1'|,$$

where now w is the largest edgelength of R_1'. The last inequality has been established for rational rectangles R_1' containing x', but, in view of (b), it holds, by continuity, for all rectangles R' containing x' and of diameter $\leqslant \delta$. This completes the proof of Theorem 9.

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On some properties of a class of singular integrals

by

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Introduction. Our purpose is to extend some known properties of the singular integrals of Calderón and Zygmund to a more general class of operators introduced in [5]. These singular integrals are convolution operators by quasihomogeneous kernels having mean value zero on certain differentiable manifold surrounding the origin (in the case of parabolic kernels, see [6]).

The aim of this paper is twofold. Firstly, we study the pointwise convergence of the quasi-homogeneous singular integrals and the behaviour of their maximal operators. Similar questions have been considered in our joint paper with E. B. Fabes (cf. [9]) for the different kind of parabolic singular integrals introduced by Jones in [4]. The same argument of [9], that is essentially a suitable modification of the method used by Calderón and Zygmund in [1], could be repeated for this general case, changing the computations to adequate them to the truncation of the kernels used here. Nevertheless it may be of interest to reconsider the question since an adaptation of the general method of "subordination of operators" given by Cotlar in [3], that can be used for the singular integrals of Calderón and Zygmund, enables us to get also a complementary result for the case $p = 1$ not considered in [9] and the pointwise convergence even for integrable functions.

Secondly, we consider the classes $T_u^p(x_0)$ studied by Calderón and Zygmund in [2], conveniently generalized, and prove that they are preserved under quasi-homogeneous singular integral operators.

In § 1 we give the definition of quasi-homogeneous functions and kernels and state some results about the singular integrals given by convolution with those kernels.

In § 2 we study the maximal operators of these integrals and obtain as a consequence that the quasi-homogeneous singular integrals converge in the pointwise sense for functions in L^p, $p \geqslant 1$.

In § 3 we give a generalization of the classes $T_u^p(x_0)$ to the case where a different number of derivations may be taken in each variable and prove some basic properties of these classes.

In § 4 we prove that the $T_u^p(x_0)$ classes are preserved by quasi-homogeneous singular integrals. The content of this paragraph was announced in [8] for the particular case of parabolic singular integrals.

Thanks are due to Prof. Zygmund and Prof. Calderón who proposed the questions considered here, to Prof. Cotlar for suggestions concerning the use of his method and to P. Krée for kindly making available his unpublished results.

1. In this paragraph we give some definitions and results which will be needed in the sequel.

By $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n), \ldots$ we denote points of the n-dimensional Euclidean space E_n. All functions we consider are complex-valued unless otherwise stated. C^m denotes the class of m times differentiable functions and C_0^m its subclass of those with compact support.

We give an n-tuple $a = (a_1, \ldots, a_n)$, *fixed* with $a_i \geqslant 1$, throughout the paper; as usual, we write $|a| = a_1 + \ldots + a_n$.

Quasi-homogeneous functions. We say that a function f is *quasi-homogeneous of degree* α, α any complex number, if

$$f(\lambda^{a_1}x_1, \ldots, \lambda^{a_n}x_n) = \lambda^\alpha f(x), \quad \forall \lambda > 0. \tag{1.1}$$

For these functions we have an analogue of Euler's formula

$$\sum_{i=1}^{n} a_i x_i \partial_i f = \alpha f, \quad \text{where} \quad \partial_i f = \partial f / \partial x_i, \tag{1.2}$$

and $\partial_i f$ is homogeneous of degree $\alpha - a_i$ for each $i = 1, \ldots, n$.

Distance. Let now $\varrho(x) > 0$ be a real-valued function defined for all $x \neq 0$, $\varrho \in C^1$ almost everywhere and quasi-homogeneous of degree 1, $\varrho(x+y) \leqslant \varrho(x) + \varrho(y)$. In particular, we shall consider

$$\varrho_0(x) = \sup_i |x_i|^{1/a_i} \quad \text{or} \quad \varrho_m(x) = \Big(\sum_{i=1}^{n} |x_i|^{m/a_i}\Big)^{1/m},$$

$m > 0$ being the least integer such that a_i divise $m/2$, as such functions $\varrho(x)$.

Then taking $[x-y] = \varrho(x-y)$ we have a distance defined on E_n, invariant by translations.

We shall use that $|x_i| \leqslant [x]^{a_i}$.

Differential form and element of volume in "polar" coordinates. Considering the differential form (cf. [5])

$$\sigma(x) = \sum_{i=1}^{n} a_i x_i d_i x, \tag{1.3}$$

$$\text{where} \quad d_i x = (-1)^{i-1} dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n,$$

we get by (1.2) that, for f quasi-homogeneous of degree α and differentiable,

$$df \wedge \sigma = \alpha f dx. \tag{1.4}$$

In particular

$$d\varrho \wedge \sigma = \varrho dx. \tag{1.4'}$$

Now we set $x_1 = \varrho(x)^{a_1} y_1, \ldots, x_n = \varrho(x)^{a_n} y_n$, where $y = (y_1, \ldots, y_n)$ is the point corresponding to x on the unit "sphere" $\varrho^{-1}(1)$, depending on $n-1$ variables, say $\theta_1, \ldots, \theta_{n-1}$. The point $x = (x_1, \ldots, x_n)$ can be given by its "polar" coordinates $(\varrho, \theta_1, \ldots, \theta_{n-1})$, where $\varrho = \varrho(x)$ and $\theta = (\theta_1, \ldots, \theta_{n-1})$ gives the position of y on $\varrho^{-1}(1)$. Then, by (1.3) and (1.4') the element of volume may be given as

$$dx = \varrho(x)^{|a|-1} d\varrho(x) \wedge \sigma(y). \tag{1.5}$$

Quasi-homogeneous kernels and singular integrals. In § 2 we shall deal with singular integrals, given as convolutions with kernels k, defined in $E_n - \{0\}$, with the following properties:

(A) $k(x)$ is a quasi-homogeneous function of degree $-|a|$, i.e.

$$k(\lambda^{a_1}x_1, \ldots, \lambda^{a_n}x_n) = \lambda^{-|a|} k(x), \quad \forall \lambda > 0.$$

(B_m) $k \in C^m(E_n - \{0\})$, $m \geqslant 1$.

(C) there exist a function ϱ, as characterized above, such that

$$\int_{\varrho^{-1}(1)} k\sigma = 0.$$

Taking into account (A), (3) and (4), it is $d(k\sigma) = 0$, so, using Stoke's theorem, condition (C) is satisfied independently of a particular ϱ.

P. Krée has introduced in [5] the operator given by convolution with such kernels and has proved that they are continuous from the Lebesgue space L^p into itself. That is, if

$$Kf(x) = \text{v.p.}\, k * f = \lim_{\varepsilon \to 0} K_\varepsilon f(x), \tag{1.6}$$

where

$$K_\varepsilon f(x) = \int_{[x-y] > \varepsilon} k(x-y) f(y)\, dy, \tag{1.7}$$

then for all $f \in L^p$, there exists a constant C, independent of f, such that

$$\|K_\varepsilon f\|_p \leqslant C\|f\|_p \text{ independently of } \varepsilon, \text{ and } \|Kf\|_p \leqslant C\|f\|_p, \tag{1.8}$$

for $1 < p < \infty$, where $\|\cdot\|_p$ stands for the usual norm in L^p.

Using the vocabulary of [10], we shall say that an operator T is *of type* (p, p) when it is continuous from L^p to L^p, $1 \leqslant p \leqslant \infty$, and that it is of *weak* type (1,1) when the measure of the set $\{x: |Tf(x)| > \lambda\}$ is bounded by $C(\|f\|_1/\lambda)$, for all $f \in L^1$, with C independent of f.

With these terminology, (1.8) means that K_ε and K are of type (p, p), $1 < p < \infty$ (K_ε independently of ε). It is also true that K_ε and K are of weak type (1,1) (cf. [5]).

In §§ 2 and 4 we shall deal with the maximal operator of these singular integrals, defined by

$$K_* f(x) = \sup_\varepsilon |K_\varepsilon f(x)|. \tag{1.9}$$

The following standard notation will be used throughout the paper: for any $\beta = (\beta_1, \ldots, \beta_n)$, $x^\beta = x_1^{\beta_1}, \ldots, x_n^{\beta_n}$, $\alpha\cdot\beta = \alpha_1\beta_1 + \ldots + \alpha_n\beta_n$, $\beta! = \beta_1! \ldots \beta_n!$, $(\partial^\beta f)(x) = (\partial_1^{\beta_1}) \ldots (\partial_n^{\beta_n}) f(x_1, \ldots, x_n)$. C will stand for a constant, not necessarily the same at each occurrence. Integrals without specification of the domain of integration will be understood as taken over the entire E_n.

2. Pointwise convergence and maximal operator. It is well known that the maximal operator of Hardy-Littlewood is of type (p, p), $1 < p \leqslant \infty$, and of weak type (1,1) (see [10]). It is not more difficult to see that the same is true for the maximal operator Λ defined by

$$\Lambda f(x) = \sup_\varrho \varrho^{-|\alpha|} \int_{[x-y]\leqslant\varrho} |f(y)|\, dy \tag{2.1}$$

(cf. [7], chap. I, lemma 2).

Lemma 2.1. *Let $\varphi(x)$ be a function defined in E_n, such that $|\varphi(x)| \leqslant \Phi([x])$, $\Phi(\varrho)$ being a non-increasing function and*

$$\int_0^\infty \Phi(\varrho)\varrho^{|\alpha|-1}\, d\varrho < \infty.$$

Let $\varphi_\varepsilon(x) = \varepsilon^{-|\alpha|}(\varepsilon^{-\alpha_1}x_1, \ldots, \varepsilon^{-\alpha_n}x_n)$.
Then, for $f \in C_0^\infty$, there exist a constant C, independent of f, such that

$$\sup_\varepsilon |\varphi_\varepsilon * f| \leqslant C\Lambda f.$$

Proof. We have

$$|\varphi_\varepsilon * f| = \varepsilon^{-|\alpha|} \int \varphi\big(\varepsilon^{-\alpha_1}(x_1-y_1), \ldots, \varepsilon^{-\alpha_n}(x_n-y_n)\big) f(y_1, \ldots, y_n)\, dy_1 \ldots dy_n$$
$$\leqslant \varepsilon^{-|\alpha|} \int \Phi(\varepsilon^{-1}[x-y])\, |f(y)|\, dy.$$

Let be

$$I(\delta) = \int_{[x-y]\leqslant\delta} |f(y)|\, dy.$$

By (2.1) it is

$$I(\delta) \leqslant \delta^{|\alpha|}\Lambda f(x). \tag{2.2}$$

The last integral becomes

$$\varepsilon^{-|\alpha|}\int_0^\infty \Phi(\delta/\varepsilon)\, dI(\delta) = \varepsilon^{-|\alpha|}\Phi(\delta/\varepsilon)I(\delta)\big|_0^\infty - \varepsilon^{-|\alpha|}\int_0^\infty I(\delta)\, d\Phi(\delta/\varepsilon).$$

By (2.2) the integrated part is bounded by $(\delta/\varepsilon)^{|\alpha|}\Phi(\delta/\varepsilon)(\Lambda f)(x)$. As by hypothesis over Φ we have $\Phi(\varrho)\varrho^{|\alpha|} \to 0$ when $\varrho \to 0$ and when $\varrho \to \infty$, this part vanishes.

Then

$$-\varepsilon^{-|\alpha|}\int_0^\infty I(\delta)\, d\Phi(\delta/\varepsilon) \leqslant -\varepsilon^{-|\alpha|}\Lambda f(x)\int_0^\infty \delta^{|\alpha|}\, d\Phi(\delta/\varepsilon)$$
$$= -\varepsilon^{-|\alpha|}(\Lambda f)(x)\Big\{\delta^{|\alpha|}\Phi(\delta/\varepsilon)\big|_0^\infty - |\alpha|\int_0^\infty \Phi(\delta/\varepsilon)\,\delta^{|\alpha|-1}\, d\delta\Big\}$$
$$= (\Lambda f)(x)\Big(\int_0^\infty \Phi(\varrho)\,\varrho^{|\alpha|-1}\, d\varrho\Big).$$

Since the last expression is independent of ε, the assertion follows.

Proposition. *For $f \in C_0^\infty$, let Kf and $K_\varepsilon f$ be defined as in* (1.6) *and* (1.7), *and let $k(x)$ satisfy conditions* (A), (B_1) *and* (C).
If

$$D_\varepsilon(f) = K_\varepsilon f(x) - Kf(x^1) + K(f\chi_Q)(x^1)$$

where χ_Q is the characteristic function of $Q = Q_\varepsilon = \{y: [x-y] \leqslant \varepsilon\}$ and $x^1 \in Q_{\varepsilon/2}$, then

$$D(f) = \sup_\varepsilon |D_\varepsilon(f)| \leqslant C\Lambda f(x).$$

Proof. We have

$$D_\varepsilon(f) = \int_{[x-y]>\varepsilon} k(x-y)f(y)\, dy - \text{v.p.}\int k(x^1-y)f(y)\, dy + \text{v.p.}\int_{[x^1-y]\leqslant\varepsilon} k(x^1-y)f(y)\, dy$$
$$= \int \big(k_\varepsilon(x-y) - k_\varepsilon(x^1-y)\big)f(y)\, dy + \int_{\substack{[x-y]>\varepsilon\\ [x^1-y]\leqslant\varepsilon}} k(x^1-y)f(y)\, dy +$$
$$+ \int_{\substack{[x^1-y]>\varepsilon\\ [x-y]\leqslant\varepsilon}} k(x^1-y)f(y)\, dy.$$

As $[x-x^1] \leqslant \varepsilon/2$,

$$|D_\varepsilon(f)| \leqslant \int k_\varepsilon(x-y)-k_\varepsilon(x^1-y)|\,|f(y)|\,dy + M \int_{\varepsilon/2<[x^1-y]\leqslant\varepsilon} [x^1-y]^{-|a|}\,|f(y)|\,dy +$$

$$+M \int_{\varepsilon/2<[x-y]\leqslant\varepsilon} [x^1-y]^{-|a|} f(y)\,dy = I_1+I_2+I_3.$$

It is easy to see that $I_2+I_3 \leqslant C\Lambda f(x)$.

To show that the same is true for I_1 we consider:

$$|k_1(x)-k_1(x^1)| \leqslant \sum_{i=1}^{n} |x_i-x_i^1|\,|\partial_i k_1(x-\theta x^1)| \qquad (0<\theta<1)$$

$$\leqslant CM \sum_{i=1}^{n} [x_i]^{-|a|-a_i}.$$

By these inequalities and the boundedness of $k(x)$, we get that

$$\varphi(x) = k_1(x)-k_1(x^1)$$

is in the conditions of lemma 2.1 and the proposition follows.

THEOREM 2.1. *The operator K_* acting on $f \in L^p$, $1 \leqslant p \leqslant \infty$, as defined in* (1.9) *for a kernel $k(x)$ satisfying* (A), (B_1) *and* (C), *is of type (p,p) for $1<p<\infty$ and of weak type* (1,1).

Proof. Let ε be such that $K_*f(x) \leqslant 2|K_\varepsilon f(x)|$.

For this chosen ε we take x^1, $[x-x^1] \leqslant \varepsilon/2$, and by the Proposition it is

$$K_*f(x) \leqslant 2|D_\varepsilon(f)|+2|Kf(x^1)|+2|K(f\chi_Q(x^1))|$$
$$\leqslant C\Lambda f(x)+2|Kf(x^1)|+2|K(f\chi_Q(x^1))|.$$

In [3], M. Cotlar proved that a linear operator T is of type (p,p) and of weak type (1,1) if it satisfies the inequality

$$(2.3)\qquad |Tf(x)| \leqslant C|T_1f(x)|+C|T_2f(x^1)|+C|T_3(f_Q)(x^1)|, \qquad \forall f \in D,$$

where T_1, T_2 and T_3 are operators of type (p,p) and of weak type (1,1) and D is any dense subset of all L^p, $1 \leqslant p < \infty$. As Λ and K are operators, both of type (p,p), $1<p<\infty$, and of weak type (1,1), and C_0^∞ is dense in all L^p, $1 \leqslant p < \infty$, the thesis follows by (2.3).

The next result may be of independent interest.

COROLLARY (Pointwise convergence). *If K_ε is defined as in* (1.7) *and $k(x)$ satisfies conditions* (A), (B_1) *and* (C) *then, for $f \in L^p$, $1 \leqslant p < \infty$,*

$$\lim_{\varepsilon\to 0} K_\varepsilon f(x) = Kf(x)$$

exists for almost every x.

Proof. Consider $\Delta(x;f) = \Delta(f) = \lim\sup K_\varepsilon f(x) - \lim\inf K_\varepsilon f(x)$. It is $|\Delta(f)| \leqslant 2K_*f(x)$.

For $f \in L^p$ let $f = g+h$, $g \in C_0^\infty$ and $h \in L^p$, $\|h\|_p < \delta$.

It is easy to see that

$$\lim_{\varepsilon\to 0} K_\varepsilon g(x) = Kg(x)$$

exists in the pointwise sense. So $\Delta(f) = \Delta(h)$.

If $p>1$, we get, by theorem 1, that

$$\|\Delta(h)\|_p \leqslant 2\|K_*h\|_p \leqslant C\delta,$$

δ arbitrarily small.

If $p=1$, for any $\lambda>0$, we get by theorem 1 that

$$\operatorname{meas}\{|\Delta(h)|>\lambda\} \leqslant \operatorname{meas}\{K_*f(x)>\lambda/2\} \leqslant C(\|h\|_1/\lambda) \leqslant C\delta/\lambda,$$

δ arbitrarily small.

Hence, in both cases, $\Delta(h) = 0$ a.e., and the assertion is true.

3. Now we characterize the $T_u^p(x_0)$ spaces.

Definition. If $u \geqslant -|a|/p$, we denote as $T_u^p(x_0)$ the class of functions $f \in L^p$, $1 \leqslant p \leqslant \infty$, such that there exists a polynomial

$$P(x-x_0) = \sum_{a\beta<u} c_\beta (x-x_0)^\beta \qquad (P \equiv 0 \text{ if } u \leqslant 0),$$

with the property that

$$(3.1)\qquad \sup_\varrho \varrho^{-u-|a|/p}\Big(\int_{[x-x_0]\leqslant\varrho} |f(x)-P(x-x_0)|^p dx\Big)^{1/p} < \infty.$$

PROPOSITION. *For each $f \in T_u^p(x_0)$ the polynomial P is unique.*

Proof. Considering two polynomials P_1 and P_2 satisfying (3.1), its difference

$$P(x-x_0) = P_1-P_2 = \sum_{a\beta<u} C_\beta (x-x_0)^\beta$$

will satisfy, for $0<\varrho<\infty$,

$$(3.2)\qquad \Big(\int_{[h]\leqslant\varrho} |P(h)|^p dh\Big)^{1/p} = O(\varrho^{u+|a|/p}), \quad \text{where} \quad h = x-x_0.$$

We must show that any such P vanish identically.

If $a\cdot\beta = \gamma < u$, that is, if $P(h)$ is quasi-homogeneous of degree γ, using the "polar" change of variables (1.5), we have

$$(3.3)\qquad \Big(\int_{[h]\leqslant\varrho} |P(h)|^p dh\Big)^{1/p} = C\Big(\int_{[h]=1} |P(h)|^p dh\Big)^{1/p} \varrho^{\gamma+|a|/p} = A\varrho^{\gamma+|a|/p}.$$

By (3.2), A must be zero, which implies that the polynomial vanishes.

If $a\cdot\beta \leqslant \gamma < u$, we use induction on γ, with the assumption that (3.2) implies $P\equiv 0$ as inductive hypothesis.

It is immediate that P vanishes for $\gamma = 0$. If $\gamma > 0$, then

$$P(h) = \sum_{a\beta<\gamma} C_\beta h^\beta + \sum_{a\beta=\gamma} C_\beta h^\beta = Q(h)+R(h)$$

and, by (3.2) and (3.3),

$$\Big(\int_{[h]\leqslant\varrho} |Q(h)|^p dh\Big)^{1/p} \leqslant \Big(\int_{[h]\leqslant\varrho} |P(h)|^p dh\Big)^{1/p} + \Big(\int_{[h]\leqslant\varrho} |R(h)|^p dh\Big)^{1/p} = O(\varrho^{u+|a|/p}) + O(\varrho^{\gamma+|a|/p}).$$

So, Q satisfies the inductive hypothesis and vanishes identically. The same is true for R, being quasi-homogeneous and the assumption follows.

Norm. The space $T_u^p(x_0)$ is linear and we introduce in it the norm

$$T_u^p(x_0; f) = \|f\|_p + \sum_{a\beta<u} |C_\beta| + \sup_{0<\varrho<\infty} \varrho^{-u-|a|/p} \Big(\int_{[x-x_0]\leqslant\varrho} |f(x) - P(x-x_0)|^p dx\Big)^{1/p}.$$

When there is no possibility of confusion we shall write $T_u^p(x_0; f) = T_u^p(f)$.

LEMMA 3.1. *The spaces $T_u^p(x_0)$, $1\leqslant p\leqslant\infty$, $u\geqslant -|a|/p$ are complete.*

Proof. Consider a sequence $\{f_j\}$ of functions such that $T_u^p(f_j - f_i)\to 0$ as j, i tend to infinity. Then the sequence converges in L^p to an f.

Let $P = \lim_{j\to\infty} P_j$, where P_j is the polynomial corresponding to f_j; the existence of such limit follows from the fact that the coefficients of P_j converge. For each ϱ we have

$$\varrho^{-u}\Big(\varrho^{-|a|}\int_{[h]\leqslant\varrho} \Big|\big(f(x_0+h) - f_j(x_0+h) - (P(h) - P_j(h)\big)\Big|^p dh\Big)^{1/p}$$
$$= \lim_{i\to\infty} \varrho^{-u}\Big(\varrho^{-|a|}\int_{[h]\leqslant\varrho} |(f_i - f_j) - (P_j - P_i)|^p dh\Big)^{1/p}$$
$$\leqslant \liminf_{i\to\infty} T_u^p(f_j - f_i) < \infty.$$

From this we see that $f\in T_u^p(x_0)$ and making j tend to infinity, we find

$$\sup_\varrho \varrho^{-u-|a|/p}\Big(\int_{[h]\leqslant\varrho} |(f-f_i) - (P-P_i)|^p dh\Big)^{1/p} \to 0.$$

From this follows $T_u^p(f-f_i)\to 0$ when $i\to\infty$, which completes the proof.

LEMMA 3.2. *For $1\leqslant p\leqslant\infty$, if $v\geqslant u\geqslant -|a|/p$, then $T_v^p(x_0)$ is continuously included in $T_u^p(x_0)$.*

Proof. We consider first the case $u\geqslant 0$. Let

$$P_u(h) = \sum_{a\beta<u} C_\beta h^\beta, \quad h = x - x_0,$$

be the polynomial that corresponds to f in the definition of the class $T_u^p(x_0)$ and let R_u be the corresponding remainder. Let P_v and R_v be similarly defined. For $\varrho\leqslant 1$,

$$|P_v(h) - P_u(h)| = \Big|\sum_{u\leqslant a\beta<v} C_\beta h^\beta\Big| \leqslant T_v^p(f)[h]^u$$

and

$$\Big(\int_{[h]\leqslant\varrho} |R_u(h)|^p dh\Big)^{1/p} \leqslant \Big(\int_{[h]\leqslant\varrho} |P_v - P_u|^p dh\Big)^{1/p} + \Big(\int_{[h]\leqslant\varrho} |R_v|^p dh\Big)^{1/p}$$
$$\leqslant C T_v^p(f)\varrho^{|a|/p+u} + T_v^p(f)\varrho^{|a|/p+v} \leqslant C T_v^p(f)\varrho^{|a|/p+u}.$$

For $\varrho > 1$, as

$$|P_u(h)| = \Big|\sum_{a\beta<u} C_\beta h^\beta\Big| \leqslant \sum_{a\beta<u} |C_\beta||h^\beta| \leqslant T_v^p(f)\varrho^u,$$

it holds

$$\Big(\int_{[h]\leqslant\varrho} |R_u(h)|^p dh\Big)^{1/p} \leqslant \|f\|_p + \Big(\int_{[h]\leqslant\varrho} |P_u(h)|^p dh\Big)^{1/p}$$
$$\leqslant T_v^p(f) + C T_v^p(f)\varrho^{|a|/p+u} \leqslant C T_v^p(f)\varrho^{|a|/p+u}.$$

Thus,

$$\sup_\varrho \varrho^{-u-|a|/p}\Big(\int_{[h]\leqslant\varrho} |R_u(h)|^p dh\Big)^{1/p} \leqslant C T_v^p(f)$$

which is the assertion.

Second case, $u < 0$. If $\varrho\geqslant 1$,

$$\varrho^{-u}\Big(\varrho^{-|a|}\int_{h\leqslant\varrho} |f(x_0+h)|^p dh\Big)^{1/p} \leqslant \|f\|_p$$

and if $\varrho < 1$,

$$\varrho^{-u}\Big(\varrho^{-|a|}\int_{[h]\leqslant\varrho} |f(x_0+h)|^p dh\Big)^{1/p} \leqslant \varrho^{-v}\Big(\varrho^{-|a|}\int_{[h]\leqslant\varrho} |f(x_0+h)|^p dh\Big)^{1/p}$$

and again the thesis follows.

LEMMA 3.3. *Given $f\in T_u^p(x_0)$, $u\geqslant -|a|/p$, let $g(x)$ be such that*

$$\sup_\varrho \varrho^{-u-|a|/p}\Big(\int_{[x-x_0]\leqslant\varrho} |g(x)|^p dx\Big)^{1/p} \leqslant T_u^p(f).$$

Then

(a) $$\int_{[x-x_0]\leqslant\varrho} [x-x_0]^{-s}|g(x)|\,dx \leqslant CT_u^p(f)\,\varrho^{|a|+u-s} \quad \text{if} \quad |a|+u>s$$

and

(b) $$\int_{[x-x_0]\geqslant\varrho} [x-x_0]^{-s}|g(x)|\,dx \leqslant CT_u^p(f)\,\varrho^{|a|+u-s} \quad \text{if} \quad |a|+u<s.$$

Proof. We set

$$\Phi(\varrho) = \int_{[x-x_0]\leqslant\varrho} |g(x)|\,dx.$$

By Hölder's inequality,

$$|\Phi(\varrho)| \leqslant C\Big(\int_{[x-x_0]\leqslant\varrho} |g|^p\,dx\Big)^{1/p}\varrho^{|a|/p'} \leqslant CT_u^p(f)\,\varrho^{u+|a|(1/p+1/p')}.$$

Hence, if $|a|+u>s$, then

$$\int_{\delta<[x-x_0]<\varrho} [x-x]^{-s}|g(x)|\,dx = \int_\delta^\varrho r^{-s}\,d\Phi(r) \leqslant \varrho^{-s}\Phi(\varrho)+s\int_0^\varrho r^{-s-1}\Phi(r)\,dr$$
$$\leqslant T_u^p(f)\,C\Big(\varrho^{|a|+u-s}+s\int_0^\varrho r^{|a|+u-s-1}\,dr\Big) \leqslant CT_u^p(f)\,\varrho^{|a|+u-s},$$

so that

$$\int_{[x-x_0]\leqslant\varrho} [x-x_0]^{-s}|g(x)|\,dx \leqslant CT_u^p(f)\,\varrho^{|a|+u-s}, \quad \text{if} \quad |a|+u>s,$$

and (a) is proved.

Similarly, if $|a|+u<s$, then

$$\int_{[x-x_0]\geqslant\varrho} [x-x_0]^{-s}|g(x)|\,dx = \int_0^\infty r^{-s}\,d\Phi(r) \leqslant s\int_\varrho^\infty r^{-s-1}\Phi(r)\,dr \leqslant CT_u^p(f)\,\varrho^{|a|+u-s},$$

and (b) is proved.

4. Convolution of quasi-homogeneous kernels and $T_u^p(x_0)$ functions. Now we are in condition to state and prove the theorem about quasi-homogeneous singular integrals acting on $T_u^p(x_0)$.

THEOREM. *Let K be an operator as defined in* (1.6) *given by convolution with a kernel $k(x)$ satisfying conditions* (A), (B$_\infty$) *and* (C). *Let M be a bound for $|\partial^\beta k(x)|$ on the set when $[x]=1$ for $0\leqslant a\cdot\beta\leqslant u+1$ if $u>0$ and for $|\beta|=0$ if $u\leqslant 0$.*

Then, if $1<p<\infty$, $u\geqslant -|a|/p$ and $f\in T_u^p(x_0)$,

(I) *if $u\neq 0,1,2,\ldots$, then $Kf\in T_u^p(x_0)$ and $T_u^p(Kf)\leqslant CT_u^p(f)$.*

(II) *if $u=1,2,\ldots$, then $Kf\in T_u^p(x_0)$ provided that*

$$\int [x-x_0]^{-|a|-u}|f(x)|\,dx = N<\infty, \quad \textit{and} \quad T_u^p(Kf)\leqslant CT_u^p(f)+MN.$$

(III) *if $u=0$ and $K_*f(x_0)<\infty$, then $Kf\in T_0^p(x_0)$ and*

$$T_0^p(Kf)\leqslant CT_0^p(f)+CK_*f(x_0).$$

Proof. (I) Let be a fixed $\varphi\in C_0^\infty(E_n)$ such that $\varphi(x)=1$ if $[x-x_0]\leqslant 1$. We set $f=f_1+f_2$, where $f_1=\varphi P$,

$$P(x-x_0) = \sum_{a\beta<u} C_\beta(x-x_0)^\beta$$

being the polynomial corresponding to $f\in T_u^p(x_0)$.

Then $f_1\in T_u^p(x_0)$ and $T_u^p(f_1)\leqslant CT_u^p(f)$. Consequently $f_2\in T_u^p(x_0)$ and $T_u^p(f_2)\leqslant CT_u^p(f)$. Then it will be enough to apply the operator K to f_1 and f_2 and prove the result for each of them.

First we consider $\psi\in C_0^\infty(E_n)$, any such function vanishing outside $[x-x_0]\leqslant R$.

$$K\psi(x) = \lim_{\varepsilon\to 0}\int_{[y]>\varepsilon} k(y)\psi(x-y)\,dy = \lim_{\varepsilon\to 0}\int_{[y]>\varepsilon} k(y)\big(\psi(x-y)-\psi(x)\big)\,dy$$

is an integral which converges uniformly for $\varepsilon\to 0$ and $|K\psi|\leqslant C(\psi)M$. Since $\partial_i(K\psi)=K(\partial_i\psi)$, then $K\psi\in C^\infty$ and $|\partial^\beta K\psi|\leqslant C(\psi)M$. Also we see that, for $[x-x_0]>2R$, $|K\psi(x)|\leqslant C(\psi)M[x-x_0]^{-|a|}$. These inequalities show that $\|K\|_p\leqslant C(\psi)M$ and that

(4.1) $$T_u^p(K\psi)\leqslant C(\psi)M.$$

Considering now the explicit form of f_1,

$$f_1(x) = \varphi(x)\sum_{a\beta<u} C_\beta(x-x_0)^\beta,$$

we have

(4.2) $$T_u^p(Kf_1)\leqslant \sum_{a\beta<u}|C_\beta|\,T_u^p\Big(K\big(\varphi(x)(x-x_0)^\beta\big)\Big).$$

But $\varphi(x)(x-x_0)^\beta$ is in the conditions of $\psi(x)$ and we apply (4.1) to it. Then (4.2) becomes

$$T_u^p(Kf_1)\leqslant \sum_{a\beta<u}|C_\beta|\,CM\leqslant CMT_u^p(f).$$

We consider now the case $u>0$.

f_2 is a function that satisfies the hypothesis of Lemma 4.

We consider Kf_2 when $u > 0$ expanding $k(x)$ by Taylor's formula at the point x_0:

$$Kf_2(x) = \text{v.p.}\int k(x-y)f(y)\,dy = \text{v.p.}\int_{[y-x_0]\leqslant\varrho} + \int_{[y-x_0]>\varrho}$$

$$= \text{v.p.}\int_{[y-x_0]\leqslant\varrho} k(x-y)f_2(y)\,dy + \sum_{\alpha\beta<u}\frac{(x-x_0)^\beta}{\beta!}\int \partial^\beta k(x_0-y)f_2(y)\,dy -$$

$$-\sum_{\alpha\beta<u}\frac{(x-x_0)^\beta}{\beta!}\int_{[y-x_0]\leqslant\varrho}\partial^\beta k(x_0-y)f_2(y)\,dy +$$

$$+\sum_{\substack{\alpha\beta\geqslant u\\|\beta|\leqslant u}}\frac{(x-x_0)^\beta}{\beta!}\int_{[y-x_0]>\varrho}\partial^\beta k(x_0-y)f_2(y)\,dy +$$

$$+\sum_{u<|\beta|\leqslant u+1}\frac{(x-x_0)^\beta}{\beta!}\int_{[y-x_0]>\varrho}\partial^\beta k\big(x_0+\theta(x-x_0)-y\big)f_2(y)\,dy$$

$$= I_1+I_2+I_3+I_4+I_5 \qquad (0<\theta<1).$$

I_2 is a sum of integrals which, on account of

$$|\partial^\beta k(x)| \leqslant M[x]^{-|\alpha|-\alpha\beta} \tag{4.3}$$

and part (a) of Lemma 3.3, are absolutely convergent near x_0. Using Hölder's inequality and $\|f_2\|_p \leqslant T_u^p(f)$, we see that they are convergent at infinity. So, I_2 is a polynomial, that we call $P(x-x_0)$, whose coefficients are dominated by $CMT_u^p(f)$.

For $[y-x_0] > \varrho$,

$$\big|\partial^\beta k\big(x_0+\theta(x-x_0)-y\big)\big| \leqslant M[x_0-y]^{-|\alpha|-\alpha\beta}$$

if we assume $[x-x_0]\leqslant\varrho/2$. So I_5 is dominated by $CMT_u^p(f)\,\varrho^{u-\alpha\beta}[x-x_0]^{\alpha\beta}$, by part (b) of Lemma 3.3, as $|\beta|>u$ implies that, a fortiori, $\alpha\cdot\beta>u$.

Similarly, by (4.3) and Lemma 3.3, I_3 and I_4 are bounded by $CMT_u^p(f)\,\varrho^{u-\alpha\beta}[x-x_0]^{\alpha\beta}$.

Finally, consider I_1. Its norm in L^p is dominated by a constant times

$$\Big(\int_{[x_0-y]\leqslant\varrho}|f_2(y)|^p dy\Big)^{1/p},$$

so

$$\|I_1\|_p \leqslant CT_u^p(f)\,\varrho^{u+|\alpha|/p}.$$

Combining this estimates, we obtain that

$$\Big(\int_{[x-x_0]\leqslant\varrho/2}|f_2(x)-P(x-x_0)|^p dx\Big)^{1/p} \leqslant CT_u^p(f)\,\varrho^{u+|\alpha|/p}.$$

Since $\|Kf_2\|_p\leqslant C\|f_2\|_p$, in account of the estimates for the coefficients of $P(x-x_0)$, we see that $T_u^p(Kf_2)\leqslant CT_u^p(f)$. As the same holds for Kf_1 the proof of (I) is completed for $u>0$.

For $u<0$, we obtain that $f_1 = 0$ and $f=f_2$. We write

$$Kf(x) = \text{v.p.}\int_{[x_0-y]\leqslant\varrho} k(x-y)f(y)\,dy + \int_{[x_0-y]>\varrho} k(x-y)f(y)\,dy = \mathrm{I}+\mathrm{II}.$$

I is estimated as before. On account of part (b) of Lemma 3.3 and as, for $[y-x_0]>\varrho$, $[x-x_0]\leqslant\varrho/2$ it is $|k(x-y)|\leqslant CM[y_0-x]^{-|\alpha|}$, we infer that II is dominated by $CMT_u^p(f)\,\varrho^u$. So, part (I) follows for $u<0$.

(II) The above proof can be mantained for $u = 1,2,3,\ldots$ except at the point where we consider the integrals of the functions $\partial^\beta k(x_0-y)f_2(y)$ for $\alpha\cdot\beta = u$, because these will no longer be convergent. Under the additional assumption the polynomial $P(x-x_0)$ vanishes, so $f_2 = f$ and the integrals converge.

(III) Case $u=0$. Let $f = g+h$, where $g(x) = f(x)$ if $[x-x_0]<2\varrho$ (ϱ fixed) and $g(x)=0$ otherwise. Then

$$\Big(\int_{[x-x_0]\leqslant\varrho}|Kf(x)|^p dx\Big)^{1/p} \leqslant \Big(\int_{[x-x_0]\leqslant\varrho}|Kg(x)|^p dx\Big)^{1/p} + \Big(\int_{[x-x_0]\leqslant\varrho}|Kh(x)|^p dx\Big)^{1/p}$$

$$\leqslant C\|g\|_p + \Big(\int_{[x-x_0]\leqslant\varrho}\Big|\int_{[x_0-y]>2\varrho}k(x-y)f(y)\,dy\Big|^p dx\Big)^{1/p} \leqslant CT_0^p(f)\,\varrho^{|\alpha|/p} +$$

$$+\Big(\int_{[x-x_0]\leqslant\varrho}\Big|\int_{[x_0-y]>2\varrho}k(x-y)-k(x_0-y)f(y)\,dy\Big|^p dx\Big)^{1/p} + C\varrho^{|\alpha|/p}K_*f(x_0).$$

We get that $|k(x-y)-k(x_0-y)|\leqslant CM\sum_{i=1}^n|x_i-x_{0i}|\,[x_0-y]^{-|\alpha|-\alpha_i}$ for $[x_0-y]>2\varrho$ and $[x-x_0]\leqslant\varrho$. So,

$$\Big|\int_{[x_0-y]>2\varrho}\big(k(x-y)-k(x_0-y)\big)f(y)\,dy\Big| \tag{4.4}$$

$$\leqslant \sum CM|x_i-x_{0i}|\int_{[x_0-y]>2\varrho}|f(y)|\,[x_0-y]^{-|\alpha|-\alpha_i}dy.$$

Let

$$\Phi(\varrho) = \int_{[x_0-y]\leqslant\varrho}|f(y)|\,dy.$$

By Hölder's inequality, $\Phi(\varrho)\leqslant CT_0^p(f)\,\varrho^{|\alpha|}$. Then the last term of (4.4) is less than or equal to

$$CM|x_i-x_{0i}|\int_{2\varrho}^\infty r^{-|\alpha|-\alpha_i}d\Phi(r)$$

$$= \sum CM|x_i-x_{0i}|\Big(\Phi(r)r^{-|\alpha|-\alpha_i}\Big|_{2\varrho}^\infty + 2|\alpha|\int_{2\varrho}^\infty\Phi(r)r^{-|\alpha|-\alpha_i-1}\,dr\Big)$$

$$\leqslant \sum CMT_0^p(f)\,|x_i-x_{0i}|\,\varrho^{-\alpha_i}.$$

Applying this last inequality it results

$$\Big(\int\limits_{[x-x_0]\leqslant\varrho}\Big|\int\limits_{[x_0-y]>2\varrho}\big(k(x-y)-k(x_0-y)\big)f(y)\,dy\Big|^p\,dx\Big)^{1/p}$$

$$\leqslant\Big(\int\limits_{[x-x_0]\leqslant\varrho}\Big|\sum CMT_0^p(f)\,\varrho^{-a_i}|x_i-x_{0i}|\Big|^p\,dx\Big)^{1/p}\leqslant CMT_0^p(f)\,\varrho^{|a|/p}.$$

This completes the proof of (III).

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