References

[1] F. Bruhat, *Sur les représentations induites des groupes de Lie*, Bull. Soc. Math. France 84 (1956), p. 97-205.

[2] Mahlon M. Day, *Amenable semigroups*, Illinois J. of Math. 1 (1957), p. 509-544.

[3] — *Convolutions, means, and spectra*, ibidem 8 (1964), p. 100-111.

[4] — *Fixed-point theorems for compact convex sets*, ibidem 4 (1961), p. 583-590.

[5] J. Dieudonné, *Sur le produit de composition II*, J. Math. Pures et Appl. (9) 39 (1960), p. 275-292

[6] — *Sur une propriété des groupes libres*, J. Reine Angew. Math. 204 (1960), p. 30-34.

[7] J. Dixmier, *Les moyennes invariantes sans les semi-groupes et leur application*, Acta Sci. Math. 12 (1950), p. 213-227.

[8] J. M. G. Fell, *The dual spaces of C*-algebras*, Trans. Amer. Math. Soc. 94 (1960), p. 365-403.

[9] — *Weak containment and induced representations of groups*, Canadian J. Math. 14 (1962), p. 237-268.

[10] E. Følner, *On groups with full Banach mean value*, Math. Scand. (1955), p. 243-254.

[11] R. Godement, *Les fonctions de type positif et la théorie des groupes*, Trans. Amer. Math. Soc. 63 (1948), p. 1-84.

[12] P. R. Halmos, *Measure theory*, 1950.

[13] E. Hewitt and Kenneth A. Ross, *Abstract harmonic analysis I*, 1963.

[14] A. Hulanicki, *Groups whose regular representation weakly contains all unitary representations*, Studia Math. 24 (1964), p. 37-59.

[15] J. L. Kelly and I. Namioka, *Linear topological spaces*, 1963.

[16] H. Kesten, *Full Banach mean value on countable groups*, Math. Scand. 7 (1959), p. 146-156.

[17] I. Namioka, *Partially ordered linear topological spaces*, Memoirs of the Amer. Math. Soc. 24 (1957), p. 19-50.

[18] — *Følner's conditions for amenable semi-groups*, Scand. Math. 15 (1964), p. 18-28.

[19] H. Reiter, *The convex hull of translates of functions in L¹*, J. London Math. Soc. 35 (1960), p. 5-16.

[20] — *Une propriété analytique d'une certaine classe de groupes localement compacts*, C. R. Acad. Sci. Paris 254 (1962), p. 3627-3629.

[21] — *Sur les groupes de Lie semi-simples connexes*, ibidem 255 (1962), p. 2883-2884.

[22] — *Sur la propriété (P₁) et les fonctions de type positif*, ibidem 258 (1964), p. 5134-5135.

[23] Neil W. Rickert, *Groups with the fixed point property*, Dissertation, Yale University (1965).

[24] Osanu Takenouchi, *Sur une classe de fonctions de type positif sur un groupe localement compact*, Math. J. of Okayama University 4 (1955), p. 143-173.

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Exponentially convex functions on a cone in a Lie group*

by

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1. Introduction. Necessary and sufficient conditions for a real sequence $\{f(n); n = 0, 1, 2, \ldots\}$ to be expressible as an integral

$$f(n) = \int_0^\infty t^n \, d\alpha(t),$$

where $d\alpha(t)$ is a bounded non-negative measure, are

(A) $$\sum_{j,k=0}^m a_j a_k f(j+k) \geqslant 0 \quad \text{and} \quad \sum_{j,k=0}^m a_j a_k f(j+k+1) \geqslant 0$$

for any set $\{a_n; n = 0, 1, \ldots, m\}$ of real numbers. This is known as the Stieltjes moment problem. (Cf. [13; 15] and for a brief history [7].) For a continuous real function $f(x)$ on the real line the representation becomes

$$f(x) = \int_{-\infty}^\infty e^{-xt} \, d\alpha(t)$$

and (A) becomes

(B) $$\sum_{j,k=0}^m a_j a_k f(x_j + x_k) \geqslant 0,$$

where $\{x_n; n = 0, 1, \ldots, m\}$ is any finite set of points on the line. Such functions were called *exponentially convex* by Bernstein [3].

In the case of the Hausdorff moment problem

$$f(n) = \int_0^1 t^n \, d\alpha(t),$$

where $d\alpha(t)$ is a bounded non-negative measure, if and only if

(C) $$0 \leqslant \sum_{j,k=0}^m a_j a_k f(j+k+1) \leqslant \sum_{j,k=0}^m a_j a_k f(j+k).$$

For a real function $f(x)$ defined for $x \geqslant 0$ the representation is replaced by

$$f(x) = \int_0^\infty e^{-xt}\, d\alpha(t)$$

and (C) is replaced by

(D) $$0 \leqslant \sum_{j,k=0}^m a_j a_k f(x_j + x_k + x) \leqslant \sum_{j,k=0}^m a_j a_k f(x_j + x_k).$$

This is the Hausdorff-Bernstein-Widder theorem. It was extended to higher dimensions by Hildebrandt and Schoenberg [9] and to $f(x)$ defined on an open connected semi-group of a topological group by Devinatz and Nussbaum ([8], § 5, Def. 4, Lemma 2, and Cor. 3, p. 231-234). (D) remains as the necessary and sufficient condition for this general case. See also Devinatz [6].

On the other hand, (A) or (B) alone appears to be inadequate for such a general extension of the Stieltjes moment problem [16] although Widder ([15], p. 273-275) and, more generally, Devinatz [6] have shown (B) to be sufficient for continuous real functions in Euclidean space.

The purpose of the present paper is to give an extension to Lie groups of the representation theorem for exponentially convex functions. (See [4] for general properties of Lie groups.) The function $f(x)$ will be defined on a special type of semi-group which we shall call a cone.

DEFINITION 1. A subset $\mathfrak{S}$ of a Lie group $\mathscr{G}$ is called a *cone* if it satisfies the following conditions:

1. $\mathfrak{S}$ is open.
2. $x, y \,\epsilon\, \mathfrak{S} \Rightarrow xy \,\epsilon\, \mathfrak{S}$.
3. $e^X \,\epsilon\, \mathfrak{S} \Rightarrow e^{tX} \,\epsilon\, \mathfrak{S}$ for $0 < t \leqslant 1$, where X is in the Lie algebra. (X is also called an *infinitesimal right translation*.)
4. There exists an open neighborhood V of e (the *identity* of $\mathscr{G}$) such that (a) every point of V is of the form e^X and (b) $\mathfrak{S} = \bigcup_{n=1}^\infty W^n$, where $W = \mathfrak{S} \cap V$; i.e., every element of $\mathfrak{S}$ can be expressed in at least one way as a finite product of elements of W.

EXAMPLE. A cone can be constructed in the following fashion. For sufficiently small $\varepsilon > 0$ there is a homeomorphism between a neighborhood V of e and a neighborhood of the origin in N-dimensional Euclidean space E^N (N = dimension of $\mathscr{G}$) such that every element of V is of the form

$$\exp\Big(\sum_{k=1}^N t_k X_k\Big)$$

for $-\varepsilon < t_k < \varepsilon$, where $\{X_k;\, k = 1, \ldots, N\}$ are linearly independent elements of the Lie algebra.

Let

$$W = \Big\{\exp\Big(\sum_{k=1}^N t_k X_k\Big);\, 0 < t_k < \varepsilon\Big\}.$$

Then

$$\mathfrak{S} = \bigcup_{n=1}^\infty W^n$$

is a cone.

SPECIFIC EXAMPLE. Consider the 2-dimensional group of all matrices of the form

$$A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix},$$

where $a \neq 0$ and b are real, $\det A = a \neq 0$;

$$A^{-1} = \begin{bmatrix} 1/a & -b/a \\ 0 & 1 \end{bmatrix}.$$

Let $\mathfrak{S}$ be the semi-group of those matrices for which $\det A = a > 1$. $\mathfrak{S}$ is a cone.

Note that, once one V has been found meeting requirement (4) of the definition of a cone, any open neighborhood V_1 of e with $V_1 \subseteq V$ will also serve the same purpose. For, there is an open neighborhood U of e, $U \subseteq V_1$, such that U is homeomorphic to a neighborhood $\mathscr{M}$ of 0 in E^N. Set $W_1 = V_1 \cap \mathfrak{S} \subseteq V \cap \mathfrak{S} = W$. Any point in W is of the form e^X. For sufficiently small $\eta > 0$, $e^{tX} \,\epsilon\, U \cap \mathfrak{S} \subseteq W_1$ for $0 < t < \eta$ since this interval corresponds to a set in $\mathscr{M}$. Thus e^X can be expressed as an n-fold product of $e^{(1/n)X} \,\epsilon\, W_1$ where n is an integer $> 1/\eta$. Hence

$$\mathfrak{S} = \bigcup_{n-1}^\infty W_1^n.$$

In fact

$$\mathfrak{S} = \bigcup_{n=1}^\infty (U \cap \mathfrak{S})^n$$

also.

It is also worth noticing that *a cone is connected*. For let p and q be any two points in $\mathfrak{S}$,

$$p = \prod_{j=1}^k \exp(X_j) \quad \text{and} \quad q = \prod_{j=k+1}^L \exp(X_j)$$

for some set $\{\exp(X_j);\, j = 1, \ldots, L\} \subseteq W$. Form

$$F(t) = \prod_{j=1}^k \exp(tX_j) \prod_{j=k+1}^L \exp[(1-t)X_j] \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

$F(t)$ is a continuous function of t by the continuity of products in $\mathscr{G}$. $F(0) = q$, $F(1) = p$. $F(t)$ is thus a path in $\mathfrak{S}$ joining p and q. Therefore $\mathfrak{S}$ is arc-wise connected and hence connected.

It would be interesting to discover whether or not connectedness is an intrinsic property of a cone. That is, does a connected set S in $\mathscr{G}$ satisfying conditions (1), (2), and (3) of the above definition also satisfy condition (4)?

Next let us denote by $\mathfrak{S}^{\wedge}$ the set of all non-zero continuous homomorphisms (real characters) χ on the cone $\mathfrak{S}$ into the multiplicative semi-group of real numbers, $\mathfrak{S}^{\wedge}$ possessing the topology of uniform convergence on compact sets of $\mathfrak{S}$. (See § 2 for more detailed definitions.)

EXAMPLE. Consider the 2-dimensional Lie group of matrices given earlier. Let $\chi(A) = |\det A|$. $\chi(A)$ is actually a character on $\mathscr{G}$. $\chi(A) = \det A > 1$ on $\mathfrak{S}$. There are infinitely many characters as may be seen by considering $[\chi(A)]^{\lambda}$, where λ runs through the real numbers.

The main result of this paper can now be stated.

THEOREM 1. *Let $\mathfrak{S}$ be a cone in a Lie group $\mathscr{G}$. Let f be a continuous real-valued function defined on $\mathfrak{S}$ satisfying*

$$(1) \qquad \sum_{j,k=1}^{m} a_j \bar{a}_k f(xx_j x_k) \geqslant 0$$

for all sets $\{a_j; j = 1, \ldots, m\}$ of complex numbers and all sets $\{x_j; j = 1, \ldots, m\} \subseteq \mathfrak{S}$ and $x \in \mathfrak{S}$, where m takes on all positive integral values.

Then there exists a unique non-negative regular Borel measure μ (on the Borel field generated by the closed sets in $\mathfrak{S}^{\wedge}$) such that

$$(2) \qquad f(x) = \int_{\mathfrak{S}^{\wedge}} \chi(x) d\mu(\chi) \quad \text{for all} \quad x \in \mathfrak{S}.$$

EXAMPLE. Refer to the preceding example. Let ν be a measure such that

$$g(x) = \int_{-\infty}^{\infty} e^{xt} d\nu(t) < \infty$$

at least for $x > 0$. Let $f(A) = g(\log \chi(A))$ for $A \in \mathfrak{S}$. Then

$$\sum_{j,k=1}^{m} a_j \bar{a}_k f(A A_j A_k) = \sum_{j,k=1}^{m} a_j \bar{a}_k \int_{-\infty}^{\infty} \chi(A)^t \chi(A_j)^t \chi(A_k)^t d\nu(t) \geqslant 0.$$

Thus $f(A)$ satisfies (1).

The proof of Theorem 1 will be given in § 4. § 3 will be devoted to a new proof of a lemma of Nelson ([11], p. 583) and § 2 to preliminary information essential to the proof of the theorem.

At first glance it might seem that this problem can be reduced to the corresponding problem for functions defined on cones in Euclidean space by dividing out the commutator subgroup Q of $\mathscr{G}$ and restricting the function to the corresponding cone in $\mathscr{G}/Q$. However this leads to serious difficulties, and it is not clear that such a procedure will work.

2. Preliminaries. We now give a few known concepts and results that will be needed later. (Cf. [1; 2; 5; 8; 12; 14].)

i. DEFINITION A. A *semi-group* $\mathfrak{S}$ is a set in which an operation (multiplication) is defined with the property that $x, y \in \mathfrak{S} \Rightarrow xy \in \mathfrak{S}$.

DEFINITION B. An *open semi-group* $\mathfrak{S}$ is an open set in a topological group $\mathscr{G}$ which is a semi-group under the operation (multiplication) in $\mathscr{G}$.

LEMMA A (Devinatz and Nussbaum [8], Prop. 1, p. 223). *Let $\mathfrak{S}$ be an open semi-group, $e \in [\mathfrak{S}]$ (closure of $\mathfrak{S}$), and χ a homomorphism of $\mathfrak{S}$ into the multiplicative semi-group of real numbers. Then $\mathfrak{S}_0 = \{x; \chi(x) = 0\}$ and $\mathfrak{S}_1 = \{x; \chi(x) \neq 0\}$ are open and $e \in [\mathfrak{S}_1]$ if $\mathfrak{S}_1 \neq \emptyset$. Hence, if $\mathfrak{S}$ is connected, χ cannot take on the value zero unless it vanishes identically.*

The proof of this lemma actually shows that, if $\mathfrak{S}_1 \neq \emptyset$, there is an open neighborhood U of e such that the homomorphism χ differs from 0 everywhere in $U \cap \mathfrak{S}$ (hence $e \notin [\mathfrak{S}_0]$). In the case where $\mathfrak{S}$ is a cone this result shows that χ cannot be 0 anywhere in $\mathfrak{S}$ unless $\chi \equiv 0$ on $\mathfrak{S}$. For we may suppose that U also satisfies condition (4) of Definition 1 (since, if V satisfies this condition, then $U \cap V$ does also, and we may replace U by $U \cap V$).

$$\mathfrak{S} = \bigcup_{n=1}^{\infty} (U \cap \mathfrak{S})^n$$

so that each element of $\mathfrak{S}$ is a finite product of elements in U. This proves the statement. (Alternately we may use the fact that the cone $\mathfrak{S}$ is connected.)

DEFINITION C. A *real character* χ of an open semi-group $\mathfrak{S}$ is a continuous homomorphism of $\mathfrak{S}$ into the multiplicative semi-group of real numbers.

The set of all real characters of an open semi-group $\mathfrak{S}$ is itself a semi-group with respect to point-wise multiplication $((\chi_1 \chi_2)(x) = \chi_1(x) \chi_2(x)$ for each $x \in \mathfrak{S})$.

We denote by K this set of real characters, excluding the zero character. In case $\mathfrak{S}$ is connected, K is actually a group.

For each $\chi \in K$ the sets $\{x; \chi(x) > 0\}$ and $\{x; \chi(x) < 0\}$ are open since χ is continuous. If $\mathfrak{S}$ is connected (in particular if $\mathfrak{S}$ is a cone), the union of these two sets is $\mathfrak{S}$ and hence each χ can have only one sign on $\mathfrak{S}$. But $\chi(x^2) = [\chi(x)]^2 > 0$ for any $x \in \mathfrak{S}$. Hence $\chi(x) > 0$ for every $\chi \in K$ and every $x \in \mathfrak{S}$ if $\mathfrak{S}$ is connected.

DEFINITION D. This set K together with the topology of uniform convergence on compact sets of $\mathfrak{S}$ is denoted by $\mathfrak{S}^{\wedge}$, the real character semi-group of $\mathfrak{S}$.

That is, a neighborhood basis is the collection of all sets of the form

$$\mathcal{N}(\chi_0, C, \varepsilon) = \{\chi; |\chi(x) - \chi_0(x)| < \varepsilon, \ \varepsilon > 0, \ C \text{ compact, all } x \in C\}.$$

Since multiplication of real characters is continuous in this topology, $\mathfrak{S}^{\wedge}$ is a topological semi-group. If $\mathfrak{S}$ is connected, $\mathfrak{S}^{\wedge}$ is actually a topological group since the zero character is omitted and since by continuity each χ is bounded below and above on any compact set $C \subseteq \mathfrak{S}$ by positive numbers (dependent on χ and C).

Nussbaum [12], p. 133, gives the following definition:

DEFINITION E. A semi-group $\mathfrak{S}$ is a locally compact full semi-group if:

(a) $\mathfrak{S}$ can be embedded in a locally compact group $\mathcal{G}$.

(b) $\mathfrak{S}$ is locally compact in the relative topology of $\mathcal{G}$.

(c) Every non-empty bounded open set in $\mathfrak{S}$ has non-zero measure with respect to the left (or right) Haar measure of $\mathcal{G}$.

DEFINITION F. Furthermore Nussbaum [12], p. 134, considers a set

$$\{T_x; \ x \in \mathfrak{S} \ (\text{a locally compact full semi-group})\}$$

which is a *weakly continuous semi-group* of self-adjoint operators acting in a Hilbert space $\mathcal{H}$. That is:

(1) T_x is a self-adjoint operator acting in a Hilbert space $\mathcal{H}$ for every $x \in \mathfrak{S}$.

(2) $T_{xy} \subseteq T_x T_y$ for all $x, y \in \mathfrak{S}$.

(3) $(T_x u \mid v)$ is continuous in x for all $v \in \mathcal{H}$ and all $u \in \bigcap \{\vartheta(T_x); \ x \in \mathfrak{S}\}$, where $\vartheta(T_x) = $ domain of T_x.
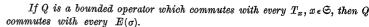
In case the operators T_x are unbounded he enunciates the following additional condition for $\mathfrak{S}$ ([12], p. 136):

(d) There exists a denumerable set $D = \{x_n\}$ in $\mathfrak{S}$ such that for each element x in $\mathfrak{S}$ there are an element y in $\mathfrak{S}$ and an element x_n in D satisfying $x_n = xy$ or $x_n = yx$. That is, for every $x \in \mathfrak{S}$, $x\mathfrak{S} \cap D \neq \emptyset$ or $\mathfrak{S}x \cap D \neq \emptyset$.

His important result is the following:

THEOREM B (Nussbaum [12], Th. 6, p. 137). *Let $\mathfrak{S}$ be a locally compact full semi-group and $\{T_x\}$ a weakly continuous semi-group of self-adjoint operators over $\mathfrak{S}$. If the operators T_x are unbounded, we assume that $\mathfrak{S}$ satisfies condition* (d). *Then there exists a spectral measure $\{E(\sigma)\}$ relative to the Borel subsets of $\mathfrak{S}^{\wedge}$ such that*

$$T_x = \int_{\mathfrak{S}^{\wedge}} \chi(x) E(d\chi).$$

If Q is a bounded operator which commutes with every T_x, $x \in \mathfrak{S}$, then Q commutes with every $E(\sigma)$.

(Note: Nussbaum does not exclude the zero character from the set K.) (Cf. also Tulcea [14], Th. 3, p. 107.)

A cone $\mathfrak{S}$ in a Lie group $\mathcal{G}$ satisfies the conditions (a), (b), (c), (d). The first three of these are readily met since $\mathcal{G}$ is already locally compact and $\mathfrak{S}$ is open. (Haar measure with condition (c) exists for $\mathcal{G}$.)

With regard to (d) consider an open neighborhood U of e which is homeomorphic to a neighborhood $\mathcal{M}$ of 0 in E^N and which satisfies condition (4) of Definition 1. (See the discussion following that definition and the examples.) Let $W = U \cap \mathfrak{S}$.

$$\mathfrak{S} = \bigcup_{n=1}^{\infty} W^n.$$

U contains a countable dense set since $\mathcal{M}$ does (points with rational coordinates). Since W is open, it also contains a countable dense set D_1. Any element of $z \in W^2$ is of the form $z = xy$ with $x, y \in W$. Any neighborhood of z contains a neighborhood $U_x U_y$, where U_x and U_y are neighborhoods of x and y respectively and lie in W. There are points $x' \in U_x$ and $y' \in U_y$ where $x', y' \in D_1$. Thus D_1^2 is dense in W^2. Furthermore, $D_1^2 \subseteq W^2$ and D_1^2 is countable. By induction W^n contains the countable dense set D_1^n and hence

$$\mathfrak{S} = \bigcup_{n=1}^{\infty} W^n$$

contains the countable dense set

$$D = \bigcup_{n=1}^{\infty} D_1^n.$$

For any $x \in \mathfrak{S}$, $x\mathfrak{S}$ is an open neighborhood of any of its points and hence $x\mathfrak{S} \cap D \neq \emptyset$. $\mathfrak{S}$ satisfies (d).

Condition (2) of Theorem B (see Definition F) implies that $T_{xy} = T_x T_y = T_y T_x = T_{yx}$ along with $E_x(\lambda) E_y(\nu) = E_y(\nu) E_x(\lambda)$ for all $x, y \in \mathfrak{S}$ and real numbers λ, ν where $\{E_x(\lambda)\}$ is the canonical resolution of the identity for T_x ([12], p. 134). In his proof of Theorem B Nussbaum introduces a family of projection operators $\{E_\sigma\}$ on $\mathcal{H}$ where each σ is a certain clopen set in the space $\mathcal{M}$ of all maximal ideals in the closure (strong operator topology) of the set of all complex polynomials in

$$\{E_x(\lambda); \ x \in \mathfrak{S}, \ -\infty < \lambda < \infty\}.$$

$\{T_x E_\sigma\}$ is a semi-group of bounded self-adjoint operators on $\mathcal{H}$ ([12], p. 136, Th. 5, proof; see also p. 137, Th. 6, proof). In place of condition (2) and $T_{xy} = T_x T_y = T_y T_x = T_{yx}$ he actually needs and uses throughout

$$(2') \qquad (T_x E_\sigma)(T_y E_\sigma) = T_{xy} E_\sigma = T_{yx} E_\sigma = (T_y E_\sigma)(T_x E_\sigma)$$

and

(2'')
$$E_x(\lambda)E_y(\nu) = E_y(\nu)E_x(\lambda).$$

But (2') is implied by (2''). (2'') also implies $[T_x T_y] = T_{xy} = T_{yx} = [T_y T_x]$, where [ ] signifies closure; this follows from the 2-parameter integral representations of T_x and T_y.

Furthermore in place of the weak continuity condition of Theorem B one can use the condition

(3') $(T_x E_\sigma \psi \mid \psi)$ is continuous in x for all $\psi \epsilon \mathscr{H}$.

In this connection the following result will be required. We quote only part of the theorem.

LEMMA C (Devinatz and Nussbaum [8], p. 229, Cor 2.) *Let* $\mathfrak{S}$ *be an open semi-group in a locally compact group* $\mathscr{G}$ *and let* $e \epsilon [\mathfrak{S}]$. *If* $\{T_x; x \epsilon \mathfrak{S}\}$ *is a weakly measurable semi-group of self-adjoint operators on a Hilbert space* $\mathscr{H}$, *then* $\|T_\theta \theta\|$ *is locally bounded for every* $\theta \epsilon \bigcap \{\vartheta(T_x); x \epsilon \mathfrak{S}\}$. *If each operator* T_x *is bounded, then* $\|T_x\|$ *is locally bounded.*

We apply this lemma below to $\{T_x E_\sigma; x \epsilon \mathfrak{S}\}$ in place of $\{T_x\}$.

When (2') holds, then for the class of sets σ needed in the proof of Theorem B, condition (3') is implied by condition

(3'') For each $\psi \epsilon \mathscr{H}$ there exists a sequence $\{h_m\}$ such that $h_m \to \psi$ in norm and $(T_x \xi \mid h_m)$ is continuous in x for every h_m and all ξ in $\bigcap \{\vartheta(T_x); x \epsilon \mathfrak{S}\}$.

For, since $\vartheta(T_x E_\sigma) = \mathscr{H}$ by (2') ([12], p. 136, Th. 5, proof; see also p. 137, Th. 6, proof), let $\theta \epsilon \mathscr{H}$ and take $\xi = E_\sigma \theta$. Thus $(T_x E_\sigma \theta \mid h_m)$ is continuous in x and hence $(T_x E_\sigma \theta \mid \psi)$ is measurable in x. ($T_x E_\sigma$ is weakly measurable. In fact so is T_x.) Select any $y \epsilon \mathfrak{S}$. $\|T_x E_\sigma \theta\|$ is bounded in a compact neighborhood C of y by a number $M = M(\theta, C) > 0$ (Lemma C)

$$\big| (T_x E_\sigma \theta \mid \psi) - (T_y E_\sigma \theta \mid \psi) \big|$$
$$\leqslant \big| (T_x E_\sigma \theta \mid \psi) - (T_x E_\sigma \theta \mid h_m) \big| + \big| (T_x E_\sigma \theta \mid h_m) - (T_y E_\sigma \theta \mid h_m) \big| +$$
$$+ \big| (T_y E_\sigma \theta \mid h_m) - (T_y E_\sigma \theta \mid \psi) \big|$$
$$\leqslant \{\|T_x E_\sigma \theta\| + \|T_y E_\sigma \theta\|\} \cdot \|\psi - h_m\| + \big| (T_x E_\sigma \theta \mid h_m) - (T_y E_\sigma \theta \mid h_m) \big|$$
$$\leqslant 2M \cdot \|h_m - \psi\| + \big| (T_x E_\sigma \theta \mid h_m) - (T_y E_\sigma \theta \mid h_m) \big|.$$

Given $\varepsilon > 0$, select h_m so that $\|h_m - \psi\| < \varepsilon/(3M)$ and hold it fixed. Then choose a neighborhood C_1 of y, $C_1 \subseteqq C$, such that

$$\big| (T_x E_\sigma \theta \mid h_m) - (T_y E_\sigma \theta \mid h_m) \big| < \varepsilon/3 \quad \text{if} \quad x \epsilon C_1.$$

Therefore

$$\big| (T_x E_\sigma \theta \mid \psi) - (T_y E_\sigma \theta \mid \psi) \big| < \varepsilon \quad \text{if} \quad x \epsilon C_1.$$

That is, $\{T_x E_\sigma\}$ is weakly continuous in x.

Hence Theorem B is valid when (1), (2''), (3'') and (a), (b), (c), (d) are satisfied.

We could have made matters easier for ourselves by supplanting (3'') with condition

(3''') $\{T_x E_\sigma\}$ is weakly measurable. (This is implied by (3'').) $\mathscr{H}$ is separable.

For, (3') is implied by (3''') also. (See [8], Th. 4, p. 229. The proof of this theorem depends on Lemma C.)

ii. Next consider a *real-valued* function f defined on a semi-group S. If

(i)
$$\sum_{j,k=1}^{m} a_j \bar{a}_k f(x_j x_k) \geqslant 0$$

for all finite sets $\{a_j; j = 1, \ldots, m\}$ of complex numbers and all sets $\{x_j; j = 1, \ldots, m\} \subseteqq S$, then it is clear that

(ii) $f(yz) = f(zy)$ for all $y, z \epsilon S.$

For take $m = 2$ and set $a_1 = 1, a_2 = i, x_1 = y, x_2 = z$. Then (i) becomes

$$f(y^2) - if(yz) + if(zy) + f(z^2) \geqslant 0$$

which implies that $i[f(yz) - f(zy)]$ is real, hence $f(yz) - f(zy) = 0$.

On the other hand, if

(iii)
$$\sum_{j,k=1}^{m} a_j \bar{a}_k f(x x_j x_k) \geqslant 0$$

for all finite sets $\{a_j; j = 1, \ldots, m\}$ of complex numbers and all sets $\{x_j; j = 1, \ldots, m\} \subseteqq S$ and all $x \epsilon S$, then with the same substitutions and reasoning as before

(iv) $f(xyz) = f(xzy)$ for all $x, y, z \epsilon S.$

Consider next $f(x_1 \ldots x_n)$ for a finite set $\{x_j; j = 1, \ldots, n\} \subseteqq S$. Now (iv) in turn implies

(v) $f([x_1 \ldots x_{k-1}] x_k [x_{k+1} \ldots x_n]) = f([x_1 \ldots x_{k-1}] \cdot [x_{k+1} \ldots x_n] x_k)$

for x_k any one of the factors except x_1. Thus $f(x_1 \ldots x_n)$ is unchanged in value under the shift to the last position of any factor other than the first. Since any rearrangement of the order of the factors other than the first can be accomplished by a finite number of shifts to and from the last position, f is unchanged under such a rearrangement.

(ii) implies

(vi) $f(x_1 [x_2 \ldots x_n]) = f([x_2 \ldots x_n] x_1)$

so that f is unchanged by a shift of the first factor to the last position, leaving the order of the other factors undisturbed.

Thus (ii) and (iv) together (hence (i) and (iii) together) imply that f is unchanged by a shift of any factor to the last position, hence under any rearrangement of the factors.

It is interesting to observe that, if S is a cone in a Lie group, then (i) and (iv) together imply (iii). For x can be expressed in the form

$$x = \exp X_1 \cdot \exp X_2 \ldots \exp X_L = y_1 y_1 y_2 y_2 \ldots y_L y_L$$

where we have set $y_i = \exp(1/2) X_i$.

$$\sum_{j,k=1}^m a_j \bar{a}_k f(x x_j x_k) = \sum_{j,k=1}^m a_j \bar{a}_k f(y_1 y_1 \ldots y_L y_L x_j x_k)$$

$$= \sum_{j,k=1}^m a_j \bar{a}_k f([y_1 \ldots y_L x_j] \cdot [y_1 \ldots y_L x_k]) \geqslant 0.$$

Note that, if S contains an identity e, then (iii) implies (i) since the substitution of $x = e$ into (iii) yields (i).

iii. The following definition and theorems contain the essential properties on reproducing spaces needed in this paper.

DEFINITION G. A function $K(x, y)$, real or complex, defined on a set $E \times E$, is a *reproducing kernel* for a Hilbert space $\mathscr{F}$ of functions defined on E if $\big(g \mid K(\cdot, y)\big) = g(y)$ for every $g \in \mathscr{F}$ and any $y \in E$. (Note that $K(\cdot, y) \in \mathscr{F}$ for each $y \in E$.)

THEOREM D (Moore-Aronszajn [1], Th. 2, p. 143). *A function* $K(x, y)$, *real or complex, defined on a set* $E \times E$, *is a reproducing kernel for a Hilbert space* $\mathscr{F}$ *of functions defined on* E *if and only if*

$$\sum_{j,k=1}^m a_j \bar{a}_k K(x_j, x_k) \geqslant 0$$

for every finite set $\{a_j; j = 1, \ldots, m\}$ *of complex numbers and every finite set* $\{x_j; j = 1, \ldots, m\} \subseteq E$. *The class* $\mathscr{F}$, *if it exists, is unique.*

THEOREM E (Aronszajn [2], p. 351). *If* $K(x, y)$, *defined on a set* $E \times E$, *is the reproducing kernel of the Hilbert space* $\mathscr{F}$ *of functions defined on* E *with norm* $\|f\|$, *then* $K_1(x, y)$, *the restriction of* $K(x, y)$ *to* $E_1 \times E_1 \subseteq$ $\subseteq E \times E$ *where* $E_1 \subseteq E$, *is the reproducing kernel of the Hilbert space* $\mathscr{F}_1$ *of all restrictions of functions of* $\mathscr{F}$ *to the subset* E_1. *For any such restriction* $f_1 \in \mathscr{F}_1$ *the norm* $\|f_1\|_1$ *is the minimum of* $\|f\|$ *for all* $f \in \mathscr{F}$ *whose restriction to* E_1 *is* f_1.

LEMMA F (see Devinatz [5], Lemma 2, p. 459.) *Suppose* $f(x)$ *is defined on the interval* $0 \leqslant x \leqslant a$ *such that*

$$\sum_{j,k=1}^m a_j \bar{a}_k f(x_j + x_k) \geqslant 0$$

for all finite sets $\{a_j; j = 1, \ldots, m\}$ *of complex numbers and all sets* $\{x_j;$ $j = 1, \ldots, m\}$, $0 \leqslant x_j \leqslant a/2$. *Then there exists an analytic function* $F(z)$ *defined in the strip* $0 < x = \mathrm{Re}\, z < a$ *which coincides with* $f(x)$ *on* $0 < x < a$ *and such that*

$$\sum_{j,k=1}^m a_j \bar{a}_k F(z_j + \bar{z}_k) \geqslant 0$$

for all sets $\{a_j; j = 1, \ldots, m\}$ *of complex numbers and all sets* $\{z_j; j = 1, \ldots,$ $\ldots, m\}$, $0 < \mathrm{Re}\, z_j < a/2$.

The interval may be $0 \leqslant x < \infty$. This lemma applies also to functions defined with the appropriate properties on N-dimensional intervals.

3. Lemma on analytic vectors. Nelson [11] has given the following definition and lemma.

DEFINITION H ([11], p. 572). Let B be an operator acting in a Banach space $\mathscr{B}$. An element g in $\mathscr{B}$ is an *analytic vector* for B if $g \in \bigcap \{\vartheta(B^k);$ $k = 1, 2, \ldots\}$ and

$$\sum_{k=0}^\infty \frac{\|B^k g\|}{k!} s^k < \infty \quad \text{for some} \quad s > 0.$$

LEMMA G ([11], Lemma 5.1, p. 583). *Let* B *be a closed symmetric operator acting in a Hilbert space* $\mathscr{H}$. *Then* B *is self-adjoint if and only if it has a dense set of analytic vectors.*

We shall give a somewhat simpler proof of the sufficiency of Nelson's lemma in the form given below. (The necessity is trivial.) Absolute, strong, and weak convergence of $\sum (B^k g) s^k/(k!)$ are all equivalent in $\mathscr{B}$, the only difficult proof being that of 'weak' $\Rightarrow$ 'strong'. However for our purposes an easy Hilbert space computation, given at the start of the demonstration of the lemma, will suffice.

LEMMA 1. *Let* B *be a closed symmetric operator acting in a Hilbert space* $\mathscr{H}$. *Suppose there exists a dense set* $\vartheta \subseteq \mathscr{H}$ *such that* $g \in \vartheta$ *implies that* $g \in \bigcap \{\vartheta(B^k); k = 1, 2, \ldots\}$ *and that*

$$\sum_{k=0}^\infty (B^k g \mid h) s^k/(k!)$$

converges for all $h \in \mathscr{H}$ *and each* s *in some interval* $|s| < r_g$, $r_g > 0$. (r_g *is independent of* h.) *Then* B *is self-adjoint.*

Proof. i. Fix $g \in \vartheta$ and select $\varepsilon > 0$. By Cauchy diagonalization

$$\left\| \sum_{k=N}^{M} \frac{B^k g}{k!} z^k \right\|^2 = \sum_{j,k=N}^{M} (B^j B^k g \mid g) \frac{z^j \bar{z}^k}{j! \, k!} \leqslant \sum_{n=2N}^{2M} \sum_{j+k=n} |(B^j B^k g \mid g)| \frac{|z|^j \cdot |z|^k}{j! \, k!}$$

$$= \sum_{n=2N}^{2M} |(B^n g \mid g)| \frac{(2\,|z|)^n}{n!} < \varepsilon$$

for $M, N > n_0(\varepsilon, g, z)$ and for $|z| < \frac{1}{2} r_g = r$.

Therefore

$$\sum_{k=0}^{N} (B^k g) z^k / (k!)$$

converges strongly to an element $G(z)$ in $\mathscr{H}$, hence weakly to this element.

On account of the absolute convergence just shown a useful expression for $\|G(z)\|$ may be found by the diagonal method. Specifically $\|G(z)\|^2$ is the limit as $N \to \infty$ of

$$\left\| \sum_{k=0}^{N} \frac{(B^k g)}{k!} z^k \right\|^2 = \sum_{j,k=0}^{N} (B^j B^k g \mid g) \frac{z^j \bar{z}^k}{j! \, k!}.$$

$$\left| \sum_{j,k=0}^{N} (B^j B^k g \mid g) \frac{z^j \bar{z}^k}{j! k!} - \sum_{n=0}^{N} \sum_{j+k=n} (B^j B^k g \mid g) \frac{z^j \bar{z}^k}{j! \, k!} \right|$$

$$\leqslant \sum_{n=N+1}^{2N} \sum_{j+k=n} |(B^j B^k g \mid g)| \frac{|z|^j \cdot |z|^k}{j! \, k!} = \sum_{n=N+1}^{2N} |(B^n g \mid g)| \cdot \frac{(2\,|z|)^n}{n!} < \varepsilon$$

for $N > n_1(\varepsilon, g, z)$ and $|z| < \frac{1}{2} r_g = r$. Hence

$$\|G(z)\|^2 = \lim_{N \to \infty} \left\| \sum_{k=0}^{N} \frac{B^k g}{k!} z^k \right\|^2 = \lim_{N \to \infty} \sum_{n=0}^{N} \sum_{j+k=n} (B^j B^k g \mid g) \frac{z^j \bar{z}^k}{j! \, k!}$$

$$= \sum_{n=0}^{\infty} (B^n g \mid g) \frac{(z+\bar{z})^n}{n!} = \sum_{n=0}^{\infty} (B^n g \mid g) \frac{(2\operatorname{Re} z)^n}{n!}$$

for $|z| < r$.

In particular set $z = it$, t real and $i = \sqrt{(-1)}$. Then $\operatorname{Re} z = 0$ and only the first term in the last sum is not zero. Hence $\|G(it)\|^2 = \|g\|^2$ for $|t| < r$.

ii. Set

$$U(t) g = \sum_{k=0}^{\infty} (B^k g) \cdot (it)^k (k!) = G(it)$$

for t real or complex. $U(t)g = G(it)$ is both the strong and the weak limit of

$$\sum_{k=0}^{N} (B^k g) \cdot (it)^k / (k!)$$

for t real or complex, $|t| < r$. Also

(3.1) $\|U(t) g\| = \|g\|$ for t real, $|t| < r$.

Since

$$\sum_{k=0}^{\infty} (B^k g \mid h) \cdot (it)^k / (k!)$$

converges for $|t| < r_g$ for each $h \in \mathscr{H}$, it may be differentiated term-by-term there:

$$\frac{d}{dt} \big(U(t) g \mid h \big) = \sum_{k=1}^{\infty} (B^k g \mid h) i^k t^{k-1} / (k-1)!$$

$$= \sum_{k=0}^{\infty} (B^k B g \mid h) i (it)^k / (k!) = i \big(U(t) B g \mid h \big) \quad \text{for} \quad |t| < r.$$

Therefore $U(t) B g \cdot i$ is the weak derivative of $U(t) g$. Actually

$$\sum_{k=0}^{\infty} (B^k B g) i (it)^k / (k!)$$

converges weakly for $|t| < r_g$ and, as before, strongly for $|t| < r$. The mere existence of

$$U(t) B g = \lim \sum_{k=0}^{N} (B^k B g) \cdot (it)^k / (k!) = \lim B \sum_{k=0}^{N} (B^k g) \cdot (it)^k / (k!)$$

as $N \to \infty$ and of

$$U(t) g = \lim \sum_{k=0}^{N} (B^k g) \cdot (it)^k / (k!)$$

shows on account of the closure of B that $B U(t) g$ exists and $B U(t) g = U(t) B g$ for $|t| < r$; $U(t) g \in \vartheta(B)$.

Repeat the argument, taking the weak derivative of $U(t) B g$. Briefly $B^2 U(t) g = B U(t) B g = U(t) B^2 g$ and of course $U(t) g \in \vartheta(B^2)$ for $|t| < r$.

By induction $U(t) g \in \vartheta(B^n)$ and $B^n U(t) g = U(t) B^n g$ for every positive integer n and for t real or complex, $|t| < r$. Thus (3.1) holds also for

$$\sum_{k=p}^{q} (B^k g) \cdot (i\tau)^k / (k!)$$

in place of g. Here t is real ($|t| < r$) and τ is real or complex ($|\tau| < r$).

iii. Given $\varepsilon > 0$. There is an $N > 0$ such that

$$\left\| \sum_{k=p}^{q} \frac{B^k U(t_0) g}{k!} (it)^k \right\| = \left\| U(t_0) \sum_{k=p}^{q} \frac{B^k g}{k!} (it)^k \right\|$$

$$= \left\| \sum_{k=p}^{q} \frac{B^k g}{k!} (it)^k \right\| < \varepsilon \quad \text{for} \quad p, q > N$$

and t_0 real $(|t_0| < r)$, t real or complex $(|t| < r)$.

Therefore

$$\sum_{k=0}^{\infty} B^k U(t_0) g \cdot (it)^k / (k!)$$

converges strongly (and weakly) in $|t| < r$ for each real t_0 satisfying $|t_0| < r$ and hence $= U(t) U(t_0) g$. Thus $U^2(t_0) g$ is well defined for real t_0, $|t_0| < r$. By induction, $U^n(t_0) g$ is well defined for real t_0, $|t_0| < r$, and for any positive integer n. Also (3.1) holds for $U^{n-1}(t_0) g$ in place of g:

$$\| U^n(t) g \| = \| U^{n-1}(t) g \| \quad \text{for } t \text{ real}, \quad |t| < r.$$

Note that first it was necessary to reduce the range from $|t| < r^g$ to $|t| < \frac{1}{2} r_g = r$ in order to obtain strong convergence of

$$\sum_{k=0}^{|\infty} (B^k g) \cdot (it)^k / (k!)$$

from its weak convergence. But the strong convergence of

$$\sum_{k=0}^{\infty} B^k U^n(t_0) g \cdot (it)^k / (k!)$$

for each $n \geqslant 1$ is already present without further reduction of the range.

iv. Let $h \in \vartheta(B^*)$ such that $B^* h = ih$. $(B^*)^n h = i^n h$. We have

$$\big(U(t) g \mid h \big) = \sum_{k=0}^{\infty} \left(\frac{B^k g}{k!} \mid h \right) \cdot (it)^k = \sum_{k=0}^{\infty} \left(g \mid \frac{B^{*k} h}{k!} \right) \cdot (it)^k$$

$$= \sum_{k=0}^{\infty} (g \mid h) \frac{(-i)^k (it)^k}{k!} = (g \mid h) \sum_{k=0}^{\infty} \frac{t^k}{k!}$$

$$= (g \mid h) e^t \quad \text{for} \quad |t| < r, \quad t \text{ real or complex.}$$

This result holds for $U^{n-1}(t_0) g$ in place of g with t_0 as above.

Let s be any real number. There is a positive integer n such that $|s/n| < r$,

$$\left(U^n \left(\frac{s}{n} \right) g \mid h \right) = \left(U \left(\frac{s}{n} \right) U^{n-1} \left(\frac{s}{n} \right) g \mid h \right) = \left(U^{n-1} \left(\frac{s}{n} \right) g \mid h \right) e^{s/n} = \ldots = (g \mid h) e^{(s/n) \cdot n}.$$

Also

$$\left| \left(U^n \left(\frac{s}{n} \right) g \mid h \right) \right| \leqslant \left\| U^n \left(\frac{s}{n} \right) g \right\| \cdot \|h\| = \left\| U^{n-1} \left(\frac{s}{n} \right) g \right\| \cdot \|h\| = \ldots = \|g\| \cdot \|h\|.$$

Combining these two results, we have

$$\left| (g \mid h) \right| e^s \leqslant \|g\| \cdot \|h\| \quad \text{or} \quad \left| (g \mid h) \right| \leqslant \|g\| \cdot \|h\| \cdot e^{-s}.$$

Let $s \to \infty$. $(g \mid h) = 0$ for all $g \in \vartheta$, a dense set in $\mathscr{H}$. Thus $h = 0$.

Similarly for $h \in \vartheta(B^*)$ such that $B^* h = -ih$. Therefore the deficiency index of B is $(0, 0)$ and B is self-adjoint.

The technique of the last part (iv) of the proof is analogous to that used earlier by Devinatz [6], p. 188, in a special situation.

4. Proof of Theorem 1. The demonstration will run its course through at most a finite number of stages, some of which will be labelled formally as lemmas. The nature of the proof is similar in part to that used by Devinatz in [6]. (1) and (2) will refer to the items so numbered in the statement of Theorem 1.

i. *Translate of f. Reproducing Hilbert space.* By continuity of f, (1) holds for $\{x_j; j = 1, \ldots, m\} \subseteq \mathfrak{S} \cup \{e\}$, but with $x \in \mathfrak{S}$ only. Select any element $v \in \mathfrak{S}$ and define f_v by

$$(3) \qquad\qquad f_v(x) = f(vx).$$

f_v is certainly defined and continuous on $\mathfrak{S} \cup V_1$, where V_1 is some open neighborhood of e such that $v V_1 \subseteq \mathfrak{S}$. Furthermore

$$(4) \qquad \sum_{j,k=1}^{m} a_j \bar{a}_k f_v(x x_j x_k) = \sum_{j,k=1}^{m} a_j \bar{a}_k f(v x x_j x_k) \geqslant 0$$

for all sets $\{a_j; j = 1, \ldots, m\}$ of complex numbers and all sets $\{x_j; j = 1, \ldots, m\} \subseteq \mathfrak{S} \cup \{e\}$ and also $x \in \mathfrak{S} \cup \{e\}$.

In particular, with $x = e$, (4) becomes

$$(5) \qquad\qquad \sum_{j,k=1}^{m} a_j \bar{a}_k f_v(x_j x_k) \geqslant 0.$$

By means of (5) form the reproducing Hilbert space of functions

defined on $\mathfrak{S} \cup \{e\}$ with kernel $f_v(xy)$. Let $\mathscr{F}'$ be the pre-Hilbert space of all functions g of the form given by

$$g(x) = \sum_{j=1}^{m} \alpha_j f_v(xx_j)$$

where $\{\alpha_j; j = 1, \ldots, m\}$ is again any set of complex numbers, $\{x_j; j = 1, \ldots, m\}$ any set of elements in $\mathfrak{S} \cup \{e\}$, and $x \epsilon \mathfrak{S} \cup \{e\}$. If

$$h(x) = \sum_{k=1}^{n} \beta_k f_v(xy_k),$$

define an inner product on $\mathscr{F}' \times \mathscr{F}'$ by

$$(g \mid h) = \sum_{j=1}^{m} \sum_{k=1}^{n} \alpha_j \bar{\beta}_k f_v(y_k x_j) = \sum_{j=1}^{m} \sum_{k=1}^{n} \alpha_j \bar{\beta}_k f_v(x_j y_k).$$

The inner product is independent of the particular representations of g and h. The reproducing property of f_v is expressed by

$$(g \mid f_v(\cdot y)) = \sum_{j=1}^{m} \alpha_j f_v(yx_j) = g(y).$$

A pseudo-norm is given by

$$\|g\|^2 = (g \mid g) = \sum_{j,k=1}^{m} \alpha_j \bar{\alpha}_k f_v(x_j x_k) \geqslant 0.$$

Since

$$|g(y)| = |(g \mid f_v(\cdot y))| \leqslant \|g\| \cdot \|f_v(\cdot y)\|,$$

$\|g\| = 0$ implies $g = 0$ and hence the pseudo-norm is actually a norm. If $\{g_n; n = 1, 2, \ldots\}$ forms a Cauchy sequence, then

$$|g_n(y) - g_m(y)| \leqslant \|g_n - g_m\| \cdot \|f_v(\cdot y)\|$$

shows that $\{g_n(y); n = 1, 2, \ldots\}$ is a point-wise Cauchy sequence. But, since $\|f_v(\cdot y)\|^2 = f_v(y^2)$ is bounded on any compact set, $\{g_n(y); n = 1, 2, \ldots\}$ is a uniform Cauchy sequence on such a set, hence it converges uniformly there to a continuous function $g(y)$. Thus $\mathscr{F}'$ can be completed to a Hilbert space $\mathscr{F}$ by means of continuous functions. The reproducing property of f_v also extends to $\mathscr{F}$:

$$(g \mid f_v(\cdot y)) = g(y) \quad \text{for all} \quad g \epsilon \mathscr{F}, y \epsilon \mathfrak{S} \cup \{e\}.$$

For details concerning the uniqueness of $(g \mid h)$ in $\mathscr{F}'$, the completion of $\mathscr{F}'$ and extension of $(g \mid h)$ and $\|g\|$ to $\mathscr{F}$, see [1; 2].

It should be pointed out that functions of type

$$\sum_{j=1}^{m} \alpha_j f_v(\cdot x_j), \quad \{x_j; j = 1, \ldots, m\} \subseteq \mathfrak{S},$$

are dense in $\mathscr{F}$ [so that $\mathscr{F}$ must be the completion of the set of such functions].

ii. *Analyticity*. Let N be the dimension of $\mathscr{G}$.

LEMMA 2. *There are N linearly independent elements X_k of the Lie algebra such that* $\{\exp X_k; k = 1, \ldots, N\} \subseteq \mathfrak{S}$.

Proof. Consider a canonical chart ([4], p. 109-110) at e in $\mathscr{G}$ with coordinates $v_1, \ldots, v_N$ in E^N mapping an open neighborhood U of e in $\mathscr{G}$ homeomorphically onto an open neighborhood $\mathscr{M}$ of 0 in E^N. $U \cap \mathfrak{S}$ is an open set whose image $\mathscr{N}$ in E^N is then also open. Hence $\mathscr{N}$ contains an N-dimensional sphere. Thus we can find N linearly independent points (vectors) $\{p_k; k = 1, \ldots, N\}$ in $\mathscr{N}$. These determine a new system of coordinates $\{u_k; k = 1, \ldots, N\}$ related to the old system $\{v_k; k = 1, \ldots, N\}$ by a non-singular linear transformation. If

$$\exp\left(\sum_{k=1}^{N} v_k Z_k\right) \leftrightarrow (v_1, \ldots, v_N),$$

where the Z_k's are linearly independent elements of the Lie algebra, then substituting in $\sum_{k=1}^{N} v_k Z_k$ for v_k's in terms of u_k's and rearranging gives

$$\exp\left(\sum_{k=1}^{N} u_k Y_k\right) \leftrightarrow (u_1, \ldots, u_N),$$

a new canonical chart, the Y_k's being linear combinations of the Z_k's. If $p_k = (0, \ldots, 0, u_k^0, 0, \ldots, 0)'$ in the new system, then by setting $X_k = u_k^0 Y_k$ we have N linearly independent X_k's with $\{\exp X_k; k = 1, \ldots, N\}$ in $U \cap \mathfrak{S}$. (Alternately, if $p_k = (v_1^k, \ldots, v_N^k)$ in the old system, then setting $X_k = \sum_{j=1}^{N} v_j^k Z_j$ yields our result.)

LEMMA 3. *Every function $g \epsilon \mathscr{F}$ is analytic on* $\mathfrak{S}$.

Proof. Let $\{X_j; j = 1, \ldots, N\}$ be as in the preceding lemma. There is a homeomorphism (chart) between an open neighborhood U_1 of e in $\mathscr{G}$ and an open neighborhood $\{(t_1, \ldots, t_N); \text{all } |t_j| < a_1, a_1 > 0\}$ of 0 in E^N such that U_1 consists of all elements $\prod_{j=N}^{1} \exp(t_j X_j)$ for $|t_j| < a_1$. (See [4], p. 110.)

Fix $y_0 \in \mathfrak{S}$. There is an open neighborhood U_2 of e, $U_2 \subseteq U_1$, such that $y_0 U_2 \subseteq \mathfrak{S}$; i.e., there is an α_2, $0 < \alpha_2 \leqslant \alpha_1$, such that

$$y_0 \cdot \prod_{j=N}^{1} \exp(t_j X_j) \in \mathfrak{S}$$

for $|t_j| < \alpha_2$. (Note order of appearance of factors in the product.)

Select fixed $\{t_j^0; j = 1, \ldots, N\}$ with $0 < t_j^0 < \alpha_2$. Then

$$z_0 = y_0 \cdot \prod_{j=N}^{1} \exp(-t_j^0 X_j) \in \mathfrak{S} \quad \text{and} \quad y_0 = z_0 \cdot \prod_{j=1}^{N} \exp(t_j^0 X_j).$$

Now for each j, $\exp(t_j X_j)$ lies in a 1-parameter subgroup for every real t_j. If $t_j > 0$, $t_j = n(j) + a_j$ where $0 \leqslant a_j < 1$ and $n(j)$ is some non-negative integer. Hence

$$\exp(t_j X_j) = \{\exp X_j\}^{n(j)} \exp(a_j X_j) \in \mathfrak{S} \quad \text{for all} \quad t_j > 0.$$

The reproducing kernel for the space $\mathscr{F}$ is $K(x, y) = f_v(xy)$ on $E \times E$, $E = \mathfrak{S} \cup \{e\}$. Let

$$E_1 = \left\{ z_0 \cdot \prod_{j=1}^{N} \exp t_j X_j; \, t_j \geqslant 0 \right\} \subseteq E$$

and consider the restriction $K_1(x, y)$ of $K(x, y)$ to $E_1 \times E_1$. Let $\mathscr{F}_1$ be the reproducing space corresponding to $K_1(x, y)$. Consider any function $g \in \mathscr{F}$. Its restriction to E_1, namely

$$g \mid_{E_1} = g \left(z_0 \cdot \prod_{j=1}^{N} \exp t_j X_j \right),$$

is an element of $\mathscr{F}_1$ (Theorem E).

Set

$$F(z_0; t_1, \ldots, t_N) = f_v \left(z_0^2 \cdot \prod_{j=1}^{N} \exp t_j X_j \right)$$

for all $t_j \geqslant 0$. With obvious substitutions

$$K_1(x, y) = K(x, y)|_{E_1 \times E_1} = K \left(z_0 \cdot \prod_{j=1}^{N} \exp t_j X_j, \, z_0 \cdot \prod_{j=1}^{N} \exp s_j X_j \right)$$

$$= f_v \left(z_0 \cdot \prod_{j=1}^{N} \exp t_j X_j \cdot z_0 \cdot \prod_{j=1}^{N} \exp s_j X_j \right) = f_v \left(z_0^2 \cdot \prod_{j=1}^{N} \exp(t_j + s_j) X_j \right)$$

$$= F(z_0; t_1 + s_1, \ldots, t_N + s_N).$$

There is a function $F(\tau_1, \ldots, \tau_N)$ of the complex variables $\{\tau_j; j = 1, \ldots, N\}$, analytic for all $t_j = \operatorname{Re} \tau_j > 0$, whose restriction to $t_j = \operatorname{Re} \tau_j > 0$ is the function $F(z_0; t_1, \ldots, t_N)$. $F(\tau_1 + \bar{\sigma}_1, \ldots, \tau_N + \bar{\sigma}_N)$ is the reproducing kernel of a Hilbert space $\mathscr{F}_2$ for $\operatorname{Re} \tau_j > 0$, $\operatorname{Re} \sigma_j > 0$

(Lemma F, Theorem D). All members of the linear manifold generated from $F(\tau_1 + \bar{\sigma}_1, \ldots, \tau_N + \bar{\sigma}_N)$ by treating $(\tau_1, \ldots, \tau_N)$ as the indeterminate and selecting sets of values of $(\sigma_1, \ldots, \sigma_N)$ are analytic in the N-dimensional half-plane $t_j = \operatorname{Re} \tau_j > 0$. Hence every element of $\mathscr{F}_2$ as a limit of such members, uniformly in any compact subset, is analytic for $t_j > 0$. The Taylor series expansion for any $G(\tau_1, \ldots, \tau_N) \in \mathscr{F}_2$ about the point $(t_1^0, \ldots, t_N^0)$ is valid for $|\tau_j - t_j^0| < t_j^0$.

$K_1(x, y)$ is the reproducing kernel for the class $\mathscr{F}_3$ of all restrictions of functions of $\mathscr{F}_2$ to E_1 (Theorem E). But by the uniqueness of the reproducing space corresponding to $K_1(x, y)$ (Theorem D) $\mathscr{F}_1 = \mathscr{F}_3$; that is,

$$g \left(z_0 \cdot \prod_{j=1}^{N} \exp t_j X_j \right)$$

is the restriction of some $G(\tau_1, \ldots, \tau_N)$ to real variables $t_j > 0$. Therefore

$$g \left(z_0 \cdot \prod_{j=1}^{N} \exp t_j X_j \right)$$

equals its Taylor series expansion about $(t_1^0, \ldots, t_N^0)$ for $|t_j - t_j^0| < t_j^0$; or

$$g \left(z_0 \cdot \prod_{j=1}^{N} \exp[t_j^0 + t_j] X_j \right)$$

equals its Taylor series expansion about the origin for $|t_j| < t_j^0$.

The proof of Lemma 3 has been completed. Note that a similar result is obtained for

$$g \left(y_0 \cdot \prod_{j=1}^{N} \exp t_j X_j \right)$$

under the further restriction that all $t_j \geqslant 0$ since in such a case

$$g \left(z_0 \cdot \prod_{j=1}^{N} \exp t_j^0 X_j \cdot \prod_{j=1}^{N} \exp t_j X_j \right) = g \left(z_0 \cdot \prod_{j=1}^{N} \exp[t_j^0 + t_j] X_j \right).$$

However later (part vi) we shall need the Taylor series representation for

$$g \left(y_0 \cdot \prod_{j=1}^{N} \exp t_j X_j \right)$$

without this additional restriction. For this purpose we now consider two charts at y_0. The first chart maps an open neighborhood U_3 of points

$$\left\{ z_0 \cdot \prod_{j=1}^{N} \exp(t_j^0 + t_j) X \right\}$$

in $\mathfrak{S}$ onto an open neighborhood $\{(t_1, \ldots, t_N); \text{ all } |t_j| < \alpha_3, \alpha_3 > 0\}$ in E^N. What has already been shown is that there is an α', $0 < \alpha' \leqslant \alpha_3$, such that

$$g\left(z_0 \cdot \prod_{j=1}^{N} \exp\left[t_j^0 + t_j\right] X_j\right)$$

equals its Taylor series expansion about the origin for $|t_j| < \alpha'$. The second chart maps an open neighborhood U_4 of points

$$\left\{y_0 \cdot \prod_{j=1}^{N} \exp t_j X_j\right\}$$

in $\mathfrak{S}$ onto an open neighborhood $\{(t_1, \ldots, t_N); \text{ all } |t_j| < \alpha_4, \alpha_4 > 0\}$ in E^N. Since these two charts at y_0 are analytically related, there is an α, $0 < \alpha \leqslant \alpha_4$, such that

$$g\left(y_0 \cdot \prod_{j=1}^{N} \exp t_j X_j\right)$$

equals its Taylor series expansion about the origin for $|t_j| < \alpha$. The proof shows that $\alpha = \alpha(y_0)$ depends on y_0 and $\{t_j^0; j = 1, \ldots, N\}$ but not on g.

For any $y_1 \epsilon \mathfrak{S}$ obviously these results apply to

$$g\left(y_0 y_1 \cdot \prod_{j=1}^{N} \exp t_j X_j\right)$$

for an $\alpha(y_0 y_1)$ depending on y_1 as well as y_0. However for later purposes let us eliminate this dependence on y_1 at least for $t_j \geqslant 0$. Note that

$$y_1 y_0 = y_1 z_0 \cdot \prod_{j=1}^{N} \exp\left(t_j^0 X_j\right)$$

for the same z_0 and $\{t_j^0; j = 1, \ldots, N\}$ as before. These depend only on y_0. The charts at y_0 are now translated to charts at $y_1 y_0$; that is, they are mappings of neighborhoods of $y_1 y_0$ onto the same neighborhoods in E^N as formerly for neighborhoods of y_0. Thus we have the same $\alpha_1, \alpha_2, \alpha_3, \alpha', \alpha_4, \alpha$ as before and they all depend on y_0 only, not on y_1. Hence

$$g\left(y_1 y_0 \cdot \prod_{j=1}^{N} \exp t_j X_j\right)$$

equals its Taylor series expansion about the origin for $|t_j| < \alpha = \alpha(y_0)$. Now for all $t_j \geqslant 0$ we have

$$g\left(y_1 y_0 \cdot \prod_{j=1}^{N} \exp t_j X_j\right) = g\left(y_0 y_1 \cdot \prod_{j=1}^{N} \exp t_j X_j\right).$$

Therefore

$$g\left(y_0 y_1 \cdot \prod_{j=1}^{N} \exp t_j X_j\right)$$

equals its Taylor series expansion with respect to the origin for $0 \leqslant t_j < \alpha = \alpha(y_0)$, $\alpha(y_0)$ being independent of y_1.

Note that we could restrict E_1 in the proof of Lemma 3 to

$$E_1 = \left\{z_0 \cdot \prod_{j=1}^{N} \exp t_j X_j; \, 0 \leqslant t_j \leqslant \gamma\right\}$$

where $2\gamma > t_j^0$ without altering our essential result. The values of α_3, α', α_4, α might be changed.

Similar results hold for

$$g\left(\left\{\prod_{j=1}^{N} \exp t_j X_j\right\} \cdot y_0\right).$$

iii. *The derivative operator.* Let X be an element of the Lie algebra such that $e^X \epsilon \mathfrak{S}$. For any $g \epsilon \mathscr{F}$ we have just seen that $Xg(x) = (d/dt) g\left(x e^{tX}\right)\big|_{t=0}$ exists for all $x \epsilon \mathfrak{S}$. Since $f_v(xy) = f_v(yx)$ for all $x, y \epsilon \mathfrak{S}$ implies $g\left(x e^{tX}\right) = g\left(e^{tX} x\right)$ for $x \epsilon \mathfrak{S}$ and $t > 0$, denoting right derivatives by means of $t = 0+$, we have

$$\frac{d}{dt} g\left(x e^{tX}\right)\bigg|_{t=0} = \frac{d}{dt} g\left(x e^{tX}\right)\bigg|_{t=0+} = \frac{d}{dt} g\left(e^{tX} x\right)\bigg|_{t=0+} = \frac{d}{dt} g\left(e^{tX} x\right)\bigg|_{t=0}.$$

To complete the picture there remains only the question of what happens at e. Here we call $Xg(e)$ the right derivative $(d/dt) g\left(e^{tX}\right)\big|_{t=0+}$ if it exists. If there is an $h \epsilon \mathscr{F}$ such that $h(x) = Xg(x)$ for every $x \epsilon \mathfrak{S}$, then $Xg(e)$ exists and $= h(e)$ by continuity of the functions in $\mathscr{F}$.

Define the operator A by $Ag = Xg$ with domain $\vartheta(A)$ all $g \epsilon \mathscr{F}$ such that $Xg \epsilon \mathscr{F}$ also. (Occasionally we write $Ag(x)$ for $(Ag)(x)$).

In the next several sections we examine various properties of A, the aim being to show that A is self-adjoint.

iv. *A is densely defined.* For any $\varepsilon > 0$ let $k(t) = k_\varepsilon(t) \geqslant 0$ be a real function with derivatives of all orders on $(-\infty, \infty)$, vanishing together with all its derivatives outside the interval $(-\varepsilon, \varepsilon)$, and such that

$$\int_{-\varepsilon}^{\varepsilon} k_\varepsilon(t) \, dt = 1.$$

For each $x \epsilon \mathfrak{S}$ there is a $\beta(x) > 0$ such that $e^{tX} x \epsilon \mathfrak{S}$ for $t > -\beta(x)$. For a fixed set $\{x_j; j = 1, \ldots, m\} \subseteq \mathfrak{S}$ let $\beta = \min\{\beta(x_j); 1 \leqslant j \leqslant m\}$. That is, $\{e^{tX} x_j; j = 1, \ldots, m\} \subseteq \mathfrak{S}$ for $t > -\beta$.

For $0 < \varepsilon < \beta$ define g_ε (depending on $\{x_j; j = 1, \ldots, m\} \subseteq \mathfrak{S}$) by

(6)
$$g_\varepsilon(x) = \int_{-\varepsilon}^{\varepsilon} k_\varepsilon(t) \sum_{j=1}^{m} \alpha_j f_v(xe^{tX} x_j)\, dt,$$

where $\{\alpha_j; j = 1, \ldots, m\}$ is a set of complex numbers, $x \in \mathfrak{S} \cup \{e\}$. $g_\varepsilon \in \mathscr{F}$ by the continuity of the integrand in the topology of $\mathscr{F}$.

For s satisfying $s - \varepsilon > -\beta$

$$\frac{g_\varepsilon(xe^{sX}) - g_\varepsilon(x)}{s} = \int_{-\varepsilon}^{\varepsilon} k(t) \sum_{j=1}^{m} \alpha_j \frac{f_v(xe^{(s+t)X} x_j) - f_v(xe^{tX} x_j)}{s}\, dt$$

$$= \int_{-\varepsilon}^{\varepsilon} k(t) \sum_{j=1}^{m} \alpha_j \frac{d}{du} f_v(xe^{uX} x_j) \Big|_{u = t + \theta s}\, dt$$

$$\to \int_{-\varepsilon}^{\varepsilon} k(t) \sum_{j=1}^{m} \alpha_j \frac{d}{dt} f_v(xe^{tX} x_j)\, dt \quad \text{as} \quad s \to 0$$

$$= \int_{-\varepsilon}^{\varepsilon} -k'(t) \sum_{j=1}^{m} \alpha_j f_v(xe^{tX} x_j)\, dt.$$

Here the mean value theorem ($0 < \theta < 1$) was applied; the limit procedure is based on the uniform convergence of the integrand resulting from the uniform continuity of the derivative, $k(t)$ being bounded.

Note that actually g_ε is defined in a neighborhood of e and its two-sided derivative exists at $x = e$ also.

Now

$$\int_{-\varepsilon}^{\varepsilon} -k'(t) \sum_{j=1}^{m} \alpha_j f_v(\cdot e^{tX} x_j)\, dt$$

belongs to $\mathscr{F}$. Therefore $g_\varepsilon \in \vartheta(A)$ and

(7)
$$Ag_\varepsilon = \int_{-\varepsilon}^{\varepsilon} -k'(t) \sum_{j=1}^{m} \alpha_j f_v(\cdot e^{tX} x_j)\, dt.$$

As $\varepsilon \to 0$,

$$k_\varepsilon(t) \to \begin{cases} \infty & \text{at} \quad t = 0, \\ 0 & \text{elsewhere,} \end{cases}$$

in such a fashion that

$$\int_{-\varepsilon}^{\varepsilon} k(t) f_v(x_j e^{tX} x_i)\, dt \to f_v(x_j x_i)$$

and

$$\int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} k(t) k(\tau) f_v(e^{\tau X} x_j e^{tX} x_i)\, dt\, d\tau \to f_v(x_j x_i).$$

(Note. In any neighborhood of $x_j x_i$ there are always points of type $e^{\tau X} x_j e^{tX} x_i$ by the continuity of multiplication in $\mathscr{G}$.)

$$\Big\| g_\varepsilon - \sum_{j=1}^{m} \alpha_j f_v(\cdot x_j) \Big\|^2$$

$$= \sum_{i,j=1}^{m} \alpha_i \bar{\alpha}_j \Big(\int_{-\varepsilon}^{\varepsilon} k(t) f_v(\cdot e^{tX} x_i)\, dt - f_v(\cdot x_i) \Big| \int_{-\varepsilon}^{\varepsilon} k(\tau) f_v(\cdot e^{\tau X} x_j)\, d\tau - f_v(\cdot x_j) \Big).$$

The inner product on the right side is

$$\int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} k(t) k(\tau) f_v(e^{\tau X} x_j e^{tX} x_i)\, dt\, d\tau + f_v(x_j x_i) - \int_{-\varepsilon}^{\varepsilon} k(t) f_v(x_j e^{tX} x_i)\, dt -$$

$$- \int_{-\varepsilon}^{\varepsilon} k(\tau) f_v(e^{\tau X} x_j x_i)\, d\tau \to 0.$$

Therefore

$$g_\varepsilon \to \sum_{j=1}^{m} \alpha_j f_v(\cdot x_j)$$

in the topology of $\mathscr{F}$. Since functions of type

$$\sum_{j=1}^{m} \alpha_j f_v(\cdot x_j), \quad \{x_j; j = 1, \ldots, m\} \subseteq \mathfrak{S},$$

are dense in $\mathscr{F}$, so are functions of type g_ε given by (6). Therefore $\vartheta(A)$ is dense in $\mathscr{F}$ and A^* exists.

v. *Closure of A, symmetry of A^*.* If in $\mathscr{F}$ we have a sequence $g_m \to g$ and $Ag_m \to h$ as $m \to \infty$ (hence $Ag_m(x) \to h(x)$ uniformly on a compact set), then $Ag(x) = h(x)$, or $Ag = h$ ([10], Th. 4, p. 342).

Therefore A is closed. $A = A^{**}$; A^* is densely defined.

Next let g be any element in $\vartheta(A^*)$. Let $z \in \mathfrak{S}$ and choose ε, $0 < \varepsilon < \beta(z)$, and $k(t) = k_\varepsilon(t)$ as before. We get

$$\Big(A^* g \Big| \int_{-\varepsilon}^{\varepsilon} k(t) f_v(\cdot e^{tX} z)\, dt \Big) = \Big(g \Big| A \int_{-\varepsilon}^{\varepsilon} k(t) f_v(\cdot e^{tX} z)\, dt \Big)$$

$$= \Big(g \Big| \int_{-\varepsilon}^{\varepsilon} -k'(t) f_v(\cdot e^{tX} z)\, dt \Big) = \int_{-\varepsilon}^{\varepsilon} -k'(t) \big(g | f_v(\cdot e^{tX} z) \big)\, dt$$

$$= \int_{-\varepsilon}^{\varepsilon} -k'(t) g(e^{tX} z)\, dt = \int_{-\varepsilon}^{\varepsilon} k(t) \frac{d}{dt} g(e^{tX} z)\, dt.$$

Let $\varepsilon \to 0$;

$$(A^*g)(z) = (A^*g)|f_v(\,\cdot\,z)) = \frac{d}{ds}g(e^{tX}z)|_{s=0} = Xg(z)$$

for all $z \in \mathfrak{S}$.

Since a priori $A^*g \in \mathscr{F}$, g is in $\vartheta(A)$, $A^*g = Ag$. Hence $A^* \subseteq A$ and A^* is symmetric.

vi. *A is self-adjoint.* Let g_ε be given as in (6). The class of functions of this type is dense in $\mathscr{F}$ as was noted before. We use induction to show that

$$g_\varepsilon \in \bigcap \{\vartheta(A^{*p}); \; p = 1, 2, \ldots\}.$$

Suppose $g_\varepsilon \in \vartheta(A^{*p})$ for any $p \geqslant 0$. Then

$$A^{*p}g_\varepsilon = A^p g_\varepsilon = \int_{-\varepsilon}^{\varepsilon} (-1)^p k^{(p)}(t) \sum_{j=1}^{m} \alpha_j f_v(\,\cdot\, e^{tX}x_j)\, dt$$

by repeated application of (7). Let $\psi \in \vartheta(A)$.

$$(A\psi | A^{*p}g_\varepsilon) = (A\psi | A^p g_\varepsilon) = \int_{-\varepsilon}^{\varepsilon} (-1)^p k^{(p)}(t) \sum_{j=1}^{m} \bar\alpha_j \big(A\psi | f_v(\,\cdot\, e^{tX}x_j)\big)\, dt$$

$$= \int_{-\varepsilon}^{\varepsilon} (-1)^p k^{(p)}(t) \sum_{j=1}^{m} \bar\alpha_j (A\psi)(e^{tX}x_j)\, dt = \int_{-\varepsilon}^{\varepsilon} (-1)^{p+1} k^{(p+1)}(t) \sum_{j=1}^{m} \bar\alpha_j \psi(e^{tX}x_j)\, dt$$

$$= \int_{-\varepsilon}^{\varepsilon} (-1)^{p+1} k^{(p+1)}(t) \sum_{j=1}^{m} \bar\alpha_j \big(\psi|f_v(\,\cdot\, e^{tX}x_j)\big)\, dt = (\psi|A^{p+1}g_\varepsilon).$$

Therefore $A^{*p}g_\varepsilon \in \vartheta(A^*)$, or $g_\varepsilon \in \vartheta(A^{*p+1})$.

Now take any $h \in \mathscr{F}$. Let s be such that $s - \varepsilon > -\beta$. We have

$$\sum_{p=0}^{P} \left(\frac{A^{*p}g_\varepsilon}{p!} \Big| h\right) s^p = \int_{-\varepsilon}^{\varepsilon} \sum_{j=1}^{m} \alpha_j \sum_{p=0}^{P} \frac{(-1)^p k^{(p)}(t)}{p!} s^p \big(f_v(\,\cdot\, e^{tX}x_j)|h\big)\, dt$$

$$= \int_{-\varepsilon}^{\varepsilon} \sum_{j=1}^{m} \alpha_j \sum_{p=0}^{P} \frac{(-1)^p k^{(p)}(t)}{p!} s^p \bar h(e^{tX}x_j)\, dt = \sum_{j=1}^{m} \alpha_j \int_{-\varepsilon}^{\varepsilon} k(t) \sum_{p=0}^{\infty} \frac{d^p}{dt^p} \bar h(e^{tX}x_j) \frac{s^p}{p!}\, dt$$

$$\xrightarrow[\text{as } P \to \infty]{} \sum_{j=1}^{m} \alpha_j \int_{-\varepsilon}^{\varepsilon} k(t) \sum_{p=0}^{\infty} \frac{d^p}{dt^p} \bar h(e^{tX}x_j) \frac{s^p}{p!}\, dt$$

for $|s| < \alpha$ for some α independent of h (part ii) with $0 < \alpha < \beta - \varepsilon$, by the analyticity of $\bar h$, provided ε is sufficiently small; and this last member incidentally

$$= \int_{-\varepsilon}^{\varepsilon} k(t) \sum_{j=1}^{m} \alpha_j \bar h(e^{(s+t)X}x_j)\, dt = \int_{-\varepsilon}^{\varepsilon} k(t) \sum_{j=1}^{m} \alpha_j \big(f_v(\,\cdot\, e^{(s+t)X}x_j)|h\big)\, dt = \big(g_\varepsilon(\,\cdot\, e^{sX})|h\big).$$

The limit process above is justified by the uniform convergence of the integrand which is seen as follows. Select α, $0 < 4\alpha < \beta$, such that its Taylor series is a valid representation for each $\bar h(e^{tX}x_j)$, $j = 1, \ldots, m$, in $|t| < 4\alpha$. There is a function $H_j(\tau)$, analytic in the circle $|\tau| < 4\alpha$, which reduces to $\bar h(e^{tX}x_j)$ for $\tau = t$ (real). Let $M = M(3\alpha, H_j) = \max |H_j(\tau)|$ on the circle $|\tau| \leqslant 3\alpha$. Take $0 < \varepsilon < \alpha$. For each real t in $|t| \leqslant \varepsilon$ apply Cauchy's inequality to the circle C_t with center at the point $(t, 0)$ and radius 2α. We get

$$|(d/dt)^p \bar h(e^{tX}x_j)| \cdot |s|^p/(p!) \leqslant M \cdot |s|^p/(2\alpha)^p \leqslant M \cdot (1/2)^p \quad \text{for} \quad |s| \leqslant \alpha.$$

Then the series in the integrand converges uniformly for $|t| \leqslant \varepsilon$ and for $|s| \leqslant \alpha$. Thus

$$\sum_{p=0}^{\infty} (A^{*p}g_\varepsilon|h) s^p/(p!)$$

converges for all $h \in \mathscr{F}$ and each s in $|s| < \alpha$ where α depends on $\{x_j; \; j = 1, \ldots, m\}$ but not on h, while $0 < \varepsilon < \alpha$.

Apply Lemma 1 with $B = A^*$. A^* is self-adjoint. $A^* = A^{**} = A$. Hence A is self-adjoint.

vii. *The exponential (or translation) operator.* Let E be the spectral resolution of the identity for A. Define

$$(8) \qquad T_t(X) \text{ or } e^{tiA} = \int_{-\infty}^{\infty} e^{ti\lambda}\, dE(\lambda) \quad \text{for} \quad -\infty < t < \infty.$$

Let $g_c \in \vartheta(A)$ for which there is a number c, $0 < c < \infty$, such that

$$Ag_c = \int_{-c}^{c} \lambda\, dE(\lambda)\, g_c.$$

(E.g., set $\Delta = \{\lambda; -c \leqslant \lambda \leqslant c\} = \{\lambda; e^{-|t|c} \leqslant e^{|t|\lambda} \leqslant e^{|t|c}\}$ and take $g_c \in \mathscr{R}\big(E(\Delta)\big)$, $\mathscr{R} = \text{range}$.) $g_c \in \vartheta\big(T_t(X)\big)$ and

$$T_t(X)g_c = \int_{-c}^{c} e^{ti\lambda}\, dE(\lambda)\, g_c = \int_{-c}^{c} \sum_{p=0}^{\infty} \frac{t^p \lambda^p}{p!}\, dE(\lambda)\, g_c$$

$$= \sum_{p=0}^{\infty} \frac{t^p}{p!} \int_{-c}^{c} \lambda^p\, dE(\lambda)\, g_c = \sum_{p=0}^{\infty} \frac{t^p}{p!} A^p g_c.$$

Pick any $y_0 \in \mathfrak{S}$. Since g_c is analytic, there is an $\alpha = \alpha(y_0)$ (independent of g_c) such that

$$T_t(X)g_c(y_0) = \sum_{p=0}^{\infty} \frac{t^p}{p!} A^p g_c)(y_0) = \sum_{p=0}^{\infty} \frac{t^p}{p!} \frac{d^p}{ds^p} g_c(y_0 e^{sX})|_{s=0}$$

$$= g_c(y_0 e^{tX}) \quad \text{for} \quad |t| < \alpha.$$

Also, if $y_1 \epsilon \mathfrak{S} \cup \{e\}$, then for the same $\alpha = \alpha(y_0)$ (independent of y_1 — see part ii)

$$T_t(X)g_c(y_0 y_1) = g_c(y_0 y_1 e^{tX}) \quad \text{for} \quad 0 \leqslant t < \alpha.$$

Now $T_t(X)$, as well as A, is the closure of its restriction to the class of such g_c. Therefore

$$(9) \qquad T_t(X)g(y_0) = g(y_0 e^{tX}) \quad \text{for} \quad |t| < \alpha,$$

$$T_t(X)g(y_0 y_1) = g(y_0 y_1 e^{tX}) \quad \text{for} \quad 0 \leqslant t < \alpha$$

for all $g \epsilon \vartheta\big(T_t(X)\big)$.

viii. *Study of domain of* $T_t(X)$. It turns out that $f_v(\cdot y_0 y_1) \epsilon \vartheta\big(T_t(X)\big)$ and

$$(10) \qquad T_t(X)f_v(\cdot y_0 y_1) = f_v(\cdot y_0 y_1 e^{tX}) \quad \text{for} \quad 0 \leqslant t < \alpha.$$

For, let $\{\mathcal{M}_k; \, k = 1, 2, \ldots\}$ be a sequence of mutually orthogonal subspaces such that

$$\mathscr{F} = \sum_{k=1}^{\infty} \oplus \mathcal{M}_k$$

and such that each $\mathcal{M}_k$ reduces $T_t(X)$ to a bounded self-adjoint operator $T_{k,t}(X)$. Let $f_k(\cdot, y_0 y_1)$ be the projection of $f_v(\cdot y_0 y_1)$ on $\mathcal{M}_k$. Then for each $g \epsilon \mathcal{M}_k$

$$\big(g | f_k(\cdot, y_0 y_1 e^{tX})\big) = \big(g | f_v(\cdot y_0 y_1 e^{tX})\big) = g(y_0 y_1 e^{tX})$$

$$= T_t(X)g(y_0 y_1) = T_{k,t}(X)g(y_0 y_1) = \big(T_{k,t}(X)g | f_v(\cdot y_0 y_1)\big)$$

$$= \big(T_{k,t}(X)g | f_k(\cdot, y_0 y_1)\big) = \big(g | T_{k,t}(X)f_k(\cdot, y_0 y_1)\big)$$

for $0 \leqslant t < \alpha$. Therefore $T_{k,t}(X)f_k(\cdot, y_0 y_1) = f_k(\cdot, y_0 y_1 e^{tX})$ and

$$\sum_{k=1}^{\infty} \|T_{k,t}(X)f_k(\cdot, y_0 y_1)\|^2 = \sum_{k=1}^{\infty} \big(f_k(\cdot, y_0 y_1 e^{tX}) | f_k(\cdot, y_0 y_1 e^{tX})\big)$$

$$= \sum_{k=1}^{\infty} \big(f_k(\cdot, y_0 y_1 e^{tX}) | f_v(\cdot y_0 y_1 e^{tX})\big)$$

$$= \sum_{k=1}^{\infty} f_k(y_0 y_1 e^{tX}, y_0 y_1 e^{tX}) = f_v(y_0 y_1 e^{tX} y_0 y_1 e^{tX}) < \infty.$$

Thus, for $0 \leqslant t < \alpha$,

$$(10') \qquad T_t(X)f_v(\cdot y_0 y_1) = \sum_{k=1}^{\infty} T_{k,t}(X)f_k(\cdot, y_0 y_1) = f_v(\cdot y_0 y_1 e^{tX}).$$

(Similarly $T_t(X)f_v(\cdot y_0) = f_v(\cdot y_0 e^{tX})$ for $|t| < \alpha$.)

ix. *The translation operator* $T(X)$. Let $M = [1/\alpha] + 1$ (so that M is the least integer $> 1/\alpha$) and let $\tau = 1/M$. Hence $0 < \tau < \alpha$.

$T_1(X) = [\{T_\tau(X)\}^M]$, the closure of the M-fold product of the same factors $T_\tau(X)$. If $M = 1$, set $y_1 = e$ and $t = 1$ in (10). If $M > 1$, then

$$T_1(X)f_v(\cdot y_0) = \{T_\tau(X)\}^M f_v(\cdot y_0) = \{T_\tau(X)\}^{M-1}f_v(\cdot y_0 e^{\tau X})$$

$$= \{T_\tau(X)\}^{M-2}f_v(\cdot y_0 e^{2\tau X}) = \cdots = T_\tau(X)f_v(\cdot y_0 e^{(M-1)\tau X})$$

$$= (f_v(\cdot y_0 e^{M\tau X}) = f_v(\cdot y_0 e^X)$$

where (10) has been applied successively with $y_1 = e, e^{\tau X}, \ldots, e^{(M-1)\tau X}$.

Set $T(X) = T_1(X)$. Then

$$(11) \qquad T(X)f_v(\cdot y) = f_v(\cdot y e^X) \quad \text{for any} \quad y \epsilon \mathfrak{S}.$$

But $f_v(\cdot y y_1) \to f_v(\cdot e y_1)$ strongly as $y \to e$, $y \epsilon \mathfrak{S}$, $y_1 \epsilon \mathfrak{S} \cup \{e\}$. Since $T(X)$ is closed, (11) holds for any $y \epsilon \mathfrak{S} \cup \{e\}$.

x. *Permutability of exponential operators corresponding to different elements of the Lie algebra*. Suppose that X_n and X_m are any elements of the Lie algebra such that $\exp X_n$, $\exp X_m \epsilon \mathfrak{S}$. Consider the corresponding self-adjoint operators A_n and A_m, and their spectral resolutions E_n and E_m. Let a_n and a_m, depending on y_0 only, have the same meaning as assigned to $\alpha(y_0)$ earlier. Finally consider $T_t(X_n)$ and $T_s(X_m)$ for $0 \leqslant t < a_n, 0 \leqslant s < a_m$:

$$T_t(X_n)T_s(X_m)f_v(\cdot y_0) = T_t(X_n)f_v(\cdot y_0 \exp s X_m) = f_v(\cdot y_0 \exp s X_m \exp t X_n),$$

where (10) has been applied successively with $y_1 = e$, $\exp s X_m$. Also

$$T_s(X_m)T_t(X_n)f_v(\cdot y_0) = f_v(\cdot y_0 \exp t X_n \exp s X_m) = f_v(\cdot y_0 \exp s X_m \exp t X_n).$$

Hence

$$T_t(X_n)T_s(X_m)f_v(\cdot y_0) = T_s(X_m)T_t(X_n)f_v(\cdot y_0).$$

For any $g \epsilon \vartheta\big(T_s(X_m)\big)$

$$\int_{-\infty}^{\infty} e^{tv}\,d\big(E_n(v)T_s(X_m)f_v(\cdot y_0) | g\big)$$

$$= \big(T_t(X_n)T_s(X_m)f_v(\cdot y_0) | g\big) = \big(T_s(X_m)T_t(X_n)f_v(\cdot y_0) | g\big)$$

$$= \big(T_t(X_n)f_v(\cdot y_0) | T_s(X_m)g\big)$$

$$= \int_{-\infty}^{\infty} e^{tv}\,d\big(E_n(v)f_v(\cdot y_0) | T_s(X_m)g\big).$$

Since t ranges over the interval $(0, a_n)$ for each s,

$$\big(T_s(X_m)f_v(\cdot y_0) | E_n(v)g\big) = \big(E_n(v)T_s(X_m)f_v(\cdot y_0) | g\big)$$

$$= \big(E_n(v)f_v(\cdot y_0) | T_s(X_m)g\big)$$

by uniqueness of measure for Laplace-Stieltjes integrals ([15], p. 243, Th. 6a). Further

$$\int_{-\infty}^{\infty} e^{s\lambda} d\big(E_n(\nu)E_m(\lambda)f_v(\cdot y_0)|g\big) = \int_{-\infty}^{\infty} e^{s\lambda} d\big(E_m(\lambda)f_v(\cdot y_0)|E_n(\nu)g\big)$$

$$= \int_{-\infty}^{\infty} e^{s\lambda} d\big(E_n(\nu)f_v(\cdot y_0)|E_m(\lambda)g\big) = \int_{-\infty}^{\infty} e^{s\lambda} d\big(E_m(\lambda)E_n(\nu)f_v(\cdot y_0)|g\big).$$

Therefore

$$\big(E_n(\nu)E_m(\lambda)f_v(\cdot y_0)|g\big) = \big(E_m(\lambda)E_n(\nu)f_v(\cdot y_0)|g\big)$$

again by uniqueness of measure since s ranges over the interval $(0, a_m)$. Since $\vartheta\big(T_s(X_m)\big)$ is dense in $\mathscr{F}$, the last equality holds for all $g \in \mathscr{F}$ and hence

$$E_n(\nu)E_m(\lambda)f_v(\cdot y) = E_m(\lambda)E_n(\nu)f_v(\cdot y)$$

for any $y \in \mathfrak{S}$. Finally, since the linear manifold generated by $\{f_v(\cdot y); y$ varying over $\mathfrak{S}\}$ is dense in $\mathscr{F}$,

$$E_n(\nu)E_m(\lambda) = E_m(\lambda)E_n(\nu).$$

$d\{E_n(\nu)E_m(\lambda)\}$ is thus an orthogonal spectral measure. Consider any finite set $X_1, \ldots, X_J$ of elements of the Lie algebra with all $\exp X_j \in \mathfrak{S}$. By induction

$$d\{E_1(\lambda_1)E_2(\lambda_2)\ldots E_J(\lambda_J)\} = d\{[E_1(\lambda_1)\ldots E_{J-1}(\lambda_{J-1})]E_J(\lambda_J)\}$$

is an orthogonal spectral measure and indeed the E_j's may be rearranged in any order. (Apply consecutive pair-wise interchanges or let the induction hypothesis be that this result is true for a product of any $j-1$ of the E's, $3 \leqslant j \leqslant J$.)

Thus the closure of $T(X_1)T(X_2)\ldots T(X_J)$ is a self-adjoint operator. (This is a known result in Hilbert space theory.)

xi. *New translation operator* T_u. Every $u \in \mathfrak{S}$ can be expressed in at least one way as a product of the form

$$u = \prod_{j=1}^{J} \exp X_j$$

where the X_j's need not all be distinct, all $\exp X_j \in \mathfrak{S}$. Define a new operator $T_u = [T(X_1)\ldots T(X_J)]$, where $[\;\;]$ means closure:

$$T_u f_v(\cdot y) = T(X_1)\ldots T(X_J)f_v(\cdot y)$$

$$= T(X_1)\ldots T(X_{J-1})f_v(\cdot y \exp X_J) = \ldots = f_v\Big(\cdot y \prod_{j=J}^{1} \exp X_j\Big)$$

$$= f_v\Big(\cdot y \prod_{j=1}^{J} \exp X_j\Big) = f_v(\cdot yu).$$

for $y \in \mathfrak{S} \cup \{e\}$, where (11) has been applied successively with y, $y \exp X_J$, …, $y \prod_{j=J}^{2} \exp X_j$ taking the role of y. Defining $T_e = I$, the identity operator, we have

$$(12) \qquad T_u f_v(\cdot y) = f_v(\cdot yu) \quad \text{for} \quad u \in \mathfrak{S} \cup \{e\}, y \in \mathfrak{S} \cup \{e\}.$$

(12) shows that T_u is uniquely defined on the linear manifold generated by $\{f_v(\cdot y); y \in \mathfrak{S} \cup \{e\}\}$ irrespective of the particular product representation of u. Let us suppose that T_u and T'_u arise from different representations

$$u = \prod_{j=1}^{J} \exp X_j \quad \text{and} \quad u = \prod_{j=J+1}^{K} \exp X_j$$

respectively. We must show that $T_u = T'_u$. This follows as a corollary of the next lemma and the above remarks since T_u and T'_u permute.

LEMMA 4. *Let B and C be permutable self-adjoint operators and let $Bx = Cx$ for all x in a dense set ϑ in $\mathscr{H}$. Then $B = C$.*

Proof. Let dE and dF be the spectral measures corresponding to B and C respectively. Let

$$G(\varDelta) = \int_{\varDelta} dE(\lambda)dF(\nu)$$

where $\varDelta$ is any Borel set in the plane R^2. Select a sequence $\{\varDelta_k; k = 1, 2, \ldots\}$ of mutually disjoint finite rectangles such that

$$R^2 = \bigcup_{k=1}^{\infty} \varDelta_k.$$

Set $\mathscr{M}_k = \mathscr{R}\big(G(\varDelta_k)\big)$, $\mathscr{R} = $ range, and $B_k = B \mid \mathscr{M}_k$, $C_k = C \mid \mathscr{M}_k$ (restrictions of B and C to $\mathscr{M}_k$).

$\mathscr{M}_k$ reduces B and C to the bounded self-adjoint operators B_k and C_k respectively. $B = \sum \oplus B_k$ and $C = \sum \oplus C_k$. Let $y \in \mathscr{H}$ and y_k its projection on $\mathscr{M}_k$. Let $x \in \vartheta$. Then

$$(B_k y_k|x) = (By_k|x) = (y_k|Bx) = (y_k|Cx) = (Cy_k|x) = (C_k y_k|x).$$

Therefore $B_k y_k = C_k y_k$ since ϑ is dense in $\mathscr{H}$;

$$\sum_{k=1}^{\infty} \|B_k y_k\|^2 = \sum_{k=1}^{\infty} \|C_k y_k\|^2.$$

Since both sides are finite or infinite together, $y \in \vartheta(B)$ if and only if $y \in \vartheta(C)$. Therefore

$$B = \sum \oplus B_k = \sum \oplus C_k = C.$$

Incidentally, if $g \epsilon \vartheta(T_u)$,

$$(T_u g)(y) = \big(T_u g | f_v(\cdot y)\big) = \big(g | T_u f_v(\cdot y)\big) = \big(g | f_v(\cdot yu)\big) = g(yu).$$

Thus $T_u g = g(\cdot u)$ and $g(\cdot u) \epsilon \mathscr{F}$.

xii. *Semi-group of operators; representation of f_v.* Suppose

$$u = \prod_{j=1}^{J} \exp X_j \quad \text{and} \quad r = \prod_{j=J+1}^{K} \exp X_j.$$

T_u and T_r permute; i.e., their spectral resolutions permute, $E_r(\lambda) E_u(\nu) = E_u(\nu) E_r(\lambda)$. From the comments following $(2'')$ of §2, $[T_u T_r] = [T_r T_u] = T_{ru} = T_{ur}$. Thus $\{T_u; u \epsilon \mathfrak{S} \cup \{e\}\}$ forms what we may call a *semi-group of operators*.

Let $\xi \epsilon \bigcap \{\vartheta(T_u); u \epsilon \mathfrak{S} \cup \{e\}\}$. Consider a function of the form

$$h = \sum_{k=1}^{m} \beta_k f_v(\cdot x_k),$$

where $\{x_k; k = 1, \ldots, m\} \subseteq \mathfrak{S} \cup \{e\}$:

$$(T_u \xi | h) = (\xi | T_u h) = \sum_{k=1}^{m} \bar{\beta}_k \big(\xi | T_u f_v(\cdot x_k)\big)$$

$$= \sum_{k=1}^{m} \bar{\beta}_k \big(\xi | f_v(\cdot x_k u)\big) = \sum_{k=1}^{m} \bar{\beta}_k \, \xi(x_k u).$$

Then $(T_u \xi | h)$ is continuous in u since $\xi(x_k u)$ is. Since $\mathscr{F}$ is the completion of the pre-Hilbert space $\mathscr{F}'$ generated by functions of type h, $(3'')$ of §2 is satisfied.

Alternately $(3''')$ of §2 is satisfied. For by the discussion following Theorem B the cone $\mathfrak{S}$ contains a countable dense set D. Hence the class of functions of type given by h above but with $\{x_k; k = 1, \ldots, m\} \subseteq D$ and β_k's having rational real and imaginary parts is dense in $\mathscr{F}$. Hence $\mathscr{F}$ is separable.

Now Theorem B may be applied for $u \epsilon \mathfrak{S}$. There exists a spectral measure $E(d\chi)$ such that

$$T_u = \int_{\mathfrak{S}^\wedge} \chi(u) E(d\chi) \quad \text{for all} \quad u \epsilon \mathfrak{S},$$

(13)
$$f_v(u) = \big(f_v(\cdot u) | f_v(\cdot e)\big) = \big(T_u f_v(\cdot e) | f_v(\cdot e)\big)$$

$$= \int_{\mathfrak{S}^\wedge} \chi(u) \big(E(d\chi) f_v(\cdot e) | f_v(\cdot e)\big)$$

$$= \int_{\mathfrak{S}^\wedge} \chi(u) \, d\nu_v(\chi) \quad \text{for } u \epsilon \mathfrak{S}.$$

xiii. *Representation of f.* Take $x = vu \, \epsilon v \mathfrak{S}$, $u = v^{-1}x$:

$$f(x) = f(vu) = f_v(u) = \int_{\mathfrak{S}^\wedge} \chi(v^{-1}x) \, d\nu_v(\chi),$$

$$\chi(x) = \chi(vv^{-1}x) = \chi(v) \chi(v^{-1}x),$$

$\chi(r) > 0$ for all $r \epsilon \mathfrak{S}$ and all $\chi \epsilon \mathfrak{S}^\wedge$. (See discussions following Lemma A and Definition C.) Hence

$$\chi(v^{-1}x) = \chi(v)^{-1} \chi(x),$$

$$f(x) = \int_{\mathfrak{S}^\wedge} \chi(x) \chi(v)^{-1} \, d\nu_v(\chi) \quad \text{for all} \quad x \epsilon v \mathfrak{S}.$$

Let μ_v be the positive measure defined on $\mathfrak{S}^\wedge$ by

$$\mu_v(h) = \int_{\mathfrak{S}^\wedge} h(\chi) \, d\mu_v(\chi) = \int_{\mathfrak{S}^\wedge} h(\chi) \chi(v)^{-1} \, d\nu_v(\chi),$$

where h is a continuous function vanishing off a compact set. μ_v will now be shown to be independent of the choice of v for v in a certain subset of $\mathfrak{S}$.

Let $v, w \epsilon \mathfrak{S}$ and choose a symmetric neighborhood W of e such that $Wv \subseteq \mathfrak{S}$ and $Ww \subseteq \mathfrak{S}$. Let $Z = W \cap \mathfrak{S}$. $Z \neq \emptyset$ since $e \epsilon [\mathfrak{S}]$. Take $z \epsilon Z$; $z^{-1}v \epsilon Wv \subseteq \mathfrak{S}$, $z^{-1}w \epsilon \mathfrak{S}$, and $v\mathfrak{S} = zz^{-1}v\mathfrak{S} \subseteq z\mathfrak{S}$, $w\mathfrak{S} \subseteq z\mathfrak{S}$. Therefore

(14)
$$\int_{\mathfrak{S}^\wedge} \chi(x) \, d\mu_v(\chi) = f(x) = \int_{\mathfrak{S}^\wedge} \chi(x) \, d\mu_z(\chi)$$

for all $x \epsilon v \mathfrak{S}$.

Now select linearly independent elements $\{X_j; j = 1, \ldots, N\}$ of the Lie algebra such that all $\exp X_j \epsilon \mathfrak{S}$ (Lemma 2 in part ii). Let

$$U = \Big\{\prod_{j=1}^{N} \exp t_j X_j; \text{ all } t_j \geqslant 0, \sum_{j=1}^{N} t_j > 0\Big\}.$$

For

$$x = \prod_{j=1}^{N} \exp t_j X_j \, \epsilon \, U$$

and for $\chi \epsilon \mathfrak{S}^\wedge$,

$$\chi(x) = \prod_{j=1}^{N} \exp t_j \lambda_j$$

for certain numbers $\{\lambda_j; j = 1, \ldots, N\}$. Set $\lambda = (\lambda_1, \ldots, \lambda_N)$. The equation

$$\chi(x) = \prod_{j=1}^{N} \exp t_j \lambda_j$$

defines a map ψ of $\mathfrak{S}^\wedge$ into E^N indicated as $\psi(\chi) = \lambda = (\lambda_1, \ldots, \lambda_N)$. This map is one-to-one. For, if

$$\prod_{j=1}^{N} \exp t_j \lambda_j = \chi(x) = \prod_{j=1}^{N} \exp t_j \eta_j$$

on U, then

$$\prod_{j=1}^{N} \exp t_j(\lambda_j - \eta_j) \equiv 1, \quad \text{or} \quad \sum_{j=1}^{N} t_j(\lambda_j - \eta_j) \equiv 0,$$

for all $t_j \geqslant 0$. Fix i; set $t_i = 1$ and set all $t_j = 0$ for $j \neq i$; then $\lambda_i - \eta_i = 0$ for each i. Thus χ is mapped into only one λ.

To show the converse let

$$U_\varepsilon = \left\{ \prod_{j=1}^{N} \exp t_j X_j; \ 0 < t_j < \varepsilon \right\}.$$

For sufficiently small ε, U_ε is an open set on account of the homeomorphism between some neighborhood of e in $\mathscr{G}$ and one of 0 in E^N, while $\{(t_1, \ldots, t_N); 0 < t_j < \varepsilon\}$ is open. Fix such an ε. $U_\varepsilon \subseteq U \subseteq \mathfrak{S}$. If $\chi_1(x) = \chi_2(x)$ on U, then the character $\chi(x) = \chi_2(x)^{-1} \cdot \chi_1(x) = 1$ on U. Select $x_1 \epsilon U_\varepsilon$ and an open neighborhood V of e such that $x_1 V \subseteq U_\varepsilon$ and such that V satisfies condition (4) of Definition 1. Let $W_1 = V \cap \mathfrak{S}$ and take any $h \epsilon W_1$. $1 = \chi(x_1 h) = \chi(x_1)\chi(h) = \chi(h)$. Since W_1 generates $\mathfrak{S}$, $\chi \equiv 1$ on $\mathfrak{S}$. Hence, if

$$\chi_1(x) = \prod_{j=1}^{N} \exp t_j \lambda_j = \chi_2(x)$$

on U, then $\chi_1 \equiv \chi_2$ on $\mathfrak{S}$ so that only one χ can be mapped by ψ into any λ.

If a point $\lambda^0 \neq 0$ is in $\mathscr{R}(\psi)$, then $\alpha \lambda^0 \epsilon \mathscr{R}(\psi)$ for all real α. (That is, if a non-zero point is in the range of ψ, then so is the entire line determined by 0 and this point.) For, if

$$\chi(x) = \prod_{j=1}^{N} \exp t_j \lambda_j^0,$$

then

$$\chi^\alpha(x) = \prod_{j=1}^{N} \exp t_j \alpha \lambda_j^0.$$

Also

$$(\chi_1 \chi_2)(x) = \prod_{j=1}^{N} \exp t_j \lambda_j^{(1)} \cdot \prod_{j=1}^{N} \exp t_j \lambda_j^{(2)} = \prod_{j=1}^{N} \exp t_j(\lambda_j^{(1)} + \lambda_j^{(2)})$$

so that $\psi(\chi_1 \chi_2) = \lambda^{(1)} + \lambda^{(2)}$. By induction similar results hold for any number of χ's. Hence ψ is a homomorphism mapping $\mathfrak{S}^\wedge$ onto a "plane" P of dimension $n \leqslant N$. (The dimension n is possibly 0.)

In the "plane" P select a neighborhood of the origin,

$$\mathscr{M} = \{\lambda = (\lambda_1, \ldots, \lambda_N); |\lambda_j| < \varepsilon, 1 \leqslant j \leqslant N\} \cap P.$$

Let

$$C_i = \left\{ x = \prod_{j=1}^{N} \exp t_j X_j; \ 0 \leqslant t_j \leqslant 1/\varepsilon \ \text{for all} \ j \neq i, 1/(2\varepsilon) \leqslant t_i \leqslant 1/\varepsilon \right\}.$$

Let

$$C = \bigcup_{i=1}^{N} C_i.$$

Since all the sets

$$\{\exp t_j X_j; 0 \leqslant t_j \leqslant 1/\varepsilon\} \quad \text{and} \quad \{\exp t_i X_i; 1/(2\varepsilon) \leqslant t_i \leqslant 1/\varepsilon\}$$

are compact, so are the sets C_i and their union C. Set $\chi_0 \equiv 1$. Consider the neighborhood

$$\mathscr{N} = \{\chi; |\chi(x) - 1| < 1/2, \text{all} \ x \epsilon C\} \subseteq \mathfrak{S}^\wedge.$$

Let $\chi \epsilon \mathscr{N}$;

$$\chi(x) = \exp \sum_{j=1}^{N} t_j \lambda_j \quad \text{for} \quad x = \prod_{j=1}^{N} \exp t_j X_j \epsilon C.$$

If $|\lambda_i| \geqslant \varepsilon$ for any i, take the corresponding $t_i = 1/\varepsilon$ and the remaining t_j's $= 0$; but then $|\chi\{\exp(1/\varepsilon)X_i\} - 1|$ is either $\geqslant 1 - e^{-1} > 1/2$ (if $\lambda_i \leqslant -\varepsilon$) or $\geqslant e - 1 > 1/2$ (if $\lambda_i \geqslant \varepsilon$), in either event a contradiction. Hence all $|\lambda_j| < \varepsilon$ and so $\psi(\chi) \epsilon \mathscr{M}$. $\psi(\mathscr{N}) \subseteq \mathscr{M}$. Therefore ψ is continuous.

Now we can define a measure μ_v^0 on the Borel sets in the "plane" P by setting $\mu_v^0(B) = \mu_v(\psi^{-1}(B))$ since $\psi^{-1}(B)$ is a Borel set in $\mathfrak{S}^\wedge$ if B is a Borel set in P. Now let us suppose

$$v = \prod_{j=1}^{N} \exp s_j X_j \epsilon U.$$

Take any

$$\prod_{j=1}^{N} \exp \tau_j X_j \epsilon U$$

and set $t_j = s_j + \tau_j$. Then

$$\chi\left(v \prod_{j=1}^{N} \exp \tau_j X_j\right) = \prod_{j=1}^{N} \exp s_j \lambda_j \cdot \prod_{j=1}^{N} \exp \tau_j \lambda_j = \prod_{j=1}^{N} \exp t_j \lambda_j.$$

From (14)

$$(15) \qquad \int_P \exp\left(\sum_{j=1}^{N} t_j \lambda_j\right) d\mu_v^0(\lambda) = \int_P \exp\left(\sum_{j=1}^{N} t_j \lambda_j\right) d\mu_z^0(\lambda)$$

for a range of values of each t_j which surely includes an interval since τ ranges over $(0, \infty)$.

If $0 < n < N$, then the coordinates of the points in P are so related that we can find $N - n$ of the λ_j's as linear combinations of n independent ones, say $\{\lambda_j; j = n+1, \ldots, N\}$ in terms of $\{\lambda_j; j = 1, \ldots, n\}$. Thus $\lambda_k = a_{k1}\lambda_1 + \ldots + a_{kn}\lambda_n$, a_{kj} real, $n < k \leqslant N$. The sum in (15) becomes

$$\sum_{j=1}^{N} t_j \lambda_j = \sum_{j=1}^{n} \left\{ t_j + \sum_{k=n+1}^{N} t_k a_{kj} \right\} \lambda_j.$$

If momentarily we take fixed $\{t_k^0; k = n+1, \ldots, N\}$ and allow t_j to vary over an interval, we see that

$$\left\{ t_j + \sum_{k=n+1}^{N} t_k^0 a_{kj} \right\}$$

varies over an interval and so the range of

$$\left\{ t_j + \sum_{k=n+1}^{N} t_k a_{kj} \right\}$$

certainly contains an interval for each j.

Therefore by the uniqueness of measure for Laplace-Stieltjes integrals

$$\mu_v\big(\psi^{-1}(B)\big) = \mu_v^0(B) = \mu_z^0(B) = \mu_z\big(\psi^{-1}(B)\big)$$

for every Borel set B in the "plane" P.

If $n = N$, there is no need to alter the form of the sum in (15) and the same result holds. If $C^\wedge$ is a compact set in $\mathfrak{S}^\wedge$, then $\psi(C^\wedge)$ is a compact set, hence a closed set in P. Thus $\mu_v(C^\wedge) = \mu_z(C^\wedge)$. Since $\mathfrak{S}^\wedge = \psi^{-1}(P)$, also $\mu_v(\mathfrak{S}^\wedge) = \mu_z(\mathfrak{S}^\wedge)$ (possibly ∞). Therefore $\mu_v = \mu_z$ on the ring generated by the compact sets in $\mathfrak{S}^\wedge$ provided $v \epsilon U$. If $n = 0$ — i.e., if $\psi(\mathfrak{S}^\wedge) = \{0\}$ — then $\mathfrak{S}^\wedge = \{\chi_0 \equiv 1\}$ by the one-to-one nature of ψ; then $\mu_v = \mu_z$ trivially by (14).

Similarly $\mu_w = \mu_z$ for $w \epsilon U$ and hence $\mu_v = \mu_w = \mu$ on this ring if $v, w \epsilon U$.

Now let $x \epsilon \mathfrak{S}$. There is a symmetric neighborhood Q of e such that $Qx \subseteq \mathfrak{S}$ and $Q \cap U \neq \emptyset$. Select any $v \epsilon Q \cap U$. Then $v^{-1}x \epsilon \mathfrak{S}$, $x \epsilon v\mathfrak{S}$, and

$$f(x) = \int_{\mathfrak{S}^\wedge} \chi(x) \, d\mu_v(\chi) = \int_{\mathfrak{S}^\wedge} \chi(x) \, d\mu(\chi).$$

Consequently

$$f(x) = \int_{\mathfrak{S}^\wedge} \chi(x) \, d\mu(\chi) \quad \text{for all} \quad x \epsilon \mathfrak{S}.$$

Finally, if there were two such measures μ_1 and μ_2 with the property that

$$\int_{\mathfrak{S}^\wedge} \chi(x) \, d\mu_1(\chi) = f(x) = \int_{\mathfrak{S}^\wedge} \chi(x) \, d\mu_2(\chi)$$

for all $x \epsilon \mathfrak{S}$, then proceed essentially as above. (14) is replaced by our present equation. The details following (14) are practically the same with μ_1 and μ_2 in place of μ_v and μ_z, but with no mention of μ_w. We show $\mu_1 = \mu_2$ in the same fashion as $\mu_v = \mu_z$ except that now we omit

$$v = \prod_{j=1}^{N} \exp s_j X_j$$

entirely. Thus μ is unique on the ring generated by the compact sets in $\mathfrak{S}^\wedge$.

Note, since $\mathfrak{S}$ is locally compact and satisfies countability axiom II, that $\mathfrak{S}^\wedge$ is locally compact ([14], p. 98, Prop. 2) and satisfies countability axiom II (Bourbaki, III, Ch. 10, 2nd ed., p. 41). Hence the σ-ring generated by the compact sets is the Borel field.

References

[1] N. Aronszajn, *La théorie des noyaux reproduisants et ses applications*, Première Partie, Proc. Cambridge Phil. Soc. 39 (1943), p. 133-153.

[2] — *Theory of reproducing kernels*, Trans. Amer. Math. Soc. 68-69 (1950), p. 337-404.

[3] S. Bernstein, *Sur les fonctions absolument monotones*, Acta Math. 52 (1928-29), p. 1-66.

[4] P. M. Cohn, *Lie groups*, Tract 46, Cambridge 1957.

[5] A. Devinatz, *Integral representations of positive definite functions, II*, Trans. Amer. Math. Soc. 77 (1954), p. 455-480.

[6] — *A representation of functions as Laplace-Stieltjes integrals*, Duke Math. J. 22 (1955), p. 185-191.

[7] — *Two parameter moment problems*, ibidem 24 (1957) p. 481-498.

[8] — and A. E. Nussbaum, *Real characters of certain semigroups with applications*, ibidem 28 (1961), p. 221-237.

[9] T. H. Hildebrandt and I. J. Schoenberg, *On linear functional operations and the moment problem for a finite interval in one or several dimensions*, Ann. of Math. (2) 34 (1933), p. 317-328.

[10] K. Knopp, *Theory and application of infinite series*, Glasgow 1928.

[11] E. Nelson, *Analytic vectors*, Ann. of Math. (2) 70 (1955), p. 572-615.

[12] A. E. Nussbaum, *Integral representation of semi-groups of unbounded self-adjoint operators*, ibidem (2) 69 (1959), p. 133-141.

[13] J. A. Shohat and J. D. Tamarkin, *The problem of moments*, Amer. Math. Soc. Surveys 1, New York 1943.

[14] C. I. Tulcea, *Spectral representation of certain semigroups of operators*, J. Math. Mech. 8 (1959), p. 95-109.

[15] D. V. Widder, *The Laplace transform*, Princeton 1941.

[16] R. B. Zarhina, *On a two-dimensional problem of moments*, Dokl. Akad. Nauk SSSR 124 (1959), p. 743-746 (Russian); cf. Math. Reviews 21, 2161.

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