We now let

$$b_n = 2^{1-g}\sum_{\chi} b(n,\chi) \tag{15}$$

where, in the summation, χ runs through all the real characters of the group of ideal classes of k. We now notice that the real characters of the ideal class group in the restricted sense of k are exactly the characters of the group of genera in the restricted sense of k. By the theorem of Gauss the number of genera in k is 2^{g-1}. It follows that $b_n = 0$ unless n is the norm of an ideal in the principal genus, i.e., the norm of a totally positive number of k, in which case $b_n = 1$. Thus the number $B_k(x)$ of rational integers $\leqslant x$ which are norms of totally positive numbers of k is given by

$$B_k(x) = \sum_{n\leqslant x} b_n. \tag{16}$$

Summing (14) over all real χ, we get

$$\sum_{n\leqslant x} b_n \log\frac{x}{n} = \frac{2^{1-g}}{\pi}\cdot\frac{x}{\sqrt{\log x}} \times \tag{17}$$

$$\times\left[a\Gamma\left(\frac{1}{2}\right)+a_1\frac{\Gamma(3/2)}{\log x}+\ldots+a_{2m-1}\frac{\Gamma(m+1/2)}{(\log x)^m}\right]+O\left(\frac{x}{(\log x)^{m+3/2}}\right).$$

Taking $m = 4$, we easily obtain the following

THEOREM. *Let k be a quadratic field with discriminant d and let $B_k(x)$ denote the number of rational integers $\leqslant x$ which are norms of totally positive numbers of k. Then*

$$B_k(x) = \frac{2^{1-g}}{\sqrt{\pi}}\cdot\frac{x}{\sqrt{\log x}}\left[a+\frac{a_1-a}{2\log x}\right]+O\left(\frac{x}{\log^{5/2}x}\right).$$

Here g denotes the number of distinct rational primes dividing the discriminant d and the constant a is given by (9).

We remark that for imaginary quadratic fields we can give an explicit expression for a_1 using the first limit formula of Kronecker.

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Estimates of $\sum_{k=1}^{N} k^{-s}\langle kx\rangle^{-t}$

by

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1. Introduction. Throughout this paper, ω is the set of all nonnegative integers, R is the set of all real numbers, and R_{irr} is the set of all irrational real numbers. For each $x\in R$, $[x]$ is the largest integer not greater than x, and $\mathrm{fr}(x) = x-[x]$ is the fractional part of x. For each $x\in R$, we define

$$\langle x\rangle = \min\big(\mathrm{fr}(x),\ \mathrm{fr}(-x)\big),$$

whence $\langle x\rangle$ is the distance between x and the set of integers.

If, in some context, "A" and "B" are expressions and C is a condition (perhaps a conjunction of several conditions) on whatever variables may appear in "A" and "B", then "$A = O(B)$ *under* (or *relative to*, or *for*) C" means as usual that A/B subject to C is bounded (here $0/0 = 0$ and $A/0 = \infty$ if $A\neq 0$), and "$A \asymp B$ *under* (or *relative to*, or *for*) C" will mean that $A = O(B)$ under C and $B = O(A)$ under C. Sometimes instead of "$A = O(B)$ under C" or "$A\asymp B$ under C" we shall write "$A = O(B)$ (C)" or "$A\asymp B$ (C)" respectively.

Throughout this paper, k, m, n, N, and p are restricted without additional mention to be integers.

(Consider the condition C: $x\in R_{\mathrm{irr}}$, $N\geqslant 1$, $s\in R$, and $t\in T$.) This paper is concerned with estimating sums of the form

$$\sum_{k=1}^{N} k^{-s}\langle kx\rangle^{-t}\ \Big(\asymp \sum_{k=1}^{N} k^{-s}|\sin k\pi x|^{t}\ \text{under C if } T \text{ is a bounded subset of } R\Big) \tag{1}$$

with s and t nonnegative real numbers. The writer knows of no general treatment of this problem. Hardy and Littlewood ([3], pp. 216-217) showed that if $s\geqslant t\geqslant 1$, then, for each $x\in R_{\mathrm{irr}}$ such that

$$1 = O(j^s\langle jx\rangle^t)\qquad (j\in\omega\setminus\{0\}), \tag{2}$$

the sum in (1) is $O\big((\log N)^2\big)$ for $N\geqslant 2$. As remarked in [3], p. 214, (2) is equivalent to

$$q_{n+1}(x) = O\big(q_n(x)^{s/t}\big)\qquad (n\in\omega), \tag{3}$$

the sequence $(q_n(x))_{n=0}^{\infty}$ being the sequence of denominators of the convergents of the usual simple continued fraction expansion of x. (This notation, which is somewhat standard, is reviewed in § 2). Moreover, the equivalence of (2) and (3) may be proved easily from (44) and (49) of § 3 and the discussion following (**) of § 3. In the notation of § 2, it is easily shown that (3) is equivalent to

$$a_{n+1}(x) = O\big(q_n(x)^{(s-t)/t}\big) \quad (n \in \omega). \tag{4}$$

It follows easily from the metric theory of simple continued fractions (see (51) and (A) of § 2) that (2) holds for almost every $x \in R$ if $s > t \geqslant 1$ and that (2) holds for almost no $x \in R$ if $s = t \geqslant 1$. In (b) of Theorem 2 (see § 4), we shall show that, if $s = t = 1$, the sum in (1) is $O\big((\log N)^2\big)$ under $N \geqslant 2$ for a much larger class of $x \in R_{\mathrm{irr}}$ than characterized by (2) or (3). In (c) of Theorem 2, we shall show that, if $s > 1$ and $s \geqslant t \geqslant 0$, the sum in (1) is $O(\log N)$ under $N \geqslant 2$ for each $x \in R_{\mathrm{irr}}$ satisfying (3). Hardy and Littlewood ([3], pp. 216-218) showed that, if $s \geqslant t \geqslant 1$, then

$$\sum_{k=1}^{\infty} k^{-(s+\varepsilon)} \langle kx \rangle^{-t} < \infty \tag{5}$$

for each $x \in R_{\mathrm{irr}}$ satisfying (2) (or (3)). In (c) of Theorem 2 we shall prove a somewhat stronger conclusion.

Behnke ([1], pp. 289-290) showed that for each $x \in R_{\mathrm{irr}}$, where $q_n(x) \leqslant N < q_{n+1}(x)$,

$$\sum_{\substack{k=1 \\ q_n(x) \nmid k}}^{N} \langle kx \rangle^{-1} = O(N \log N) \quad (N \geqslant 2). \tag{6}$$

(In (6), contrary to [1] and to [6], p. 109, one cannot replace $\log N$ by $\log q_n(x)$; this is pointed out in the discussion before Theorem 1 in § 4.) We shall make a closer examination of the sum in (6) in (d) of Theorem 1 (see § 4). Walfisz ([10], p. 787) used (6) to show that, for each real $\varepsilon > 0$,

$$\sum_{k=1}^{N} \langle kx \rangle^{-1} = O\big(N (\log N)^{1+\varepsilon}\big) \quad (N \geqslant 2) \tag{7}$$

for almost every $x \in R_{\mathrm{irr}}$; we shall obtain more precise information in (b) of Theorem 4 (see § 4). The sum in (7) is the sum in (1) with $s = 0$ and $t = 1$.

A number of additional results on the magnitude of the sum in (1) will be obtained in § 4.

As is implicit in [10] in view of [9], one may use (7) and summation by parts to estimate the sum in [1] with $t = 1$. The estimates so obtained are not the best for two reasons. One reason is that (7) itself is not a sharp estimate. In fact, in the sense of (b) of Theorem 4 (see § 4), no single replacement of (7) is sharp. The other reason is that in the summation by parts formula

$$\text{(SP)} \qquad \sum_{k=1}^{N} u_k v_k = \sum_{k=1}^{N-1} (u_k - u_{k+1}) \sum_{j=1}^{k} v_j + u_N \sum_{j=1}^{N} v_j$$

with $(u_k)_{k=1}^{\infty}$ positive and decreasing and with $(v_k)_{k=1}^{\infty}$ positive, an upper bound$_k$ on $\sum_{j=1}^{k} v_j$ which is sharp on an infinite set of k need not result in an upper bound$_N$ on $\sum_{k=1}^{N} u_k v_k$ which is sharp on an infinite set of N. On the other hand, in the same context, an upper bound$_k$ on $\sum_{j=1}^{k} v_j$ which is sharp on the set of all k (too much to expect in estimating the left member of (7) for almost every $x \in R_{\mathrm{irr}}$) should yield an upper bound$_N$ on $\sum_{k=1}^{N} u_k v_k$ which is sharp on the set of all N.

Using (7) and summation by parts, one may prove (as is implicit in [10], § 3, in view of [9], p. 571) that for each real $\varepsilon > 0$, for almost every $x \in R_{\mathrm{irr}}$,

$$\sum_{k=1}^{N} k^{-1} \langle kx \rangle^{-1} = O\big((\log N)^{2+\varepsilon}\big) \quad (N \geqslant 2).$$

In (b) of Theorem 6 (see § 4) we show that for almost every $x \in R_{\mathrm{irr}}$,

$$\sum_{k=1}^{N} k^{-1} \langle kx \rangle^{-1} \asymp (\log N)^2 \quad (N \geqslant 2).$$

In § 4 we present many more results and introduce most of the theorems by commenting on the existing literature.

The writer thanks the referee for suggesting the use of (15) (the referee used $c = \sqrt{2}$ in (15)) instead of (15′) in § 2, for shortening the proof of the note following (I) in § 2 (by using the first inequality of (17)), for pointing out several mathematical inaccuracies, and for an extensive list of errata.

2. Preliminaries. For each real $t \geqslant 0$ and each $n \geqslant -1$, let

$$S_t(n) = \sum_{j=1}^{n} j^{-t}, \tag{8}$$

whence $S_t(-1) = 0$ and, if $n \in \omega$,

$$\int_1^{n+1} u^{-t}du \leqslant S_t(n) \leqslant 1+\int_1^n u^{-t}du,$$

(9)
$$\begin{cases} \dfrac{(2^{1-t}-1)n^{1-t}}{1-t} \leqslant S_t(n) \leqslant \dfrac{n^{1-t}}{1-t}, & \text{if} \quad t<1, \\ \log(n+1) \leqslant S_1(n) \leqslant 1+\log n, & \\ \dfrac{1-(n+1)^{-(t-1)}}{t-1} \leqslant S_t(n) \leqslant \dfrac{t-n^{-(t-1)}}{t-1}, & \text{if} \quad t>1. \end{cases}$$

For all real $s \geqslant 0$ and $t \geqslant 0$ and all $n, p \in \omega$ with $0 \leqslant n < p$, let

(10)
$$S(n, p; s, t) = \sum_{j=1}^{n} j^{-s}(p-j)^{-t}.$$

Now consider such s, t, n, p. Suppose that $n > 0$. Now

$$p^{-t}S_s([n/2]) \leqslant \sum_{j=1}^{[n/2]} j^{-s}(p-j)^{-t} \leqslant 2^t p^{-t} S_s([n/2]),$$

$$n^{-s}\sum_{j=[n/2]+1}^{n}(p-j)^{-t} \leqslant \sum_{j=[n/2]+1}^{n} j^{-s}(p-j)^{-t} \leqslant ([n/2]+1)^{-s}\sum_{j=[n/2]+1}^{n}(p-j)^{-t},$$

$$n^{-s}\big(S_t(p-[n/2]-1)-S_t(p-n-1)\big) \leqslant \sum_{j=[n/2]+1}^{n} j^{-s}(p-j)^{-t}$$
$$\leqslant ([n/2]+1)^{-s}\big(S_t(p-[n/2]-1)-S_t(p-n-1)\big).$$

Hence

(11)
$$p^{-t}S_s([n/2])+n^{-s}\big(S_t(p-[n/2]-1)-S_t(p-n-1)\big) \leqslant S(n, p; s, t)$$
$$\leqslant (p/2)^{-t}S_s([n/2])+([n/2]+1)^{-s}\big(S_t(p-[n/2]-1)-S_t(p-n-1)\big)$$
$$\leqslant (p/2)^{-t}S_s([n/2])+([n/2]+1)^{-s}S_t(n-[n/2]) < 2S_{s+t}(n-[n/2]).$$

In the rest of this section we shall present a few preliminaries on simple continued fractions.

We define a set R_n and a function $A_n\colon R_n \to R$ for each $n \in \omega$. Let $R_0 = R$, and let $A_0(x) = x$ for each $x \in R$. Let R_1 be the set of all real nonintegers, and let

$$A_1(x) = (x-[x])^{-1} \quad (x \in R_1).$$

Consider any integer $n > 1$. Let R_n be the set of all $x \in R_{n-1}$ such that $A_{n-1}(x) \in R_1$, and let

$$A_n(x) = A_1\big(A_{n-1}(x)\big) \quad (x \in R_1).$$

For each $n \in \omega$, let $a_n\colon R_n \to R$ be defined by

$$a_n(x) = [A_n(x)] \quad (x \in R_n).$$

Now $R_{n+1} \subset R_n \subset R_0 = R$ for each $n \in \omega$. It is well-known that $\bigcap_{n\in\omega} R_n = R_{\text{irr}}$ and, in the notation of continued fractions, for each $x \in R_{\text{irr}}$,

$$x = [a_0(x), a_1(x), \ldots] = a_0(x)+\cfrac{1}{a_1(x)+\cdots}.$$

Similarly, for each $n \in \omega$, for each $x \in R_n \setminus R_{n+1}$,

$$x = [a_0(x), a_1(x), \ldots, a_n(x)].$$

For each $n \in \omega$, define the functions p_n, q_n, θ_n on R_n by letting, for each $x \in R_n$,

$$p_n(x)/q_n(x) = \theta_n(x) = [a_0(x), a_1(x), \ldots, a_n(x)]$$

with $p_n(x), q_n(x)$ relatively prime integers and $q_n(x) > 0$. It is known that for each $n \in \omega$, (12) and(13) below hold, where $q_{-1}(x) = 0$ for each $x \in R$.

(12)
$$x-\theta_n(x) = \frac{(-1)^n}{q_n(x)\big(A_{n+1}(x)q_n(x)+q_{n-1}(x)\big)} \quad (x \in R_n).$$

(13)
$$p_n(x)q_{n-1}(x)-p_{n-1}(x)q_n(x) = (-1)^{n-1} \quad (x \in R_n).$$

We may rewrite (13) as

(14)
$$q_{n-1}(x)\theta_n(x) = p_{n-1}(x)+\frac{(-1)^{n-1}}{q_n(x)} \quad (x \in R_n).$$

By induction, where $c = (1+\sqrt{5})/2$, for each $x \in R_{\text{irr}}$,

(15)
$$\max\Big(c^{n-1}, \prod_{j=1}^{n} a_j(x)\Big) \leqslant q_n(x) \leqslant 2^{n-1}\prod_{j=1}^{n} a_j(x) \quad (n > 0).$$

(In [7], p. 75, the somewhat harder inequalities (with c as just defined)

(15′)
$$c^{n-1}\prod_{j=1}^{n}\Big(\big(a_j(x)+2\big)/3\Big) \leqslant q_n(x) \leqslant c^n \prod_{j=1}^{n} a_j(x) \quad (n \in \omega)$$

are proved.) It follows from (15) (alternatively, (15′), see [7], p. 78, (1) and (2)) that

(16)
$$\log q_n(x) \asymp n-1+\sum_{j=1}^{n}\log a_j(x) \quad (x \in R_{\text{irr}} \text{ and } n > 0).$$

For each integer $N > 0$, we define functions h_n, Q_n, and Q'_n on R_{irr} as follows. Consider any such N. Consider any $x \in R_{\mathrm{irr}}$. Let $h_N(x)$ be that integer $n > 0$ such that $q_{n-1}(x) \leqslant N < q_n(x)$. Thus

(17) $$q_{h_N(x)-1}(x) \leqslant N < q_{h_N(x)}(x).$$

Let

(18) $$Q_N(x) = [N/q_{h_N(x)-1}(x)],$$

whence

(19) $$1 \leqslant Q_N(x) \leqslant a_{h_N(x)}(x).$$

Let

(20) $$Q'_N(x) = \min\big(Q_N(x),\, a_{h_N(x)}(x) - 1\big).$$

Much of the content of the following propositions (A)-(O) is well-known.

(A) *Suppose that* $(c_n)_{n=1}^{\infty}$ *is a sequence of positive real numbers. Then* $\sum_{n=1}^{\infty} c_n^{-1} < \infty$ *(resp.,* $= \infty$*) is a necessary and sufficient condition that for almost every (resp., almost no)* $x \in R_{\mathrm{irr}}$,

(21) $$a_n(x) = O(c_n) \quad (n \geqslant 1).$$

Proposition (A) is a variant of the classical Bernstein-Borel theorem and is usually stated under the added assumption that $(c_n)_{n=1}^{\infty}$ is nondecreasing. For a proof of (A) as it stands, see [7], pp. 98-99.

(B) *Suppose that* $s \geqslant 1$ *is a real number and* $(c_n)_{n=1}^{\infty}$ *is a nondecreasing sequence of positive real numbers. Then* $\sum_{n=1}^{\infty} (nc_n)^{-1} < \infty$ *(resp.,* $= \infty$*) is a necessary and sufficient condition that for almost every (resp., almost no)* $x \in R_{\mathrm{irr}}$,

(22) $$\sum_{j=1}^{n} a_j(x)^s = O\big((nc_n)^s\big) \quad (n \geqslant 1).$$

Proof. Khintchine [4] proved that case of (B) in which $s = 1$.

If (22) holds for every $x \in X$ for some set $X \subset R_{\mathrm{irr}}$ having positive measure, then $\sum_{n=1}^{\infty} (nc_n)^{-1} < \infty$ by (A).

Suppose that $\sum_{n=1}^{\infty} (nc_n)^{-1} < \infty$. By Khintchine's result, for almost every $x \in R_{\mathrm{irr}}$, (22) with s replaced by 1 holds. Consider any such x. For some real $K > 1$,

$$\sum_{j=1}^{n} a_j(x) \leqslant Knc_n \quad (n \geqslant 1).$$

Let K by so given. For each $n \in \omega \setminus \{0\}$,

$$\sum_{j=1}^{n} a_j(x)^s < \sum_{j=1}^{n} (Knc_n)^{s-1} a_j(x) < (Knc_n)^s,$$

whence (22) holds.

Thus (B) holds.

The proof of (B) shows that (C) below holds.

(C) *Suppose that* $(c_n)_{n=1}^{\infty}$ *is a nondecreasing sequence of positive real numbers. Then* $\sum_{n=1}^{\infty} (nc_n)^{-1} < \infty$ *(resp.,* $= \infty$*) is a necessary and sufficient condition that for almost every (resp., almost no)* $x \in R_{\mathrm{irr}}$,

(22′) $$\Big(\sum_{j=1}^{n} a_j(x)^s\Big)^{1/s} = O(nc_n) \quad (n \geqslant 1;\ s \geqslant 1).$$

(D) *There is a real number* $d > 0$ *such that for almost every* $x \in R_{\mathrm{irr}}$,

(23) $$\lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} \log a_j(x) = d.$$

Proposition (D) is due to Khintchine [5]. For an alternative proof of (D), see Ryll-Nardzewski [8].

(E) *For almost every* $x \in R_{\mathrm{irr}}$,

(24) $$h_N(x) \asymp \log N \quad (N \geqslant 2).$$

For a proof of (E), see [7], pp. 134-135; (E) is essentially an immediate consequence of (D) via (17) and (16).

(F) *Suppose that* k *is an integer. Suppose that* $(c_n)_{n=1}^{\infty}$ *is a nondecreasing sequence of positive real numbers. Then* $\sum_{n=1}^{\infty} c_n^{-1} < \infty$ *(resp.,* $= \infty$*) is a necessary and sufficient condition that for almost every (resp., almost no)* $x \in R_{\mathrm{irr}}$,

(26) $$a_{h_n(x)+k}(x) = O(c_{[\log n]}) \quad (n \geqslant 3;\ h_n(x) > -k).$$

Proof. First suppose that $\sum_{n=1}^{\infty} c_n^{-1} < \infty$. Let $b = (1+\sqrt{5})/2$. By (15), $q_n(x) \geqslant b^{n-1}$ for all $x \in R_{\mathrm{irr}}$ and $n \in \omega$. There is a real $K > 1$ such that $h_n(x) + k < K\log n$ for all $x \in R_{\mathrm{irr}}$ and $n \in \omega$ such that $n \geqslant 2$ and $h_n(x) > -k$. Let K be so given. For each integer $n < K$, let $d_n = c_1$. For each integer $n \geqslant K$, let $d_n = c_{[n/K]}$. It is easily verified that $\sum_{n=1}^{\infty} d_n^{-1} < \infty$. By (A), for almost every $x \in R_{\mathrm{irr}}$,

$$a_n(x) = O(d_n) \quad (n \geqslant 1),$$

whence, under $n \geqslant 3$ and $h_n(x) > -k$,

$$a_{h_n(x)+k}(x) = O(d_{h_n(x)+k}) = O(d_{[K\log n]}) = O(c_{[\log n]}).$$

Next, instead of supposing that $\sum_{n=1}^{\infty} c_n^{-1} < \infty$, suppose that (26) holds for each x in some subset, say X, of R_{irr} having positive measure. By (E), there are a real number $b > 0$ and a set $Y \subset X$ having positive measure such that for each $x \in Y$,

(27) $$h_n(x)+k \geqslant b\log n$$

for all sufficiently large $n \in \omega$. Let b and Y be so given. For each $x \in Y$, by (26) and (27),

$$a_{h_n(x)+k}(x) = O(c_{[\log n]}) = O(c_{[(h_n(x)+k)/b]})$$

for $n \geqslant 3$ and $h_n(x)+k \geqslant b$. For each $x \in Y$ and each $m \in \omega$, if $m > k$, then $m = h_n(x)+k$ where $n = q_{m-k-1}(x)$. Hence, for each $x \in Y$,

$$a_m(x) = O(c_{[m/b]}) \quad (m \geqslant b;\ m > k).$$

Since also Y has positive measure, by (A),

$$\sum_{m=[b]+1}^{\infty} c_{[m/b]}^{-1} < \infty.$$

It follows easily that $\sum_{n=1}^{\infty} c_n^{-1} < \infty$.

Thus (F) holds.

(G) *Suppose that $s \geqslant 1$ is a real number. Suppose that $(c_n)_{n=1}^{\infty}$ is a nondecreasing sequence of positive real numbers. Then $\sum_{n=1}^{\infty} (nc_n)^{-1} < \infty$ (resp., $= \infty$) is a necessary and sufficient condition that for almost every (resp., almost no) $x \in R_{\mathrm{irr}}$,*

(28) $$\sum_{j=1}^{h_n(x)} a_j(x)^s = O\big(([\log n]c_{[\log n]})^s\big) \quad (n \geqslant 3).$$

The proof of (G) is essentially the same as the proof of (F) with $k = 0$. One merely works with the sequences $(nc_n)_{n=1}^{\infty}$ and $\big(\sum_{j=1}^{n} a_j(x)^s\big)_{n=1}^{\infty}$ instead of the sequences $(c_n)_{n=1}^{\infty}$ and $\big(a_n(x)\big)_{n=1}^{\infty}$, and one uses (B) instead of (A). The sufficiency part of the nonparenthetical part of (G) with $s = 1$ is essentially given in [7], 4.3, p. 35. Under the added condition that $c_{2n} = O(c_n)$ for $n \in \omega\setminus\{0\}$, the sufficiency part of the parenthetical part of (G) with $s = 1$ is essentially given in [7], 4.11, pp. 194-195.

Just as (C) may be obtained from (A) and that case of (B) in which $s = 1$, one may obtain (H) below from (F) (with $k = 0$) and that case of (G) in which $s = 1$.

(H) *Suppose that $(c_n)_{n=1}^{\infty}$ is a nondecreasing sequence of positive real numbers. Then $\sum_{n=1}^{\infty} (nc_n)^{-1} < \infty$ (resp., $= \infty$) is a necessary and sufficient condition that for almost every (resp., almost no) $x \in R_{\mathrm{irr}}$,*

(28′) $$\Big(\sum_{j=1}^{h_n(x)} a_j(x)^s\Big)^{1/s} = O([\log n]c_{[\log n]}) \quad (n \geqslant 3;\ s \geqslant 1).$$

(I) *Suppose that k is an integer. Suppose that $(c_n)_{n=1}^{\infty}$ is a nondecreasing sequence of positive real numbers. Then $\sum_{n=1}^{\infty} (nc_n)^{-1} < \infty$ (resp., $= \infty$) is a necessary and sufficient condition that for almost every (resp., almost no) $x \in R_{\mathrm{irr}}$,*

(29) $$a_{h_n(x)+k}(x) = O(c_n) \quad (n \geqslant 1;\ h_n(x) > -k).$$

Proof. First suppose that $\sum_{n=1}^{\infty} (nc_n)^{-1} < \infty$. Let $d_n = c_{2^n}$ for each integer $n > 0$. Now

$$\sum_{n=1}^{\infty} d_n^{-1} = 2\sum_{n=1}^{\infty} 2^{n-1}(2^n c_{2^n})^{-1} < 2\sum_{n=1}^{\infty}\sum_{j=2^{n-1}+1}^{2^n} (jc_j)^{-1} = 2\sum_{j=1}^{\infty} (jc_j)^{-1} < \infty.$$

By (F), for almost every $x \in R_{\mathrm{irr}}$,

$$a_{h_n(x)+k}(x) = O(d_{[\log n]}) = O(c_{2^{[\log n]}}) = O(c_{[2^{\log n}]}) = O(c_{[n\log 2]}) = O(c_n)$$

for $n \geqslant 3$ and $h_n(x) > -k$. Thus (29) holds for almost every $x \in R_{\mathrm{irr}}$.

Next suppose that (29) holds for each x in some subset, say X, of R_{irr} having positive measure. For each $x \in X$,

$$a_{h_n(x)+k}(x) = O(c_n) = O(c_{3^{[\log n]}}) \quad (n \geqslant 1).$$

By (F), $\sum_{n=1}^{\infty} c_{3^n}^{-1} < \infty$. Hence

$$\sum_{n=1}^{\infty} (nc_n)^{-1} = \sum_{j=0}^{\infty}\sum_{n=3^j}^{3^{j+1}-1} (nc_n)^{-1} \leqslant \sum_{j=0}^{\infty}\sum_{n=3^j}^{3^{j+1}-1} (3^j c_{3^j})^{-1}$$
$$\leqslant \sum_{j=0}^{\infty} 3^{j+1}(3^j c_{3^j})^{-1} = 3\sum_{j=0}^{\infty} c_{3^j}^{-1} < \infty.$$

Thus (I) holds.

NOTE. *In* (I), *one may replace* (29) *by*

$$a_{h_n(x)+k}(x) = O(c_n) \quad (n \geqslant 1;\ h_n(x) > -k,\ 1;\ n = q_{h_n(x)-1}(x)) \tag{29'}$$

without impairing the validity of (I).

Proof. Suppose that (29′) holds for all x in some subset, say X, of R_{irr} having positive measure. For almost every $x \in X$, where $n' = q_{h_n(x)-1}(x) \leqslant n$,

$$a_{h_n(x)+k}(x) = a_{h_{n'}(x)+k}(x) = O(c_{n'}) = O(c_n) \quad (n \geqslant 1;\ h_n(x) > -k).$$

Now apply (I).

(J) *Suppose that $s \geqslant 1$ is a real number. Suppose that $(c_n)_{n=1}^{\infty}$ is a nondecreasing sequence of positive real numbers. Then $\sum_{n=2}^{\infty}\big(n(\log n)c_n\big)^{-1} < \infty$ (resp., $= \infty$) is a necessary and sufficient condition that for almost every (resp., almost no) $x \in R_{\mathrm{irr}}$,*

$$\sum_{j=1}^{h_n(x)} a_j(x)^s = O\big(((\log n)c_n)^s\big) \quad (n \geqslant 2). \tag{30}$$

The proof of (J) is essentially the same as the proof of (I) with $k = 0$. One merely works with the sequences $\big((\log n)c_n\big)_{n=1}^{\infty}$ and $\big(\sum_{j=1}^{h_n(x)} a_j(x)^s\big)_{n=1}^{\infty}$ instead of the sequences $(c_n)_{n=1}^{\infty}$ and $\big(a_{h_n(x)}(x)\big)_{n=1}^{\infty}$, and one uses (G) instead of (F). Proposition (J) with $s = 1$ is essentially given in [7], 4.4, p. 136, and 4.13, p. 196.

Just as (C) may be obtained from (A) and that case of (B) in which $s = 1$, one may obtain (K) below from (I) (with $k = 0$) and that case of (J) in which $s = 1$.

(K) *Suppose that $(c_n)_{n=1}^{\infty}$ is a nondecreasing sequence of positive real numbers. Then $\sum_{n=2}^{\infty}\big(n(\log n)c_n\big)^{-1} < \infty$ (resp., $= \infty$) is a necessary and sufficient condition that for almost every (resp., almost no) $x \in R_{\mathrm{irr}}$,*

$$\Big(\sum_{j=1}^{h_n(x)} a_j(x)^s\Big)^{1/s} = O\big((\log n)c_n\big) \quad (n \geqslant 2;\ s \geqslant 1). \tag{30'}$$

(L) *Suppose that $n \in \omega$, $x \in R_n$, $t \in R$, and $0 < t$. Then*

$$\sum_{j=0}^{n} a_{j+1}(x)^t q_j(x)^t \leqslant \sum_{j=0}^{n} q_{j+1}(x)^t < (2^t-1)^{-1}2^{t+1}q_{n+1}(x)^t. \tag{31}$$

Proof. $q_{j+1}(x) \geqslant 2q_{j-1}(x)$ for each j among $1, 2, \ldots, n$. Hence

$$\begin{aligned}\sum_{j=0}^{n} a_{j+1}(x)^t q_j(x)^t &\leqslant \sum_{j=0}^{n} q_{j+1}(x)^t = \sum_{h=0}^{[n/2]} q_{n+1-2h}(x)^t + \sum_{h=0}^{[(n-1)/2]} q_{n-2h}(x)^t \\ &< \sum_{h=0}^{\infty}\big(q_{n+1}(x)2^{-h}\big)^t + \sum_{h=0}^{\infty}\big(q_n(x)2^{-h}\big)^t \leqslant 2q_{n+1}(x)^t\sum_{h=0}^{\infty}(2^{-t})^h \\ &= (2^t-1)^{-1}2^{t+1}q_{n+1}(x)^t.\end{aligned}$$

(M) *Suppose that f is a nondecreasing real-valued function on the set of all nonnegative real numbers. Suppose that $f(0) > 1$ and that $f(2u) = O\big(f(u)\big)$ for $u > 1$. (Then there is $c \in R$ such that $f(u) = O(u^c)$ for $u > 1$* (cf. [7], 5.3, p. 147).) *Let X be the set of all $x \in R_{\mathrm{irr}}$ such that*

$$a_j(x) \leqslant f(j) \quad (j \geqslant 1). \tag{32}$$

Relative to $x \in X$ and $n > 1$,

$$\sum_{j=1}^{h_n(x)} a_j(x) = O\left(\frac{(\log n)f(\log n)}{\log f(\log n)}\right). \tag{33}$$

There is a set $Y \subset X$ of cardinal of the continuum such that, relative to $x \in Y$ and $n > 1$,

$$\sum_{j=1}^{h_n(x)} a_j(x) \asymp \frac{(\log n)f(\log n)}{\log f(\log n)}. \tag{34}$$

To prove (M), apply [7], 5.6, p. 152 and [7], 5.8, p. 154. See also [7], top of p. 134 and observe that the right members of (33) and (34) are $o(n)$.

(N) *Suppose that $n_0 \in \omega$, $\delta \in R$, and $\delta > 0$. Let f and X be given as in* (M). *Let Z be the set of all $x \in X$ such that*

$$\Big(\prod_{j=1}^{n} a_j(x)\Big)^{1/n} \geqslant f(n)^{\delta} \quad (n \geqslant n_0). \tag{35}$$

(The left member of (35) is the geometric mean of $a_1(x), \ldots, a_n(x)$. If $\delta < 1$ and f is unbounded, for sufficiently large n_0, Z has the cardinal of the continuum (cf. [7], 5.10, p. 157).) *Relative to $x \in Z$ and $n \geqslant 2$,*

$$\sum_{j=1}^{h_n(x)} a_j(x) \asymp \frac{(\log n)}{\log f(\log n)} f\left(\frac{\log n}{\log f(\log n)}\right). \tag{36}$$

To prove (N), apply [7], 5.10, p. 157.

(O) *Suppose that* $K \in R$, $1 \leqslant K$, *and* S *is the set of all* $x \in R_{\rm irr}$ *such that* $a_j(x) \leqslant K$ *for each positive* $j \in \omega$. *Relative to* $x \in S$ *and* $n \geqslant 2$,

$$\sum_{j=1}^{h_n(x)} a_j(x) \asymp \log n. \tag{37}$$

To prove (O) (which is elementary and, except for notation, well-known), apply [7], 5.9, p. 156.

3. Some basic estimates. The general approach of this section is very close to one used by Behnke [1]. The discussion concerned with (47)-(49) is fairly close to Walfisz [10], p. 787.

Consider any real numbers $s \geqslant 0$ and $t \geqslant 0$.

Consider any positive integer n. Consider any $x \in R_{n+1}$. For brevity, for each j among $0, 1, \ldots, n+1$, instead of $a_j(x), p_j(x), q_j(x), \theta_j(x), \Lambda_j(x)$ we shall write $a_j, p_j, q_j, \theta_j, \Lambda_j$ respectively.

Suppose that N is an integer and $q_n \leqslant N < q_{n+1}$. Let $Q = [N/q_n]$ and $r = N - Qq_n$, whence

$$N = Qq_n + r, \quad 0 \leqslant r < q_n.$$

Now $q_{n+1} = a_{n+1}q_n + q_{n-1}$, and $0 < q_{n-1} \leqslant q_n$. Hence $Q \leqslant a_{n+1}$. Let

$$Q' = \min(Q, a_{n+1} \dot{-} 1).$$

If $x \in R_{\rm irr}$, then $n+1 = h_N(x)$, $Q = Q_N(x)$, and $Q' = Q'_N(x)$.

Consider any nonnegative integer $h \leqslant Q$. (Throughout the rest of § 3, h is so given.) Let r_h be $q_n - 1$ or r according as $h \leqslant Q$ or $h = Q$. The numbers

$$(hq_n+1)\theta_n,\ (hq_n+2)\theta_n,\ (hq_n+3)\theta_n,\ \ldots,\ \big(hq_n+(q_n-1)\big)\theta_n \tag{*}$$

are congruent modulo 1 in some order to the numbers

$$1/q_n,\ 2/q_n,\ 3/q_n,\ \ldots,\ (q_n-1)/q_n. \tag{\#}$$

Consider the numbers

$$(hq_n+1)x,\ (hq_n+2)x,\ (hq_n+3)x,\ \ldots,\ \big(hq_n+(q_n-1)\big)x. \tag{**}$$

From (12) and (14), observe that if $h < Q$ the numbers of (**) other than $(hq_n+q_{n-1})x$ lie, modulo 1, one in each of the $q_n - 2$ successive intervals determined by the numbers in (#). Similarly, observe that if $h = Q$ the first r numbers of (**) other than $(Qq_n+q_{n-1})x$ lie, modulo 1, in some order one in each of an appropriate number of the $q_n - 2$ successive intervals determined by the numbers in (#). If $h = Q$, then $(Qq_n+q_{n-1})x$ is among the first r numbers of (**) if and only if $q_{n-1} \leqslant r$. If $h = Q = a_{n+1}$, then

$$a_{n+1}q_n + r = N < q_{n+1} = a_{n+1}q_n + q_{n-1},$$

whence $r < q_{n-1}$. Thus, if $h = Q = a_{n+1}$, then $(Qq_n+q_{n-1})x$ is not among the first r numbers of (**). It follows from these observations that, if $h < Q$, then

$$\sum_{k=1}^{q_n-2} \big((k+1)/q_n\big)^{-t} < \sum_{\substack{j=1 \\ j \neq q_{n-1}}}^{q_n-1} \langle (hq_n+j)x \rangle^{-t}.$$

Thus, by (8),

$$2^{-t} q_n^t S_t(q_n-2) \leqslant \sum_{\substack{j=1 \\ j \neq q_{n-1}}}^{q_n-1} \langle (hq_n+j)x \rangle^{-t} \quad \text{if} \quad h < Q. \tag{38}$$

Regardless of whether $h < Q$, by the observations following (**),

$$\sum_{\substack{j=1 \\ j \neq q_{n-1}}}^{r_h} \langle (hq_n+j)x \rangle^{-t} < 2 \sum_{k=1}^{r_h} (k/q_n)^{-t} = 2q_n^t S_t(r_h). \tag{39}$$

Now (where $0^{-s} = \infty$ if $s > 0$)

$$(h+1)^{-s} q_n^{-s} \sum_{\substack{j=1 \\ j \neq q_{n-1}}}^{r_h} \langle (hq_n+j)x \rangle^{-t} \leqslant \sum_{\substack{j=1 \\ j \neq q_{n-1}}}^{r_h} (hq_n+j)^{-s} \langle (hq_n+j)x \rangle^{-t} \tag{40}$$
$$\leqslant h^{-s} q_n^{-s} \sum_{\substack{j=1 \\ j \neq q_{n-1}}}^{r_h} \langle (hq_n+j)x \rangle^{-t}.$$

By (40) and (38),

$$2^{-t}(h+1)^{-s} q_n^{t-s} S_t(q_n-2) \tag{41}$$
$$\leqslant \sum_{\substack{j=1 \\ j \neq q_{n-1}}}^{q_n-1} (hq_n+j)^{-s} \langle (hq_n+j)x \rangle^{-t} \quad \text{if} \quad h < Q.$$

By (40) and (39), regardless of whether $h < Q$,

$$\sum_{\substack{j=1 \\ j \neq q_{n-1}}}^{r_h} (hq_n+j)^{-s} \langle (hq_n+j)x \rangle^{-t} < 2h^{-s} q_n^{t-s} S_t(r_h). \tag{42}$$

Assume tentatively (until the end of (50)) that $q_n > 1$. If $h < a_{n+1}$, then $hq_n + q_{n-1} < q_{n+1} < \Lambda_{n+1}q_n + q_{n-1}$, and, by (12) and (14),

$$\langle (hq_n+q_{n-1})x \rangle = \frac{1}{q_n} - \frac{hq_n+q_{n-1}}{q_n(\Lambda_{n+1}q_n+q_{n-1})} = \frac{\Lambda_{n+1}-h}{\Lambda_{n+1}q_n+q_{n-1}}. \tag{43}$$

If $h < a_{n+1}$, then, by (43),

$$(44)\qquad (a_{n+1}-h)/q_{n+1} < \langle (hq_n+q_{n-1})x\rangle < 2(a_{n+1}-h)/q_{n+1}.$$

If $0 < h < a_{n+1}$, then $hq_{n+1} < a_{n+1}(hq_n+q_{n-1}) < 2ha_{n+1}q_n$, and, by (44),

$$(45)\qquad \frac{q_{n+1}^t q_n^{-s}}{2^{s+t}h^s(a_{n+1}-h)^t} \leqslant (hq_n+q_{n-1})^{-s}\langle (hq_n+q_{n-1})x\rangle^{-t} \leqslant \frac{a_{n+1}^s q_{n+1}^{t-s}}{h^s(a_{n+1}-h)^t}.$$

By (44) with $h = 0$,

$$2^{-t}a_{n+1}^{-t}q_{n+1}^t q_{n-1}^{-s} \leqslant q_{n-1}^{-s}\langle q_{n-1}x\rangle^{-t} \leqslant a_{n+1}^{-t}q_{n+1}^t q_{n-1}^{-s},$$

$$(46)\qquad 2^{-t}q_n^t q_{n-1}^{-s} \leqslant q_{n-1}^{-s}\langle q_{n-1}x\rangle^{-t} \leqslant 2^t q_n^t q_{n-1}^{-s}.$$

Now $h \leqslant a_{n+1}$; hence, by (12),

$$(47)\qquad h|q_nx-p_n| < a_{n+1}/q_{n+1} < q_n^{-1}.$$

By (47)

$$(48)\qquad \langle hq_nx\rangle = \langle h|q_nx-p_n|\rangle = h|q_nx-p_n|.$$

By (48) and (12),

$$(49)\qquad h/(q_{n+1}+q_n) < \langle hq_nx\rangle < h/q_{n+1}.$$

By (49), if $h > 0$,

$$(50)\qquad a_{n+1}^t q_n^{t-s}h^{-(s+t)} \leqslant q_{n+1}^t q_n^{-s}h^{-(s+t)} \leqslant (hq_n)^{-s}\langle hq_nx\rangle^{-t}$$
$$\leqslant (q_{n+1}+q_n)^t q_n^{-s}h^{-(s+t)} \leqslant (a_{n+1}+2)^t q_n^{t-s}h^{-(s+t)}.$$

Next assume tentatively that $q_n = 1$. Then $n = 1$, and $q_1 = q_0 = 1$. By (12),

$$(51)\qquad \langle hq_1x\rangle = \langle hx\rangle = \min(\theta_1-x,\, x-\theta_1+1) = \min(h,\, A_2+1-h)/(A_2+1).$$

We no longer assume that q_n is restricted in any way.

By (41) and (42),

$$(52)\qquad 2^{-t}q_n^{t-s}S_t(q_n-2)\big(S_s(Q)-1\big) \leqslant \sum_{\substack{k=q_n\\ k\not\equiv 0,\, q_{n-1} \pmod{q_n}}}^{N} k^{-s}\langle kx\rangle^{-t}$$
$$\leqslant 2q_n^{t-s}S_t(q_n-1)\,S_s(Q).$$

If $q_n > 1$, by (45) (applicable since $N < a_{n+1}q_n+q_{n-1}$) and (10),

$$(53)\qquad 2^{-s-t}q_n^{-s}q_{n+1}^t S(Q-1,\, a_{n+1};\, s,\, t) \leqslant \sum_{\substack{k=q_n\\ k\equiv q_{n-1} \pmod{q_n}}}^{N} k^{-s}\langle kx\rangle^{-t}$$
$$\leqslant a_{n+1}^s q_{n+1}^{t-s}S(Q',\, a_{n+1};\, s,\, t).$$

If $q_n > 1$, then, by (50),

$$(54)\qquad a_{n+1}^t q_n^{t-s}S_{s+t}(Q) \leqslant \sum_{\substack{k=q_n\\ k\equiv 0 \pmod{q_n}}}^{N} k^{-s}\langle kx\rangle^{-t} \leqslant (a_{n+1}+2)^t q_n^{t-s}S_{s+t}(Q).$$

If $q_n = 1$, then, by (51) (since $Q = Qq_1+0 = N \leqslant a_2$)

$$(55)\qquad (a_2+1)^t S_{s+t}(Q) \leqslant \sum_{k=1}^{N} k^{-s}\langle kx\rangle^{-t} \leqslant 2(a_2+2)^t S_{s+t}(Q).$$

For each integer M, if $1 \leqslant M < q_1$, then $q_1 = a_1$, and, by (12) with n replaced by 0,

$$(56)\qquad \langle kx\rangle = \min(k,\, A_1-k)/A_1$$

for each positive integer $k \leqslant a_1$, and

$$(57)\qquad a_1^t S_{s+t}(M) \leqslant \sum_{k=1}^{M} k^{-s}\langle kx\rangle^{-t} \leqslant 2(a_1+1)^t S_{s+t}(M).$$

By (52), (53), (54) and (11) if $q_n > 1$ and by (55) (in (55), $q_1 = 1$ and $q_2 = a_2+1$) and (11) if $q_n = 1$ (whence $n = 1$),

$$(58)\qquad 2^{-t}q_n^{t-s}S_t(q_n-2)\big(S_s(Q)-1\big)+a_{n+1}^t q_n^{t-s}S_{s+t}(Q) \leqslant \sum_{k=q_n}^{N} k^{-s}\langle kx\rangle^{-t}$$
$$\leqslant 2q_n^{t-s}S_t(q_n-1)\,S_s(Q)+a_{n+1}^t q_n^{t-s}\big(2^{\max(t-s,0)}S(Q',\, a_{n+1};\, s,\, t)+2\cdot 3^t S_{s+t}(Q)\big)$$
$$\leqslant 2q_n^{t-s}S_t(q_n-1)\,S_s(Q)+4\cdot 3^t a_{n+1}^t q_n^{t-s}S_{s+t}(Q).$$

By (38) and (39),

$$(59)\qquad 2^{-t}Qq_n^t S_t(q_n-2) \leqslant \sum_{\substack{k=1\\ k\not\equiv 0,\, q_{n-1} \pmod{q_n}}}^{N} \langle kx\rangle^{-t} \leqslant 2(Q+1)q_n^t S_t(q_n-1).$$

If $q_n > 1$, by (45) and (46) with $s = 0$,

$$(60)\qquad 2^{-t}q_n^t a_{n+1}^t\big(S_t(a_{n+1})-S_t(a_{n+1}-Q)\big)$$
$$\leqslant 2^{-t}\Big(q_n^t+q_{n+1}^t\big(S_t(a_{n+1}-1)-S_t(a_{n+1}-Q)\big)\Big)$$
$$\leqslant \sum_{\substack{k=1\\ k\equiv q_{n-1} \pmod{q_n}}}^{N} \langle kx\rangle^{-t}$$
$$\leqslant 2^t q_n^t+q_{n+1}^t\big(S_t(a_{n+1}-1)-S_t(a_{n+1}-Q'-1)\big)$$
$$\leqslant 2^t q_n^t a_{n+1}^t\big(S_t(a_{n+1})-S_t(a_{n+1}-Q'-1)\big).$$

If $q_n > 1$, by (59) and (60),

$$(61)\quad q_n^t\big(2^{-t}QS_t(q_n-2)+2^{-t}a_{n+1}^t\big(S_t(a_{n+1})-S_t(a_{n+1}-Q)\big)\big)$$

$$\leqslant \sum_{\substack{k=1\\ k\not\equiv 0 \pmod{q_n}}}^{N} \langle kx\rangle^{-t}$$

$$\leqslant q_n^t\big(2(Q+1)S_t(q_n-1)+2^t a_{n+1}^t\big(S_t(a_{n+1})-S_t(a_{n+1}-Q'-1)\big)\big).$$

If $q_n > 1$, by (50) with $s = 0$,

$$(62)\quad a_{n+1}^t q_n^t S_t(Q) \leqslant q_{n+1}^t S_t(Q) \leqslant \sum_{\substack{k=1\\ k\equiv 0 \pmod{q_n}}}^{N} \langle kx\rangle^{-t} \leqslant (a_{n+1}+2)^t q_n^t S_t(Q).$$

If $q_n > 1$, by (61) and (62) (since

$$S_t(a_{n+1})-S_t(a_{n+1}-Q'-1) = \sum_{j=1}^{Q'+1}(a_{n+1}-Q'-1+j)^{-t} \leqslant 2S_t(Q)),$$

$$(63)\quad q_n^t\big(2^{-t}QS_t(q_n-2)+a_{n+1}^t S_t(Q)\big) \leqslant \sum_{k=1}^{N}\langle kx\rangle^{-t}$$

$$\leqslant q_n^t\big(2(Q+1)S_t(q_n-1)+3^{t+1}a_{n+1}^t S_t(Q)\big).$$

Since $S_t(-1) = 0$, observe from (55) with $s = 0$ that (63) holds also if $q_n = 1$ (whence $n = 1$).

By replacing N, n, Q, Q' in (58) by $q_{j+1}-1, j, a_{j+1}, a_{j+1}-1$ respectively for all j among $1, 2, \ldots, n-1$, summing and taking (57) into account with $M = q_1-1$ ((57) is applicable if $q_1 > 1$; if $q_1 = 1$, then $S_{s+t}(q_1-1) = 0$), we obtain

$$(64)\quad a_1^t S_{s+t}(q_1-1)+2^{-t}\sum_{j=1}^{n-1} q_j^{t-s}S_t(q_j-2)\big(S_s(a_{j+1})-1\big)+$$

$$+\sum_{j=1}^{n-1} a_{j+1}^t q_j^{t-s} S_{s+t}(a_{j+1})$$

$$\leqslant \sum_{k=1}^{q_n-1} k^{-s}\langle kx\rangle^{-t} \leqslant 2(a_1+1)^t S_{s+t}(q_1-1)+2\sum_{j=1}^{n-1} q_j^{t-s}S_t(q_j-1)S_s(a_{j+1})+$$

$$+4\cdot 3^t\sum_{j=1}^{n-1} a_{j+1}^t q_j^{t-s}S_{s+t}(a_{j+1}).$$

By (64) and (58)

$$(65)\quad a_1^t S_{s+t}(q_1-1)+2^{-t}\sum_{j=1}^{n-1} q_j^{t-s}S_t(q_j-2)\big(S_s(a_{j+1})-1\big)+$$

$$+\sum_{j=1}^{n-1} a_{j+1}^t q_j^{t-s}S_{s+t}(a_{j+1})+2^{-t}q_n^{t-s}S_t(q_n-2)\big(S_s(Q)-1\big)+a_{n+1}^t q_n^{t-s}S_{s+t}(Q)$$

$$\leqslant \sum_{k=1}^{N} k^{-s}\langle kx\rangle^{-t}$$

$$\leqslant 2\cdot 2^t a_1^t S_{s+t}(q_1-1)+2\sum_{j=1}^{n-1} q_j^{t-s}S_t(q_j-1)S_s(a_{j+1})+$$

$$+4\cdot 3^t\sum_{j=1}^{n-1} a_{j+1}^t q_j^{t-s}S_{s+t}(a_{j+1})+2q_n^{t-s}S_t(q_n-1)S_s(Q)+4\cdot 3^t a_{n+1}^t q_n^{t-s}S_{s+t}(Q).$$

4. The main theorems. In Theorem 1 below we shall apply some of the estimates in § 3. In Theorem 1, (a) applies generally, (b)-(e) apply to the case $s = 0$, (f) applies to the case $0 < t < s$ and $1 < s$, (g) applies to the case $1 < t$, and (h) applies to the case $s = t = 1$.

In Theorem 1, the second equation of (78) in its context in (d) was proved essentially by Behnke ([1], pp. 289-290). Behnke stated his result with what would amount to $N\log N$ replaced by $N\log q_n$ in the second equation of (78), and a similar statement occurs in Koksma ([6], Satz 16, p. 109). The result so stated is incorrect; to see this, in (75) take $N = q_n a_{n+1}$, whence $Q' = a_{n+1}-1$ and the right side of (75) is $N\log q_n + N\log a_{n+1} (= N\log N)$, whereas for some $x\in R_{\mathrm{irr}}$, $\log a_{n+1} = O(\log q_n)$ is false. Behnke's argument proves the second equation of (78), and another proof of the second equation of (78) has been given by Walfisz ([9], pp. 587-589).

Behnke ([1], p. 282) proved that case of (f) of Theorem 1 in which $s \geqslant 2$ and $t = 1$.

THEOREM 1. *Suppose that s_0, t_0, and t_1 are positive real numbers. Then* (a)-(h) *below hold where*

$$n = h_N(x)-1, \qquad Q = Q_N(x), \qquad Q' = Q'_N(x),$$

and, for each $j\in\omega$, we write a_j and q_j for $a_j(x)$ and $q_j(x)$ respectively.

(a) *Where s and t are nonnegative real numbers, let*

$$\Gamma = a_1^t S_{s+t}(q_1-1)+\sum_{j=1}^{n-1} q_j^{t-s}S_t(q_j)\big(S_s(a_{j+1})-1\big)+\sum_{j=1}^{n-1} a_{j+1}^t q_j^{t-s}S_{s+t}(a_{j+1})+$$

$$+q_n^{t-s}S_t(q_n)\big(S_s(Q)-1\big)+a_{n+1}^t q_n^{t-s}S_{s+t}(Q),$$

$$\Delta = \sum_{j=1}^{n} q_j^{t-s}S_t(q_j).$$

Relative to $x\in R_{\mathrm{irr}}$, $N\in\omega$, $n>0$, $0\leqslant s$, *and* $0\leqslant t\leqslant t_1$,

(66) $$\Gamma=O\Big(\sum_{k=1}^{N}k^{-s}\langle kx\rangle^{-t}\Big),\qquad \sum_{k=1}^{N}k^{-s}\langle kx\rangle^{-t}=O(\Gamma+\Delta),$$

(67) $$\sum_{k=1}^{N}\langle kx\rangle^{-t}\asymp q_n^t\big(QS_t(q_n)+a_{n+1}^tS_t(Q)\big).$$

(b) *Relative to* $x\in R_{\mathrm{irr}}$, $N\in\omega$, $q_n>1$ (*or* $n>0$ *for* (69)), *and* $0\leqslant t\leqslant t_1$,

(68) $$\sum_{\substack{k=1\\ q_n\nmid k}}^{N}\langle kx\rangle^{-t}\asymp q_n^t\Big(QS_t(q_n)+a_{n+1}^t\big(S_t(a_{n+1})-S_t(a_{n+1}-Q)\big)\Big),$$

(69) $$\sum_{\substack{k=1\\ q_n\mid k}}^{N}\langle kx\rangle^{-t}\asymp a_{n+1}^tq_n^tS_t(Q),$$

(70) $$\begin{cases} q_n^tQS_t(q_n)=O\Big(\sum\limits_{\substack{k=1\\ q_n\nmid k}}^{N}\langle kx\rangle^{-t}\Big),\\ \sum\limits_{\substack{k=1\\ q_n\nmid k}}^{N}\langle kx\rangle^{-t}=O\Big(q_n^t\big(QS_t(q_n)+Q^tS_t(Q)\big)\Big).\end{cases}$$

Moreover, there are positive real numbers α *and* β *such that for each* $x\in R_{\mathrm{irr}}$, *for* $0\leqslant t\leqslant t_1$

(70′) $$\begin{cases} \varliminf\limits_{N\to\infty}\Big(\sum\limits_{\substack{k=1\\ q_n\nmid k}}^{N}\langle kx\rangle^{-t}\Big)\Big/\big(q_n^tQS_t(q_n)\big)<\beta,\\ \alpha<\varlimsup\limits_{N\to\infty}\Big(\sum\limits_{\substack{k=1\\ q_n\nmid k}}^{N}\langle kx\rangle^{-t}\Big)\Big/\Big(q_n^t\big(QS_t(q_n)+Q^tS_t(Q)\big)\Big).\end{cases}$$

(c) *Suppose that* $t_1<1$. *Relative to* $x\in R_{\mathrm{irr}}$, $N\in\omega$, $q_n>1$, *and* $0\leqslant t\leqslant t_1$,

(71) $$\sum_{\substack{k=1\\ q_n\nmid k}}^{N}\langle kx\rangle^{-t}\asymp N,$$

(72) $$\sum_{\substack{k=1\\ q_n\mid k}}^{N}\langle kx\rangle^{-t}\asymp q_{n+1}^tQ^{1-t}\asymp N^ta_{n+1}^tQ^{1-2t},$$

(73) $$\sum_{k=1}^{N}\langle kx\rangle^{-t}\asymp N+N^ta_{n+1}^tQ^{1-2t}.$$

Moreover, there are positive real numbers α *and* β *such that for each* $x\in R_{\mathrm{irr}}$, *for* $0\leqslant t\leqslant t_1$,

(74) $$\begin{aligned} \alpha&<\varliminf_{N\to\infty}\Big(\sum_{k=1}^{N}\langle kx\rangle^{-t}\Big)\Big/\big(N+N^ta_{n+1}^{\min(t,1-t)}\big)<\beta,\\ \alpha&<\varlimsup_{N\to\infty}\Big(\sum_{k=1}^{N}\langle kx\rangle^{-t}\Big)\Big/\big(N+N^ta_{n+1}^{\max(t,1-t)}\big)<\beta.\end{aligned}$$

(d) *Relative to* $x\in R_{\mathrm{irr}}$, $N\in\omega$, *and* $q_n>1$ (*or* $n>0$ *for* (76) *and* (77)),

(75) $$\sum_{\substack{k=1\\ q_n\nmid k}}^{N}\langle kx\rangle^{-1}\asymp N\log q_n+q_na_{n+1}\big(\log a_{n+1}-\log(a_{n+1}-Q')\big),$$

(76) $$\sum_{\substack{k=1\\ q_n\mid k}}^{N}\langle kx\rangle^{-1}\asymp q_na_{n+1}(1+\log Q),$$

(77) $$\sum_{k=1}^{N}\langle kx\rangle^{-1}\asymp N\log N+q_na_{n+1}(1+\log Q),$$

(78) $$\begin{cases} N\log q_n=O\Big(\sum\limits_{\substack{k=1\\ q_n\nmid k}}^{N}\langle kx\rangle^{-1}\Big),\\ \sum\limits_{\substack{k=1\\ q_n\nmid k}}^{N}\langle kx\rangle^{-1}=O(N\log N).\end{cases}$$

Moreover, there are positive real numbers α *and* β *such that for each* $x\in R_{\mathrm{irr}}$,

(78′) $$\begin{cases} \alpha<\varliminf\limits_{N\to\infty}\Big(\sum\limits_{\substack{k=1\\ q_n\nmid k}}^{N}\langle kx\rangle^{-1}\Big)\Big/(N\log q_n)<\beta,\\ \alpha<\varlimsup\limits_{N\to\infty}\Big(\sum\limits_{\substack{k=1\\ q_n\nmid k}}^{N}\langle kx\rangle^{-1}\Big)\Big/(N\log N)<\beta.\end{cases}$$

(e) *Suppose that* $1<t_0<t_1$. *Relative to* $x\in R_{\mathrm{irr}}$, $N\in\omega$, $n>0$, *and* $t_0\leqslant t\leqslant t_1$,

(79) $$\sum_{k=1}^{N}\langle kx\rangle^{-t}\asymp\sum_{\substack{k=1\\ q_n\mid k}}^{N}\langle kx\rangle^{-t}\asymp(q_na_{n+1})^t\asymp q_{n+1}^t.$$

(f) *Suppose that* $0<t_0<t_1$ *and* $1<s_0$. *Relative to* $x\in R_{\mathrm{irr}}$, $N\in\omega$, $n>0$, $0\leqslant t\leqslant t_1$, $s_0\leqslant s$, *and* $t_0\leqslant s-t$,

(80) $$\sum_{k=1}^{N}k^{-s}\langle kx\rangle^{-t}\asymp\sum_{j=0}^{n}a_{j+1}^tq_j^{t-s}\asymp\sum_{j=0}^{n}q_{j+1}^tq_j^{-s}.$$

(g) *Suppose that* $1 < t_0 < t_1$. *Relative to* $x \in R_{\mathrm{irr}}$, $N \in \omega$, $n > 0$, $0 \leqslant s$, *and* $t_0 \leqslant t \leqslant t_1$, (80) *holds.*

(h) *In the notation of* (a) *with* $s = t = 1$, *relative to* $x \in R_{\mathrm{irr}}$, $N \in \omega$, *and* $n > 0$,

$$\Gamma \asymp \sum_{j=1}^{n+1} a_j + \sum_{j=1}^{n-1} (\log q_j)(\log a_{j+1}) + (\log q_n)(\log Q), \tag{81}$$

$$\Delta \asymp 1 + \sum_{j=1}^{n} \log q_j, \tag{82}$$

$$\sum_{k=1}^{N} k^{-1} \langle kx \rangle^{-1} = O\Big((\log N)(\log q_n) + \sum_{j=1}^{n+1} a_j\Big). \tag{83}$$

Proof. In their context in (a), (66) and (67) follow from (65) and (63) respectively. (Observe that if $1 \leqslant j < n$ and $q_j \leqslant 2$, then

$$q_j^{t-s} S_t(q_j) S_s(a_{j+1}) \leqslant 2a_{j+1}^t q_j^{t-s} S_{s+t}(a_{j+1});$$

observe similarly that if $q_n \leqslant 2$, then

$$q_n^{t-s} S_t(q_n) S_s(Q) \leqslant 2a_{n+1}^t q_n^{t-s} S_{s+t}(Q).$$

Observe that $Q \leqslant a_{n+1}^t S_t(Q)$.)

In their contexts in (b), (68) and (69) follow from (61) and (62) (and (67) with $q_n = 1$) respectively. (It is easily verified that

$$Q \leqslant a_{n+1}^t \big(S_t(a_{n+1}) - S_t(a_{n+1} - Q)\big).$$

This observation is pertinent in (61) if $q_n = 2$.) As N increases from q_n to $q_{n+1} - 1$ (recall that $q_n \leqslant N < q_{n+1}$), Q increases from 1 to a_{n+1}, and $S_t(a_{n+1}) - S_t(a_{n+1} - Q)$ increases from a_{n+1}^{-t} to $S_t(a_{n+1})$. Moreover,

$$a_{n+1}^t \big(S_t(a_{n+1}) - S_t(a_{n+1} - Q)\big) = \sum_{j=a_{n+1}-Q+1}^{a_{n+1}} (a_{n+1}/j)^t$$
$$< \sum_{j=a_{n+1}-Q+1}^{a_{n+1}} \Big(\big(a_{n+1} - (a_{n+1} - Q)\big) / \big(j - (a_{n+1} - Q)\big)\Big)^t = Q^t S_t(Q).$$

Because of these considerations, (70) and (70′) follow from (68).

Suppose that $t_1 < 1$. By (9), (17), and (18), relative to $x \in R_{\mathrm{irr}}$, $N \in \omega$, $n > 0$, and $0 \leqslant t \leqslant t_1$,

$$q_n^t Q S_t(q_n) \asymp q_n^t Q q_n^{1-t} \asymp Q q_n \asymp N,$$
$$q_n^t Q^t S_t(Q) \asymp q_n^t Q^t Q^{1-t} \asymp Q q_n^t \leqslant N.$$

Hence, in their contexts in (c), (71) follows from (70), and similarly (recall (9)) (72) follows from (69), and (73) follows from (71) and (72). As N increases from q_n to $q_{n+1} - 1$, Q increases from 1 to a_{n+1}, and $a_{n+1}^t Q^{1-2t}$ $(= Q(a_{n+1}/Q^2)^t)$ varies monotonically from a_{n+1}^t to a_{n+1}^{1-t}. Thus the sentence containing (74) follows from the sentence containing (73).

In their context in (d), (75) and (76) follow from (68) and (69) respectively with $t = 1$ (cf. (9)), and, with $n > 0$ instead of $q_n > 1$,

$$\sum_{k=1}^{N} \langle kx \rangle^{-1} \asymp N \log q_n + q_n a_{n+1}(1 + \log Q)$$

by (67) with $t = 1$ (cf. (9)). Also in this context,

$$N \log q_n \leqslant N \log N \asymp 1 + N \log(q_n Q) = 1 + N(\log q_n + \log Q)$$
$$< N \log q_n + q_n a_{n+1}(1 + \log Q).$$

Thus, in its context in (d), (77) holds. Using

$$\log N \asymp \log q_n + \log Q,$$

one may prove (78) and (78′) in their context in (d) from (70) and (70′) respectively (cf. (9)).

If $q_n = 1$, the first and second members of (79) are the same. In its context in (e), (79) follows from (67) and (69) with $t = 1$ (cf. (8) and (9))

Let $c = (1 + \sqrt{5})/2$ and $b = \min(t_0, s_0 - 1)/2$. For all real u and v, if $0 \leqslant u < v$ and $q \in \omega \setminus \{0\}$, then

$$q^u S_u(q) = \sum_{j=1}^{q} (q/j)^u \leqslant \sum_{j=1}^{q} (q/j)^v = q^v S_v(q).$$

Hence, in the context of (80) in (f), by (15) and (9),

$$\sum_{j=1}^{n} q_j^{t-s} S_t(q_j) \leqslant \sum_{j=1}^{n} q_j^{-s} q_j^{s-b} S_{s-b}(q_j) \leqslant (s_0 - b)(s_0 - b - 1)^{-1} \sum_{j=1}^{n} q_j^{-b}$$
$$\leqslant (s_0 - b)(s_0 - b - 1)^{-1} \sum_{j=1}^{\infty} (c^b)^{-j+1} < \infty.$$

Because of this, one may prove (f) from (a) (cf. (9)).

For each nonnegative $s \in R$ and each $a \in \omega \setminus \{0\}$, $S_s(a) \leqslant a$ by (8). Because of this, one may prove (g) from (a) (cf. (9)).

In their context in (h), (81) and (82) follow from the definitions of Γ and Δ and (8) and (9). In this same context, by (66), (81), (82), and (16),

$$\sum_{k=1}^{N} k^{-1} \langle kx \rangle^{-1} = O(\Gamma + \Delta) = O\Big(\Big(n + \sum_{j=2}^{n} \log a_j + \log Q\Big) \log q_n + \sum_{j=1}^{n+1} a_j\Big)$$
$$= O\Big((\log q_n + \log Q) \log q_n + \sum_{j=1}^{n+1} a_j\Big),$$

whence (since also $q_n Q \leqslant N$) (83) holds.

The proof of Theorem 1 is complete.

In Theorem 1 we gave estimates of the sum in (1) for all $x \in R_{\mathrm{irr}}$ in terms of the simple continued fraction expansion of x. In Theorems 2 and 3 we shall give estimates of the sum in (1) for certain subclasses of R_{irr}; the estimates will not be in terms of the simple continued fraction expansions, but the subclasses of R_{irr} will be defined in such terms. Theorems 2 and 3 will be proved by using Theorem 1.

Theorem 2 sharpens the results of Hardy and Littlewood mentioned in connection with (2)-(5) in § 1.

Walfisz ([9], p. 571) showed that for almost every $x \in R_{\mathrm{irr}}$, for every real $\varepsilon > 0$,

$$\sum_{k=1}^{N} k^{-1}\langle kx\rangle^{-1} = O\big((\log N)^{3+\varepsilon}\big) \qquad (N \geqslant 2),$$

and it is implicit in Walfisz ([10], § 3) that the exponent $3+\varepsilon$ could be replaced by $2+\varepsilon$. It is easy to bring the (strengthened) result of Walfisz to within the compass of (a) of Theorem 2. (The strengthened result of Walfisz follows from (7) via summation by parts.) See (b) of Theorem 6 also.

Walfisz ([9], p. 585) showed that for almost every $x \in R_{\mathrm{irr}}$ and for every $x \in R_{\mathrm{irr}}$ for which $\big(a_j(x)\big)_{j=1}^{\infty}$ is bounded,

$$\sum_{k=1}^{\infty} k^{-3/2}\langle kx\rangle^{-1} < \infty.$$

It is easy to bring this result (which in fact is covered by (5) in its context) to within the compass of (c) of Theorem 2 (e.g., let $t = 1$, $s = 5/4$, and $\varepsilon = 1/4$). (Walfisz's result follows from (7) via summation by parts; the exponent $-3/2$ could be replaced by any real number < -1.) In fact, this will be done in (c) of Theorem 5.

THEOREM 2. *Let $f, X, Y, n_0, \delta, Z, K$, and S be given as in* (M), (N), *and* (O) *of* § 2. *Then* (a)-(c) *below hold.*

(a) *Suppose that $u\log u = O\big(f(u)\big)$ for $u \geqslant 2$. Relative to $x \in X$ and $N \geqslant 3$,*

$$\sum_{k=1}^{N} k^{-1}\langle kx\rangle^{-1} = O\left(\frac{(\log N)f(\log N)}{\log\log N}\right). \tag{84}$$

Relative to $x \in Y$ and $N \geqslant 2$,

$$\sum_{k=1}^{N} k^{-1}\langle kx\rangle^{-1} \asymp \frac{(\log N)f(\log N)}{\log\log N}. \tag{85}$$

If also $u(\log u)^2 = O\big(f(u)\big)$ for $u \geqslant 2$, then, relative to $x \in Z$ and $N \geqslant 3$,

$$\sum_{k=1}^{N} k^{-1}\langle kx\rangle^{-1} \asymp \frac{\log N}{\log\log N} f\left(\frac{\log N}{\log\log N}\right). \tag{86}$$

(b) *Let X_0 be the set of all $x \in R_{\mathrm{irr}}$ such that*

$$a_j(x) \leqslant Kj(1+\log j) \qquad (j \geqslant 1). \tag{87}$$

(Thus X_0 has measure 0.) Relative to $x \in X_0$ and $N \geqslant 2$,

$$\sum_{k=1}^{N} k^{-1}\langle kx\rangle^{-1} = O\big((\log N)^2\big). \tag{88}$$

There is a set $Y_0 \subset X_0$ such that Y_0 has the cardinal of the continuum and such that, relative to $x \in Y_0$ and $N \geqslant 2$,

$$\sum_{k=1}^{N} k^{-1}\langle kx\rangle^{-1} \asymp (\log N)^2. \tag{89}$$

(c) *Consider the conditions*

$$a_{j+1}(x) \leqslant Kq_j(x)^{(s-t)/t} \qquad (j \in \omega), \tag{90}$$

$$q_{j+1}(x) \leqslant Kq_j(x)^{s/t} \qquad (j \in \omega). \tag{91}$$

[Note. *If $t = 0 < s$, then each of (90) and (91) is considered to be equivalent to $x \in R_{\mathrm{irr}}$. If $0 < t < s$, then for almost every $x \in R_{\mathrm{irr}}$ there is a choice of $K \in R$ such that (90) and (91) hold. Observe that if $0 < t \leqslant s$, then (91) implies (90), and (90) implies (91) with K replaced by $K+1$.*] *Suppose that $s_0, t_1 \in R$ and $1 < s_0$ and $0 < t_1$. Relative to $s_0 \leqslant s$, $0 \leqslant t \leqslant t_1$, $x \in R_{\mathrm{irr}}$, $N \geqslant 2$, and (90) or (91),*

$$\sum_{k=1}^{N} k^{-s}\langle kx\rangle^{-t} = O(\log N). \tag{92}$$

Relative to $s_0 \leqslant s \leqslant t_1$, $N \geqslant 2$, and $x \in S$ (i.e., (90) with $t = s$),

$$\sum_{k=1}^{N} k^{-s}\langle kx\rangle^{-s} \asymp \log N. \tag{93}$$

Suppose also that $\varepsilon \in R$ and $\varepsilon > 0$. Relative to $s_0 \leqslant s$, $0 \leqslant t \leqslant t_1$, $x \in R_{\mathrm{irr}}$, and (90) or (91),

$$\sum_{k=1}^{\infty} k^{-s-\varepsilon}\langle kx\rangle^{-t} = O(1). \tag{94}$$

Proof of (a). For some real $c > 1$, relative to $u > 1$, $u = O(f(u))$ and $f(u) = O(u^c)$ (cf. (M)). Hence, relative to $u > 2$, $\log f(u) \asymp \log u$. By (M), relative to $x \in X$ and $N \geqslant 3$,

$$\sum_{j=1}^{h_N(x)} a_j(x) = O\left(\frac{(\log N) f(\log N)}{\log\log N}\right). \tag{95}$$

By (M) relative to $x \in Y$ and $N \geqslant 3$,

$$\sum_{j=1}^{h_N(x)} a_j(x) \asymp \frac{(\log N) f(\log N)}{\log\log N}. \tag{96}$$

In Theorem 1, $n+1 = h_N(x)$, and, relative to $N \geqslant 3$,

$$\begin{aligned}(\log N)(\log q_n) &\leqslant (\log N)^2 \\ &= \frac{(\log N)}{\log\log N}(\log N)(\log\log N) = O\left(\frac{(\log N) f(\log N)}{\log\log N}\right).\end{aligned} \tag{97}$$

Now (84) in its context follows from (83), (97), and (95). By (81) and (66) (with $s = t = 1$) in their contexts, relative to $x \in X$ and $h_N(x) > 1$,

$$\sum_{j=1}^{h_N(x)} a_j(x) = O\Big(\sum_{k=1}^{N} k^{-1} \langle kx\rangle^{-1}\Big). \tag{98}$$

Now (85) in its context follows from (84), (96), and (98). (It should be remarked that, where $N' = [f(1)f(2)+2]$, $h_{N'}(x) > 1$ for each $x \in X$, and positive real upper and lower bounds of $\sum_{j=1}^{N} k^{-1}\langle kx\rangle^{-1}$ for $x \in X$ and $1 \leqslant N \leqslant N'$ are easily computed.) If $u(\log u)^{\frac{1}{2}} = O(f(u))$ for $u \geqslant 2$, then, relative to $N \geqslant 3$,

$$(\log N)^2 = O\left(\frac{\log N}{\log\log N} f\left(\frac{\log N}{\log\log N}\right)\right). \tag{99}$$

Now (86) in its context follows from (83), (99), (36), and (98).

Proof of (b). In (a), let $f(u) = K$ for $0 \leqslant u < 1$, and let $f(u) = Ku(1+\log u)$ for each real $u \geqslant 1$. Then $X_0 = X$; let $Y_0 = Y$. (Here recall (M).) Now (88) and (89) in their contexts reduce to (84) and (85) in their contexts.

Proof of (c). (92) and (93) in their contexts follow from (16) and (f) of Theorem 1. (Here recall the definition of S from (O).) Let $c = (1+\sqrt{5})/2$. For each $x \in R_{\mathrm{irr}}$ and each $n \in \omega$, $q_n(x) \geqslant c^{n-1}$ by (15). In the context of (94), since (91) implies $t \leqslant s$ and (90), for each integer $n > 0$,

$$\sum_{j=0}^{n} a_{j+1}(x)^t q_j(x)^{t-s-\varepsilon} \leqslant \sum_{j=0}^{n} q_j(x)^{-\varepsilon} K^t,$$

$$K^t\Big(1+\sum_{j=1}^{n} (c^{-\varepsilon})^{n-1}\Big) < K^{t_1}\big(1+(1-c^{-\varepsilon})^{-1}\big).$$

Thus, in its context, (94) holds by (f) of Theorem 1.

The proof of Theorem 2 is complete.

Chowla ([2], p. 546) showed that for each $x \in R_{\mathrm{irr}}$ for which $(a_j(x))_{j=1}^{\infty}$ is bounded,

$$\sum_{k=1}^{N} \langle kx\rangle^{-1} = O(N\log N) \qquad (N \geqslant 2).$$

A more general result is presented in (b) of Theorem 3 below. The summation by parts formula which may be used in the proof of parts of Theorem 3 is (SP) (with $N \geqslant 1$) of § 1.

Theorem 3. *Let K and S be given as in* (O) *of* § 2. *Suppose that s_0, $s_1 \in R$ and $0 < s_0 < s_1$. Then* (a)-(c) *below hold.*

(a) *Consider the conditions*

$$a_{j+1}(x) \leqslant K q_j(x)^{(1-t)/t} \qquad (j \in \omega), \tag{100}$$

$$q_{j+1}(x) \leqslant K q_j(x)^{1/t} \qquad (j \in \omega). \tag{101}$$

[Note. *Recall the note within* (c) *of Theorem* 2 *with* $s = 1$.] *Suppose that $t_1 \in R$ and $t_1 < 1$. Relative to $x \in R_{\mathrm{irr}}$, $N \in \omega$, $0 \leqslant t \leqslant t_1$, $s_0 \leqslant s \leqslant s_1$, and* (100) *or* (101),

$$N \leqslant \sum_{k=1}^{N} \langle kx\rangle^{-t} \leqslant \sum_{k=1}^{N} \langle kx\rangle^{-t_1} = O(N), \tag{102}$$

$$S_s(N) \leqslant \sum_{k=1}^{N} k^{-s}\langle kx\rangle^{-t} \leqslant \sum_{k=1}^{N} k^{-s}\langle kx\rangle^{-t_1} = O(S_s(N)). \tag{103}$$

(b) *Consider the condition*

$$a_{j+1}(x) \leqslant K\big(1+\log q_j(x)\big) \qquad (j \in \omega). \tag{104}$$

[Note. *It follows from* (16) *and* (D) *that for almost every $x \in R_{\mathrm{irr}}$, $\log q_n(x) \asymp n$ $(n \geqslant 2)$. Hence, by* (A), (104) *holds for almost no $x \in R_{\mathrm{irr}}$. By* (16), *if $x \in R_{\mathrm{irr}}$ and $a_n(x) = O(n)$ $(n \geqslant 1)$, then* (104) *holds for some choice of K.*]

Relative to $x \in R_{\mathrm{irr}}$, $N \geqslant 2$, $s_0 \leqslant s \leqslant s_1$, *and* (104),

(105) $$\sum_{k=1}^{N} \langle kx\rangle^{-1} \asymp N \log N,$$

(106) $$\sum_{k=1}^{N} k^{-s} \langle kx\rangle^{-1} \asymp \int_1^N u^{-s}(\log u)\,du.$$

Relative to $x \in R_{\mathrm{irr}}$, $N \geqslant 2$, $s_0 \leqslant s < 1$, *and* (104),

(107) $$\sum_{k=1}^{N} k^{-s} \langle kx\rangle^{-1} \asymp (1-s)^{-1} N^{1-s} \log N,$$

(108) $$\sum_{k=1}^{N} k^{-1} \langle kx\rangle^{-1} \asymp (\log N)^2.$$

(c) *Suppose that* $t_0, t_1 \in R$ *and* $1 < t_0 < t_1$. *Relative to* $x \in S$, $N \geqslant 2$, $t_0 \leqslant t \leqslant t_1$, *and* $s_0 \leqslant s \leqslant s_1$,

(109) $$\sum_{k=1}^{N} \langle kx\rangle^{-t} \asymp N^t,$$

(110) $$\sum_{k=1}^{N} k^{-s} \langle kx\rangle^{-t} \asymp \int_1^N u^{t-s-1}\,du.$$

Relative to $x \in S$, $N \in \omega$, $s_0 \leqslant s \leqslant s_1$, $t_0 \leqslant t \leqslant t_1$, *and* $s \neq t$,

(111) $$\sum_{k=1}^{N} k^{-s} \langle kx\rangle^{-t} \asymp (t-s)^{-1}(N^{t-s}-1).$$

(*Also, see* (93).)

Proof of (a). In the notation of Theorem 1, relative to $x \in R_{\mathrm{irr}}$, $N \in \omega$, $q_n > 1$, $0 \leqslant t < 1$, and (100) or (101),

$$N^t a_{n+1}^t Q^{1-2t} = O(N^t q_n^{1-t} Q^{1-2t}) = O\big(N^t (q_n Q)^{1-t} Q^{-t}\big) = O(N^t N^{1-t} Q^{-t}) = O(N).$$

Thus (102) in its context follows from (73) in its context. (Use (100) or (101) to handle those x and N for which $q_n = 1$.) The reader may prove (103) in its context from (102) via summation by parts.

Proof of (b). In the notation of Theorem 1, relative to $x \in R_{\mathrm{irr}}$, $N \in \omega$, $q_n > 1$, and (104),

$$q_n a_{n+1}(1+\log Q) = O\big(q_n(\log q_n)(1+\log Q)\big) = O\big(q_n Q(\log q_n)\big) = O(N \log N).$$

Thus (105) in its context follows from (77) in its context. (Use (104) to handle those x and N for which $q_n = 1$.) The reader may prove (106) in its context from (105) via summation by parts. The reader may then prove (107) and (108) in their context from (106).

Proof of (c). The proof of (c) is similar to the proofs of (a) and (b) and uses (79) instead of (73) and (77).

The proof of Theorem 3 has been sufficiently discussed.

We now pass to "almost everywhere" results.

In Theorem 4 we shall estimate the sum in (1) with $s = 0$.

Walfisz ([10], p. 787) showed that for almost every $x \in R_{\mathrm{irr}}$, for every real $\varepsilon > 0$, (7) holds. A more precise result is given in (b) of Theorem 4.

THEOREM 4. *Suppose that* $t_0, t_1 \in R$. *Suppose that* $(c_N)_{N=1}^{\infty}$ *is a non-decreasing sequence of positive real numbers. Then* (a)-(c) *below hold.*

(a) *There is a positive real number* β *such that for almost every* $x \in R_{\mathrm{irr}}$, *for* $0 \leqslant t < 1$,

(112) $$\varlimsup_{N\to\infty} \sum_{k=1}^{N} \langle kx\rangle^{-t}/N < \beta.$$

[*Moreover, trivially, for each* $x \in R_{\mathrm{irr}}$,

$$N < \sum_{k=1}^{N} \langle kx\rangle^{-u} < \sum_{k=1}^{N} \langle kx\rangle^{-t}$$

for all $N \in \omega \setminus \{0\}$ *and* $u, t \in R$ *with* $0 < u < t$.]

(b) *There are positive real numbers* α *and* β *such that for almost every* $x \in R_{\mathrm{irr}}$,

(113) $$\alpha < \varliminf_{N\to\infty} \Big(\sum_{k=1}^{N} \langle kx\rangle^{-1}\Big)/(N \log N) < \beta.$$

Moreover, $\sum_{N=1}^{\infty} (Nc_N)^{-1} < \infty$ (*resp.,* $= \infty$) *is a necessary and sufficient condition that for almost every* (*resp., almost no*) $x \in R_{\mathrm{irr}}$,

(114) $$\sum_{k=1}^{N} \langle kx\rangle^{-1} = O(Nc_N) \quad (N \geqslant 1).$$

(c) *Suppose that* $1 < t_0 \leqslant t_1$. *Relative to* $x \in R_{\mathrm{irr}}$, $t_0 \leqslant t \leqslant t_1$, $N \geqslant 1$, *and* $h_N(x) > 1$,

(115) $$\Big(\sum_{k=1}^{N} \langle kx\rangle^{-t}\Big)^{1/t} \asymp \Big(\sum_{k=1}^{N} \langle kx\rangle^{-2}\Big)^{1/2}.$$

Suppose that $t \in R$ *and* $1 < t$. *There are positive real numbers* α *and* β *such that for every* $x \in R_{\mathrm{irr}}$,

(116) $$\alpha < \varliminf_{N\to\infty} \Big(\sum_{k=1}^{N} \langle kx\rangle^{-t}\Big)/N^t < \beta.$$

Moreover, $\sum_{N=1}^{\infty}(Nc_N)^{-1}<\infty$ (*resp.*, $=\infty$) *is a necessary and sufficient condition that for almost every* (*resp.*, *almost no*) $x\in R_{\mathrm{irr}}$,

$$\sum_{k=1}^{N}\langle kx\rangle^{-t}=O\big((Nc_N)^t\big)\quad (N\geqslant 1). \tag{117}$$

Proof. We shall use the notation of Theorem 1. Thus, if $x\in R_{\mathrm{irr}}$, then $n+1=h_N(x)$. To prove (a), apply (73) (in its context) and (I) (with $k=0$ and $c_j=(1+\log j)^2$ for each $j\in\omega\setminus\{0\}$).

By (D), for almost every $x\in R_{\mathrm{irr}}$, $\varlimsup_{j\to\infty} a_j(x)\leqslant e^{d}$. Hence the existence of α and β as specified in (b) follows from (77) in its context. For $x\in R_{\mathrm{irr}}$,

$$q_n a_{n+1}(1+\log Q)\leqslant q_n Q a_{n+1}\leqslant N a_{n+1},$$

and, if $N=q_n$, then also

$$N=q_{h_n(x)-1}(x)\quad\text{and}\quad q_n a_{n+1}(1+\log Q)=N a_{n+1}.$$

It now follows from (I) (with $k=0$) and the note following the proof of (I) that $\sum_{N=1}^{\infty}(Nc_N)^{-1}<\infty$ (resp., $=\infty$) is a necessary and sufficient condition that for almost every (resp., almost no) $x\in R_{\mathrm{irr}}$, relative to $N\in\omega$ and $n>0$,

$$q_n a_{n+1}(1+\log Q)=O(Nc_N).$$

By [7], 3.7, p. 52, if $\sum_{N=1}^{\infty}(Nc_N)^{-1}<\infty$, then $\log N=o(c_N)$ as $N\to\infty$. Because of these considerations, that part of (b) involving (114) follows from (77) in its context.

Now (115) in its context follows from (e) of Theorem 1. The existence of α and β as specified in (c) follows from (e) of Theorem 1. For $x\in R_{\mathrm{irr}}$, $q_{n+1}\leqslant 2Na_{n+1}$, and, if $N=q_n$, then also

$$N=q_{h_N(x)-1}\quad\text{and}\quad Na_{n+1}\leqslant q_{n+1}.$$

It now follows from (I) (with $k=0$) and the note following the proof of (I) that $\sum_{N=1}^{\infty}(Nc_N)^{-1}<\infty$ (resp., $=\infty$) is a necessary and sufficient condition that for almost every (resp., almost no) $x\in R_{\mathrm{irr}}$, relative to $N\in\omega$ and $n>0$, $q_{n+1}=O(Nc_N)$. Because of this, that part of (c) involving (117) follows from (e) of Theorem 1.

The proof of Theorem 4 is complete.

In Theorem 5 we shall estimate the sum in (1) with $0\leqslant t<1$. (Observe that in (c) of Theorem 5, some pairs (s,t) with $1<t$ are relevant also.)

THEOREM 5. (a)-(c) *below hold.*

(a) *There is a positive real number* β *such that for almost every* $x\in R_{\mathrm{irr}}$, *for* $0\leqslant s<1$ *and* $0\leqslant t<1$,

$$\varlimsup_{N\to\infty}\Big(\sum_{k=1}^{N}k^{-s}\langle kx\rangle^{-t}\Big)\Big/N^{1-s}\leqslant(1-s)^{-1}\beta. \tag{118}$$

[*Moreover, trivially* (*see* (9)), *for each* $x\in R_{\mathrm{irr}}$,

$$\frac{2^{1-s}-1}{1-s}N^{1-s}\leqslant S_s(N)<\sum_{k=1}^{N}k^{-s}\langle kx\rangle^{-u}<\sum_{k=1}^{N}k^{-s}\langle kx\rangle^{-t}$$

for all $N\in\omega\setminus\{0\}$ *and* $u,t\in R$ *with* $0<u<t$.]

(b) *Suppose that* $0\leqslant t<1$. *There is a positive real number* β *such that for almost every* $x\in R_{\mathrm{irr}}$,

$$\varlimsup_{N\to\infty}\Big(\sum_{k=1}^{N}k^{-1}\langle kx\rangle^{-t}\Big)\Big/\log N<\beta. \tag{119}$$

[*Moreover, trivially* (*see* (9)), *for each* $x\in R_{\mathrm{irr}}$,

$$\log N\leqslant S_1(N)<\sum_{k=1}^{N}k^{-1}\langle kx\rangle^{-u}<\sum_{k=1}^{N}k^{-1}\langle kx\rangle^{-t}$$

for all $N\in\omega\setminus\{0\}$ *and* $u\in R$ *with* $0<u<t$.]

(c) *For almost every* $x\in R_{\mathrm{irr}}$, *for all* $s,t\in R$, *if* $1<s$ *and* $0\leqslant t<s$, *then*

$$\sum_{k=1}^{\infty}k^{-s}\langle kx\rangle^{-t}<\infty. \tag{120}$$

Proof of (a). Use (a) of Theorem 4 and summation by parts (i.e., (SP) of § 1).

Proof of (b). We shall use the notation of Theorem 1. By (8) and (9), relative to $x\in R_{\mathrm{irr}}$ and $m\in\omega\setminus\{0\}$,

$$\sum_{j=1}^{m}q_j^{t-1}S_t(q_j)\big(S_1(a_{j+1})-1\big)\asymp\sum_{j=1}^{m}\log a_{j+1},$$

$$\sum_{j=1}^{m}q_j^{t-1}S_t(q_j)\asymp\sum_{j=1}^{m}1=m.$$

Let $c=(1+\sqrt{5})/2$, whence $1<c$. By (8), (15), and (A), for almost every $x\in R_{\mathrm{irr}}$, relative to $m\in\omega\setminus\{0\}$,

$$\sum_{j=1}^{\infty}a_{j+1}^{t}q_j^{t-1}S_{1+t}(a_{j+1})\leqslant\sum_{j=1}^{\infty}a_{j+1}^{t+1}c^{(t-1)(j-1)}<\infty.$$

In the notation of Theorem 1 and (a) of Theorem 1 with $s = 1$, it follows from the considerations just made that for some real $b > 0$, for almost every $x \in R_{\mathrm{irr}}$,

$$\Gamma + \Delta < bn + b \sum_{j=1}^{n} \log a_{j+1},$$

whenever N is sufficiently large (recall that $n+1 = h_N(x)$). Hence (b) follows from (a) of Theorem 1, (D), and (E).

Proof of (c). By (A), for almost every $x \in R_{\mathrm{irr}}$, $a_j(x) = O(j^2)$ for $j > 0$. For each such x, for all $s, t \in R$, if $0 \leqslant t < s$, then (by (15)) (90) holds for some real $K > 1$. For each such x apply (94) with s and ε replaced by s' and ε' respectively where $1 < s' < s$, $t < s'$, and $\varepsilon' = s - s'$.

The proof of Theorem 5 is complete.

In Theorem 6 below we shall estimate the sum in (1) with $t = 1$ and $0 < s \leqslant 1$. (The case $t = 1$ and $s > 1$ is covered by (c) of Theorem 5.)

Walfisz ([9], p. 584) showed that for almost every $x \in R_{\mathrm{irr}}$, for every real $\varepsilon > 0$,

$$\sum_{k=1}^{N} k^{-1/2} \langle kx \rangle^{-1} = O\big(N^{1/2} (\log N)^{2+\varepsilon}\big) \qquad (N \geqslant 2),$$

and it is implicit in Walfisz ([10], § 3) that the exponent $2+\varepsilon$ could be replaced by $1+\varepsilon$. The strengthened result of Walfisz follows from (7) via summation by parts; here one could replace $k^{-1/2}$ and $N^{1/2}$ by k^{-s} and N^{1-s} respectively with $0 < s < 1$. A more precise result is given in (a) of Theorem 6 (cf. also the note following the proof of Theorem 7).

Recall from the discussion introducing Theorem 2 the (strengthened) estimate of Walfisz for $\sum_{k=1}^{N} k^{-1} \langle kx \rangle^{-1}$. A more precise result is given in (b) of Theorem 6.

THEOREM 6. *Suppose that $s_0, s_1 \in R$ and $(c_N)_{N=1}^{\infty}$ is a nondecreasing sequence of positive real numbers. Then* (a) *and* (b) *below hold.*

(a) *Suppose that $0 < s_0 < s_1 < 1$. If $\sum_{N=2}^{\infty} \big(N (\log N) c_N\big)^{-1} < \infty$, then for almost every $x \in R_{\mathrm{irr}}$, relative to $s_0 \leqslant s \leqslant s_1$ and $N \geqslant 2$,*

$$\sum_{k=1}^{N} k^{-s} \langle kx \rangle^{-1} = O\big(N^{1-s} (\log N) c_N\big). \tag{121}$$

If $0 < s < 1$ and $\sum_{N=2}^{\infty} \big(N (\log N) c_N\big)^{-1} = \infty$, then for almost no $x \in R_{\mathrm{irr}}$, relative to $N \geqslant 2$, (121) *holds.* (*Cf. the note following the proof of Theorem* 7.)

(b) *For almost every $x \in R_{\mathrm{irr}}$,*

$$\sum_{k=1}^{N} k^{-1} \langle kx \rangle^{-1} \asymp (\log N)^2 \qquad (N \geqslant 2). \tag{122}$$

(*Cf. the note following the proof of Theorem* 6.)

Proof of (a). We shall use the notation of Theorem 1. By (a) of Theorem 1, (8), (9), and (L), relative to $x \in R_{\mathrm{irr}}$, $N \in \omega$, $n > 0$, and $s_0 \leqslant s \leqslant s_1$,

$$a_{n+1} q_n^{1-s} = O\Big(\sum_{k=1}^{N} k^{-s} \langle kx \rangle^{-1}\Big),$$

$$\begin{aligned}\sum_{k=1}^{N} k^{-s} \langle kx \rangle^{-1} &= O\Big(\sum_{j=1}^{n-1} q_j^{1-s} (\log q_j) a_{j+1}^{1-s} + q_n^{1-s} (\log q_n) Q^{1-s} + \sum_{j=1}^{n} a_{j+1} q_j^{1-s}\Big) \\ &= O\Big((\log q_{n-1}) q_n^{1-s} + N^{1-s} \log q_n + \sum_{j=1}^{n} a_{j+1} q_j^{1-s}\Big) \\ &= O\Big(N^{1-s} \log q_n + N^{1-s} \sum_{j=2}^{n+1} a_j\Big).\end{aligned}$$

If $\sum_{N=1}^{\infty} \big(N (\log N) c_N\big)^{-1} < \infty$, then $\lim_{N\to\infty} c_N = \infty$ by [7], 3.7, p. 52, and, by (J) (take $s = 1$ in (J)), for almost every $x \in R_{\mathrm{irr}}$, relative to $s_0 \leqslant s \leqslant s_1$ and $N \geqslant 2$, (121) holds. If $0 < s < 1$ and $\sum_{N=1}^{\infty} \big(N (\log N) c_N\big)^{-1} = \infty$, then, by the note (with c_n replaced by $(1+\log n) c_n$) following the proof of (I), for almost every $x \in R_{\mathrm{irr}}$, it is false that

$$a_{n+1} q_n^{1-s} = O\big((\log N) c_N N^{1-s}\big) \qquad (N \geqslant 2),$$

and hence (121) fails relative to $N \geqslant 2$.

Proof of (b). We shall use the notation of Theorem 1 with $s = t = 1$, whence $n+1 = h_N(x)$. By (J) with $s = 1$ and $c_N = (1+\log N)^{1/2}$, for almost every $x \in R_{\mathrm{irr}}$,

$$\sum_{j=1}^{n+1} a_j = o\big((\log N)^2\big) \qquad (N \to \infty). \tag{123}$$

Hence, by (83) in its context, for almost every $x \in R_{\mathrm{irr}}$,

$$\sum_{k=1}^{N} k^{-1} \langle kx \rangle^{-1} = O\big((\log N)^2\big) \qquad (N \geqslant 2). \tag{124}$$

By (16) and (D), for almost every $x \in R_{\mathrm{irr}}$,

$$\log q_m(x) \asymp m \qquad (m \geqslant 2). \tag{125}$$

By (82) and (125) in their contexts and (E), for almost every $x \in R_{\mathrm{irr}}$,

(126) $$\Delta \asymp n^2 \asymp (\log N)^2 \quad (n > 1).$$

By (126), (125), (81), and (66) in their contexts and (D), for almost every $x \in R_{\mathrm{irr}}$, for some integer $n_x > 0$, relative to $n \geqslant n_x$,

(127) $$(\log N)^2 \asymp n^2 = O\Big((\log q_{[n/2]}) \sum_{j=[n/2]}^{n-1} \log a_{j+1}\Big)$$
$$= O(\Gamma) = O\Big(\sum_{k=1}^{N} k^{-1}\langle kx\rangle^{-1}\Big).$$

By (124) and (127) in their contexts, (b) holds.

The proof of Theorem 6 is complete.

Note. *Examination of the proof of* (b) *of Theorem* 6 *shows that there are positive real numbers α and β such that for almost every $x \in R_{\mathrm{irr}}$, for all sufficiently large $N \in \omega$,*

(128) $$\alpha(\log N)^2 < \sum_{k=1}^{N} k^{-1}\langle kx\rangle^{-1} < \beta(\log N)^2.$$

For all real $s \geqslant 0$ and $t \geqslant 0$, if either $t \leqslant 1$ or $t < s$, then one of the results in Theorems 4, 5, and 6 provides estimates for the sum in (1) for almost every $x \in R_{\mathrm{irr}}$. The case $t > 1$ and $0 \leqslant s \leqslant t$ is treated in Theorem 7 below.

THEOREM 7. *Suppose that $t_0, t_1 \in R$ and $1 < t_0 < t_1$. Suppose that $(c_N)_{N=1}^{\infty}$ is a nondecreasing sequence of positive real numbers. If $\sum_{N=2}^{\infty} (N(\log N)c_N)^{-1} < \infty$, then for almost every $x \in R_{\mathrm{irr}}$, relative to $t_0 \leqslant t \leqslant t_1$, $0 \leqslant s \leqslant t$, and $N \geqslant 2$,*

(129) $$\sum_{k=1}^{N} k^{-s}\langle kx\rangle^{-t} = O\Big(N^{t-s}\big((\log N)c_N\big)^t\Big).$$

If $1 < t$, $0 \leqslant s$, and $\sum_{n=2}^{\infty} (N(\log N)c_N)^{-1} = \infty$, then for almost no $x \in R_{\mathrm{irr}}$, relative to $N \geqslant 2$, (129) *holds.* (*Cf. the note following the proof of Theorem* 7.)

Proof. In (g) of Theorem 1, if $0 \leqslant s \leqslant t$,

$$\sum_{j=0}^{n} a_{j+1}^t q_j^{t-s} \leqslant N^{t-s} \sum_{j=0}^{n} a_{j+1}^t.$$

Hence the first conclusion involving (129) follows from (g) of Theorem 1 and (K). In (g) of Theorem 1, if $N = q_n$, then

$$N = q_{h_N(x)-1}(x) \quad \text{and} \quad N^{t-s} a_{n+1}^t \leqslant \sum_{j=0}^{n} a_{j+1}^t q_j^{t-s}.$$

Hence the second conlusion of Theorem 7 follows from (g) of Theorem 1 and the note (with $k = 0$ and with c_n replaced by $(1+\log n)c_n$) following the proof of (I).

Note. *A glance at the relevant proofs shows that in those parts of Theorems* 6 *and* 7 *using the hypothesis*

$$\sum_{N=2}^{\infty} (N(\log N)c_N)^{-1} = \infty,$$

one may in this hypothesis, (121), *and* (129) *delete $\log N$ without impairing the validity of those parts of Theorems* 6 *and* 7.

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