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Means and Følner condition on locally compact groups

by

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In the last years several papers by different authors appeared, all dealing with some conditions imposed on a locally compact topological group which imply various properties of spaces of functions intrinsically connected with the group which are not valid for all locally compact topological groups in general. Let us mention only three of the results of this type.

In 1948 Godement [11] defined the class (R) of locally compact topological groups such that any continuous positive definite function f on G is the limit of a net of functions $\sum_i x_i * \tilde{x}_i$ with $x_i \in L_2(G)$ convergent to f uniformly on compact sets. In 1960 Reiter [19] considered the class $\mathscr{R}$ of groups G such that for any compact subset A of G and any $\varepsilon > 0$ there exists a function $f \in L_1(G)$ such that $f \geqslant 0$ and $\|f\|_1 = 1$ and, moreover,

$$\int |f(t^{-1}s) - f(s)|\, ds < \varepsilon \quad \text{for all} \quad t \in A.$$

Dieudonné and Reiter in a series of papers [5], [6], [19], [20], [21] showed that the groups of class $\mathscr{R}$ have many interesting properties concerning the convex hulls of translations of functions from L_1 and the behavior of convolution operators defined by symmetric probability measures on G. They investigated also the class $\mathscr{R}$ in terms of subgroups, homomorphic images, extensions. We should also mention here several attempts made towards generalization of the many well-known results concerning invariant means on bounded functions on a discrete group to locally compact topological groups (with L_∞ in place of the space of bounded functions) [6] and [14]. Very recently Reiter [23] has shown that $(R) = \mathscr{R}$.

The aim of this paper is to propose a new generalization of the notion of an invariant mean value on $L_\infty(G)$ for a locally compact group. The class $\mathscr{I}$ of the groups for which such an invariant mean exists ap-

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pears to coincide with the classes (R) and $\mathscr{R}$ and the fact that it contains Abelian and compact groups as well as that it is closed with respect to taking subgroups, homomorphic images and extensions becomes as simple to prove as it is in the case of discrete groups. We also obtain Følner's conditions whose combinatorial character shows that the class $\mathscr{I}$ can be defined intrinsically not appealing to any fonction space on the group.

Our indebtedness to the celebrated method of Day of treating invariant means on discrete groups in terms of weak and strong invariance is obvious. We also benefited from some recent ideas of Namioka as presented in [18]. We want to express our gratitude to Issac Namioka for conversations about the subject of this paper as well as to Fred Greenleaf and Czesław Ryll-Nardzewski for many corrections.

Preliminaries. All the groups considered here are locally compact topological groups. The class $\mathscr{B}$ of Baire subsets is the least σ-ring of subsets of the group containing all the sets of the form $\{s: f(s) \geqslant 0\}$ where f are real continuous functions on the group vanishing outside compact sets. The left-invariant Haar measure defined on $\mathscr{B}$ is denoted by $|A|$ for $A \in \mathscr{B}$. In what follows, we shall not use any other Haar measurable sets than those of $\mathscr{B}$. If $A, B \in \mathscr{B}$, then the function

$$f(s, t) = |sA \Delta tB|$$

is a $\mathscr{B}\times\mathscr{B}$-measurable function on the product of the group with itself. The differential of the left-invariant Haar measure is denoted by ds and the Radon-Nikodym derivative of the right-invariant Haar measure with respect to the left-invariant Haar measure by $\Delta(s)$. We have

$$\int f(s^{-1})\Delta(s^{-1})\,ds = \int f(s)\,ds$$

for any left-integrable function f.

By L (or $L(G)$) we denote the linear space of continuous functions on a group G vanishing outside compact sets. For any $1 \leqslant p < \infty$ we denote by L_p (or $L_p(G)$) the space of $\mathscr{B}$-measurable functions x integrable with the p-th power equiped with the norm $\|x\|_p$. If $p = 1$ we shall often write $\|x\|$ instead of $\|x\|_1$. The space L_∞ (or $L_\infty(G)$) consists of the essentially bounded locally Baire measurable functions on G. For any $1 \leqslant p < \infty$ the space L is norm dense in L_p. We consider also the space of Borel measures M (or $M(G)$). By δ_s, $s \in G$, we denote the measure concentrated at the point s and the total mass 1. If $1 \leqslant p < \infty$, the space L_q with $1/p + 1/q = 1$ is the space of linear bounded functionals on L_p. The value of the functional defined by $f \in L_q$ on the element $x \in L_p$ is denoted by (f, x). For a function $f \in L_p$, $1 \leqslant p \leqslant \infty$, and a measure $\mu \in M$, we define the convolution $f*\mu(s) = \int f(st^{-1})\Delta(t^{-1})\,d\mu(t)$.

Clearly, $f*\mu \in L_p$. If $d\mu(s) = x(s)\,ds$, where $x \in L_1$, then

$$f*x(s) = \int f(t)x(t^{-1}s)\,dt = \int f(st^{-1})x(t)\Delta(t^{-1})\,dt.$$

For any function $x \in L_p$, $1 \leqslant p \leqslant \infty$, and any $\mu \in M$ we denote by x^*, $x^\sim$ and μ^*, $\mu^\sim$ the functions and measures

$$x(s) = \overline{x(s^{-1})}\Delta(s^{-1}), \qquad x^\sim(s) = \overline{x(s^{-1})},$$

$$\mu^*(E) = \mu(E^{-1}), \qquad \mu^\sim(E) = \int_{E^{-1}} \Delta(t^{-1})\,d\mu(t).$$

We have

$$(x, y) = \overline{(y, x)}, \qquad (x, y*z) = (y*x, z) = (x*z^\sim, y),$$

whatever functions or measures x, y, z are such that the corresponding integrals are absolutely convergent. If $f \in L_p$, $1 \leqslant p \leqslant \infty$, $x \in L_1$, $\mu \in M$, then

$$(f*x)*\mu = f*(x*\mu).$$

We also have

$$f*\delta_s^\sim = f_s, \quad \text{where} \quad f_s(t) = f(ts),$$
$$\delta_s*f = {}_sf, \quad \text{where} \quad {}_sf(t) = f(s^{-1}t).$$

We shall use many other properties of the convolution of functions and measures which are not listed here, for the general reference we send the reader e.g. to [13].

A net $\{x_\gamma\}$, $x_\gamma \in L_\infty$, is said to be *convergent almost uniformly to a function* x on a set A if there is a fixed number M such that $\|x\|_\infty < M$ and for any $\varepsilon > 0$ and $\delta > 0$ there exists a γ_0 such that if $\gamma > \gamma_0$, then

$$|x_\gamma(s) - x(s)| < \varepsilon \text{ for all } s \in A \setminus B_\gamma, \qquad \text{where} \qquad |B_\gamma| < \delta.$$

PROPOSITION 0.1. *If a net $\{x_\gamma\}$, $x_\gamma \in L_\infty$, is almost uniformly convergent to a function x on a compact set A, then for any $\alpha, \beta \in L$ the net $\{\alpha*x*\beta\}$ is uniformly convergent to the function $\alpha*x*\beta$ on A.*

A continuous function f on G is called *positive definite* if for any $x \in L_1(G)$

$$\int\int f(s^{-1}f)x(s)\overline{x(t)}\,ds\,dt \geqslant 0.$$

A continuous positive definite function is necessarily bounded on G and $|f(s)| \leqslant |f(1)|$, $s \in G$. For any $x \in L_2$ the function $x*x^\sim$ is continuous positive definite. If $x \in L$, then $x*x^\sim \in L_2$ and is positive definite.

PROPOSITION 0.2 (cf. [11]). *For any continuous positive definite function $f \in L_2$ and any $\varepsilon > 0$ there exists a function $x \in L$ such that*

$$|x*x^\sim(s) - f(s)| < \varepsilon \quad \textit{for any} \quad s \in G.$$

PROPOSITION 0.3. *For any $u \in L_2$ and any $\varepsilon > 0$ there exists a function $x \in L$ such that*

$$|x*x^{\sim}(s) - u*u^{\sim}(s)| < \varepsilon \quad \text{for all} \quad s \in G.$$

Let Φ denote the class of functions α such that

(0.1) $$\alpha(s) \geqslant 0 \quad \text{for} \quad s \in G,$$

(0.2) $$\int \alpha(s)\,ds = 1.$$

Let G be a group and H a normal subgroup of G. Denote by π the homomorphism $\pi: G \to G/H$ and let $\pi(s) = \dot{s}$. Let $ds, \overline{dh}, d\dot{s}$ denote the differentials of the left-invariant Haar measures on G, H and G/H, respectively, such that for any $x \in L_1(G)$ we have

(0.3) $$\int x(s)\,ds = \int_{G/H} d\dot{s} \int_H x(sh)\,\overline{dh}.$$

Let $\Delta(s)$ and $\varrho(h)$ denote the Radon-Nikodym derivatives of the right-invariant Haar measure with respect to the left-invariant measure on G and H, respectively.

We now define a mapping of $L_1(G)$ into $L_1(G/H)$,

$$x \to \dot{x},$$

where

$$\dot{x}(\dot{s}) = \int x(sh)\,\overline{dh} \quad (\dot{s} = \pi(s)).$$

It is clear that if $x \in \Phi(G)$, then $\dot{x} \in \Phi(G/H)$. We show that any $\alpha \in \Phi(G/H)$ is of this form. Let F be a continuous function on G such that for any $s \in G \int F(sh)\,\overline{dh} = 1$ and $F(s) \geqslant 0$. (For the proof of the existence of such a function see [1].)

Let

(0.4) $$\beta(s) = \alpha(\pi(s)) \cdot F(s).$$

We then have

$$\int \beta(sh)\,\overline{dh} = \int \alpha(\pi(sh)) F(sh)\,\overline{dh} = \alpha(\dot{s})$$

and hence, by (0.3),

$$\int \beta(s)\,ds = 1.$$

PROPOSITION 0.4. *Let $f \in L_\infty(G)$ and suppose that f is constant on the cosets $sH = Hs$, $s \in G$. Then f defines a function $\dot{f} \in L_\infty(G/H)$ and $f = \dot{f} \circ \pi$. Let $\alpha \in \Phi(G)$. Then $\alpha * f$ is constant on the cosets $sH = sH$ and*

$$(\alpha * f)^{\cdot} = \dot{\alpha} * \dot{f}, \quad (f * \alpha^{\sim})^{\cdot} = \dot{f} * \dot{\alpha}^{\sim}.$$

In fact,

$$(\alpha * f)^{\cdot}(\dot{s}) = \alpha * f(s) = \int \alpha(t) f(\pi(t^{-1}s))\,dt$$
$$= \int d\dot{t} \int \alpha(th) \dot{f}(\dot{t}^{-1}\dot{s})\,\overline{dh} = \int \dot{\alpha}(\dot{t}) \dot{f}(\dot{t}^{-1}\dot{s})\,d\dot{t} = \dot{\alpha} * \dot{f}(\dot{s}).$$

The second of the equalities is verified in the same way.

PROPOSITION 0.5. *Let*

$$p(s) = \begin{cases} \Delta^{-1}(s)\varrho(s) & \text{for} \quad s \in H, \\ 0 & \text{for} \quad s \in G \setminus H. \end{cases}$$

Then for any $f \in L_\infty(G)$ and $x \in L_1(G)$ we have

$$\sup_{h \in H} |p(h) \cdot f * x^{\sim}(h)| \leqslant \|f\|_\infty \|x\|_1.$$

In fact,

$$|p(h) \cdot f * x^{\sim}(h)| \leqslant \|f\|_\infty \int |x(h^{-1}s) \Delta^{-1}(h) \varrho(h)|\,ds$$
$$= \|f\|_\infty \int |x(sh) \varrho(h)|\,ds = \|f\|_\infty \int |x(sgh) \varrho(h)|\,\overline{dg}\,d\dot{s}$$
$$= \|f\|_\infty \int |x(sg)|\,\overline{dg}\,d\dot{s} = \|f\|_\infty \|x\|_1.$$

1. Means. Let G be a group, X a closed subspace of $L_\infty(G)$ with the property that if $f \in X$, then the complex conjugate $\bar{f} \in X$.

A linear functional m on X is called a *mean* on X if it has the following properties:

(i) $\overline{mf} = m\bar{f}$.

(ii) $\operatorname*{ess\,inf}_{s \in G} f(s) \leqslant mf \leqslant \operatorname*{ess\,sup}_{s \in G} f(s)$ for any real $f \in X$.

Any mean is necessarily bounded. A mean m is called *finite* if m is defined by a Borel measure μ on G, i.e. if

$$mf = \int f(s)\,d\mu(s) = (f, \mu).$$

Clearly μ must be a probability measure. In the case $X = L_\infty$, any finite mean is defined by a probability measure μ which is absolutely continuous with respect to the Haar measure and consequently $d\mu(s) = \alpha(s)\,ds$, where α is real function belonging to L_1 such that

(1.1) $$\alpha(s) \geqslant 0, \quad \int \alpha(s)\,ds = 1.$$

Denote by $\Phi(G)$ the class of finite means on $L_\infty(G)$. We shall identify $\Phi(G)$ with the class of functions α satisfying (1.1). Note that Φ forms a semi-group under convolution. The class Φ is to play the utmost important role in all what follows.

If X is the space $C(G)$ of continuous bounded functions on G, then the set of finite means on $C(G)$ is precisely the set M_0 of all Borel probability measures on G. M_0 forms a semi-group under convolution and Φ is a two-sided ideal of it.

The following proposition is an immediate consequence of a more general theorem on linear lattices (cf. e.g. [17], p. 16, Theorem 4.3):

PROPOSITION 1.1. *The set of finite means is w^*-dense in the set of means on L_∞.*

In fact, by the definition of a mean, any mean m is a non-negative functional on $L_\infty(G)$. By Theorem 4.3 of [17], p. 16, there exists a net $\{x_\gamma\}$ of non-negative functions $x_\gamma \in L_1(G)$ which is w*-convergent to m. Thus if $f = 1$, we have

$$\lim_\gamma \|x_\gamma\| = \lim_\gamma (f, x_\gamma) = mf = 1$$

and, hence, the net $x_\gamma/\|x_\gamma\|$ is w*-convergent to m and, moreover, $x_\gamma/\|x_\gamma\| \in \Phi$.

PROPOSITION 1.2. *The set of means on $L_\infty(G)$ is convex and w^*-compact.*

2. Invariant means. Class $\mathscr{J}$. In this section we propose a new definition of an invariant mean on $L_\infty(G)$ for a locally compact group G. In the case when G is discrete our definition coincides with the usual one.

DEFINITION. We say that a mean m on L_∞ is *topologically left* (*right*) *invariant* if for any $\alpha \in \Phi$ and $f \in L_\infty$

$$m(\alpha * f) = mf \qquad (m(f * \beta^\sim) = mf).$$

We say that a mean is *topologically invariant* if

$$m(\alpha * f * \beta^\sim) = mf \quad \text{for any } f \in L_\infty \text{ and any } \alpha, \beta \in \Phi.$$

The class of topological locally compact groups for which exists a topologically left or right invariant or invariant mean is denoted by $\mathscr{J}_l$, $\mathscr{J}_r$ or $\mathscr{J}$, respectively.

PROPOSITION 2.1. *A mean m is topologically left* (*right*) *invariant or topologically invariant if for any $f \in L_\infty$ we have*

$$m(\mu * f) = mf \quad (m(f * \nu^\sim) = mf), \qquad m(\mu * f * \nu^\sim) = mf$$

for any $\mu, \nu \in M_0$, respectively.

In fact, let $\alpha \in \Phi$, then $\alpha * \mu$, $\alpha * \nu \in \Phi$; therefore

$$m(\mu * f * \nu^\sim) = m\big(\alpha * (\mu * f * \nu^\sim) * \alpha^\sim\big) = m\big((\alpha * \mu) * f * (\alpha * \nu)^\sim\big) = mf.$$

Since $\Phi \subset M_0$, the converse implication is trivial.

COROLLARY 2.2. $\mathscr{J} \subset \mathscr{J}_l \cap \mathscr{J}_r$.

In fact, if m is topologically invariant, then

$$m(f * \mu) = m(\delta_1 * f * \mu) = mf \quad \text{for any} \quad \mu \in M_0.$$

Hence, m is topologically right invariant. Similarly, m is topologically left invariant.

COROLLARY 2.3. *If a mean m is topologically left* (*right*) *invariant, then it is left* (*right*) *invariant, i.e.*

$$m_s f = m(\delta_s * f) = mf, \qquad mf_s = m(f * \delta_s^\sim) = mf.$$

PROPOSITION 2.4. *Any topologically left* (*right*) *invariant mean m defined on the class $C(G)$ of bounded continuous functions on a group G is uniquely extendable to a topologically left* (*right*) *invariant mean m' on $L_\infty(G)$.*

Proof. Let m be a topologically left invariant mean on $C(G)$. Let $f \in L_\infty$ and let $\alpha \in \Phi$. Then we have $\alpha * f \in C(G)$. We put

$$m'f = m(\alpha * f) \tag{2.1}$$

and we see that any extension to a topologically left invariant mean on L_∞ must be of this form.

It is easy to see that m' as defined by (2.1) does not depend on the choice of $\alpha \in \Phi$. In fact, we have

$$m(\alpha * f) = m(\beta * f) \quad \text{for any} \quad \alpha, \beta \in \Phi, \tag{2.2}$$

because for any $\varepsilon > 0$ there is an element $\varrho \in \Phi$ such that $\|\alpha * \varrho - \alpha\| < \varepsilon$ and $\|\beta * \varrho - \beta\| < \varepsilon$, hence, since m is topologically left invariant,

$$|m(\alpha * f) - m(\varrho * f)| = |m(\alpha * f) - m(\alpha * \varrho * f)| = \big|m\big((\alpha * \varrho - \alpha) * f\big)\big| \leqslant \|f\|_\infty \varepsilon,$$

$$|m(\beta * f) - m(\varrho * f)| = |m(\beta * f) - m(\beta * \varrho * f)| = \big|m\big((\beta * \varrho - \beta) * f\big)\big| \leqslant \|f\|_\infty \varepsilon,$$

which shows that

$$|m(\alpha * f) - m(\beta * f)| \leqslant \varepsilon \|f\|_\infty$$

and, since ε is arbitrary, (2.2) follows.

The following conditions are obvious translation of Dixmier's conditions as presented in [7].

PROPOSITION 2.5. *In order that $G \in \mathscr{J}_r$, $G \in \mathscr{J}_l$ or $G \in \mathscr{J}$ the following conditions are necessary and sufficient, respectively. For any $\alpha_1, \ldots, \alpha_n$; $\beta_1, \ldots, \beta_n \in \Phi$ and any $f_1, \ldots, f_n \in L_\infty(G)$*

$$\operatorname*{ess\,inf}_{s \in G} \sum_{i=1}^{n} (f_i * \alpha_i^\sim - f_i)(s) \leqslant 0, \tag{D_r}$$

$$\operatorname*{ess\,inf}_{s \in G} \sum_{i=1}^{n} (\alpha_i * f_i - f_i)(s) \leqslant 0, \tag{D_l}$$

$$\operatorname*{ess\,inf}_{s \in G} \sum_{i=1}^{n} (\alpha_i * f_i * \beta_i^\sim - f_i)(s) \leqslant 0. \tag{D}$$

3. Weak and strong invariance. The notion of weak and strong invariance is an adaptation of the corresponding notions for discrete groups due to Day (cf. e.g. [2]).

DEFINITION. We say that a net $\{a_\gamma\}$, $a_\gamma \in \Phi$, is *w-[strongly] convergent to left (right) invariance* if, for any $a \in \Phi$,

$$\underset{\gamma}{\text{w-lim}}(a_\gamma * a_\gamma - a_\gamma) = 0 \qquad [\lim_\gamma \|a_\gamma * a_\gamma - a_\gamma\| = 0],$$

$$(\underset{\gamma}{\text{w-lim}}(a_\gamma * a - a_\gamma) = 0 \qquad [\lim_\gamma \|a_\gamma * a - a_\gamma\| = 0]).$$

We say that a net $\{a_\gamma\}$ is *w-[strongly] convergent to invariance* if for any $a, \beta \in \Phi$ we have

$$\underset{\gamma}{\text{w-lim}}(a * a_\gamma * \beta - a_\gamma) = 0 \qquad [\lim_\gamma \|a * a_\gamma * \beta - a_\gamma\| = 0].$$

PROPOSITION 3.1. *A group $G \in \mathcal{J}_l$ $(G \in \mathcal{J}_r)$ or $G \in \mathcal{J}$ if, and only if, there exists a net $\{a_\gamma\}$, $a_\gamma \in \Phi$, w-convergent to left (right) invariance or to invariance, respectively.*

Proof. Let m be a topologically right invariant mean on L_∞. By proposition 1.1, there exists a net $\{a_\gamma\}$, $a_\gamma \in \Phi$, w-convergent to m. For any $a \in \Phi$ and each $f \in L_\infty$ we have

$$(3.1) \qquad 0 = m(f * a^{\sim} - f) = \lim_\gamma (f * a^{\sim} - f, a_\gamma) = \lim_\gamma \big((f * a^{\sim}, a_\gamma) - (f, a_\gamma)\big)$$
$$= \lim_\gamma (f, a_\gamma * a - a_\gamma),$$

which proves that $\{a_\gamma\}$ is w-convergent to right invariance.

Conversely, if $\{a_\gamma\}$ is a net, $a_\gamma \in \Phi$, which is w-convergent to right invariance, since the set of the means is w*-compact, there exists a mean m which is a cluster point of the net $\{a_\gamma\}$. The sequence of equalities (3.1) read from the right to the left shows that m is topologically right invariant.

It is clear that if for a group G there exists a net $\{a_\gamma\}$, $a_\gamma \in \Phi$, strongly convergent to right invariance, then there exists also a net $\{a'_\gamma\}$, $a'_\gamma \in \Phi$, w-convergent to right invariance (we may, of course, put $a'_\gamma = a_\gamma$). It is of importance that the converse is also true. In the case of discrete group this was originally proved by Day [2]. Recently Namioka has found a very simple and elegant proof of this fact. His proof with few formal changes only applies to our case and we reproduce it here for the sake of completeness.

PROPOSITION 3.2. *If for a group G there exists a net $\{a_\gamma\}$, $a_\gamma \in \Phi$, w-convergent to right invariance, then there exists also a net $\{a'_\gamma\}$, $a'_\gamma \in \Phi$, strongly convergent to right invariance.*

Proof (Namioka [18], proof of theorem 2.2). Let E be the product $(L_1(G))^\Phi$. Then E is a locally convex linear topological space under the product of norm topologies. Define a linear map $T\colon L_1(G) \to E$ as follows. For $x \in L_1(G)$ and $a \in \Phi$,

$$T(x)(a) = x * a - x.$$

Now the weak topology on E coincides with the product of the weak topologies (see e.g. [15], p. 160). Then, since

$$\underset{\gamma}{\text{w-lim}}(a_\gamma * a - a_\gamma) = 0 \quad \text{for any } a \in \Phi,$$

we see that 0 is in the weak closure of $T(\Phi)$. Since E is locally convex and $T(\Phi)$ is a convex set, the weak closure of $T(\Phi)$ is identical with the closure $T(\Phi)^-$ of $T(\Phi)$ relative to the product of norm topologies. Hence $0 \in T(\Phi)^-$ which means that there is a net $\{a'_\gamma\}$, $a'_\gamma \in \Phi$, such that for any fixed $a \in \Phi$

$$\lim_\gamma \|a'_\gamma * a - a_\gamma\| = 0,$$

which proves proposition 3.2.

PROPOSITION 3.3. *If a net $\{a_\gamma\}$, $a_\gamma \in \Phi$, is strongly convergent to right (left) invariance, then the net $\{a^* * a\}$ is strongly convergent to invariance.*

Proof. Note that if $\{a_\gamma\}$ is strongly convergent to right invariance, then the net $\{a_\gamma^*\}$ is strongly convergent to left invariance. Thus, for fixed $a, \beta \in \Phi$ and any $\varepsilon > 0$ and any $\gamma > \gamma_0$ we have

$$\|a * a_\gamma^* - a_\gamma^*\| < \varepsilon \quad \text{and} \quad \|a_\gamma * \beta - a_\gamma\| < \varepsilon.$$

Hence, since $\|a_\gamma\| = \|a_\gamma^*\| = \|\beta\| = 1$,

$$\|a * a_\gamma^* * a_\gamma * \beta - a_\gamma^* * a_\gamma * \beta\| < \varepsilon, \qquad \|a_\gamma^* * a_\gamma * \beta - a_\gamma^* * a_\gamma\| < \varepsilon$$

and so

$$\|a * a_\gamma^* * a_\gamma * \beta - a_\gamma^* * a_\gamma\| < 2\varepsilon$$

as required.

COROLLARY 3.4. *We have $\mathcal{J}_r = \mathcal{J} = \mathcal{J}_l$.*

COROLLARY 3.5. *$G \in \mathcal{J}$ if, and only if, there exists a net $\{a_\gamma\}$, $a_\gamma \in \Phi$, w-convergent to right invariance.*

COROLLARY 3.6. *Conditions* (D_r), (D_l) *and* (D) *are equivalent.*

4. Reiter's condition. In 1960 Reiter [19] proposed the following condition for a locally compact group G:

(R) For every compact set $A \subset G$ and any $\varepsilon > 0$ there exists a function $a \in \Phi$ such that

$$\|{}_s a - a\| < \varepsilon \quad \text{for any } s \in A.$$

It is almost obvious that (R) implies the existence of a net $\{a_\gamma\}$, $a_\gamma \in \Phi$, strongly convergent to left invariance. In fact, if $\gamma = (\varepsilon, A)$ with the partial order $(\varepsilon', A') \geqslant (\varepsilon, A)$ if $\varepsilon' \leqslant \varepsilon$, $A' \supset A$, and α_γ is the element of Φ the existence of which (R) asserts, then for any $\alpha \in \Phi$, if A is a compact set such that $\gamma \geqslant \gamma_0$ we have $\|{}_s\alpha_\gamma - \alpha_\gamma\| < \varepsilon/2$ for all $s \in A$, while $\|{}_s\alpha_s - \alpha_\gamma\| \leqslant 2$ for all $s \in G$. Thus $\gamma \geqslant \gamma_0$ implies

$$\int_A \|{}_s\alpha_\gamma - \alpha_\gamma\| \alpha(s)\, ds \leqslant \frac{\varepsilon}{2} \int_A \alpha(s)\, ds \leqslant \frac{\varepsilon}{2},$$

$$\int_{G\setminus A} \|{}_s\alpha_\gamma - \alpha(s)\|\, ds \leqslant 2 \int_{A\setminus G} \alpha(s)\, ds < \frac{\varepsilon}{2}$$

and so

$$\begin{aligned}\varepsilon &> \int_{A\setminus G} \|{}_s\alpha_\gamma - \alpha_\gamma\| \alpha(s)\, ds + \int_A \|{}_s\alpha_\gamma - \alpha_\gamma\| \alpha(s)\, ds \\ &= \int_G \|{}_s\alpha_\gamma - \alpha_\gamma\| \alpha(s)\, ds \geqslant \int \Big| \int \alpha(s)\, \alpha_\gamma(s^{-1}t) - \alpha(s)\, \alpha_\gamma(t)\, ds \Big|\, dt \\ &= \|\alpha * \alpha_\gamma - \alpha_\gamma\|.\end{aligned}$$

It is also not difficult to verify the converse implication.

Let A be a compact set and let $\varepsilon' > 0$. We are going to find a function α such that $\alpha \in \Phi$ and

$$\|{}_s\alpha - \alpha\| < \varepsilon' \quad \text{for all } s \in A. \tag{4.1}$$

Put $\varepsilon = \varepsilon'/5$ and take any $\beta \in \Phi$. There exists a neighborhood U of the unity of G such that

$$\|{}_s\beta - \beta\| < \varepsilon \text{ for all } s \in U \text{ and } \|e_U * \beta - \beta\| < \varepsilon, \tag{4.2}$$

where $e_X = |X|^{-1}\chi_X$, for any Baire set $X \subset U$ of finite positive measure.

Let

$$A \subset \bigcup_{i=0}^{n} s_i U \quad \text{with} \quad s_0 = 1 \tag{4.3}$$

and let $e_i = e_{s_i U}$. If $\{\alpha_\gamma\}$, $\alpha_\gamma \in \Phi$, is a net strongly convergent to left invariance, there exists an α_γ such that

$$\|\beta * \alpha_\gamma - \alpha_\gamma\| < \varepsilon \tag{4.4}$$

$$\|e_i * \alpha_\gamma - \alpha_\gamma\| < \varepsilon \quad \text{for} \quad i = 0, 1, \ldots, n. \tag{4.5}$$

Let $\alpha = \beta * \alpha_\gamma$. Clearly $\alpha \in \Phi$. Then, by (4.2) and the fact that ${}_s\beta * \alpha_\gamma = {}_s(\beta * \alpha_\gamma)$, we have $\|{}_s\alpha - \alpha\| < \varepsilon$ for $s \in U$ and also $\|e_u * \alpha - \alpha\| < \varepsilon$ whence

$$\|e_u * \alpha - {}_s\alpha\| < 2\varepsilon \quad \text{for} \quad s \in U.$$

Since $\|\cdot\|$ is left invariant,

$$\|e_{s_i U} * \alpha - {}_{s_i t}\alpha\| < 2\varepsilon \text{ for all } t \in U \text{ and } i = 0, 1, \ldots, n. \tag{4.6}$$

By (4.4),

$$\|\alpha - \alpha_\gamma\| < \varepsilon, \tag{4.7}$$

whence $\|e_i * \alpha_\gamma - e_i * \alpha_\gamma\| < \varepsilon$ for all $i = 0, 1, \ldots, n$. Hence, by (4.5),

$$\|e_i * \alpha - \alpha_\gamma\| < 2\varepsilon \quad \text{for all} \quad i = 0, 1, \ldots, n$$

and, by (4.7), $\|e_i * \alpha - \alpha\| < 3\varepsilon$. Therefore, by (4.6), $\|{}_{s_i t}\alpha - \alpha\| < 5\varepsilon = \varepsilon'$ for all $i = 0, \ldots, n$, $t \in U$ which, in virtue of (4.3), gives (4.1). Thus we have proved the following

PROPOSITION 4.1. *A group G has Reiter's property if and only if $G \in \mathscr{J}$.*

Clearly enough if G is discrete, i.e. when compact sets are finite, Reiter's property is another formulation for the existence of a net of finite means strongly convergent to left invariance.

5. Følner's condition. In 1955 Følner formulated a condition which he proved to be necessary and sufficient for a discrete group to belong to class $\mathscr{J}$.

For any finite set A and every $\varepsilon > 0$ there exists a finite set E such that

$$|sE \Delta E| < \varepsilon |E| \quad \text{for any } s \in A.$$

Recently Namioka [18] has shown that this condition is a straight forward consequence of the existence of a net of finite means strongly convergent to invariance. In this section we formulate Følner's condition for topological locally compact groups and we prove that it is a simple consequence of Reiter's condition. Our proof is somewhat modeled on that of Namioka but, oddly enough, is simpler.

We say that a *group G satisfies Følner condition* if it has the following property:

(F) For any positive Borel measure μ on G, any set A of finite μ measure: $0 < \mu(A) < \infty$, and any $\varepsilon > 0$, $\delta > 0$ there exists a compact Baire set E and a Borel set $B \subset A$ with $\mu(B) < \delta$ such that

$$|sE \Delta E| < \varepsilon |E| \tag{5.1}$$

for any $s \in A \setminus B$.

PROPOSITION 5.1. *If $G \in \mathscr{J}$, then G satisfies* (F).

Since compact Baire sets approximate in measure Baire sets of finite measure, it is enough to show that G satisfies a version of (F), E being a Baire set of finite measure. First we show a simple

LEMMA. *Let α, β be two Baire measurable non-negative functions on G both in $L_1(G)$. Let $E_\lambda = \{s : \alpha(s) \geqslant \lambda\}$ and $F_\lambda = \{s : \beta(s) \geqslant \lambda\}$. Then*

$$\|\alpha - \beta\|_1 = \int_0^\infty |E_\lambda \Delta F_\lambda|\, d\lambda. \tag{5.2}$$

Proof. Write $\chi_\lambda = \chi_{E_\lambda}$ and $\eta_\lambda = \chi_{F_\lambda}$. Then, for any $s \in G$,

$$\alpha(s) = \int_0^{\alpha(s)} 1 d\lambda = \int_0^\infty \chi_\lambda(s) d\lambda$$

and similarly

$$\beta(s) = \int_0^{\beta(s)} 1 d\lambda = \int_0^\infty \eta_\lambda(s) d\lambda.$$

Hence

$$|\alpha(s)-\beta(s)| = \int_0^\infty \big(\chi_\lambda(s)-\chi_\lambda(s)\eta_\lambda(s)\big) d\lambda + \int_0^\infty \big(\eta_\lambda(s)-\chi_\lambda(s)\eta_\lambda(s)\big) d\lambda. \tag{5.3}$$

In fact, if $\alpha(s) \geqslant \beta(s)$, then $\chi_\lambda(s) \geqslant \eta_\lambda(s)$ and only the first of the summands of the right hand side appears and

$$\chi_\lambda(s)-\chi_\lambda(s)\eta_\lambda(s) = \chi_\lambda(s)-\eta_\lambda(s).$$

Similarly, if $\alpha(s) \leqslant \beta(s)$, then only the second summand of the right of (5.3) appears. By (5.3) we have

$$\|\alpha-\beta\| = \int |\alpha(s)-\beta(s)| ds$$

$$= \int_0^\infty d\lambda \int \big(\chi_\lambda(s)-\chi_\lambda(s)\eta_\lambda(s)\big) + \big(\eta_\lambda(s)-\eta_\lambda(s)\chi_\lambda(s)\big) ds = \int_0^\infty |E_\lambda \Delta F_\lambda| d\lambda$$

as required.

Now suppose α is a Baire function belonging to Φ. Clearly, if $E_\lambda = \{s: \alpha(s) \geqslant \lambda\}$, then, for any $t \in G$, $tE_\lambda = \{s: {}_t\alpha(s) \geqslant \lambda\}$. By (5.2) we have

$$\|{}_s\alpha - \alpha\| = \int_0^\infty |sE_\lambda \Delta E_\lambda| d\lambda$$

and also

$$\int_0^\infty |E_\lambda| d\lambda = 1. \tag{5.4}$$

Now let μ be any Borel measure on G and let A be a given compact set with $0 < \mu(A) < \infty$, let $\varepsilon > 0$ and $\delta > 0$. Suppose $G \in \mathscr{J}$. Then $G \in \mathscr{R}$ and, accordingly, there is a function α in Φ such that

$$\|{}_s\alpha - \alpha\| < \varepsilon' = \varepsilon\delta\mu(A)^{-1}.$$

Then

$$\int_0^\infty |E_\lambda| d\lambda \int_A \frac{|sE_\lambda \Delta E_\lambda|}{|E_\lambda|} d\mu(s) = \int_A \|{}_s\alpha - \alpha\| ds < \varepsilon' \mu(A) = \varepsilon.$$

Hence, by (5.4), there must exist a λ such that $|E_\lambda| \neq 0$ and

$$\int_A \frac{|sE_\lambda \Delta E_\lambda|}{|E_\lambda|} d\mu(s) < \varepsilon$$

which shows that for the set

$$B = \left\{s: \frac{|sE_\lambda \Delta E_\lambda|}{|E_\lambda|} \geqslant \varepsilon\right\}$$

we have $\mu(B) < \delta$ and so

$$|sE_\lambda \Delta E| < \varepsilon |E_\lambda| \text{ for all } s \in A \setminus B \text{ with } \mu(B) < \delta,$$

as required.

It is easy to prove that if a group G satisfies (F), then $G \in \mathscr{J}$, directly (an almost trivial cancelation argument shows that (F) implies (D)), we postpone, however, the proof of this fact to the next section where it will be shown via certain property of positive definite functions on G which is of its own interest.

Clearly, if $\mu(A) = \operatorname{card} A$ we obtain the following weaker version of Følner's condition:

COROLLARY 5.2. *If $G \in \mathscr{J}$, then for any $\varepsilon > 0$ and any finite set A there exists a compact set E such that*

$$|sE \Delta E| < \varepsilon |E| \quad \text{for any } s \in A.$$

As has been proved recently by Namioka [18], this condition is equivalent to the existence of a left-invariant mean on $C(G)$.

6. Weak containment of unitary representations in the regular representation. In [11] Godement considers the class (R) of groups G such that any continuous positive definite function on G is the limit of a net of functions of the form

$$\sum_{i=1}^n x_i * x_i^\sim, \quad x_i \in L_2(G),$$

convergent uniformly on compact sets. This can be formulated equivalently as the property that any unitary representation of the group is weakly contained in the regular representation (cf. [8] and [14]). Godement himself has noticed that $G \in (R)$ if only the function identically equal to 1 can be approximated uniformly on compact sets by functions of the form $x * x^\sim$ with $x \in L_2(G)$. In the case of connected locally compact groups the class (R) has been shown by Takenouchi [22] to coincide with the class (E) of Ivasava and in the case of discrete groups it is shown in [14] that (R) coincides with $\mathscr{J}$. As a matter of fact, it is proved in [14] that if $G \in (R)$, then there exists an invariant (not necessarily topologically invariant) mean value on $L_\infty(G)$. Neil W. Rickert has pointed out that there are some mistakes in the proofs for non-discrete case. All of them can be corrected, although the proof of the fact that $(R) = \mathscr{J}$ which we present here seems to be much simpler than anything which can be derived from [14]. In view of section 7, the result of Takenouchi is also a simple corollary of the equality $(R) = \mathcal{J}$ and the fact that any semi-simple non-compact Lie group contains a non-Abelian free group as a closed discrete subgroup [1].

DEFINITION. We say that a *group G belongs to class* (R) if there exists a net $\{x_\gamma\}$, $x_\gamma \in L_2(G)$, such that for any compact set A the net $\{x_\gamma * x_\gamma^\sim\}$ converges uniformly on A to the function identically equal to 1.

PROPOSITION 6.1. *If G satisfies* (F), *then* $G \in (R)$.

Proof. First we note that it is sufficient to find a net $\{x_\gamma\}$, $x_\gamma \in L_2(G)$, such that $x_\gamma * x_\gamma^\sim$ converges almost uniformly to 1. In fact, by proposition 0.1, for any $\alpha \in \Phi \cap L$ the net $\{\alpha * x_\gamma * x_\gamma^\sim * \alpha^\sim\}$ is convergent uniformly on any compact set to $\alpha * 1 * \alpha^\sim = 1$.

Let A be a compact set, ε, δ positive numbers. By (F), there exists a compact Baire set E such that (5.1) holds. Let $\gamma = (A, \varepsilon, \delta)$ with the obvious partial order in the triples (A, ε, δ). We put $x_\gamma = \chi_E |E|^{-1/2}$. Let $s \in A \setminus B$. Then

$$|1 - x_\gamma * x_\gamma^\sim| = \left|1 - \frac{1}{|E|}\int \chi_E(t)\chi_E(s^{-1}t)\,dt\right| = \left|1 - \frac{1}{|E|}\int \chi_E(t)\chi_{sE}(t)\,dt\right|$$
$$= \left|1 - \frac{1}{|E|}\right| |E \cap sE| = \frac{1}{|E|}|E \setminus sE| < \varepsilon.$$

PROPOSITION 6.2. *If $G \in (R)$, then G satisfies Reiter's condition.*

The proof is related to one of the proofs of Day as presented in [3].

Proof. Suppose that for a compact set A and $\varepsilon > 0$ we have

$$(6.1) \qquad |x * x^\sim(s) - 1| < \varepsilon/4 \quad \text{for all } s \in A.$$

We may of course assume that the unit element 1 of G belongs to A and consequently that $1 = x * x^\sim(1) = \|x\|_2$. Let $y(s) = |x(s)|$, $s \in G$. Clearly, $|x * x^\sim(s)| \leqslant y * y^\sim(s)$. Since $y * y^\sim$ is a positive definite function,

$$y * y^\sim(s) \leqslant y * y^\sim(1) = \|y\|_2 = \|x\|_2 = 1.$$

Therefore

$$0 \leqslant 1 - y * y^\sim(s) \leqslant 1 - |x * x^\sim(s)| \leqslant |1 - x * x^\sim(s)| < \varepsilon/4$$

for $s \in A$. Hence, for $s \in A$,

$$|1 - (y, {}_s y)| = \left|1 - \int y(t)\,{}_s y(t)\,dt\right| = |1 - y * y^\sim(s)| < \varepsilon/4$$

[1] The fact that a semi-simple non-compact Lie group contains a non-Abelian free group as a discrete subgroup can be easily deduced from classical results on semi-simple Lie groups. In fact, any such non-compact group contains a factor group of the simply connected covering group of $SL(2, R)$ which, in turn, contains a non-Abelian discrete free subgroup. For the detailed exposition and references the reader is advised to consult e. g. [23].

and consequently,

$$\|y - {}_s y\|_2 = (1 - (y, {}_s y)) + (1 - (y, {}_s y)) < \varepsilon/2.$$

But, if $\alpha = y^2$, then, since $\|y\|_2 = 1$, $\alpha \in \Phi$ and so for $s \in A$ we have

$$\|\alpha - {}_s\alpha\|_1 = \int |y^2(t) - {}_s y^2(t)|\,dt$$
$$= \int |y(t) - {}_s y(t)|\,|y(t) + {}_s y(t)|\,dt \leqslant \|y - {}_s y\|_2 \|y + {}_s y\|_2 \leqslant 2\|y - {}_s y\|_2 < \varepsilon,$$

which shows that G satisfies Reiter's condition.

7. Class $\mathcal{J}$. In this section we show that $\mathcal{J}$ contains compact groups as well as locally compact Abelian groups and is closed under operations of taking subgroups, factor groups, and extensions. Since we already know that $\mathcal{J}$ coincides with the class $\mathcal{R}$ of groups satisfying Reiter's condition, all these facts become consequences of the corresponding theorems proved by Reiter and Dieudonné in a series of papers [5], [6], [19], [20], [21] for class $\mathcal{R}$. Their proofs however are more complicated than the ones we present here which are almost as simple as the proofs of corresponding theorems for discrete groups.

PROPOSITION 7.1. *If G is an Abelian group, then $G \in \mathcal{J}$.*

Proof. It is sufficient to show that there is a net $\{\alpha_\gamma\}$, $\alpha_\gamma \in \Phi(G)$, strongly convergent to right invariance. For any finite subset $\delta = \{\alpha_1, \ldots, \alpha_k\}$ of Φ and a positive integer n let $\gamma = (\delta, n)$. We partially order the pairs (δ, n) writing $(\delta', n') \geqslant (\delta, n)$ if $\delta' \supset \delta$ and $n' \geqslant n$. Let

$$\alpha_\gamma = n^{-k} \sum_{0 < j_1, \ldots, j_k \leqslant n} \alpha_1^{j_1 *} * \ldots * \alpha_k^{j_k *},$$

where $\alpha^{j*} = \alpha * \ldots * \alpha$ (j times). Then for any $\alpha \in \Phi$ if $\alpha = \alpha_i \in \delta$ and any n for $\gamma = (\delta, n)$ we have

$$\|\alpha_i * \alpha_\gamma - \alpha_\gamma\| = n^{-k}\Big\|\sum_{0 < j_1, \ldots, j_k \leqslant n} \alpha_1^{j_1 *} * \ldots * \alpha_i^{j_i *} * \ldots * \alpha_k^{j_k *}(\alpha_i^{(j_i+1)*} - \alpha_i^{j_i *})\Big\|$$
$$= n^{-1}\Big\|\sum_{j=1}^{n}(\alpha_i^{j+1} - \alpha_i^{j})\Big\| \leqslant 2n^{-1}.$$

Since n tends to infinity with γ, (7.1) follows.

PROPOSITION 7.2. *If G is compact, then $G \in \mathcal{J}$.*

Obvious.

PROPOSITION 7.3. *If H is a closed normal subgroup of G and $G \in \mathcal{J}$, then $G/H \in J$.*

Proof. Let m be a topologically invariant mean on $L_\infty(G)$. For each $f \in L_\infty(G/H)$ we put $\overline{m}f = m(f \circ \pi)$, where π is the homomorphism

$G \to G/H$. Clearly, $\overline{m}$ is a mean. By proposition 0.4, if $\alpha \in \Phi(G/H)$ and β is as in (0.4),

$$\overline{m}(\alpha * f) = m((\alpha * f)\circ\pi) = m(\beta * (f\circ\pi)) = m(f\circ\pi) = \overline{m}f,$$

which shows that $\overline{m}$ is topologically left invariant and so $G/H \in \mathscr{J}$.

PROPOSITION 7.4. *If H is a closed subgroup of G and $G \in \mathscr{J}$, then $H \in \mathscr{J}$.*

The easiest way of proving 7.4 seems to be via the equality $\mathscr{J} = (R)$. In fact, let A be a compact subset of H. Then A is a compact subset of G and, since $G \in (R)$, in view of proposition 0.3, for any $\varepsilon > 0$ there exists a function $u \in L$ such that

$$|u * u^{\sim}(h) - 1| < \varepsilon/2 \quad \text{for all } h \in A. \tag{7.3}$$

But $u * u^{\sim}$ is a continuous positive definite function vanishing outside a compact set on G consequently its restriction to H, i.e. $u * u^{\sim}|H = f$ has the same property as a function on H. By proposition 0.2 there exists a function $x \in L(H) \subset L_2(H)$ such that

$$|x * x^{\sim}(h) - f(h)| < \varepsilon/2 \quad \text{for} \quad h \in A.$$

Hence, by (7.3),

$$|x * x^{\sim}(h) - 1| < \varepsilon \quad \text{for} \quad h \in A,$$

which shows that $H \in (R) = \mathscr{J}$.

PROPOSITION 7.5. *Let H be a closed normal subgroup of a group G. Suppose that H and G/H belong to the class $\mathscr{J}$. Then $G \in \mathscr{J}$.*

Proof. Let ds, $\overline{dh}$, $d\dot{s}$ be the differentials of the left invariant Haar measures in G, H and G/H chosen as for (0.3) and let $p(s)$ be defined as in proposition 0.5. Then for any $f \in L_\infty(G)$ and $x \in L_1(G)$ the function $f * x^{\sim}$ is continuous and

$$\sup_{h \in H} |p(h)(f * x^{\sim})(h)| \leqslant \|f\|_\infty \|x\|_1 .$$

Therefore the operation

$$x \to p \cdot (f * x^{\sim})$$

is a continuous linear operator from $L_1(G)$ into $L_\infty(H)$. Let m_1 be a topologically invariant mean on $L_\infty(H)$. Then

$$F(x) = m_1(p \cdot (f * x^{\sim}))$$

is a continuous functional on $L_1(G)$. Consequently,

$$F(x) = (f', x), \quad f' \in L_\infty(G).$$

For any $h \in H$, $\delta_h \in M_0(H)$ and

$$\begin{aligned}(\delta_h * f', x) &= (f', \delta_{h^{-1}} * x) = m_1(p \cdot (f * \delta_{h^{-1}} * x)^{\sim}) \\ &= m_1(p \cdot (f * x^{\sim} * \delta_{h^{-1}}^{\sim})) = \Delta(h) m_1(p \cdot (f * x^{\sim} * \delta_h)) \\ &= \Delta(h) p(h) m_1(p \cdot (f * x^{\sim} * \delta_h)) = \varrho(h)\varrho(h)^{-1} m_1(p \cdot (f * x^{\sim})\delta_h^{\sim}) \\ &= m_1(p \cdot (f * x)^{\sim}) = (f', x),\end{aligned}$$

which shows that f' is constant on the cosets $Hs = sH$, $s \in G$. Therefore f' defines a unique function $\dot{f} \in L_\infty(G/H)$. Let m_2 be the topologically invariant mean on $L_\infty(G/H)$. We put

$$mf = m_2 \dot{f}.$$

Clearly, m is a mean on $L_\infty(G)$. It remains to prove that m is topologically invariant. Let $\alpha \in \Phi(G)$. Then

$$\begin{aligned}((f * \alpha^{\sim})', x) &= m_1(p \cdot (f * \alpha^{\sim} * x^{\sim})) = m_1(p \cdot (f * (x * \alpha)^{\sim})) \\ &= (f', x * \alpha) = (f' * \alpha^{\sim}, x),\end{aligned}$$

whence $(f * \alpha^{\sim})' = f' * \alpha^{\sim}$. But f' is constant on the cosets $Hs = sH$. Therefore, by 0.4, if $\dot{\alpha}(\dot{s}) = {}^{\cdot}\!\int \alpha(sh)\,\overline{dh}$, we have $(f' * \alpha^{\sim})^{\cdot} = \dot{f} * \dot{\alpha}$ and, consequently,

$$m(f * \alpha^{\sim}) = m_2(\dot{f} * \dot{\alpha}^{\sim}) = m_2 \dot{f} = mf,$$

which completes the proof.

We conclude by showing that a classical result of von Neumann is an immediate consequence of Følner's condition.

PROPOSITION 7.6. *If G is a non-Abelian free group with descrete topology generated by two elements a_1 and a_2, then $G \notin \mathscr{J}$.*

Proof. There is no finite set E in G such that

$$|a_i E \Delta E| < \tfrac{1}{2}|E| \quad \text{for} \quad i = 1, 2,$$

because the inequality says that more than half of the words of E begin with a_i for both $i = 1$ and $i = 2$. Thus, by 5.2, $G \notin \mathscr{J}$.

Added in proof. Recently F. P. Greenleaf, I. Namioka and H. Reiter (*) have shown that the class $\mathfrak{J}$ of locally compact groups which admit topologically invariant mean is identical with the class of groups which admit invariant mean on $L_\infty(G)$. As a matter of fact, in a brief and very elegant way I. Namioka has shown that an invariant mean on $L_\infty(G)$ defines a topologically invariant on $L_\infty(G)$.

(*) F. P. Greenleaf, *Equivalence of various types of invariant means on topological groups* (to appear); I. Namioka, *On a recent theorem by H. Reiter* (to appear); H. Reiter, *On some properties of locally compact groups*, Indagationes Mathematicae 27 (1965), p. 697-701.

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Exponentially convex functions on a cone in a Lie group*

by

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1. Introduction. Necessary and sufficient conditions for a real sequence $\{f(n); n = 0, 1, 2, \ldots\}$ to be expressible as an integral

$$f(n) = \int_0^\infty t^n \, d\alpha(t),$$

where $d\alpha(t)$ is a bounded non-negative measure, are

$$\text{(A)} \qquad \sum_{j,k=0}^{m} a_j a_k f(j+k) \geqslant 0 \quad \text{and} \quad \sum_{j,k=0}^{m} a_j a_k f(j+k+1) \geqslant 0$$

for any set $\{a_n; n = 0, 1, \ldots, m\}$ of real numbers. This is known as the Stieltjes moment problem. (Cf. [13; 15] and for a brief history [7].) For a continuous real function $f(x)$ on the real line the representation becomes

$$f(x) = \int_{-\infty}^{\infty} e^{-xt} \, d\alpha(t)$$

and (A) becomes

$$\text{(B)} \qquad \sum_{j,k=0}^{m} a_j a_k f(x_j + x_k) \geqslant 0,$$

where $\{x_n; n = 0, 1, \ldots, m\}$ is any finite set of points on the line. Such functions were called *exponentially convex* by Bernstein [3].

In the case of the Hausdorff moment problem

$$f(n) = \int_0^1 t^n \, d\alpha(t),$$

where $d\alpha(t)$ is a bounded non-negative measure, if and only if

$$\text{(C)} \qquad 0 \leqslant \sum_{j,k=0}^{m} a_j a_k f(j+k+1) \leqslant \sum_{j,k=0}^{m} a_j a_k f(j+k).$$

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