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Density and multiplicative structure of sets of integers*

by

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0. Introduction. The main purpose of this paper is to extend the theory, begun by P. Erdös together with H. Davenport, of the logarithmic density of sequences of natural numbers. However, our method will give information on natural density, as well as on certain other asymptotic densities. (The reader unfamiliar with density theory will find the necessary definitions in Section One.)

Erdös and Davenport [3], [4] proved that for any set M of natural numbers, the set $D(M)$ consisting of all numbers divisible by at least one member of M possesses a logarithmic density. Earlier, Besicovitch [2] had shown that $D(M)$ need not possess a natural density. He also proved that a sequence having the property that no member divides another need not have zero natural density. Erdös [5] and Behrend [1] showed that such a sequence does possess zero logarithmic density, hence zero lower natural density. Using their result on $D(M)$, Erdös and Davenport proved that any sequence which does not have zero logarithmic density contains a division chain, that is, an infinite subsequence for which each member divides its successor.

The papers cited above tend to show that logarithmic density is a more accurate indicator of multiplicative structure than is natural density.

Section One of this paper consists of elementary results whose application to natural density is well known. In Section Two, we introduce a decomposition of a sequence which allows us to apply the ideas of Section One to logarithmic density. Finally, in Section Three, we sharpen the results of Erdös and Davenport on $D(M)$ and on division chains, and obtain new information on special division chains which exist in sets with positive natural density. For instance, if a sequence C has positive natural density, then C contains a subsequence $q_1, q_1q_2, q_1q_2q_3, \ldots$ such that q_{i+1} is divisible only by primes greater than $q_1q_2 \ldots q_i$ raised

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to the power $\log\log(q_1 q_2 \ldots q_i)$. Also, our method will lead to a theorem, which compliments a theorem of Erdös [7], on the irregularity of the prime divisors of almost all integers.

1. Basic definitions and ideas of asymptotic density.

1.1. Definition. Let $\{\mu_i\}$ be a sequence of countably additive measures on the subsets of N, the natural numbers. Assume limit $\mu_i N = 1$. Define $\mu^* A = \limsup \mu_i A$ and $\mu_* A = \liminf \mu_i A$ for A in N. If $\mu^* A = \mu_* A$, let μA denote this number, and say that A *possesses μ-density.*

The following model will give all of the asymptotic densities which will interest us.

1.2. Definition. Let $\{c_i\}$ be an arbitrary sequence of positive real numbers. For A in N, define

$$\mu_k A = \sum (c_i\colon\ i \in A,\ i \leqslant k) \Big/ \sum_{i=1}^{k} c_i. \tag{1}$$

Natural density δ is obtained when $c_i = 1$ for each i, and logarithmic density l is obtained when $c_i = 1/i$. For l it is convenient to replace the denominator of (1) by $\log k$. We will always assume that functions involving the logarithmic function are suitably redefined for certain small values of the argument.

The following theorem is stated without proof.

1.3. Theorem. (a) *If $A \subset B$, then $\mu_* A \leqslant \mu_* B$ and $\mu^* A \leqslant \mu^* B$.*

(b) *If $C \subset A \cup B$, $\mu^* C \leqslant \mu^* A + \mu^* B$.*

(c) *If A and B are disjoint sets with μ-density, then $\mu(A \cup B) = \mu A + \mu B$.*

(d) *If A and B possess μ-density and $A \subset B$, then $\mu(B - A) = \mu B - \mu A$.*

In items 1.4 through 1.10 it will be understood that $\{E_i\}$ is a sequence of sets, each having μ-density. We will assume that $F_j = \bigcup_{i=1}^{j} E_i$ possesses μ-density for each j. Let $F = \bigcup E_i$.

1.4. Definition. The sequence $\{E_i\}$ will be called *μ-summable* if μF exists and $\mu F = \operatorname{limit} \mu F_j$.

Since asymptotic densities are far from being countably additive set functions, μ-summability is a strong condition.

1.5. Theorem. *We always have* limit $\mu F_j \leqslant \mu_* F$.

Proof. Since $\mu F_j \leqslant \mu F_{j+1}$, limit μF_j exists. By 1.3 (a), $\mu F_j \leqslant \mu_* F$ for each j. The result follows.

1.6. Corollary. *If* limit $\mu F_j = 1$, $\mu F = 1$.

Proof. Clearly, limit $\mu F_j \leqslant \mu_* F \leqslant \mu^* F \leqslant 1$.

1.7. Theorem. *Suppose that there is a sequence of constants $\{L_i\}$, with $\sum L_i$ converging, so that for all i and k, $\mu_k E_i \leqslant L_i$. Then $\{E_i\}$ is μ-summable.*

Proof. We only need show that $\mu^* F \leqslant \operatorname{limit} \mu F_j$, because of 1.5. For any j and k we have

$$\mu_k(F - F_j) = \mu_k F - \mu_k F_j \leqslant \sum_{j+1} \mu_k E_i \leqslant \sum_{j+1} L_i = R_j.$$

Thus for each j, $\mu^* F - \mu F_j \leqslant R_j$. The result follows.

1.8. Corollary. *Suppose that $\{E_i\}$ is a sequence of pairwise disjoint sets. If $\mu_k E_i \leqslant \mu E_i$ for all i and k, then $\{E_i\}$ is μ-summable.*

Proof. Certainly $\sum \mu E_i \leqslant 1$. Put $L_i = \mu E_i$ in 1.7.

1.9. Example. If $A = \{a_1 < a_2 < \ldots\}$, then $D(A)$, as defined previously, equals $\bigcup_N D(a_i)$. It is easily seen that, for any i and k, $\delta_k(D(a_i)) \leqslant 1/a_i$. Thus if $\sum 1/a_i$ converges, $\delta(D(A)) = \operatorname{limit} \delta\big(\bigcup_{i=1}^{N} D(a_i)\big)$ by 1.7. In the special case where the a_i's are pairwise relatively prime, a routine computation shows that $\delta(A) = 1 - \prod (1 - 1/a_i)$.

1.10. Theorem. *Let $\{E_i\}$ be μ-summable with G contained in F. If $\mu(G \cap E_i) = 0$ for each i, $\mu G = 0$.*

Proof. Let $G_j = G \cap F_j$. Certainly $\mu G_j = 0$. We have $\mu^* G = \mu^*(G - G_j) \leqslant \mu(F - F_j) = \mu F - \mu F_j$. Since limit $\mu F_j = \mu F$, $\mu G = 0$.

1.11. Theorem. *Let $\{c_i\}$ and $\{d_i\}$ be sequences of positive real numbers. Let μ and σ be the asymptotic densities induced by these sequences as in* 1.2. *Suppose that* (i) *$\sum c_i$ and $\sum d_i$ are divergent, and* (ii) *c_i/d_i is monotone non-increasing as i increases. Then for any A,*

$$\sigma_* A \leqslant \mu_* A \leqslant \mu^* A \leqslant \sigma^* A.$$

Proof. This is a well-known theorem on Nörlund means. A complete discussion can be found in Hardy [8].

1.12. Corollary. *For any A, $\delta_* A \leqslant l_* A \leqslant l^* A \leqslant \delta^* A$.*

2. The properties of a certain multiplicative decomposition. In this section we will always assume that $C = \{c_1 < c_2 < \ldots\}$ is an infinite set of natural numbers for which $c_1 > 1$. Any sequence of integers will be increasing unless otherwise stated.

2.1. Lemma. *If p denotes a prime number, we have the inequalities*

$$\log x < \prod \big((1 - 1/p)^{-1}\colon\ p \leqslant x,\ x \geqslant 2\big) < M \log x,$$

where M is an absolute constant.

Proof. See A. E. Ingham [9] for a proof of this classical theorem.

2.2. Definition. If $x \geqslant 1$, let $P(x)$ be the set of all natural numbers that are composed entirely of primes greater than x.

2.3. Lemma. *If a is any natural number and $x \geqslant 2$, then* $\delta(aP(x)) = (1/a)\prod(1-1/p\colon p \leqslant x)$.

Proof. Note that $E = N - P(x)$ consists of those numbers divisible by a prime not greater than x. Hence, by 1.9, $\delta E = 1-\prod(1-1/p\colon p \leqslant x)$. The result follows at once.

2.4. Definition. Let Γ be the family of all arithmetic functions f which satisfy $f(n) \geqslant g(n)$ for $n = 2, 3, \ldots$, where $g(n)$ is the greatest prime divisor of n.

2.5. Definition. Let C be a set of natural numbers and let f be an arbitrary member of Γ. The *f-primary part of C*, denoted by $A(f; C)$, is defined to be the collection $\{c_i\colon c_i \notin c_jP(f(c_j))$ for any $j\}$. The *f-secondary part of C*, denoted by $B(f, C)$, is defined to be $C-A(f, C)$. If there is no possibility of confusion, the sets just defined will be called A and B, respectively.

The decomposition of C given in 2.5 together with the following representation theorem forms the basis of our method.

2.6. Theorem. *Let c belong to C and f belong to Γ. Then either c belongs to A or c may be uniquely represented as $c = as$, where a belongs to A and s belongs to $P(f(a))$.*

Proof. Suppose that c is an element of B. Then $c = c_is_1$, where s_1 belongs to $P(f(c_i))$ for at least one c_i. Let c_i be the least member of C which satisfies this condition. If c_i does not belong to A, then $c_i = c_js_2$, where s_2 is contained in $P(f(c_j))$; and $c = c_js_2s_1$. Since $f(n) \geqslant g(n)$, s_2s_1 belongs to $P(f(c_j))$, and we have a contradiction. Thus c has at least one representation in the desired form.

To demonstrate the uniqueness of the representation, we prove that if a_i and a_j are distinct members of A, then the sets $a_iP(f(a_i))$ and $a_jP(f(a_j))$ are actually disjoint. Suppose $a_is_1 = a_js_2$ (this number need not belong to C), where s_1 is in $P(f(a_i))$ and s_2 is in $P(f(a_j))$. We may assume without loss of generality that $g(a_i)$ does not exceed $g(a_j)$. Then a_i and s_2 are relatively prime, and hence $a_j = a_is_3$. Since $s_3 \neq 1$, s_3 is contained in $P(f(a_i))$ because s_3 divides s_1. This contradicts the definition of A.

2.7. Examples. (i) Let $C = \{2, 3, 4, \ldots\}$ and $f(n) = g(n)$. We easily see that A consists of all powers of prime numbers. In fact for any C and f, A will always contain the least member of C and any prime powers which happen to lie in C.

(ii) Let C be a sequence for which no member divides another. Certainly $A = C$ for any f.

(iii) Let $C = dN$ for an integer $d > 1$ and $f(n) = n$. It is seen that A contains d together with numbers of the form ds where s is composed of primes not greater than d. However, many other numbers belong to A.

(iv) Let $C = \{2, 3, 4, \ldots\}$ and $f(n) = g(n)+2$. Here A is made up of powers of primes together with numbers $p^\alpha q^\beta$ where p and q are prime twins.

(v) Let $C = \{r+st\colon t = 0, 1, \ldots, (r, s) = 1\}$, $f = g$. In this situation A cannot be described in a simple manner; Dirichlet's theorem at least says that A contains infinitely many primes.

It is obvious that if $f_1(n) \leqslant f_2(n)$ for each n, then $A(f_1, C) \subset A(f_2, C)$ for any C. Questions about the fine structure of A are very difficult, but we can say quite a bit about the density of A under a wide range of conditions.

The next theorem exploits an idea first used by Erdös [5].

2.8. Theorem. *Let f belong to Γ, and let C be arbitrary. Then we have*

$$\sum\{(a\log f(a))^{-1}\colon a \in A\} \leqslant M,$$

where M is the constant in 2.1.

Proof. In the demonstration of 2.6, we proved that if i and j are distinct, then $a_iP(f(a_i))$ and $a_jP(f(a_j))$ are disjoint sets. By 2.3, $\delta[aP(f(a))] = (1/a)\prod(1-1/p\colon p \leqslant f(a))$. Hence by 2.1 this number is greater than $(1/a)(M\log f(a))^{-1}$. Since the sum of the densities of a collection of disjoint sets does not exceed one, we may conclude that for any n,

$$\sum_{i=1}^{n}(a_iM\log f(a_i))^{-1} \leqslant 1.$$

If need be, we allow n to tend to infinity to obtain the result.

2.9. Definition. We define Γ' to be those f in Γ for which there exists a real number $K = K(f)$ such that $f(n) \leqslant n^K$ for each n.

2.10. Theorem. *Let C be arbitrary set of natural numbers. If f belongs to Γ', then $lA = 0$.*

Proof. Assume that A is infinite, and let K satisfy $f(n) \leqslant n^K$. Then since $\log f(n) \leqslant K\log n$, it follows that

$$\sum\{(a\log a)^{-1}\colon a \in A\} \leqslant KM.$$

Next choose k so that $\sum\limits_{i=k+1}^{\infty}(a_i\log a_i)^{-1} < \varepsilon/2$. Now if n is so large that

$$(1/\log n)\sum_{i=1}^{k} 1/a_i < \varepsilon/2,$$

then

$$l_n A = (1/\log n)\sum(1/a_i\colon a_i \leqslant n) \leqslant \varepsilon/2 + \sum_{i=k+1}^{\infty}(a_i \log a_i)^{-1} < \varepsilon.$$

It follows that $lA = 0$.

In many situations we will be able to analyze a larger family in Γ than Γ'; we work in that direction.

2.11. Definition. Let Ω be the family of all arithmetic functions h for which $h(n)\log n$ is positive and increasing (eventually), for $n = 1, 2, \ldots$, and for which $\sum_{n=1}^{\infty}(h(n)n\log n)^{-1}$ is a divergent series.

2.12. Definition and remark. In a natural manner we may associate a density function $\bar{h}$ with each h in Ω. We define

$$\bar{h}_k C = \sum\{(ch(c)\log c)^{-1}\colon c \in C, c \leqslant k\}/S_k$$

where $S_k = \sum_{i=1}^{k}(ih(i)\log i)^{-1}$.

We observe that the hypotheses of 1.11 are satisfied, and hence for any sequence C and function h in Ω,

$$l_* C \leqslant \bar{h}_* C \leqslant \bar{h}^* C \leqslant l^* C.$$

The functions $h(n) = \log\log n$ and $h(n) = (\log\log\log n)(\log\log n)$ are two obvious members of Ω.

2.13. Definition. Let Γ'' be those f in Γ for which $f(n) \leqslant n^{h(n)}$ for some h in Ω.

2.14. Theorem. *Let f belong to Γ'' and let C be an arbitrary set of natural numbers. Then $\bar{h}A = 0$ and $l_* A = 0$, where h and $\bar{h}$ are as in 2.13 and 2.12.*

Proof. Let h in Ω satisfy $f(n) \leqslant n^{h(n)}$. By 2.8, $\sum\big((a\log f(a))^{-1}\colon a \in A\big) \leqslant M$, hence

$$\sum(h(a)\,a\log a)^{-1} \leqslant M.$$

The convergence of the preceding series leads to the result that $\bar{h}A = 0$, where $\bar{h}$ is as defined in 2.12. The proof follows the same line as that of 2.10 and is omitted. From the inequalities in 2.12 we conclude that $l_* A = 0$.

2.15. Lemma. *If $p_1, p_2, \ldots, p_r$ are prime numbers, then $\prod_{i=1}^{r}(1-1/p_i)^{-1} = 1+\sum 1/d$ where d represents the general integer whose prime factors are a subset of those mentioned above.*

Proof. Note that $(1-1/p)^{-1} = 1+1/p+1/p^2+\ldots$ The result follows.

The next theorem, which relates the decomposition to l-summability, is our principal result on the density of $B(f, c)$.

2.16. Theorem. *Let C be a set of natural numbers and let f be any element in Γ. Suppose that $l[C \cap a\,P(f(a))]$ exists for each a in A. Then the sequence of sets $\{C \cap a\,P(f(a))\}$ is l-summable. In other words, lB exists and is equal to $\sum l[C \cap a\,P(f(a))]$.*

Proof. If A is finite, the result is clear by the finite additivity of l. If A is infinite, let a be a fixed member of A, and consider the following series of inequalities:

$$\begin{aligned}&\sum\{1/b\colon b \in C \cap aP(f(a)),\, b \leqslant n\}\\ &\quad \leqslant \sum\{1/as\colon as \in a\,P(f(a)),\, as \leqslant n\} \leqslant (1/a)\sum\{1/s\colon s \in P(f(a)),\, s \leqslant n\}\\ &\quad \leqslant (1/a)\prod\{(1-1/p)^{-1}\colon f(a) < p \leqslant n\} \qquad (\text{by } 2.15)\\ &\quad = \frac{\prod(1-1/p\colon p \leqslant f(a))}{a\prod(1-1/p\colon p \leqslant n)} \leqslant M\log n(a\log f(a))^{-1}.\end{aligned}$$

Thus for any i and n,

$$l_n[C \cap a_i P(f(a_i))] \leqslant M(a_i \log f(a_i))^{-1} = L_i.$$

We know from 2.8 that $\sum L_i \leqslant M^2$. We may now apply 1.7 to conclude that the family of sets is l-summable.

2.17. Corollary. *If $lA = 0$ and if $l[C \cap aP(f(a))]$ exists for each a in A, then $lC = \sum l[C \cap aP(f(a))]$.*

Proof. By 2.16, lB exists and equals the above sum. Since $lA = 0$, $lB = lC$.

The following is an obvious special case.

2.18. Corollary. *If f belongs to Γ' and $l[C \cap aP(f(a))] = 0$ for each a in A, then $lC = 0$.*

2.19. Remark. The above results give us a twofold attack on a sequence C since the convergence of $\sum(a\log f(a))^{-1}$ gives information on A, and the l-summability of the sets $\{aP(f(a))\}$ gives information on B.

3. Some applications of the decomposition.

3.1. Definition. A set of natural numbers C will be called a *multiplicative set* if $C = D(M)$ for a set of natural numbers M which does not contain 1.

3.2. Definition. An infinite set C will be called a *division chain* if c_i divides c_{i+1} for each i.

3.3. THEOREM (Erdös, Behrend). *If C has the property that no member divides another, then $lC = \delta_* C = 0$.*

Proof. Let $f = g$, a member of Γ'. Here $A = C$, and $lC = 0$ by 2.10.

We can easily strengthen 3.3. The following is a simple example of the method mentioned in 2.19.

3.4. THEOREM. *If C has the property that each member of C divides only finitely many members of C, then $lC = 0$.*

Proof. Let $f = g$ so that $lA = 0$. Note that $l[C \cap aP(f(a))] = 0$ for each a in A since the set intersection must be finite. Apply 2.18.

3.5. LEMMA. *If C is a multiplicative set and f is any member of Γ, then $lB = \sum \delta\big(aP(f(A)) \colon a \in A\big)$.*

Proof. Since C is multiplicative, $C \cap aP(f(a)) = aP(f(a))$. Now $l[aP(f(a))] = \delta[aP(f(a))]$, and we may apply 2.16.

3.6. THEOREM (Erdös-Davenport). *Let C be a multiplicative set. Then lC exists and equals $\delta_* C$.*

Proof. Let $f = g$, so that $lA = 0$; lB exists by 3.5. Thus lC exists. We know $\delta_* C \leqslant lC = \sum \delta[aP(f(a))]$. Also, $\delta_* C \geqslant \delta_* B \geqslant \sum \delta[aP(f(a))]$. The last inequality follows from 1.5.

3.7. COROLLARY. *If C is a multiplicative set and f is any member of Γ, then lA exists.*

Proof. We know that lB and lC exist by 3.5 and 3.6, respectively. Hence $lA = lC - lB$.

3.8. Remark. Another proof of 3.6 can be made along the following line: Any multiplicative set can be written as a disjoint union of arithmetic progressions so that the collection of least members of these progressions has the property that any one divides at most finitely many others. This collection has zero logarithmic density by 3.3. It is an easy consequence of 1.8 that this union possesses the desired density.

3.9. THEOREM. *Let C be a multiplicative set, and let f belong to Γ''. Then $lA = 0$.*

Proof. By 2.14 we know that $l_* A = 0$. Since C is multiplicative, lA exists and therefore must be zero.

We now prove three theorems about division chains. The first two sharpen the result of Erdös and Davenport mentioned in the introduction; the third needs a stronger hypothesis, which certainly would be satisfied by a set with positive natural density.

3.10. THEOREM. *Let $K_1, K_2, \ldots$ be any sequence of positive numbers. If C does not possess zero logarithmic density, then C contains a division chain of the form $q_1, q_1 q_2, q_1 q_2 q_3, \ldots$, where q_{i+1} is composed entirely of primes greater than $(q_1 q_2 \ldots q_i)^{K_i}$.*

Proof. For each i, $f_i(n) = n^{K_i}$ belongs to Γ'. Now there exists a_1 in $A(f_1, C)$ so that $l^* C_1 \neq 0$, where $C_1 = C \cap a_1 P(f(a_1))$. Otherwise, $lC = 0$ by 2.18. Likewise, there exists a_2 in $A(f_2, C_1)$ so that $l^* C_2 \neq 0$, where $C_2 = C_1 \cap a_2 P(f_2(a_2))$. Continuing inductively, we construct the sequence $a_1, a_2, \ldots$ It is seen from the construction that $a_{i+1} = a_i s_i$, where s_i belongs to $P(f_i(a_i))$. Set $q_1 = a_1$ and $q_{i+1} = s_i$ for i greater than one.

3.11. THEOREM. *Suppose that $l^* C \neq 0$, and that $\{f_i\}$ is an arbitrary sequence of functions in Γ''. Then C contains a division chain of the form $r_1 q_1, r_2 q_2, \ldots$, where q_i belongs to $P(f_i(r_i))$ and $r_i q_i$ divides r_{i+1} for each i.*

Proof. If $M = D(E)$ is any multiplicative set, then we know that $lA(f_i, M) = 0$ for each i, by 3.9. Thus there exists a_1 in $A(f_1, D(C))$ such that $l^* C_1 \neq 0$, where $C_1 = C \cap a_1 P(f_1(a_1))$, or else $lC = 0$ by 1.10. Likewise, there exists a_2 in $A(f_2, D(C_1))$ such that $l^* C_2 = 0$ where $C_2 = C_1 \cap a_2 P(f_2(a_2))$. Inductively we form the sequence $a_1, a_2, \ldots$ By our method of construction, $a_{i+1} = a_i s_i n_i$ where $a_i s_i$ belongs to C and s_i belongs to $P(f_i(a_i))$, and n_i is some positive integer. If we put $r_i = a_i$ and $q_i = s_i$, we obtain a division chain in C that is of the desired form.

3.12. THEOREM. *If C is a set of natural numbers for which $l_* C$ is positive, and h is an arbitrary element of Ω, then C contains a division chain of the form $q_1, q_1 q_2, q_1 q_2 q_3, \ldots$, where q_{i+1} is composed of primes greater than $(q_1 q_2 \ldots q_i)$ raised to the power $h(q_1 q_2 \ldots q_i)$.*

Proof. Let $f(n) = n^{h(n)}$, and consider $A(f, C)$. In the proof of 2.14 we showed that $\bar{h} A = 0$, where $\bar{h}$ is the density function associated with h (see 2.12). Since the family of sets $\{aP(f(a)) \colon a \in A\}$ is l-summable, it must be $\bar{h}$-summable. Thus there is an a_1 in A so that $\bar{h}^* C_1 \neq 0$, where $C_1 = C \cap a_1 P(f(a_1))$. Otherwise we would conclude that $\bar{h}(C) = 0$ by 1.10; this cannot be the situation since $0 < l_* C \leqslant \bar{h}_* C$. We may now proceed by our usual inductive construction technique (repeated use of 2.14 and 1.10) to form the division chain $a_1, a_2, a_3, \ldots$, where $a_{i+1} = a_i s_i$ and s_i belongs to $P(f(a_i))$. The proper form is obtained by setting $a_1 = q_1$ and $q_{i+1} = s_i$ for $i = 1, 2, \ldots$

One might wonder whether the conclusion of 3.12 would follow from the weaker assumption that $l^* C > 0$. The following theorem shows that 3.10 is a best possible result.

3.13. THEOREM. *Let $\Psi(n)$ be any arithmetic function which tends to infinity with n. Then there is a sequence of integers with positive upper logarithmic density which contains no division chain of the form $d_1 < d_2 < \ldots$ where d_{i+1}/d_i is composed of primes greater than $d_i^{\Psi(d_i)}$ for each i.*

Proof. Let $\varepsilon > 0$ be fixed, and consider an integer x in the interval $(n, n^{1+\varepsilon})$. The density of integers xs, where s is composed of primes greater than $x^{\Psi(x)}$, is less than $M[x\Psi(x)\log x]^{-1}$, where M is as in 2.1. Thus if we let x range over $(n, n^{1+\varepsilon})$, the density of all such multiples of x is less than

$$2M\int_{n}^{n^{1+\varepsilon}} [x\Psi(x)\log x]^{-1}dx.$$

Since $\Psi(n)$ becomes large with n, the integral tends to zero as n becomes large.

Let $\delta > 0$ be fixed and choose the positive numbers $\varrho_1, \varrho_2, \dots$ so that $2\sum \varrho_i < \delta$. For each i, choose n_i, so that

(1) the density of integers of the form xs, where x belongs to $(n_i, n_i^{1+\varepsilon})$ and s is composed of primes greater than $x^{\Psi(x)}$, is less than ϱ_i;

(2) the density of integers of the form $x's'$ in $(n_i, n_i^{1+\varepsilon})$, where x' belongs to $(n_j, n_j^{1+\varepsilon})$ for some $j < i$ and s' is composed of primes greater than $x^{\Psi(x')}$, is less than $2\sum(\varrho_j\colon j < i)$.

If we form a sequence by taking for each i those members of $(n_i, n_i^{1+\varepsilon})$ which are not of the form $x's'$ described above, then the sequence does not have a division chain of the appropriate form. The upper logarithmic density of the sequence is greater than $[\varepsilon/(1+\varepsilon)]-\delta$, a number which can be made as close to one as is desired.

However, we note that the methods used previously would produce a division chain for which d_{i+1}/d_i is composed of primes greater than $d_i^{\log\log d_i}$ for any sequence $c_1 < c_2 < \dots$ where

$$\sum\{(c_i\log c_i)^{-1}\colon c_i \leqslant x\} \geqslant K\log\log x \ (i.o.)$$

for some positive K.

3.14. Definition. If $C = \{2, 3, 4, \dots\}$ and f belongs to Γ, we will denote $A(f, C)$ and $B(f, C)$ by $A(f)$ and $B(f)$, respectively. We call $A(f)$ the *f-primitive integers.*

It was noted in 2.7 that the g-primitive integers are the powers of prime integers. The next few results show that in certain respects, primitive integers are generalizations of prime (or prime power) integers.

3.15. Theorem. *Let f be any member of Γ and let n be greater than one; then n possesses a unique factorization $n = a_1 a_2 \dots a_r$ where each a_i is an f-primitive integer and a_{i+1} belongs to $P(f(a_i))$.*

Proof. Suppose that n does not belong to $A(f)$. Then $n = a_1 s_1$ where s_1 belongs to $P(f(a_1))$. If s_1 is not in $A(f)$, it may be factored as $s_1 = a_2 s_2$ with s_2 in $P(f(a_2))$. This process must terminate with some $S_{r-1} = a_r$. Thus n has at least one factorization in the disired form.

If we have two such factorizations $n = a_1 a_2 \dots a_r = a_1' a_2' \dots a_t'$, then $a_1 s = a_1' s'$ where s is in $P(f(a_1))$ and s' is in $P(f(a_1'))$. Theorem 2.6

demands that $a_1 = a_1'$. We may proceed inductively to show that $a_i = a_i'$ for $i = 1, \dots, r = t$.

3.16. Definition. The canonical factorization of n given by 3.14 will be called the *f-factorization of n.*

3.17. Theorem. *If f belongs to Γ and $l(A(f)) = 0$, then $\delta(A(f)) = 0$.*

Proof. Since $l(B(f)) = \sum\delta\{aP(f(a))\colon a\in A(f)\} = 1$, we conclude from 1.6 that the sets $\{aP(f(a))\}$ are δ-summable. Hence $\delta(B(f)) = 1$ and $\delta(A(f)) = 0$.

3.18. Theorem. *Let f be a member of Γ for which $\delta(A(f)) = 0$. If E_r is the set of all positive integers whose f-factorization has at most r factors, then $\delta(E_r) = 0$.*

Proof. Suppose that the assertion is true when $r = k$. Note that $F = E_{k+1}-E_k$ is the collection of all numbers that have precisely $k+1$ factors in their f-factorization. Now observe that for any a in $A(f)$, $F \cap aP(f(a))$ is of the form aG, where G is a subset of E_k. Hence $\delta(aG) = 0$. Thus F is contained in $\bigcup\{aP(f(a))\colon a\in A(f)\}$, the union of a δ-summable family of sets. Since F intersects each member of this family in a set of zero natural density, $\delta(F) = 0$ by 1.10. Hence $\delta(E_{k+1}) = 0$.

The next theorem concerns the irregularity of the prime factors of almost all integers. Erdös [7] has obtained a related result.

3.19. Theorem. *Let r be an arbitrary positive integer, and let $f(n) = g(n)^{h(n)}$ where h belongs to Ω. Except for a set of zero natural density, each positive integer n has the following property: If $p_1, p_2, \dots, p_t$ are the prime factors of n in order of increasing size, then p_{i+1} exceeds $f(p_i)$ for at least r values of i.*

Proof. Consider $C_r = C-E_r$, where E_r is as in 3.17. By 3.9, $l(A(f)) = 0$; hence $\delta(A(f)) = 0$ by 3.16. Therefore $\delta(E_r) = 0$ by 3.17. If n belongs to C_r, then n possesses an f-factorization $n = a_1 a_2 \dots a_s$, $s > r$. If p_j is the largest prime divisor of a_i and p_{j+1} is the smallest prime divisor of a_{i+1}, we see that p_{j+1} is contained in $P(f(p_j))$.

The final result shows that 3.19 does not hold for functions which grow faster than those in Γ''.

3.20. Theorem. *Suppose $f(n) = n^{h(n)}$ where $\sum[nh(n)\log n]^{-1}$ converges. Then $l(A(f)) > 0$.*

Proof. Suppose $lA = 0$. Then the ideas of 3.18 show that $l(\bigcup_{k=1}^{n} A_k) = 0$ where A_k consists of those integers with precisely k factors in their f-factorization. If I_n is those integers with at least n factors in their f-factorization, $A(f, I_n) = A_n$ and

$$l(B(f, I_n)) = \sum\Big\{1/a\prod\big(1-1/p\colon p \leqslant f(a)\big)\colon a\in A_n\Big\} = 1 \quad \text{for each } n.$$

However, as n becomes large, the least element in A_n becomes large; and the sum, which is dominated by $M\sum\{[ah(a)\log a]^{-1}\colon a\in A_n\}$, tends to zero. This contradiction gives the result. Professor Erdös has pointed out that it is possible to prove that $\delta(A(f))$ exists.

In conclusion, I wish to express my deep appreciation to my advisor, Professor Robert Zink. Also, I wish to thank Professor Paul Erdös for a number of helpful discussions.

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О рациональных точках некоторых кривых высшего рода

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Способ нахождения точек алгебраической кривой рода $g>1$, рациональных над заданным полем K конечной степени, неизвестен. Существует лишь предположение, что такая кривая имеет в K только конечное число точек.

Значительно большие результаты получены при исследовании кривых первого рода. Морделл [1] доказал, что совокупность точек кривой первого рода из абсолютной области рациональности $R(1)$ образует коммутативную группу с конечным числом образующих. Таким образом, существует такое конечное число рациональных точек $P_1, P_2, \ldots, P_r$, что любая рациональная точка P представима в виде

$$P = n_1P_1+n_2P_2+\ldots+n_rP_r$$

с некоторыми целыми $n_1, n_2, \ldots, n_r$. Позднее доказательство Морделла было несколько упрощено и значительно обобщено Вейлем [2].

В настоящей работе мы будем рассматривать кривые

$$x^4+y^4 = A \tag{1}$$

и

$$x^6+y^6 = A \tag{2}$$

при определённых ограничениях, накладываемых на ранги кривых первого рода:

$$u^4-A = v^2 \tag{3}$$

и

$$u^3+1 = Av^2, \quad v^2+1 = Au^3, \quad u^3+v^2 = A. \tag{4}$$

В частности, мы установим, что если ранг одной из кривых (4) над полем $R(\sqrt{-3})$ не превышает 2, то кривая (2) не имеет в этом поле точек, за исключением случаев: $A=1$, $\{x,y\}=\{\varepsilon_1, 0\}, \{0, \varepsilon_2\}$; $A=2$, $\{x,y\}=\{\varepsilon_1, \varepsilon_2\}$, $\varepsilon_1^6=\varepsilon_2^6=1$. Аналогичный результат будет также получен и для кривой (1), рассматриваемой над полем $R(\sqrt{-1})$.